Computational Finance - Lab Assignment 3
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1 PDE Methods: Binary and Barrier Options

In this assignment, you will implement and analyze numerical methods for pricing two classes of exotic options:

(a) **Binary (Digital) Options**

(b) **Barrier Options (Up-and-Out, Down-and-In)**

You will use Monte Carlo, closed-form formulas under Black–Scholes, and a finite-difference PDE approach (implicit scheme).

Binary options

Consider a price process $(S_t)_{t\in\mathbb{R}^+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \quad S_0 = 1 \tag{1}$$

under the risk-neutral probability measure $\mathbb{Q}$. The binary (or digital) call option is a contract with maturity T, strike price K and payoff

$$C_d := \mathbb{1}_{[K,\infty)}(S_T) = \begin{cases} 1 \; Eur & S_T \geq K \\ 0 \; Eur & S_T < K \end{cases} \tag{2}$$

1. Derive the Black-Scholes PDE satisfied by the pricing function $C_d(t, S_t)$ of the binary call option, together with its terminal condition.

2. Show that the solution of $C_d(t, x)$ of the Black-Scholes PDE is given by

$$\begin{aligned} C_d(t,x) &= e^{(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\mid\sigma\mid\sqrt{T-t}}\right) \\ &= e^{-(T-t)r}\Phi(d_-(T-t)) \end{aligned}$$

   where

$$d_-(T-t) := \frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\mid\sigma\mid\sqrt{T-t}}, \quad 0 \leq t < T \tag{3}$$

3. Use Monte carlo to compute the price of a binary option to its closed form solution.

4. Use the implicit discretization scheme and the Crank Nicolson scheme to price a binary/digital options. Make also a sensitivity analysis on the parameter space and plot the digital option surface $c(S, t)$. Finally compute the delta as a function of the underlying S.

Knock-out Barrier options

We are now going to consider a slightly more exotic option, the so called **Knock-out barrier** options. Let us consider an up-and-out barrier call option with maturity T, strike price K barrier (or call level B) whose payoff is given by

$$C = (S_T - K)\,\mathbb{1}_{\max_{0\leq t\leq T} S_t < B} \tag{4}$$

with $B \geq K$ so that the payoff is not zero. In this case the option expires worthless if the barrier is not triggered. If instead the barrier is hit the option becomes a vanilla option.
The goal this exercise is to compute the fair value of such path-dependent options using a closed form formula, Monte carlo methodologies and the BS PDE with an appropriate terminal condition.

1. **BONUS Exercise (1 Point):** Show that when $K \leq B$ the price

$$e^{-(T-t)r}\mathbb{1}_{\{M_0^t<B\}}\mathbb{E}\left[\left(x\frac{S_{T-t}}{S_0}-K\right)^+\mathbb{1}_{\{x\max_{0\leq u\leq T-t}\frac{S_u}{S_0}<B\}}\right]_{x=S_t} \tag{5}$$

of the up-and-out barrier call option with maturity T strike price K and barrier level B is given by

$$\begin{aligned}
C(t,T,S_T,M_0^T) &:= C(H) \\
&= e^{-(T-t)r}\mathbb{E}\left[(S_T-K)^+\mathbb{1}_{\{M_0^T<B\}} \mid \mathcal{F}_t\right] \\
&= S_t\mathbb{1}_{\{M_0^t<B\}}\{\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\} \\
&-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\
&-e^{-(T-t)}K\mathbb{1}_{\{M_0^t<B\}}\{\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\} \\
&+\left(\frac{S_t}{B}\right)^{1-\frac{2r}{\sigma^2}}e^{-(T-t)}K\mathbb{1}_{\{M_0^t<B\}}\{\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\}
\end{aligned}$$

where

$$\delta_\pm^\tau(z)=\frac{1}{\sigma\sqrt{\tau}}\left(\log z+\left(r\pm\frac{\sigma^2}{2}\right)\tau\right), \quad z>0 \tag{6}$$

2. Use the analytical formula above to compare it with the price you derive using a MC simulation. Compare the values of the option wrt to the option parameters as well as the model's parameters. For the MC simulation we are going to use an adjusted/corrected methodology proposed in the following paper (Adjusted MC price barrier option). If we define by $C(H)$ the price of the above continuous up and out call option and then we define by $C_m(H)$ the price of an otherwise identical discrete barrier option (for example the one we derive from a MC simulation). The we have that $C_m(H)=C(He^{-\beta_1\sigma\sqrt{T/m}})+o(\frac{1}{\sqrt{m}})$. Here $\beta_1 \approx -\frac{\zeta(1/2)}{\sqrt{2\pi}} \approx 0.5826$ with ζ the Rienmann zeta function. Run also a convergence analysis on the MC method and comment on the convergence rates.[1]

3. Use the implicit discretization scheme to price the barrier up and out call option. Make also a sensitivity analysis on the parameter space and plot the barrier option surface $c(S,t)$. Finally compute the delta as a function of the underlying S.

