

Sender–Receiver Exercise 1: Reading for Receivers

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The goals of this exercise are:

- to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them
- to practice reasoning about the running time of algorithms
- to see how the choice of a model of computation can affect the computational complexity of a problem

In the previous class, we have seen that there exists a sorting algorithm (namely, **MergeSort**) whose worst-case running time is $O(n \log n)$. In fact, the (worst-case) computational complexity of sorting by comparison-based algorithms is $\Theta(n \log n)$. That is, *every* (correct) comparison-based sorting algorithm (one that operates just by comparing keys to each other) has worst-case running time $\Omega(n \log n)$. This holds even when the keys are drawn from the universe $[n] = \{0, 1, \dots, n-1\}$.

In our first sender-receiver exercise, you will see that for keys drawn from the universe $[n]$, it is actually possible to sort asymptotically faster — in time $O(n)$! How is this possible in light of the $\Omega(n \log n)$ lower bound? Well, the algorithm will not be a comparison-based one; it will directly access and manipulate the keys themselves (rather than just comparing them to each other).

1 The result

Let’s precisely define the computational problem of sorting on a finite universe.

Input: A universe size $U \in \mathbb{N}$ and an array A of key-value pairs $((K_0, V_0), \dots, (K_{n-1}, V_{n-1}))$, where each key $K_i \in [U]$

Output: An array A' of key-value pairs $((K'_0, V'_0), \dots, (K'_{n-1}, V'_{n-1}))$ that is a valid sorting of A .

Computational Problem SORTINGONFINITEUNIVERSE()

We will prove:

Theorem .1. *There is an algorithm for SORTINGONFINITEUNIVERSE on arrays of size n and key-universe size U with (worst-case) running time $O(n + U)$. One such algorithm is called **SingletonBucketSort**.*

Since we have not yet precisely defined our computational model or what constitutes a “basic operation,” this theorem and its proof are still somewhat informal, but in a few weeks we will have the language to make it all completely precise. As you will see in Problem Set 2, it is possible to improve the dependence on U from linear to logarithmic with a more involved algorithm.

Note that this is a case where we measure runtime as a function of two size parameters n and U . For convenience, we include the universe size U explicitly in the input to the `SORTINGONFINITEUNIVERSE` problem, but it is also possible to omit U , allow the keys to be arbitrary natural numbers, and modify the algorithm to start by calculating $U = \max_i K_i$.

2 The Proof

1. Algorithm:

2. Correctness:

3. Runtime:

3 Food for Thought

If you and your partner have extra time after completing the exercise, here is an additional question to think about.

When there are multiple key-value pairs in A with the same key K , a direct implementation of **SingletonBucketSort** will produce an output in which those key-value pairs appear in the *opposite order* from how they appear in A . (Why?) As you will see in Problem 1, it is often useful to have *stable* sorting algorithms, which are guaranteed to maintain the input ordering for repeated keys. (See Problem 1.3 in the textbook for the stability of other sorting algorithms.)

Two approaches to make **SingletonBucketSort** stable are the following:

1. Append each new key value pair (K_i, V_i) as the *tail* of the linked list $C[K_i]$ rather than inserting it as the head, or
2. When constructing the array A' , reverse the order of each linked list $C[K]$ before adding it to A' .

Which of these two approaches will maintain runtime $O(n + U)$?