CS120: Intro. to Algorithms and their Limitations Anshu & Hesterberg

Lecture 12: Independent Sets and Interval Scheduling

Harvard SEAS - Fall 2023 *2023-10-17*

1 Announcements

Recommended Reading: CLRS Sec 16.1–16.2

- Sender-Receiver Exercise today!

2 Loose ends of Lec 11 : 2-colorable graphs

Theorem 2.1. *If G is a connected 2-colorable graph, then* `BFSColoring` *(G) will color G using 2 colors.*

We will cover two proofs of the theorem.

Proof 1. Let f^* be a 2-coloring of G. We may assume that $f^*(v_0) = 0$ without loss of generality. Let $S_0 := \{v \in V : f^*(v) = 0\}$ and $S_1 := \{v \in V : f^*(v) = 1\}$. Note that for any two vertices $v, v' \in S_0$, there is no edge between v, v' (similarly for S_1).

Now consider an execution of `BFSColoring(`G`)` starting at v_0 and let the discovered vertices be $v_0, v_1, v_2, \dots v_{n-1}$. Note that these vertices are discovered using "frontier vertex set" $F_0, F_1, \dots$. All the vertices in F_k are neighbours to some vertex in F_{k-1}. We have $F_0 = \{v_0\}$ and vertices are consecutively within some F_k. In order words, $v_1, v_2, \dots v_\ell \in F_1$ for some ℓ; $v_{\ell+1}, v_{\ell+2}, \dots v_m \in F_2$ for some $m > \ell$. Thus, $v_1, v_2, \dots v_\ell \subset S_1$, $v_{\ell+1}, v_{\ell+2}, \dots v_m \subset S_0$ and so on. `BFSColoring(`G`)` assigns the smallest color not assigned to any prior neighbor (according to the ordering above). Thus, all of $v_1, v_2, \dots v_\ell$ get assigned 1, all of $v_{\ell+1}, v_{\ell+2}, \dots v_m$ get assigned 0 and so on. This is precisely the desired 2-Coloring. □

Proof 2. Let f^* be a 2-coloring of G. We may assume that $f^*(v_0) = 0$ without loss of generality. Let f be the coloring of G found by `BFSColoring(`G`)`. We argue by (strong) induction on i that $f(v_i) = f^*(v_i)$ for $i = 0, \dots, n-1$.

For $i = 0$, we observe that `BFSColoring(`G`)` sets $f(v_0) = 0$. Now for $i > 0$, we will argue that f^* satisfies the same rule used to construct f, namely:

$$f^*(v_i) = \min \{c \in \mathbb{N} : c \neq f^*(v_j) \;\; \forall j < i \text{ s.t. } \{v_i, v_j\} \in E\}. \tag{1}$$

In other words, the value of f^* at v_i is "forced" by its values at the previously assigned vertices v_j. Since f^* is a valid 2-coloring, the value $c = f^*(v_i)$ satisfies the condition $c \neq f^*(v_j)$ for all $j < i$ such that $\{v_i, v_j\} \in E$ automatically holds. If $f^*(v_i) = 0$, then it is certainly the minimum value of c satisfying this condition. If $f^*(v_i) = 1$, we note that that by the definition of BFS, there is a previous vertex v_j (with $j < i$) with an edge to v_i. Since f^* is a valid 2-coloring, we must have $f^*(v_j) = 0$. So $c = 0$, does not satisfy the condition in Equation (1), and hence $c = 1$ must be the minimum value satisfying the condition.

By the definition of `BFSColoring`(G), we have

$$f(v_i) = \min \{c \in \mathbb{N} : c \neq f(v_j) \ \ \forall j < i \text{ s.t. } \{v_i, v_j\} \in E\} \tag{2}$$

By our (strong) induction hypothesis, the right-hand sides of (1) and (2) are equal, and thus $f(v_i) = f^*(v_i)$. ☐

Corollary 2.2. *Graph 2-Coloring can be solved in time $O(n + m)$.*

Proof. We can partition G into connected components in time $O(n+m)$. Then, for each connected component we can use BFSColoring on each component, which takes total time $O(n + m)$. ☐

3 Definitions

In the Sender-Receiver Exercise, you've seen the definition of independent sets, which are closely related to graph colorings:

Definition 3.1. Let $G = (V, E)$ be a graph. An *independent set* in G is a subset $S \subseteq V$ such that there are no edges entirely in S. That is, $\{u, v\} \in E$ implies that $u \notin S$ or $v \notin S$.

