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Approximation Methods for
Inhomogeneous Geometric Brownian Motion

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We present an accurate and easy-to-compute approximation of the transition probabilities and the associated Arrow-Debreu (AD) prices for the inhomogeneous geometric Brownian motion (IGBM) model for interest rates, default intensities or volatilities. Through this procedure, dubbed *exponent expansion*, transition probabilities and AD prices are obtained as a power series in time to maturity. This provides remarkably accurate results—for time horizons up to several years—even when truncated after the first few terms. For farther time horizons, the exponent expansion can be combined with a fast numerical convolution to obtain high-precision results.

Keywords: inhomogeneous geometric Brownian motion; constant elasticity of variance; Arrow-Debreu security, derivative pricing; power series expansions.

1. Introduction

Continuous-time diffusion processes are important modelling tools in quantitative finance with applications ranging from portfolio optimization and econometric applications to contingent claim pricing. In particular, short-rate models of the form

$$dY_t = \mu_y(Y_t)dt + \sigma_y(Y_t)dB_t, \tag{1.1}$$

are of significant importance in financial modelling, providing the foundation of many approaches used for pricing of both interest rate and credit derivatives; see

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e.g., Andersen and Piterbarg (2010), O'Kane (2008). Here $\mu_y(y)$ is the drift function, describing the trend of the process, and $\sigma_y(y) \geq 0$ is the volatility function that controls the level of randomness induced by the Brownian motion $(B_t)_{t \geq 0}$.

Among short rate models, affine models—see Duffie *et al.* (2000)—which include e.g., Vasicek (1977), Hull & White (1990) and Cox *et al.* (1985), play an important role because of their analytical tractability. Such models lead to analytical expressions for the transition probabilities and furthermore to closed-form price processes of the Arrow-Debreu (AD) prices and zero-coupon bonds. In the context of default intensity models, one may obtain survival probabilities in closed form. We here refer to, e.g., Shreve (2004) and O'Kane (2008). However, a better description of the market observables often requires richer models than those for which an analytic solution is available. These are usually tackled by means of computationally expensive numerical schemes, like Monte Carlo (MC) or partial differential equations (PDE), ultimately relying on a discretization of the diffusion processes.

In the context of econometric applications, closed-form expressions for the transition probabilities are central for maximum-likelihood estimations of the parameters in diffusion models. In the context of derivatives pricing, lacking a closed-form solution for fundamental building blocks, like zero-coupon bond prices or survival probabilities, is particularly onerous in the context of multi factor problems, notably the ones involving the calculation of the portfolio level 'valuation adjustments' known as XVA (see, e.g., Gregory (2010)), currently very prominent in financial engineering. In both cases, reliable analytical approximations are important in order to reduce the numerical burden associated with these computations.

In this paper, we develop analytical approximations for the inhomogeneous geometric Brownian motion (IGBM) satisfying

$$dY_t = a(b - Y_t)\,dt + \sigma Y_t\,dB_t, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}$ are the mean reversion speed and level, respectively, and $\sigma \geq 0$ is the volatility. The IGBM—see Bhattacharya (1978) and Zhao (2009)—is a special case of the constant elasticity of variance (CEV) process; cf. Cox & Ross (1976). It is a conspicuous example of a diffusion that is particularly suitable for financial applications (see Aït Sahalia (1999), Linetsky (2004), Kitapbayev & Leung (2017)), due to its positive definiteness and mean-reverting features, which makes it an alternative choice to the Cox *et al.* (1985) or Black and Karasinski (1991) models as a means to describe the dynamics of default intensities, volatilities or interest rates. In particular, the IGBM is used among practitioners in the context of pricing of options on credit default swaps (see, e.g., O'Kane (2008)) and as a model of asset prices in the context of mean reversion strategies, as in Kitapbayev & Leung (2017). Unfortunately, it lacks the level of analytical tractability of affine models and little work has been done to date to develop useful approximations [a].

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[a]After this manuscript was completed, we became aware of the work of Li *et al.* (2018) in which an alternative approximation scheme is presented for the same non-linear diffusion process.

By employing the so-called exponent expansion (EE) — see Capriotti (2006) — we derive analytical approximations of transition densities $\rho_y(0, t, y_0, y)$ or probabilities, that is

$$\int_A \rho_y(0, T, y_0, y)\mathrm{d}y \equiv \mathbb{P}\left[Y_T \in A \,|\, Y_0 = y_0\right], \tag{1.3}$$

important in econometric maximum likelihood estimations, and of AD prices, the fundamental building blocks for pricing contingent claims; see, e.g., Andersen and Piterbarg (2010) and Shreve (2004)

$$\psi_y(0, T, y_0, y) = \mathbb{E}\left[\delta(Y_T - y) \exp\left(-\int_0^T Y_u \mathrm{d}u\right) \Big| Y_0 = y_0\right], \tag{1.4}$$

where $\delta(\cdot)$ is the Dirac-delta function.

