The Sample Complexity (with a Generative Model)

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CS 2824: Foundations of Reinforcement Learning

Announcements

- HW1 is posted now!
  - it cover many concepts from class
- Reading assignments
  - please do the readings; they are helpful for our lectures
  - next reading assignment posted soon

Today:

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- Recap: computational complexity
  - Question: Given an MDP $\mathscr{M} = (S, A, P, r, \gamma)$ can we exactly compute $Q^\star$ (or find $\pi^\star$) in polynomial time?

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- Today: statistical complexity
  - Question: Given only sampling access to an unknown MDP $\mathscr{M} = (S, A, P, r, \gamma)$ how many observed transitions do we need to estimate $Q^\star$ (or find $\pi^\star$)?

Recap

Summary Table

	Value Iteration	Policy Iteration	LP-based Algorithms
Poly.	$S^2 A \frac{L(P,r,\gamma)\ \log \frac{1}{1-\gamma}}{1-\gamma}$	$(S^3 + S^2 A) \frac{L(P,r,\gamma) \log \frac{1}{1-\gamma}}{1-\gamma}$	$S^3 A\, L(P, r, \gamma)$
Strongly Poly.	X	$(S^3 + S^2 A) \cdot \min \left\{ \frac{A^S}{S}, \frac{S^2 A \log \frac{S^2}{1-\gamma}}{1-\gamma} \right\}$	$S^4 A^4 \log \frac{S}{1-\gamma}$

- VI: poly time for fixed γ, not strongly poly
- PI: poly and strongly-poly time for fixed γ
- LP approach: poly and strongly-poly time

  (LP approach is only logarithmic in $1/(1-\gamma)$)

Today

Sampling Models
(for learning in an unknown MDP)

- Episodic setting:
  - in every episode, $s_0 \sim \mu$.
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- Sample complexity of RL:
  how many transitions do we need observe in order to find a near optimal policy?

Two Fundamental Questions in Sample Complexity

1. The Model Size Question (Sublinear Learning)
   - How many parameters do we need to specify the transition kernel P? (and how many for the policy?)
   - Q1: Can we find an ϵ-optimal policy with **sublinear** sample complexity?

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2. The Horizon Question (Error Amplification)
   - The Scale: In discounted settings, the values scale as $1/(1-\gamma)$.
   - Target: It is natural to measure our additive error ϵ relative to this scale.
   - Q2: What is the "**horizon amplification**"? i.e. the dependence on $1/(1-\gamma)$

Attempt 1:

the naive model based approach

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  - Call our simulator N times at each state action pair.
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    $$\widehat{P}(s' \mid s, a) = \frac{\text{count}(s', s, a)}{N}$$

    where $\text{count}(s', s, a)$ is the #times (s, a) transitions to state s'.
  - we know the reward after one call.

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      (for simplicity, we often assume $r(s, a)$ is determinstic)
- The total number of calls to our generative model is SAN.

Model accuracy

Proposition: c is an absolute constant. $\epsilon > 0$. For $N \geq \frac{c\gamma}{(1-\gamma)^4} \frac{S \log(cSA/\delta)}{\epsilon^2}$

and with probability greater than $1 - \delta$,

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- Uniform value accuracy: For all policies π, $\|Q^\pi - \widehat{Q}^\pi\|_\infty \leq \epsilon/2$

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  $\max_{s,a} \|P(\cdot \mid s,a) - \widehat{P}(\cdot \mid s,a)\|_1 \leq (1-\gamma)^2\epsilon/2$.
- Uniform value accuracy: For all policies π,
  $\|Q^\pi - \widehat{Q}^\pi\|_\infty \leq \epsilon/2$
- Near optimal planning: Suppose that $\hat{\pi}^\star$ is the optimal policy in $\widehat{M}$.
  $\|Q^\star - Q^{\hat{\pi}^\star}\|_\infty \leq \epsilon$

