CS1200: Intro. to Algorithms and their Limitations Prof. Salil Vadhan

Sender–Receiver Exercise 6: Reading for Receivers

Harvard SEAS - Fall 2025 *2025-11-13*

1 Sender–Receiver Exercise: Subset Sum is $\mathsf{NP}_{\mathsf{search}}$-complete

The goals of this exercise are:

1. to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them,
2. to reinforce the definition of $\mathsf{NP}_{\mathsf{search}}$ and practice $\mathsf{NP}_{\mathsf{search}}$-completeness proofs

To prepare for this exercise as a receiver, you should try to understand the theorem statement and definition in Section 2 below, and review the relevant material from lectures. Your partner sender will communicate the proof of Theorem 2.2.

2 The Result

So far, we have seen examples of $\mathsf{NP}_{\mathsf{search}}$-complete problems in logic (e.g. SAT) and graph theory (e.g. Independent Set). Here you will see an example of a numerical $\mathsf{NP}_{\mathsf{search}}$-complete problem.

Input: Natural numbers $v_0, v_1, \ldots, v_{n-1}, t$

Output: A subset $S \subseteq [n]$ such that $\sum_{i \in S} v_i = t$, if such a subset S exists

Computational Problem Subset Sum

Example Subset Sum **instance:** Given the input $(v_0, v_1, \ldots, v_5, t) = (1, 5, 3, 8, 6, 2, 7)$, a solution would be the subset $S = \{0, 4\}$ since $v_0 + v_4 = 1 + 6 = 7 = t$. For readability, we will sometimes describe a solution by a set of input numbers instead of a subset of indices (e.g. $\{v_0, v_4\}$ instead of $\{0, 4\}$).

Theorem 2.1. Subset Sum *is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

Actually, we will focus on the *vector* version of the problem, where we replace the natural numbers v_i with vectors having entries in $\{0, 1\}$ and t with a vector of natural numbers.

Input: Vectors $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1} \in \{0, 1\}^d$, $\vec{t} \in \mathbb{N}^d$

Output: A subset $S \subseteq [n]$ such that $\sum_{i \in S} \vec{v}_i = \vec{t}$, if such a subset S exists

Computational Problem VectorSubsetSum()

We will use the notation $\vec{v}[j]$ to denote the j^{th} entry of vector $\vec{v}$, so the condition $\sum_{i \in S} \vec{v}_i = \vec{t}$ means that for every $j = 0, 1, \ldots, d-1$, we have $\sum_{i \in S} \vec{v}_i[j] = \vec{t}[j]$.

Example VECTOR SUBSET SUM **instance:** Consider the 3 vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$ and target $\vec{t}$ written in the table below:

	0	1	2	3
$\vec{v}_0$	1	0	0	1
$\vec{v}_1$	1	0	0	1
$\vec{v}_2$	0	1	0	0
$\vec{v}_3$	0	1	1	0
$\vec{t}$	2	1	1	2

A solution is $S = \{0, 1, 3\}$ since $\vec{v}_0 + \vec{v}_1 + \vec{v}_3 = \vec{t}$. Similarly to ordinary SUBSET SUM, we will sometimes describe a solution by a set of input vectprs instead of a subset of indices (e.g. $\{\vec{v}_0, \vec{v}_1, \vec{v}_3\}$ instead of $\{0, 1, 3\}$).

Theorem 2.2. VECTOR SUBSET SUM *is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

How to derive Theorem 2.1 from Theorem 2.2 will be shown in section. In today's SRE, we will prove Theorem 2.2.

3 Proof of Theorem 2.2

VECTOR SUBSET SUM **is in** $\mathsf{NP}_{\mathsf{search}}$**:**

Reduction to show VECTOR SUBSET SUM **is** $\mathsf{NP}_{\mathsf{search}}$**-hard, and example:**

Analysis of the reduction: