

# Planning in MDPs

**Sham Kakade and Kianté Brantley**

**CS 2824: Foundations of Reinforcement Learning**

# Announcements

- The first **reading assignment** is out, and it is due on Feb 4th at 23:59
- **Office Hours:**
  - Tuesday 3-4 PM (Lukas)
  - Wednesday 1-2 PM (Jay)
  - Thursday 2:15-3:15 PM (Alex)

# Recap: Value iteration

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**Theorem:**  $V^{\pi^t}(s) \geq V^*(s) - \frac{2\gamma^t}{1-\gamma} \|Q^0 - Q^*\|_\infty \forall s \in S$

Q: when will  $\pi^t$  be the optimal policy?

# Outline

1. Policy Iteration
2. Computation complexity of VI and PI
3. Linear Programming formulation

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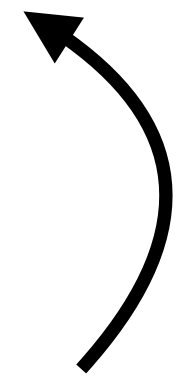
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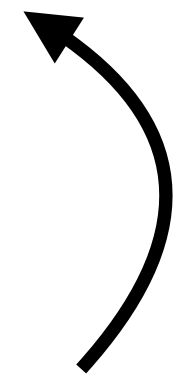
# Policy Iteration Algorithm:

Closed-form for PE  
(see 1.1.3 in Monograph)

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# Analysis of Policy Iteration

Recall: Policy Improvement  $\pi^{t+1}(s) = \arg \max_a Q^{\pi^t}(s, a), \forall s$

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$$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$$

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# Analysis of Policy Iteration

Q: what happens when  $\pi^{t+1}$  and  $\pi^t$  are exactly the same?

Show that  $\pi^t$  is an optimal policy  $\pi^\star$

Q: does this imply that the algorithm will terminate?

# Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

# Computation complexity of VI and PI

Given an MDP  $\mathcal{M} = (S, A, P, r, \gamma)$  can we **exactly** compute  $Q^\star$  (or find  $\pi^\star$ )  
in time polynomial wrt  $S, A, 1/(1 - \gamma)$ ?

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What about  $\text{poly}(S, A)$  algs?

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# The primal linear programming

Recall the Bellman consistency:

$$V(s) = \max_a \left\{ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} [V(s')] \right\}, \forall s$$

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(Proof in HW1)

# Proof of Bellman Optimality

## Theorem 1: Bellman Optimality

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$$\begin{aligned} V^{\star}(s) &= r(s, \pi^{\star}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi^{\star}(s))} V^{\star}(s') \\ &\leq \max_a \left[ r(s, a) + \gamma \mathbb{E}_{s' \sim P(s, a)} V^{\star}(s') \right] = r(s, \hat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \hat{\pi}(s))} V^{\star}(s') \\ &= r(s, \hat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \hat{\pi}(s))} \left[ r(s', \pi^{\star}(s')) + \gamma \mathbb{E}_{s'' \sim P(s', \pi^{\star}(s'))} V^{\star}(s'') \right] \\ &\leq r(s, \hat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \hat{\pi}(s))} \left[ r(s', \hat{\pi}(s')) + \gamma \mathbb{E}_{s'' \sim P(s', \hat{\pi}(s'))} V^{\star}(s'') \right] \\ &\leq r(s, \hat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \hat{\pi}(s))} \left[ r(s', \hat{\pi}(s')) + \gamma \mathbb{E}_{s'' \sim P(s', \hat{\pi}(s'))} \left[ r(s'', \hat{\pi}(s'')) + \gamma \mathbb{E}_{s''' \sim P(s'', \hat{\pi}(s''))} V^{\star}(s''') \right] \right] \\ &\leq \mathbb{E} \left[ r(s, \hat{\pi}(s)) + \gamma r(s', \hat{\pi}(s')) + \dots \right] = V^{\hat{\pi}}(s) \end{aligned}$$

# The primal linear programming

$$\begin{aligned} \min \quad & \sum_s \mu(s) V(s) \\ \text{s.t.} \quad & V(s) \geq \max_a \left[ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} V(s') \right] \quad \forall s, a \in S \times A \end{aligned}$$

**Convert the constraint to linear**

$$\begin{aligned} \min \quad & \sum_s \mu(s) V(s) \\ \text{s.t.} \quad & V(s) \geq r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} V(s') \quad \forall s, a \in S \times A \end{aligned}$$

# LP Runtime

[Ye, '05]: there is an interior point algorithm (CIPA)  
which is (“nearly”) **strongly polynomial, i.e., no poly dependence on  $1/(1 - \gamma)$**

$$S^4 A^4 \ln \left( \frac{S}{1 - \gamma} \right)$$

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  - It is also very helpful conceptually.
  - In some cases, it also provides a reasonable algorithmic approach
- Let us start by understanding the dual variables

# State action occupancy measure

$\mathbb{P}_h(s, a; s_0, \pi)$ : probability of  $\pi$  visiting  $(s, a)$  at time step  $h \in \mathbb{N}$ , starting at  $s_0$

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$$V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a)$$

**A Bellman equation like property for  $d_{s_0}^\pi(s, a)$**

$$\sum_a d_\mu^\pi(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{\bar{s}, \bar{a}} P(s | \bar{s}, \bar{a}) d_\mu^\pi(\bar{s}, \bar{a})$$

Proof:

**A Bellman equation like property for  $d_{s_0}^\pi(s)$**

# The “State-Action” Polytope

- Let us define the **state-action polytope  $K$**  as follows:

$$K_{\mu} := \left\{ d \mid d \geq 0 \text{ and } \sum_a d(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{s', a'} P(s \mid s', a') d(s', a') \right\}$$

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- This set precisely characterizes all state-action visitation distributions:

**Lemma:**  $d \in K_\mu$  if and only if there exists a (possibly randomized) policy  $\pi$

s.t.  $d_\mu^\pi = d$

# The Dual LP

$$\begin{aligned} \max \quad & \sum_{s,a} d(s,a)r(s,a) \\ \text{s.t.} \quad & d \in K_\mu \end{aligned}$$

- One can verify that this is the dual of the primal LP.

# Summary

**Notations:** Value / Q functions, state-action occupant measures,  
Bellman equation / optimality

**Planning algorithms:** VI, PI, LP (primal and dual)