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- Also,
  $Q^\pi = (I - \gamma P^\pi)^{-1} r$
  (where one can show the inverse exists)

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Proof: Using our matrix equality for Q^π, we have:

$$\begin{aligned}
Q^\pi - \widehat{Q}^\pi &= Q^\pi - (I - \gamma \widehat{P}^\pi)^{-1} r \\
&= (I - \gamma \widehat{P}^\pi)^{-1}((I - \gamma \widehat{P}^\pi) - (I - \gamma P^\pi))Q^\pi \\
&= \gamma(I - \gamma \widehat{P}^\pi)^{-1}(P^\pi - \widehat{P}^\pi)Q^\pi \\
&= \gamma(I - \gamma \widehat{P}^\pi)^{-1}(P - \widehat{P})V^\pi
\end{aligned}$$

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  - For a fixed s, a. With pr greater than $1 - \delta$,

    $$\|P(\cdot \mid s,a) - \widehat{P}(\cdot \mid s,a)\|_1 \leq c\sqrt{\frac{S\log(1/\delta)}{N}}$$

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- The first claim now follows by the union bound.

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For the second claim,

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$$\leq \frac{\gamma}{(1-\gamma)^2}\max_{s,a}\|P(\cdot \mid s,a) - \widehat{P}(\cdot \mid s,a)\|_1$$

(why is the first inequality true?)

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The proof for the Claim 3 immediately follows from the second claim.

Attempt 2:

obtaining sublinear sample complexity

idea: use concentration only on $V^\star$

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- $\widehat{Q}^\star$: optimal value in estimated model $\widehat{M}$.
  $\widehat{\pi}^\star$: optimal policy in $\widehat{M}$.
  $Q^{\widehat{\pi}^\star}$: (true) value of estimated policy.

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Proposition: (Crude Value Bound) With probability greater than $1 - \delta$,

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What about the value of the policy?

$$\|Q^\star - Q^{\hat{\pi}^\star}\|_\infty \leq \frac{\gamma}{(1-\gamma)^3}\sqrt{\frac{2\log(2SA/\delta)}{N}}$$

Sample Size Corollaries

Corollary: for $\epsilon < 1$, provided $N \geq \frac{c}{(1-\gamma)^4} \frac{\log(cSA/\delta)}{\epsilon^2}$ then $\|Q^\star - \widehat{Q}^\star\|_\infty \leq \epsilon$ (with prob. greater than $1 - \delta$)

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What about the policy?

Corollary: for $\epsilon < 1$, provided $N \geq \frac{c}{(1-\gamma)^6} \frac{\log(cSA/\delta)}{\epsilon^2}$ then

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Component-wise Bounds Lemma

Lemma: we have that

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Proof:

For the first claim, the optimality of $\pi^\star$ in M implies:

$$Q^\star - \widehat{Q}^\star = Q^{\pi^\star} - \widehat{Q}^{\widehat{\pi}^\star} \le Q^{\pi^\star} - \widehat{Q}^{\pi^\star} = \gamma(I - \gamma \widehat{P}^{\pi^\star})^{-1}(P - \widehat{P})V^\star,$$

using the simulation lemma in the final step.

See notes for the proof of second claim.

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  - By Hoeffding's inequality and the union bound,

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which holds with probability greater than $1-\delta$.

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- Proof of second claim is similar (see the book)

Attempt 3:
minimax optimal sample complexity
idea: better variance control

(“near”) Minimax Optimal Sample Complexity

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Theorem: With probability greater than $1 - \delta$,

$$\|Q^\star - \widehat{Q}^\star\|_\infty \leq \gamma \sqrt{\frac{c}{(1-\gamma)^3} \frac{\log(cSA/\delta)}{N}} + \frac{c\gamma}{(1-\gamma)^3} \frac{\log(cSA/\delta)}{N},$$

where c is an absolute constant.

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Corollary: for $\epsilon < 1$, provided $N \geq \frac{c}{(1-\gamma)^3} \frac{\log(cSA/\delta)}{\epsilon^2}$ then

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What about the policy?

