Binary options

Consider a price process $(S_t)_{t\in\mathbb{R}+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \quad S_0 = 1 \tag{1}$$

under the risk-neutral probability measure $\mathbb{Q}$. The binary (or digital) call option is a contract with maturity T, strike price K and payoff

$$C_d := \mathbb{1}_{[K,\infty)}(S_T) = \begin{cases} 1\ Eur & S_T \geq K \\ 0\ Eur & S_T < K \end{cases} \tag{2}$$

1. Derive the Black-Scholes PDE satisfied by the pricing function $C_d(t, S_t)$ of the binary call option, together with its terminal condition.

· Knowing : (1): $dS_t = r S_t dt + \sigma S_t dB_t$

· We want to determine the price of the binary options : $C_d(t, S_t)$

· By Itô's Lemma to (1):

$$dC_d(t,S_t) = \left(\frac{\partial C_d}{\partial t} + r\cdot S_t\cdot\frac{\partial C_d}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_d}{\partial S_t^2}\right)dt + \sigma S_t \frac{\partial C_d}{\partial S_t} dB_t$$

· In Black-Scholes, we want to hedge such that we set up a portfolio with value Π by selling one derivative and buy $\frac{\partial C_d}{\partial S_t}$ shares:

$$\Pi_t = -C_d(t,S_t) + \Delta_t S_t$$

Then we have:

$$d\Pi_t = -dC_d(t,S_t) + \Delta_t dS_t$$
$$= -\left(\frac{\partial C_d}{\partial t} + r\cdot S_t\cdot\frac{\partial C_d}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_d}{\partial S_t^2}\right)dt - \sigma S_t \frac{\partial C_d}{\partial S_t} dB_t$$
$$+ \Delta_t r S_t dt + \Delta_t \sigma S_t dB_t$$

Since we set $\Delta_t = \frac{\partial C_d(t,S_t)}{\partial S_t}$, then:

$$d\Pi_t = -\left(\frac{\partial C_d}{\partial t} + r\cdot S_t\cdot\frac{\partial C_d}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_d}{\partial S_t^2}\right)dt - \sigma S_t \frac{\partial C_d}{\partial S_t} dB_t$$
$$+ r S_t \frac{\partial C_d(t,S_t)}{\partial S_t} dt + \sigma S_t \frac{\partial C_d(t,S_t)}{\partial S_t} dB_t$$

$$= -\frac{\partial C_d}{\partial t}\,dt - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_d}{\partial S_t^2}\,dt \quad (*)$$

- Since we want the return of portfolio to be equal the risk-free rate (that is arbitrary-free), then we have:

$$\begin{aligned} d\Pi_t &= r\,\Pi_t\,dt \\ &= r\cdot(-C_d(t,S_t) + \Delta_t S_t)\,dt \\ &= -r\cdot C_d(t,S_t)\,dt + r\cdot\frac{\partial C_d(t,S_t)}{\partial S_t} S_t\,dt \quad (**) \end{aligned}$$

- Combine $(*)$ and $(**)$:

$$\frac{\partial C_d(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_d}{\partial S_t^2} + r\cdot\frac{\partial C_d(t,S_t)}{\partial S_t} S_t = r\cdot C_d(t,S_t)$$

- Combining with initial & boundary conditions, the Black-Scholes PDE:

$$\begin{cases} \dfrac{\partial C_d(t,S_t)}{\partial t} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 C_d}{\partial S_t^2} + r\cdot\dfrac{\partial C_d(t,S_t)}{\partial S_t} S_t = r\cdot C_d(t,S_t) \\ S_0 = 1 \\ C_d(T,S_T) = \mathbb{1}\{S_T \geq K\}. \end{cases}$$

2. Show that the solution of $C_d(t,x)$ of the Black-Scholes PDE is given by

$$C_d(t,x) = e^{(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{|\sigma|\sqrt{T-t}}\right)$$
$$= e^{-(T-t)r}\Phi(d_-(T-t))$$

where

$$d_-(T-t) := \frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{|\sigma|\sqrt{T-t}}, \quad 0 \le t < T \tag{3}$$

- Known that $C_d(T, S_T)$ is the value of a European Binary call option at time t, with the payoff : $C_d(T, S_T) = \mathbb{1}_{\{S_T \geq K\}}$.
- Under the risk-neutral measure $\mathbb{Q}$, we know that the current value of this payoff is $\mathbb{E}_{\mathbb{Q}}[C_d(t, S_t)]$ with the discount factor

$$C_d(t,x) = \mathbb{E}_{\mathbb{Q}}[e^{-(T-t)\cdot r} \cdot C_d(t, S_t) \mid S_t = x]$$
$$= e^{-(T-t)\cdot r} \cdot \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{S_T \geq K\}} \mid S_T = x]$$
$$= e^{-(T-t)\cdot r} \cdot \mathbb{Q}(S_T \geq K) \quad (***)$$

- Now we need to define the distribution of S_T under $\mathbb{Q}$:

Since we have $dS_t = rS_t\,dt + \sigma S_t\,dB_t$, $S_0 = 1$; by Itô's Lemma, this SDE has a unique strong solution given explicitly by the exponential formula : $S_t = S_0 \exp((r-\frac{\sigma^2}{2})t - \sigma B_t)$, $t \geq 0$,

This means that :

$$\log(S_t) = \log(S_0) + (r - \frac{\sigma^2}{2})t - \sigma B_t$$

$$\Leftrightarrow \log(S_t) \sim \mathcal{N}(\log(S_0) + (r-\frac{\sigma^2}{2})t, \sigma^2 T)$$

$$\Leftrightarrow \log(S_T) \sim \mathcal{N}(\log(S_t) + (r-\frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

• Now we will use the distribution $\log(S_T)$ to determine (***):

$$Q(S_T \geq K) = Q(\log(S_T) \geq \log(K)) \overset{(1)}{=} \Phi\left(\frac{\log(S_t) + (r - \frac{\sigma^2}{2})(T-t) - \log(K)}{|\sigma|\sqrt{T-t}}\right)$$

$$= \Phi\left(\frac{\log(S_t/K) + (r - \frac{\sigma^2}{2})(T-t)}{|\sigma|\sqrt{T-t}}\right)$$

$$= \Phi\left(\frac{\log(x/K) + (r - \frac{\sigma^2}{2})(T-t)}{|\sigma|\sqrt{T-t}}\right)$$

where (1) since: if $X \sim N(\mu, \sigma^2) \Rightarrow P(X \geq a) = \Phi\left(\frac{\mu - a}{\sigma}\right)$; and set $x = S_t$

• Lastly, we need to check for the boundary points:

$$\begin{cases} \lim_{t \to T} C_d(t, x) = \Phi(+\infty) = 1 & \text{if } x \geq K \\ \lim_{t \to T} C_d(t, x) = \Phi(-\infty) = 0 & \text{if } x < K \end{cases}$$

$\Rightarrow C_d(T, x) = \mathbb{1}_{\{x \geq K\}}$, which matches the definition.

• As a result, we prove that

$$C_d(t, x) = e^{-(T-t)r} \cdot \Phi\left(\frac{(r - \sigma^2/2)(T-t) + \log(x/K)}{|\sigma|\sqrt{T-t}}\right)$$

$$= e^{-(T-t)r} \cdot \Phi(d_-(T-t))$$

where $d_-(T-t) := \dfrac{(r - \sigma^2/2)(T-t) + \log(x/K)}{|\sigma|\sqrt{T-t}}$, $0 \leq t < T$

is the unique solution of PDE.