A proper k-coloring of a graph G is equivalent to a partition of V into k independent sets (each color class should be an independent set).

When we have a graph $G = (V, E)$ representing conflicts, instead of partitioning V into a small number of conflict-free subsets (as coloring would), it is sometimes useful to instead find a single, large conflict-free subset. This gives rise to the following computational problem:

Input	: A graph $G = (V, E)$
Output	: An independent set $S \subseteq V$ in G of maximum size

Computational Problem Independent Set

Example: Throwing a big party where everyone will get along.

Like with graph coloring, we can try a greedy algorithm for Independent Set:

```
GreedyIndSet(G)
Input  : A graph G = (V, E)
Output : A "large" independent set in G
Choose an ordering v_0, v_1, v_2, ..., v_{n-1} of V;
S = ∅;
foreach i = 0 to n − 1 do
    if ∀j < i s.t. {v_i, v_j} ∈ E we have v_j ∉ S then S = S ∪ {v_i};
return S
```

And, similarly to coloring, we can only prove fairly weak bounds on the performance of the greedy algorithm in general:

Theorem 3.2. *For every graph G with n vertices and m edges,* `GreedyIndSet`(G) *can be implemented in time $O(n+m)$ and outputs an independent set of size at least $n/(d_{max}+1)$, where d_{max} is the maximum vertex degree in G.*

Proof.
Omitted (and possibly covered in section). ☐

However, when there is more structure in the conflict graph, a careful ordering for the greedy algorithm can yield an optimal solution. An example of such structure comes from the Interval Scheduling problem we saw in the first lecture:

Input	: A collection of intervals $[a_0, b_0], \ldots, [a_{n-1}, b_{n-1}]$, where each $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$
Output	: YES if the intervals are disjoint (for all $i \neq j$, $[a_i, b_i] \cap [a_j, b_j] = \emptyset$) NO otherwise

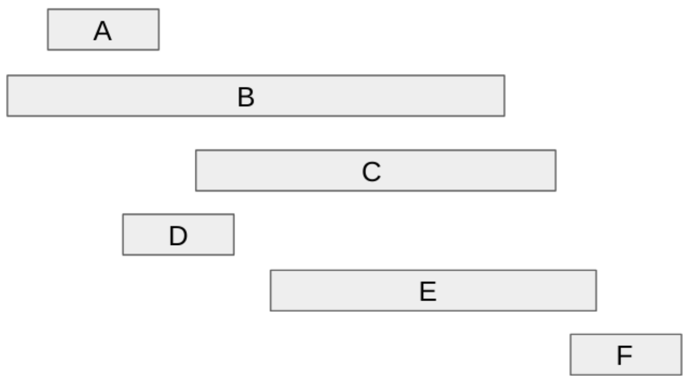
Computational Problem IntervalScheduling-Decision

We saw that we could solve this problem in time $O(n \log n)$ by reduction to Sorting. However, if the answer is NO, we might be satisfied by trying to schedule *as many* intervals *as possible*:

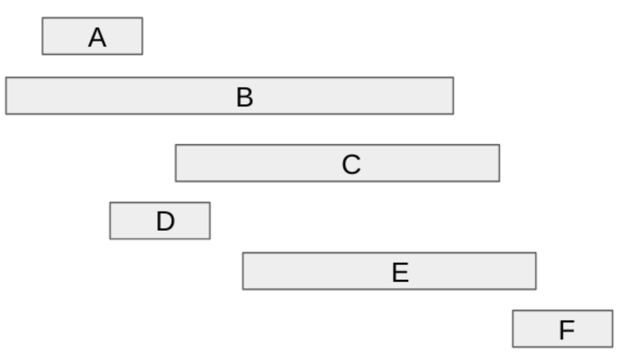
Input	: A collection of intervals $[a_0, b_0], \ldots, [a_{n-1}, b_{n-1}]$, where each $a_i, b_i \in \mathbb{Q}$ and $a_i \leq b_i$
Output	: A maximum-size subset $S \subseteq [n]$ such that $\forall i \neq j \in S$, $[a_i, b_i] \cap [a_j, b_j] = \emptyset$.