Naively, we need $N/(1-\gamma)^2$ more samples.

A different thm gives: With the same N,

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Lower Bound: We can't do better.

Proof sketch: part 1

- From "Component-wise Bounds" lemma, we want to bound:
$$Q^\star - \widehat{Q}^\star \leq \gamma \|(I - \gamma \widehat{P}^{\pi^\star})^{-1}(P - \widehat{P})V^\star\|_\infty \leq ??$$

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- From Bernstein's ineq, with pr. greater than $1 - \delta$, we have (component-wise):
$$|(P - \widehat{P})V^\star| \leq \sqrt{\frac{2\log(2SA/\delta)}{N}}\sqrt{\mathrm{Var}_P(V^\star)} + \frac{1}{1-\gamma}\frac{2\log(2SA/\delta)}{3N}\vec{1}$$

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- Therefore
  $$Q^\star - \widehat{Q}^\star \leq \gamma \sqrt{\frac{2\log(2SA/\delta)}{N}} \| (I - \gamma \widehat{P}^{\pi^\star})^{-1} \sqrt{\mathrm{Var}_P(V^\star)} \|_\infty$$
  $+$"lower order term"

Bellman Equation for the Variance

- Variance: $\mathrm{Var}_P(V)(s,a) := \mathrm{Var}_{P(\cdot|s,a)}(V)$

  Component wise variance: $\mathrm{Var}_P(V) := P(V)^2 - (PV)^2$

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- Let's keep around the MDP M subscripts.
  Define Σ_M^π as the (total) variance of the discounted reward:

$$\Sigma_M^\pi(s,a) := E\left[\left(\sum_{t=0}^{\infty}\gamma^t r(s_t,a_t) - Q_M^\pi(s,a)\right)^2 \middle| s_0 = s, a_0 = a\right]$$

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- Bellman equation for the total variance:
  $\Sigma_M^\pi = \gamma^2 \mathrm{Var}_P(V_M^\pi) + \gamma^2 P^\pi \Sigma_M^\pi$

Key Lemma

Lemma: For any policy π and MDP M,

$$\left\| (I - \gamma P^{\pi})^{-1} \sqrt{\mathrm{Var}_P(V_M^{\pi})} \right\|_{\infty} \leq \sqrt{\frac{2}{(1-\gamma)^3}}$$

Proof idea: convexity + Bellman equations for the variance.

Putting it all together

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$$\|(I - \gamma \widehat{P}^{\pi^\star})^{-1}\sqrt{\mathrm{Var}_P(V^\star)}\|_\infty = \|(I - \gamma P^{\pi^\star}_{\widehat{M}})^{-1}\sqrt{\mathrm{Var}_P(V^{\pi^\star}_{M})}\|_\infty$$

$$\leq \|(I - \gamma P^{\pi^\star}_{\widehat{M}})^{-1}\sqrt{\mathrm{Var}_P(V^{\pi^\star}_{\widehat{M}})} + \text{"lower order"}$$

$$\leq \sqrt{\frac{2}{(1-\gamma)^3}} + \text{"lower order"}$$

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$$\|(I-\gamma \widehat{P}^{\pi^\star})^{-1}\sqrt{\mathrm{Var}_P(V^\star)}\|_\infty = \|(I-\gamma P^{\pi^\star}_{\widehat{M}})^{-1}\sqrt{\mathrm{Var}_P(V^{\pi^\star}_{M})}\|_\infty$$

$$\leq \|(I-\gamma P^{\pi^\star}_{\widehat{M}})^{-1}\sqrt{\mathrm{Var}_P(V^{\pi^\star}_{\widehat{M}})} + \text{"lower order"}$$

$$\leq \sqrt{\frac{2}{(1-\gamma)^3}} + \text{"lower order"}$$

First equality above: just notation

Putting it all together

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Second step: concentration $\rightarrow$ we need to quantify:

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Last step: previous slide