The EE, originally introduced in chemical physics by Makri & Miller (1989), was introduced to finance by Capriotti (2006) and applied to the calculation of transition probabilities and AD prices of several diffusion processes, including Hull & White (1990), Cox *et al.* (1985) and Black and Karasinski (1991), demonstrating a remarkable congruence with the exact analytical solutions available for these models. By the EE, transition probabilities and AD prices are obtained as a power series in the expiry date T of the financial claim, which becomes asymptotically exact if an increasing number of terms is included. The expansion produces remarkably accurate results even truncating it to the first few (say, $n = 3$) terms.

The paper is organised as follows. In the next section we briefly review the EE-method for the transition probabilities and AD prices. In Section 3 we present the derivation of the exponent expansion for the IGBM model (1.2) up to the fourth order, and in Section 3.1 we produce numerical results, including for zero-coupon bond prices, illustrating the accuracy of the approximation. Finally, we conclude in Section 4.

2. Exponent expansion

In order to simplify the derivation, it is convenient to transform the original process into an auxiliary one, (X_t), with constant volatility σ. Following Aït Sahalia (1999), this can be achieved in general through the following integral transformation

$$X_t = \gamma(Y_t) \equiv \sigma \int_0^{Y_t} \frac{\mathrm{d}z}{\sigma_y(z)} . \tag{2.1}$$

A straightforward application of Ito's Lemma gives the stochastic differential equation satisfied by (X_t) for $t \geq 0$:

$$\mathrm{d}X_t = \mu_x(X_t)\mathrm{d}t + \sigma\mathrm{d}B_t, \tag{2.2}$$

where

$$\mu_x(x) = \sigma\left[-\frac{\mu_y(\gamma^{-1}(x))}{\sigma_y(\gamma^{-1}(x))} - \frac{1}{2}\frac{\partial\sigma_y}{\partial y}(\gamma^{-1}(x))\right]. \tag{2.3}$$

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Here, $y = \gamma^{-1}(x)$ is the inverse of the transformation (2.1). The transition probabilities and AD prices for the processes (X_t) and (Y_t) are related by the Jacobian associated with (2.1) giving

$$\rho_y(0, t, y_0, y) = \sigma \frac{\rho(0, t, x_0, \gamma(y))}{\sigma_y(y)} \ , \qquad \psi_y(0, t, y_0, y) = \sigma \frac{\psi(0, t, x_0, \gamma(y))}{\sigma_y(y)} \ . \qquad (2.4)$$

2.1. *Transition densities*

In order to find an expression for the transition probabilities associated with Eq. (2.2), which is accurate for a time T as long as possible, we make the ansatz

$$\rho(0, T, x_0, x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{(x - x_0)^2}{2\sigma^2 T} - W(x, x_0, T) \right] \ . \qquad (2.5)$$

Such transition probabilities must satisfy the Kolmogorov forward (or Fokker-Planck) equation:

$$\partial_t \rho(0, T, x_0, x) = \left[-\partial_x \mu_x(x) + \frac{1}{2}\sigma^2 \partial_x^2 \right] \rho_x(0, T, x_0, x) \ . \qquad (2.6)$$

This implies in turn that the function $W(x, x_0, t)$ satisfies the equation

$$\partial_t W = -\mu_x \partial_x W + \frac{1}{2}\sigma^2 \partial_x^2 W - \frac{1}{2}\sigma^2 \left(\partial_x W\right)^2 + \partial_x \mu_x - \frac{x - x_0}{T} \left(\partial_x W + \frac{\mu_x}{\sigma^2}\right) \ . \qquad (2.7)$$

Expanding the function $W(x, x_0, t)$ in powers of T, we obtain

$$W(x, x_0, T) = \sum_{n=0}^{\infty} W_n(x, x_0) \, T^n \ . \qquad (2.8)$$

Then, substituting in Eq. (2.7) and equating equal powers of T leads in a straightforward way to a decoupled equation for the order zero in T,

$$W_0(x, x_0) = -\frac{1}{\sigma^2} \int_{x_0}^{x} \mathrm{d}z \, \mu_x(z) \ , \qquad (2.9)$$

and to the set of recursive differential equations

$$(n+1)W_{n+1} = -(x - x_0)\partial_x W_{n+1} + \left[\frac{1}{2}\sigma^2 \partial_x^2 - \mu_x \partial_x \right] W_n$$

$$- \frac{1}{2}\sigma^2 \sum_{m=0}^{m=n} \partial_x W_m \partial_x W_{n-m} + \delta_{n,0} \partial_x \mu_x \ . \qquad (2.10)$$

