

Planning in MDPs

Sham Kakade and Kianté Brantley
CS 2824: Foundations of Reinforcement Learning

Announcements

- The first **reading assignment** is out, and it is due on Feb 4th at 23:59
- **Office Hours:**
 - Tuesday 3-4 PM (Lukas)
 - Wednesday 1-2 PM (Jay)
 - Thursday 2:15-3:15 PM (Alex)

Recap: Value iteration

$$Q^{t+1} = \mathcal{T}Q^t$$

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Theorem: $V^{\pi^t}(s) \geq V^{\star}(s) - \frac{2\gamma^t}{1 - \gamma} \|Q^0 - Q^{\star}\|_{\infty} \forall s \in S$

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Q: when will π^t be the optimal policy?

Outline

1. Policy Iteration
2. Computation complexity of VI and PI
3. Linear Programming formulation

Policy Iteration Algorithm:

1. Initialization: $\pi^0 : S \mapsto A$

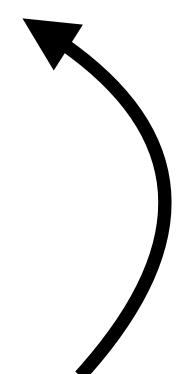
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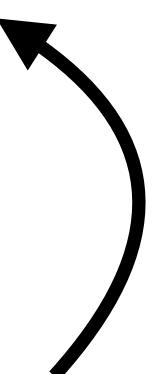
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Policy Iteration Algorithm:

Closed-form for PE

(see 1.1.3 in Monograph)

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Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg \max_a Q^{\pi^t}(s, a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s, a) \geq Q^{\pi^t}(s, a), \forall s, a$

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$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$

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Analysis of Policy Iteration

Q: what happens when π^{t+1} and π^t are exactly the same?

Show that π^t is an optimal policy π^*

Q: does this imply that the algorithm will terminate?

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2. Computation complexity of VI and PI
3. Linear Programming formulation

Computation complexity of VI and PI

Given an MDP $\mathcal{M} = (S, A, P, r, \gamma)$ can we **exactly** compute Q^\star (or find π^\star) in time polynomial wrt $S, A, 1/(1 - \gamma)$?

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What about poly(S, A) algs?

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The primal linear programming

Recall the Bellman consistency:

$$V(s) = \max_a \left\{ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} [V(s)] \right\}, \forall s$$

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$$\min \sum_s \mu(s) V(s)$$

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Proof of Bellman Optimality

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The primal linear programming

$$\min \sum_s \mu(s) V(s)$$

$$\text{s.t. } V(s) \geq \max_a [r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} V(s')] \quad \forall s, a \in S \times A$$

Convert the constraint to linear

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$$\text{s.t. } V(s) \geq r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} V(s') \quad \forall s, a \in S \times A$$

LP Runtime

[Ye, '05]: there is an interior point algorithm (CIPA)
which is (“nearly”) **strongly polynomial**, i.e., no poly dependence on $1/(1 - \gamma)$

$$S^4 A^4 \ln \left(\frac{S}{1 - \gamma} \right)$$

What about the Dual LP?

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- Let us now consider the dual LP.
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 - In some cases, it also provides a reasonable algorithmic approach
- Let us start by understanding the dual variables

State action occupancy measure

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$$V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a)$$

A Bellman equation like property for $d_{s_0}^\pi(s, a)$

$$\sum_a d_\mu^\pi(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{\bar{s}, \bar{a}} P(s \mid \bar{s}, \bar{a}) d_\mu^\pi(\bar{s}, \bar{a})$$

Proof:

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$$d_\mu^\pi(s) = (1 - \gamma) \mu(s) + \gamma \sum_{s'} P(s \mid s') d_\mu^\pi(s')$$

The “State-Action” Polytope

- Let us define the **state-action polytope K** as follows:

$$K_\mu := \left\{ d \mid d \geq 0 \text{ and} \sum_a d(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{s', a'} P(s \mid s', a')d(s', a') \right\}$$

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- This set precisely characterizes all state-action visitation distributions:
Lemma: $d \in K_\mu$ if and only if there exists a (possibly randomized) policy π
s.t. $d_\mu^\pi = d$

The Dual LP

$$\begin{aligned} \max \quad & \sum_{s,a} d(s,a)r(s,a) \\ \text{s.t.} \quad & d \in K_\mu \end{aligned}$$

- One can verify that this is the dual of the primal LP.

Summary

Notations: Value / Q functions, state-action occupant measures, Bellman equation / optimality

Planning algorithms: VI, PI, LP (primal and dual)