2 Calibration SP500 Implied Volatility using the Heston Model

The goal of this exercise is to calibrate the Heston model on implied volatility quoted data on SP500 options. The goal for this is to use two different pricing methodologies. The one is based on the scheme proposed by Andersen and the other one is using the affine property of the model and the fact that we can derive a closed form formula for the pricing of vanilla options as we have already seen under the assumption of a GBM on the underlying price process. Let us assume the following dynamics on the asset prices under the risk neutral measure $\mathbb{Q}$

$$\begin{aligned}
dS_t &= rS_tdt+S_t\sqrt{V_t}dB_t \\
dV_t &= \kappa\left(\theta-V_t\right)dt+\sigma\sqrt{V_t}dW_t
\end{aligned}$$

Denoting the price of the underlying asset by S_t, the implied volatility may be parametrized in terms of moneyness $m=K/S_t$ and time to maturity $\tau=T-t$ of the option. The implied volatility associated with

[1] One sees this result as saying that to price a discretely monitored barrier option using the continuous formula one shoudl first shift the barrier away from S_0 by a factor $e^{\beta_1\sigma\sqrt{T/m}}$. This corrects for the fact that when the discrete time process $\{S_{k\Delta}, k=0,1,...\}$ breaches the barrier it overshoots. The constant $\beta_1\sigma\sqrt{T/m}$ should be viewed as an epproximation to the overshoot in the logarithm of the price of the underlying.