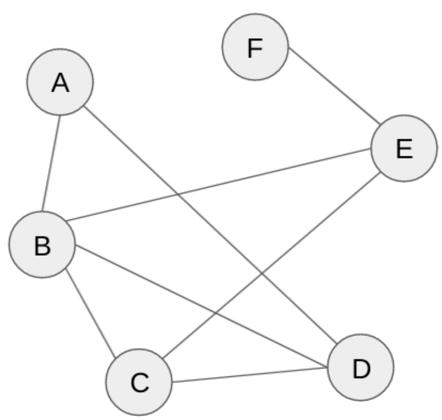
Computational Problem IntervalScheduling-Optimization

Example:



Q: How can we model IntervalScheduling-Optimization as an Independent Set problem?





A: We represent each interval as a vertex, and we place an edge between two vertices (i.e. intervals) if they conflict. Then an independent set is exactly a set of intervals which have no conflicts, so maximizing the size of this is equivalent to finding the largest set of conflict-free intervals.

With this graph-theoretic modelling, we can instantiate `GreedyIndSet()` for IntervalScheduling-Optimization:

`GreedyIntervalScheduling`(x)
Input : A list x of n intervals $[a, b]$, with $a, b \in \mathbb{Q}$
Output : A "large" subset of the input intervals that are disjoint from each other
Choose an ordering of the input intervals $[a_0, b_0], [a_1, b_1], \ldots, [a_{n-1}, b_{n-1}]$;
$S = \emptyset$;
foreach $i = 0$ *to* $n - 1$ **do**
  if $\forall j < i$ *s.t.* $j \in S$ *we have* $[a_j, b_j] \cap [a_i, b_i] = \emptyset$ **then** $S = S \cup \{i\}$;
return S

Q: What ordering of the input intervals should we use?

A: Want to first assign the intervals with the earliest *end* time.

Theorem 3.3. *If the input intervals are sorted by increasing order of end time b_i, then we have that* `GreedyIntervalScheduling`(x) *will find an optimal solution to IntervalScheduling-Optimization, and can be implemented in time $O(n \log n)$.*

Proof.
Intuitively, for any interval scheduling problem (black in Figure 1), we can modify any solution $(i_0^*, i_1^*, \ldots, i_{\ell-1}^*)$ to it (gray boxes) into the greedy solution (i_0, i_1, ..., i_{k-1} (white) by "smushing it left", replacing the first j intervals of our solution with the first j intervals of the greedy solution to get a valid solution $(i_0, i_1, \ldots, i_{j-1}, i_j^*, \ldots, i_{\ell-1}^*$. Also, $k \geq \ell$: If not, then Greedy would pick one more interval, since i_ℓ^* would be valid.

Formally, let $S^* = \{i_0^* \leq i_1^* \leq \ldots \leq i_{k^*-1}^*\}$ be an optimal solution to Interval Scheduling (where we say that $i < i'$ for intervals i and i' if i ends before i' begins). Then let $S = \{i_0 \leq i_1 \leq \ldots \leq i_{k-1}\}$ be the solution found by the greedy algorithm. Recall that b_{i_j} is the endtime of interval i_j (and above we sort both solutions on end time).