The differential equations (2.10) are all first order, linear, inhomogeneous and of the form

$$nW_n(x, x_0) = -(x - x_0)\partial_x W_n(x, x_0) + \Lambda_{n-1}(x, x_0) \ , \qquad (2.11)$$

where $\Lambda_{n-1}(x, x_0)$ is a function that is completely determined by the first $n - 1$ relations. It can be readily verified, by substitution and integration by parts, that the solution of (2.11) is given by

$$W_n(x, x_0) = \int_0^1 \xi^{n-1} \Lambda_{n-1}(x_0 + (x - x_0)\xi, x_0)\mathrm{d}\xi. \qquad (2.12)$$

Analogous to the expansion developed in Aït Sahalia (1999), the exponent expansion has in general a finite convergence radius which is a decreasing function of the volatility. As it will be shown in what follows, for the values of volatilities and T relevant for financial applications, the exponent expansion turns out to be very accurate even when truncated after the first few terms.

2.2. *Arrow-Debreu Prices*

It is well known that the AD price (1.4) satisfies the conjugate forward partial differential equation (PDE) given by

$$\partial_t \psi(0, T, x_0, x) = \left(-\gamma^{-1}(x) - \partial_x \mu_x(x) + \frac{1}{2}\sigma^2 \partial_x^2 \right) \psi(0, T, x_0, x) \tag{2.13}$$

where $\psi(0, 0, x_0, x) = \delta(x_0 - x)$ is the initial condition. As in Section 2.1, we insert the ansatz

$$\psi(0, T, x_0, x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{(x - x_0)^2}{2\sigma^2 T} - W(x, x_0, T) \right], \tag{2.14}$$

for the AD prices into Eq. (2.13), and we obtain

$$\partial_t W = -\mu_x(x)\partial_x W + \frac{1}{2}\sigma^2 \partial_x^2 W - \frac{1}{2}\sigma^2 \left(\partial_x W \right)^2$$
$$+ \gamma^{-1}(x) + \partial_x \mu_x(x) - \frac{x - x_0}{t} \left(\partial_x W + \frac{\mu_x(x)}{\sigma^2} \right). \tag{2.15}$$

Expanding the function $W(x, x_0, t)$ in powers of T—as in Eq. (2.8)— substituting it in Eq. (2.15) and matching equal powers of T, leads to Eq. (2.9) and to the following set of recursive differential equations:

$$(n + 1)W_{n+1} = -(x - x_0)\partial_x W_{n+1} + \left[\frac{1}{2}\sigma^2 \partial_x^2 - \mu_x(x)\partial_x \right] W_n$$
$$- \frac{1}{2}\sigma^2 \sum_{m=0}^{n} \partial_x W_m \partial_x W_{n-m} + \delta_{n,0}(\gamma^{-1}(x) + \partial_x \mu_x(x)) . \tag{2.16}$$

The solution is of the form (2.12).

2.3. *Extensions to long time horizons*

For long time horizons T, the accuracy of the exponent expansion can be improved by dividing the time interval into $N > 1$ subintervals and using the following property of the transition density (or AD prices) associated with (homogeneous) Markov processes:

$$\Phi(0, T, x_0, x_N) = \int \cdots \int \prod_{i=1}^{N} \Phi\left(0, \frac{T}{N}, x_{i-1}, x_i \right) dx_1 dx_2 \cdots dx_{N-1}. \tag{2.17}$$

The function $\Phi(0, t, x_j, x_i)$ stands either for the transition density in (1.3) or the AD price in (1.4). One can use the exponent expansion to approximate the transition densities (or the AD price) $\Phi(0, T/N, x_i, x_{i-1})$ on each T/N sub-interval and

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concatenate them by $N - 1$ integrations over the real line. As shown in Capriotti (2006), Eq. (2.17) can be computed efficiently by means of matrix multiplications. We note that this approach can also be followed to extend the results of this paper to the case of piece-wise constant coefficients in the IGBM diffusion (1.2). In such a case, the division of the time interval is performed in such a way that the coefficients are constant on each of the subintervals.