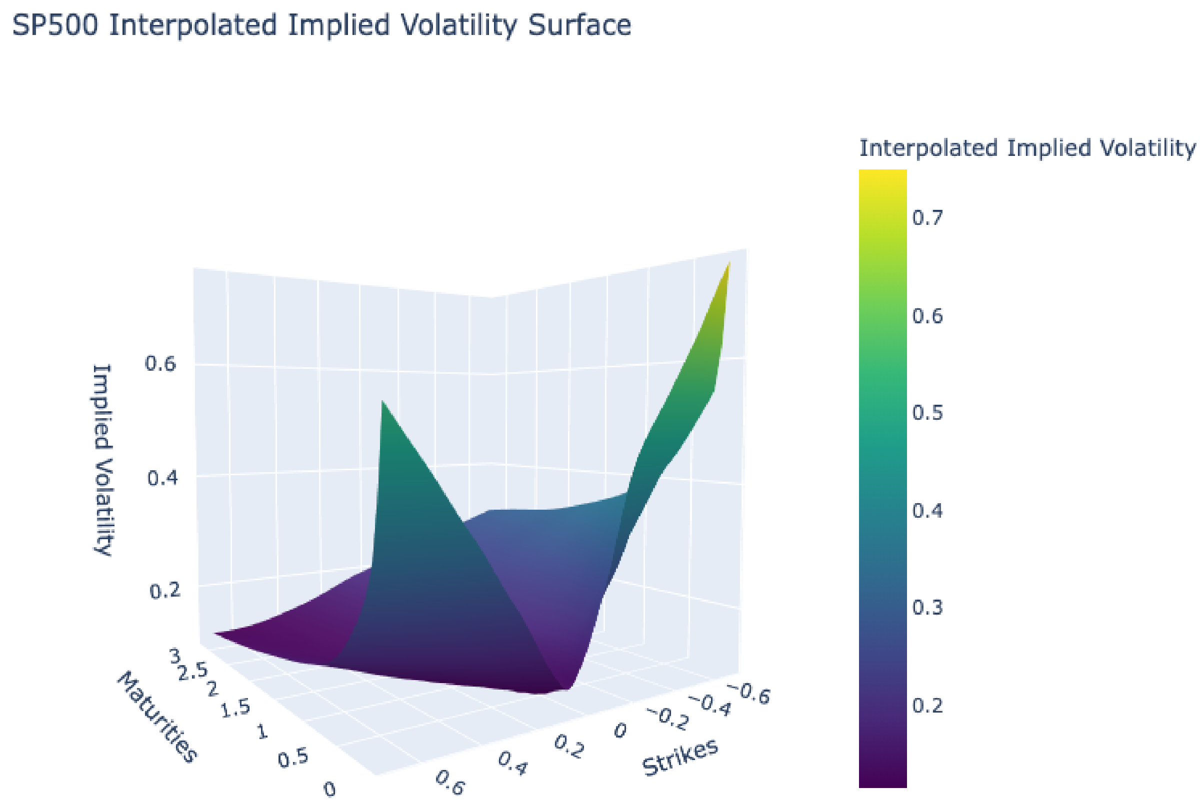

Figure 1: Implied Volatility Surface for SP500 on 2023/11/01

a call option with moneyness m and time-to-maturity τ of a non-dividend paying asset S is the unique value $\sigma_t(m,\tau)$ such that the Black-Scholes price $C_{BS}(S_t, K, \tau, \sigma_t(m,\tau))$ matches the market price $C_t(m,\tau)$ of the call:

$$
\begin{aligned}
C_t(m,\tau) = C_{BS}(S_t, K, \tau, \sigma_t(m,\tau)) &= S_t N(d_1) - K e^{-r\tau} N(d_2) \\
d_1 &= \frac{-\ln m + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \\
d_2 &= \frac{-\ln m + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}
\end{aligned}
$$

In this exercise we have quoted data of implied volatilities for call options which span a total of 10 days. Hence for a given fixed grid in moneyness and time maturities $(\mathbf{m}, \tau)$ we have for 10 different days the whole implied volatility surface of option quotes on SP500.
Our goal is to run a calibration procedure for each day in our training set. During the calibration of the Heston model we search for the model parameters such that the difference between market plain vanilla option prices C^{mkt} and the model prices $C = C^H$ will be as small as possible. The parameter set that we are interested in calibrating in this case is $\Theta = \{\rho, V_0, \theta, \kappa, \sigma\}$. In order to determine the optimal parameters a target function is defined which is based on the difference between the model and the market values. We note that there is no unique target function. Our goal will be to minimize the mean squared error. Hence the common choices include

$$\min_{\theta \in \Theta} \sqrt{\sum_i \sum_j w_{i,j} \left(C^{mkt}\left(t_0, S_0; K_i, T_j\right) - C\left(t_0, S_0; K_i, T_j, \theta\right)\right)^2} \tag{7}$$

and

$$\min_{\theta\in\Theta}\sqrt{\sum_i\sum_j w_{i,j}\left(\sigma_0^{mkt,imp}\left(t_0,S_0;K_i,T_j\right)-\sigma_0^{imp}\left(t_0,S_0;K_i,T_j,\theta\right)\right)^2} \tag{8}$$

where $C^{mkt}(t_0,S_0;K_i,T_j)$ is the call option price for strike K_i and maturity T_j in the market and $C\left(t_0,S_0;K_i,T_j,\theta\right)$ is the Heston call option value. Similarly $\sigma_0^{mkt,imp}(\cdot)$ and $\sigma_0^{imp}(;\theta)$ are the implied volatilities from the market and the Heston model, respectively and $w_{i,j}$ is some weighting function. You can use a $w_{i,j}=1$ for your implementations. The main difficulty when calibrating the Heston model is that the set Θ includes five parameters that need to be determined, and that the model parameters are not completely "independent". With this we mean that the effect of different parameters on the shape of the implied volatility smile may be quite similar. For this reason, one may encounter several "local minima" when searching for optimal parameter values. The optimization can be accelerated by a reduction of the set of parameters to be optimized.
The steps for this exercise will be the following:

1. **Dynamics in logarithmic coordinates** (theory)
   (a) Derive—via Itô's formula—the SDEs followed by

   $$X_t = \ln S_t, \qquad Y_t = \ln V_t,$$

   where (S_t,V_t) solves the standard Heston system.
   (b) Give a short argument for *why* the log-variance coordinate Y_t is convenient.
2. **Vanilla option pricing experiments** (code + plots)
   (a) Fix one parameter set $(\kappa,\theta,\sigma,\rho,v_0)$.
   (b) Price European calls on a *strike* $\times$ *maturity* grid using
      (i) one Monte-Carlo engine (*either* Euler–Maruyama *or* the Quadratic–Exponential (QE) scheme [2]),
      (ii) the semi-closed Heston formula via its characteristic function [3].
   (c) Produce two figure sets:
      i. call-price surfaces (Monte-Carlo vs. analytic),
      ii. implied-volatility surfaces (extracted from those prices).
3. **Calibration to SP500 implied volatilities** (code + discussion)
   (a) **Load the market data.** Two NumPy archives are provided in the working directory:

File	Content
`raw_ivol_surfaces.npy`	original strikes and vols
`interp_ivol_surfaces.npy`	cubic-interpolated vols

   Each archive is a pickled dictionary

   `Dict[str date]` $\longrightarrow$ `Dict[str, np.ndarray]`,

   where the inner structure is

Key	Raw archive	Interpolated archive
`tenors`	$(N,)$	$(N,)$
`strikes`	$(15,N)$ grid	$(100,)$ uniform vector
`vols`	$(15,N)$ matrix	$(N,100)$ matrix

   Example:

```
import numpy as np
raw   = np.load("raw_ivol_surfaces.npy", allow_pickle=True).item()
interp= np.load("interp_ivol_surfaces.npy", allow_pickle=True).item()

date  = "2023 11 01"
raw_vols     = raw[date]["vols"]     # 15 × N
interp_vols  = interp[date]["vols"]  # N × 100
```

[2] Andersen (2008), SSRN 946405
[3] Yiran Cui et. al, arXiv:1511.08718

(b) **Estimate parameters.** Choose and implement a calibration objective (e.g. least–squares on implied vols) and optimise the Heston parameters on at least *two* different trade dates.

(c) **Diagnostics.**

- Plot model vs. market IV surfaces (heatmap or wireframe).
- Compute error metrics — RMSE, mean absolute error, max error.
- Comment on parameter stability across dates and where the fit deteriorates (short tenor, deep OTM, etc.).
- Suggest at least one extension that could improve the calibration (e.g. time-dependent parameters, stochastic–jump volatility, SABR, etc.).

A Assignment 1 - The Black-Scholes heat equation and Exotic option pricing

The aim of this appendix is to present the canonical financial model namely the Black-Scholes Merton model, and to use it as a backbone to introduce the different numerical methods for PDE solvers. Let us first remind ourselfs the Black-Scholes PDE and how this is derived.

Derivation of the Black-Scholes PDE

This paragraph is intended to provide a rigorous derivation of the so-called Black-Scholes partial differential equation. Let $(W_t)_{t\geq 0}$ be a Brownian motion and $S := (S_t)_{t\geq 0}$ the asset price. This model assumes the following dynamics under the so-called historical (observed) probability measure $\mathbb{P}$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 > 0 \tag{9}$$

with $\mu \in \mathbb{R}$ is called the drift and $\sigma > 0$ the instanteneous volatility. Let $V(t,s)$ be the value of a European call option. The question we are interested in here is the following; assuming that the dynamics of the underlying asset is given by the above stochastic differential equation, what is the value today ($t = 0$) of a European option with payoff $f(S_T)$ at maturity $T > 0$? For clarity we shall denote V_t the value at time $t \in [0,T]$ of such financial derivative. The first step is to obtain a probabilistic representation for the option price. Under the absence absence of arbitrage there exists a probability $\mathbb{Q}$ under which we can write

$$V_0 = D_{0,T}\mathbb{E}_{\mathbb{Q}}\left[f(S)\right] \tag{10}$$

where $(D_{0,t})_{t\geq 0}$ represents the discount factor proces satisfying the (stochastic) differential equation $dD_{0,t} = -r_t D_{0,t}dt, \;\; D_{0,0} = 1$. We will simply assume a constant interest rate hence we just have a exponential growth ode which gives us an exponential discounting. An application of Itô's lemma yields that the option price satisfies the stochastic differential equation

$$dV_t = \left(\mu S_t \partial_S V_t + \partial_t V_t + \frac{\sigma^2}{2} S_t^2 \partial^2_{SS} V_t\right) dt + \sigma S_t \partial_S V_t dW_t \tag{11}$$

at any time t between inception and maturity with appropriate boundary conditions. Consider now a portfolio Π consisting at time t of a long position in the option V and long position in Δ_t shares S, i.e $\Pi_t = V_t + \Delta_t S_t$. On a small time interval $[t, t+dt]$ the profit and loss of such a portfolio is $d\Pi_t = dV_t + \Delta_t dS_t$. Using the equation (11) we obtain that

$$d\Pi_t = \{\mu\left(\Delta_t + \partial_S V_t\right) S_t + \partial_t V_t + \frac{\sigma^2}{2} S_t^2 \partial^2_{SS} V_t\}dt + \left(\Delta + \partial_S V_t\right)\sigma S_t dW_t \tag{12}$$