Claim 3.4 (greedy stays ahead)**.** *For all $j \in \{0, \ldots, k^* - 1\}$, we have:*

1. *$j < k$, i.e. the Greedy Algorithm schedules at least $j + 1$ intervals, and*
2. *$b_{i_j} \leq b_{i_j^*}$, i.e. the j'th interval scheduled by the Greedy algorithm ends no later than the j'th interval scheduled by the optimal solution.*

Proof. For the $j = 0$ base case, since greedy always picks the absolute first interval by end time, the claim follows. Then assuming it holds up to j, we have $b_{i_j} \leq b_{i_j^*} < a_{i_{j+1}^*}$. The second inequality follows since the next interval in the optimal solution must start after the prior interval ending. But this means that interval i_{j+1}^* is *available* to the greedy algorithm after it has picked interval i_j, and since we would only not pick it if there is an available interval ending even earlier, we establish the claim for $j + 1$ and conclude. □

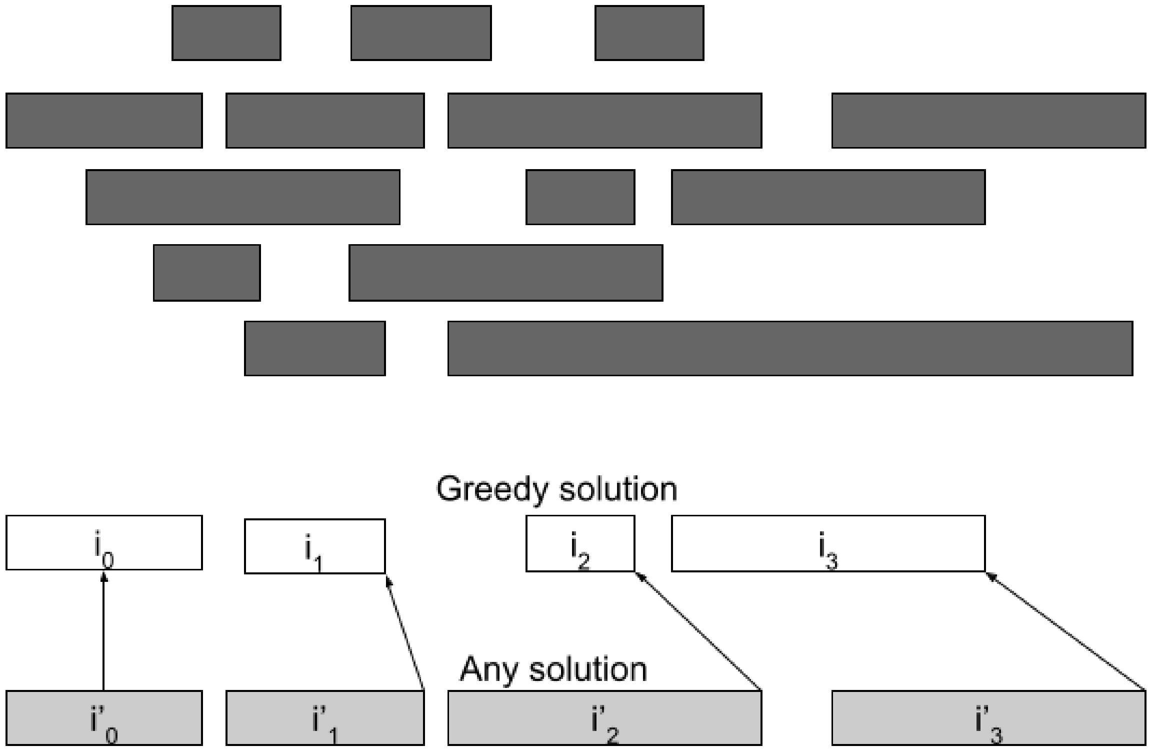


Figure 1: Transforming any interval scheduling solution into the greedy one.

Then from this claim we establish that $k^* - 1 < k$ and so the Greedy Algorithm schedules $k \geq k^*$ intervals. Since k^* is the optimal (maximum) number of intervals that can be scheduled, we conclude that $k = k^*$ and the Greedy Algorithm schedules an optimal number of intervals.

For the runtime, we can order the intervals by increasing end time by sorting in time $O(n \log n)$. Next we observe that in Line 5 we only need to check that the start time a_i of the current interval is later than the end time of b_j of the most recently scheduled interval (since all others have earlier end time), so we can carry out this check in constant time. Thus the loop can be implemented in time $O(n)$, for a total runtime of $O(n \log n) + O(n) = O(n \log n)$. $\square$