This expression makes it clear that the only way to eliminate the risk—solely present in the form of the Brownian perturbations- is to set $\Delta_t = -\partial_S V_t$. This is called Delta-hedging and you have already worked with this in the first assignment. now since we assume the absence of arbitrage, the returns of the portfolio Π_t over the period $[t, t+dt]$ are necessarily equal to the risk-free rate $r_t \equiv r$. Otherwise (assume the returns are higher than the risk-free one), it is possible to construct an arbitrage, for instance by borrowing money at time t to buy the portfolio, then invest it at rate r_t and sell it at time $t+dt$. Hence our assumption on the absence of arbitrage implies that $d\Pi_t = r\Pi_t dt$ and hence from equation (12)

$$\partial_t V_t + rS\partial_S V_t + \frac{\sigma^2}{2} S^2 \partial^2_{SS} V_t = rV_t \tag{13}$$

This equation (13) is called the Black-Scholes PDE associated with the boundary conditions given by the payoff $V_T = f(S_T)$.

Before trying to solve (numerically) a partial differential equation, it may sound sensible to simplify it. Recall that the BS PDE above is a parabolic PDE with boundary conditions $V_T(S)$ (for instance a European call option with maturity $T > 0$ and strike $K > 0$, we have $V_T(S) = (S_T - K)^+$). Define $\tau := T - t$ and the function $g_\tau(S) := V_t(S)$, then $\partial_t V_t(S) = -\partial_\tau g_\tau(S)$ and hence

$$-\partial_\tau g_\tau + rS\partial_S g_\tau + \frac{\sigma^2}{2} S^2 \partial^2_{SS} g_\tau = r g_\tau \tag{14}$$

with boundary conditions $g_0(S)$. Define now the function f by $f_\tau(S) := e^{r\tau} g_\tau(S)$. Show that

1. $$-\partial_\tau f_\tau + rS\partial_S f_\tau + \frac{\sigma^2}{2} S^2 \partial^2_{SS} f_\tau = 0 \tag{15}$$

   with boundary conditions $f_0(S)$.

2. Consider now a further trnasformation $x := \log(S)$ and the function $\psi_\tau(x) := f_\tau(S)$. Show that this transformation gives us the following PDE

   $$-\partial_\tau \psi_\tau + \left(r - \frac{\sigma^2}{2}\right) \partial_x \psi_\tau + \frac{\sigma^2}{2} \partial^2_{xx} \psi_\tau = 0 \tag{16}$$

   with boundary condition $\psi_0(x)$.

- Finally define the function ϕ_τ via $\psi_\tau(x) =: \phi_\tau(x) e^{\alpha x + \beta \tau}$. Show that equation (16) becomes

  $$\partial_\tau \phi_\tau(x) = \frac{\sigma^2}{2} \partial^2_{xx} \phi_\tau(x) \tag{17}$$

  for all real number x with (Dirichlet) boundary condition $\phi_0(x) = e^{-\alpha x} \psi_0(x)$. This as you have already seen in class is the so called *heat-equation*.

Options on Extrema

We know that Vanilla optiosn with payoff $C = \phi(S_T)$ can be priced as

$$e^{-rT} \mathbb{E}\left[\phi(S_T)\right] = e^{-rT} \int_0^\infty \phi(y) \phi_{S_T}(y) dy \tag{18}$$

where $\phi_{S_T}(y)$ is the probability density function of S_T which satisfies

$$\mathbb{Q}\left(S_T \le y\right) = \int_0^y \phi_{S_T}(\nu) d\nu, \quad y > 0 \tag{19}$$

Recall that typically we have that $\phi(x) = (x - K)^+$ for the European call option with strike K and $\phi(x) = \mathbb{1}_{[K,\infty)}(x)$ for the binary call option. On the other hand exotic options also called path-dependent options whose payoff depends on the whole path of the underlying price process. For example the payoff of an option on extrema take the form

$$C := \phi(M_0^T, S_T) \tag{20}$$

where $M_0^T = \max_{t\in[0,T]} S_t$ is the maximum of $(S_t)_{t\in\mathbb{R}_+}$ over the time interval $[0, T]$. In such situations option price at time $t = 0$ can be expressed as

$$e^{-rT} \mathbb{E}\left[\phi(M_0^T, S_T)\right] = e^{-rT} \int_0^\infty \int_0^\infty \phi(x, y) \phi_{M_0^T, S_T}(x, y) dx dy \tag{21}$$

where $\phi_{M_0^T, S_T}$ is the joint probability density function of (M_0^T, S_T) which satisfies

$$\mathbb{Q}\left(M_0^T \le x, S_T \le y\right) = \int_0^x \int_0^y \phi_{M_0^T, S_T}(u, v) du dv \tag{22}$$

Using the joint probability density function of the shifted Brownian motion $\tilde{W}_T = W_T + \mu T$ and the

$$\hat{X}_0^T = \max_{t\in[0,T]} \tilde{W} = \max_{t\in[0,T]} (W_t + \mu t) \tag{23}$$

We have that the joint probability density function of $\phi_{\hat{X}_0^T, \tilde{W}_T}$ of the drifted Brownian motion and its maximum is given by

$$\phi_{\hat{X}_0^T, \tilde{W}_T}(\alpha, b) = \mathbb{1}_{\{\alpha \ge \max(b,0)\}} \frac{1}{T} \sqrt{\frac{2}{\pi T}} (2\alpha - b)\, e^{\mu b - (2\alpha - b)^2/(2T) - \mu^2 T/2} \tag{24}$$

Using this we are able to price any exotic option with payoff $\phi(\tilde{W}_T, \hat{X_0}^T)$ as

$$c_0 = e^{-rT} \int_{-\infty}^{\infty} \int_{y \vee 0}^{\infty} \phi(x, y) d\mathbb{Q}\left(\tilde{W}_T \leq x, \hat{X_0}^T \leq y\right) \tag{25}$$

Notice that so far we have not assumed any dynamics on the underlying asset. We will now assume that S_t follows a GBM. In order to price barrier options by the above probabilistic method we will use the probability density function of the maximum

$$M_0^T = \max_{t \in [0,T]} S_t \tag{26}$$

of the GBM over a given time interval and the joint pdf $\phi_{M_0^T, S_T}(u, \nu)$.

Lemma A.1. *An exotic option with integrable claim payoff of the form*

$$C = \phi(M_0^T, S_T) = \phi(\max_{t \in [0,T]} S_t, S_T) \tag{27}$$

can be priced at time $t = 0$ as

$$\begin{aligned} e^{-rT}\mathbb{E}\left[C\right] &= \frac{e^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}} \int_0^{\infty} \int_y^{\infty} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x})(2x - y) e^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy \\ &+ \frac{e^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \int_0^{\infty} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) e^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy \end{aligned}$$

Assignment 2 - Pricing formula of a vanilla call option and representations of the characteristic function in the Heston model

For a spot price S_0 and a strike price K the price of a vanilla call option with maturity T is given by

$$\begin{aligned} C(\theta, K, T) &= e^{-rT}\mathbf{E}\left[(S_T - K)\mathbb{1}_{\{S_T \geq K\}}(S_T)\right] \\ &= e^{-rT}\left(\mathbf{E}\left[S_T \mathbb{1}_{\{S_T \geq K\}}(S_T)\right] - K\mathbf{E}\left[\mathbb{1}_{\{S_T \geq K\}}(S_T)\right]\right) \\ &= S_0 P_1(\theta, K, T) - e^{-rT} K P_2(\theta, K, T) \end{aligned}$$

In the Heston model P_1 and P_2 are solutions to certain pricing PDEs and are given as

$$\begin{aligned} P_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-iu\log K}}{iuF}\phi(\theta, u - i, T)\right) du \\ P_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-iu\log K}}{iuF}\phi(\theta, u, T)\right) du \end{aligned}$$

where i is the imaginary unit, $F = S_0 e^{rT}$ the forward price and $\phi(\theta, u, T)$ is the characteristic function of the Heston model. Thus the formula for pricing a vanicall call option becomes

$$\begin{aligned} C(\theta, K, T) &= e^{-rT}\left(S_0 P_1(\theta, K, T) - K P_2(\theta, K, T)\right) \\ &= e^{-rT}\left(S_0 \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-iu\log K}}{iuF}\phi(\theta, u - i, T)\right) du - e^{-rT} K \frac{1}{2} - e^{-rT} K \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-iu\log K}}{iuF}\phi(\theta, u, T)\right) du\right) \end{aligned}$$

The characteristic function was originally given by Heston

$$\phi(\theta; u, t) = exp\{iu\left(\log S_0 + rt\right) + \frac{\kappa\theta}{\sigma^2}\left[(\xi + d)t - 2\log\frac{1 - g_1 e^{d \cdot t}}{1 - g_1} + \frac{V_0}{\sigma^2}(\xi + d)\frac{1 - e^{d \cdot t}}{1 - g_1 e^{d \cdot t}}\right]\}$$

where

$$\begin{aligned} \xi &:= \kappa - \sigma\rho i u \\ d &:= \sqrt{\xi^2 + \sigma^2(u^2 + iu)} \\ g_1 &= \frac{\xi + d}{\xi - d} \end{aligned}$$

pointed out that when evaluating this form as a function of u for moderate to long maturities, discontinuities appear be- cause of the branch switching of the complex power function $G^{\alpha}(u) = exp(\alpha log G(u))$ with $G(u) = (1 - g_1 e^{dt})/(1 - g_1)$ and $\alpha = \frac{\kappa\theta}{\sigma^2}$. This depends on that $G(u)$ has a shape of a spiral as u increases and when it repeatedly crosses the negative real axis the phase of $G(u)$ changes from $-\alpha\pi$ to $\alpha\pi$ causing a discontinuity when α is not a natural number. Hence we can use the following alternative representation, for large maturities which is continuous and give s a numerically stable characteristic function and hence prices in the full dimensional and unrestricted parameter space.

$$\phi(\theta; u, t) = \exp\{iu\left(\log(S_0) + rt\right) + \frac{\kappa\theta}{\sigma^2}\left[(\xi - d)\,t - 2\log\frac{1 - g_2 e^{-d\cdot t}}{1 - g_2}\right] + \frac{V_0}{\sigma^2}\,(\xi - d)\,\frac{1 - e^{-d\cdot t}}{1 - g_2 e^{-d\cdot t}}\}$$

where

$$g_2 = \frac{\xi - d}{\xi + d} = \frac{1}{g_1}$$

B Appendix: Quadratic–Exponential (QE) Scheme for the Heston Model

This appendix summarises *verbatim* the key formulas of Andersen's bias-free Quadratic–Exponential (QE) scheme while reorganising them into logical blocks and an implementable algorithm.

B.1 Bias-free sampling of (S_t, V_t)

Integrate the square-root variance SDE on $[t, t + \Delta]$:

$$V_{t+\Delta} = V_t + \kappa(\theta - V_u) * du + \sigma \int_t^{t+\Delta} \sqrt{V_u}\, dW_V(u), \tag{28}$$

$$\int_t^{t+\Delta} \sqrt{V_u}\, dW_V(u) = \sigma^{-1}\Big(V_{t+\Delta} - V_t - \kappa\theta\Delta + \kappa \int_t^{t+\Delta} V_u\, du\Big). \tag{29}$$

A Cholesky decomposition of the correlated Brownian motions yields

$$d\ln X_t \;=\; -\tfrac{1}{2} V_t\, dt + \rho\sqrt{V_t}\, dW_V(t) + \sqrt{1-\rho^2}\sqrt{V_t}\, dW(t),$$

with W independent of W_V. Hence

$$\begin{aligned}\ln X_{t+\Delta} = \;& \ln X_t + \frac{\rho}{\sigma}\big(V_{t+\Delta} - V_t - \kappa\theta\Delta\big) + \Big(\tfrac{\kappa\rho}{\sigma} - \tfrac{1}{2}\Big)\int_t^{t+\Delta} V_u\, du \\ & + \sqrt{1-\rho^2}\int_t^{t+\Delta} \sqrt{V_u}\, dW(u).\end{aligned} \tag{30}$$

Conditionally on V_t and $\int_t^{t+\Delta} V_u\, du$, the final Itô–integral is Gaussian with mean 0 and variance $\int_t^{t+\Delta} V_u\, du$.

B.2 Exact one-step law of V

V follows a CIR process; its exact transition is non-central χ^2:

$$\mathbb{P}\big(V_{t+\Delta} < v \;\big|\; V_t\big) \;=\; F_{\chi'^2}\Big(\tfrac{4\kappa v}{\sigma^2\,(1-e^{-\kappa\Delta})};\, \tfrac{4\kappa\theta}{\sigma^2},\, \tfrac{4\kappa e^{-\kappa\Delta} V_t}{\sigma^2\,(1-e^{-\kappa\Delta})}\Big).$$

Its conditional moments are

$$m = \mathrm{e}(V_{t+\Delta} \mid V_t) = \theta + (V_t - \theta)e^{-\kappa\Delta}, \tag{31}$$

$$s^2 = \mathrm{Var}(V_{t+\Delta} \mid V_t) = \frac{\sigma^2 e^{-\kappa\Delta}}{\kappa}\Big[V_t(1 - e^{-\kappa\Delta}) + \tfrac{\theta}{2}(1 - e^{-\kappa\Delta})\Big]. \tag{32}$$

B.3 Quadratic–Exponential (QE) approximation

Define the shape parameter

$$\psi := \frac{s^2}{m^2}, \qquad \psi_c \in [1, 2] \text{ (e.g. 1.5)}.$$

Regime I — $\psi \le \psi_c$. Set

$$b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1}, \quad a = \frac{m}{1+b^2}$$

and simulate

$$V_{t+\Delta} = a\,(b + Z_V)^2, \qquad Z_V \sim \mathcal{N}(0,1).$$

Regime II — $\psi > \psi_c$. Set

$$p = \frac{\psi - 1}{\psi + 2}, \qquad \beta = \frac{1-p}{m} = \frac{2}{m(1+\psi)},$$

then draw $u \sim \text{Unif}[0,1]$ and use the inverse CDF

$$V_{t+\Delta} = \begin{cases} 0, & u \le p, \\ \beta^{-1} \ln\left[\frac{1-p}{1-u}\right], & u > p. \end{cases}$$

B.4 Algorithm: one QE step

Algorithm 1 Bias-free QE update from t to $t + \Delta$

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1. **Inputs:** V_t, X_t, parameters $(\kappa, \theta, \sigma, \rho)$, step Δ.
2. Compute m, s^2 and $\psi = s^2/m^2$.
3. *If* $\psi \le \psi_c$ (Regime I):
   (a) find b^2, a as above;
   (b) draw $Z_V \sim \mathcal{N}(0,1)$; set $V_{t+\Delta} = a\,(b + Z_V)^2$.

   Else (Regime II):
   (a) compute p, β;
   (b) draw $u \sim \text{Unif}[0,1]$ and set $V_{t+\Delta} = 0$ if $u \le p$, else $\beta^{-1}\ln[(1-p)/(1-u)]$.
4. **Log-asset update.** Approximate $\displaystyle\int_t^{t+\Delta} V_u\,du \approx \Delta\,[\gamma_1 V_t + \gamma_2 V_{t+\Delta}]$ with $\gamma_1 + \gamma_2 = 1,\ \gamma_i \ge 0$, and evolve

   $$\ln X_{t+\Delta} = \ln X_t + K_0 + K_1 V_t + K_2 V_{t+\Delta} + \sqrt{K_3 V_t + K_4 V_{t+\Delta}}\,Z,$$

   where $Z \sim \mathcal{N}(0,1)$ is independent of V, and

   $$K_0 = -\tfrac{\rho\kappa\theta}{\sigma}\Delta, \quad K_1 = \gamma_1\Delta\left(\tfrac{\kappa\rho}{\sigma} - \tfrac{1}{2}\right) - \tfrac{\rho}{\sigma},$$
   $$K_2 = \gamma_2\Delta\left(\tfrac{\kappa\rho}{\sigma} - \tfrac{1}{2}\right) + \tfrac{\rho}{\sigma}, \qquad K_3 = \gamma_1\Delta(1-\rho^2), \quad K_4 = \gamma_2\Delta(1-\rho^2).$$
5. **Martingale correction (optional).** Replace K_0 by K_0^*:

   $$K_0^* = \begin{cases} -\dfrac{Ab^2a}{1-2Aa} + \dfrac{1}{2}\ln(1-2Aa) - \left(K_1 + \tfrac{1}{2}\Delta\gamma_1\right), & \psi \le \psi_c, \\ -\ln\left[\frac{\beta(1-\rho)}{\beta - A}\right] - \left(K_1 + \tfrac{1}{2}\Delta\gamma_1\right), & \psi > \psi_c, \end{cases}$$

   with $A = \dfrac{\rho}{\sigma^2}\left(1 + \kappa\gamma_2\Delta\right) - \tfrac{1}{2}\gamma_2\Delta\rho^2$.

B.5 Implementation notes

- QE preserves positivity of V and avoids Euler bias.
- Choose $\psi_c \approx 1.5$ (threshold value for switching between sampling algorithms).