3. EE applied to the IGBM model

For the IGBM model (1.2), the volatility function reads $\sigma_y(y) = \sigma y$, and the function $\gamma(y)$, defined by the integral transformation (2.1), is simply $\gamma(y) = \ln(y)$. The drift of the auxiliary process (2.2), that is expression (2.3), is therefore given by

$$\mu_x(x) = -a(b - e^x) - \frac{1}{2}. \tag{3.1}$$

In the Appendix, we give the expression for the the functions $W_n(x)$ in Eqs. (2.5) and (2.14) up to order $n = 4$.

Given the expression for $W_n(x)$ and the corresponding approximations for the AD price, approximate expressions for the T-maturity zero-coupon bond price $P_{0,T}$ can be obtained by means of numerical integration applied to

$$P_{0,T} = \mathbb{E}\left[\exp\left(-\int_0^T Y_u du\right) \Big| y_0\right] = \int \psi_y(0, T, y_0, y) dy; \tag{3.2}$$

see, e.g., Andersen and Piterbarg (2010).

3.1. *Numerical results*

The accuracy of the exponent expansion for the transition density of the IGBM model (1.2) is illustrated in Figure 1. Here and in the following, in order to ascertain the accuracy of the EE we have computed the transition density, and AD prices using numerical solution by means of the standard Crank-Nicholson method; see, e.g., Andersen and Piterbarg (2010), of the PDEs (2.6), and (2.13), respectively. As previously observed for other diffusions (Capriotti (2006)) the exponent expansion is characterized by a remarkably fast convergence by including successive terms of the approximation.

As illustrated in Figure 2, similarly to the case of the transition probabilities, the exponent expansion provides a remarkably good, and fast converging approximation of the AD prices for financially sensible parametrizations, and for a sizeable value of the time step T.

In Table 1 we show the results for T maturity zero-coupon bonds (3.2) for the IGBM model as obtained by means of the EE up to $n = 4$. These illustrate how, as in the case of transition densities and AD prices, the EE is characterized by a rapid convergence with the order n, so that $n = 4$ provides already a virtually exact representation of the zero-coupon bond price, for $T \simeq 1$ years.

Table 1. IGBM T maturity zero-coupon bonds obtained with the exponent expansion (EE) up to 4-th order and PDE numerical solutions. The parameters of the IGBM process are: mean reversion speed $a = 0.1$, level $b = 0.04$, volatility $\sigma = 0.6$ and initial rate $y_0 = 0.06$.

t	EE(1)	EE(2)	EE(3)	EE(4)	PDE
0.1	0.99403	0.99403	0.99403	0.99403	0.99403
0.5	0.97066	0.97068	0.97071	0.97071	0.97071
1.0	0.94251	0.94264	0.94286	0.94286	0.94286
2.0	0.88940	0.88991	0.89143	0.89142	0.89105
3.0	0.84125	0.84174	0.84608	0.84586	0.84464

Table 2. IGBM T maturity zero-coupon bonds obtained with the 4-th order exponent expansion (EE) and N-fold convolutions Eq. (2.17), for $T/N = 5$, 2.5 and 1.0. The PDE numerical solutions are reported for comparison. The parameters of the IGBM process are: mean reversion speed $a = 0.1$, level $b = 0.04$, volatility $\sigma = 0.6$ and initial rate $y_0 = 0.06$.

t	$\text{EE}_5(4)$	$\text{EE}_{2.5}(4)$	$\text{EE}_1(4)$	PDE
5.0	0.78337	0.77137	0.77257	0.77251
10.0	0.62781	0.64212	0.64780	0.64778
15.0	0.50838	0.55135	0.56125	0.56120
20.0	0.41293	0.47743	0.49057	0.49046

As previously remarked, an important advantage of the EE is the possibility to systematically improve its accuracy over large time horizons by means of the convolution approach described in Section 2.3. This is illustrated in Table 2 showing how splitting the time interval dramatically improves the convergence. This allows the EE to produce remarkably accurate results (to four significant digits) even for zero-coupon bonds with maturities over twenty years, as also shown in Figure 3.

4. Conclusions

In this paper we have presented the application of the exponent expansion (EE) – an accurate and easy-to-compute approximation borrowed from the physical sciences – to the calculation of transition probabilities, Arrow-Debreu (AD) prices and zero-coupon bonds for the inhomogeneous geometric Brownian motion (IGBM), a model that has recently been proposed as a more realistic representation of interest rates or default intensities; see e.g., Zhao (2009), Li *et al.* (2018). The EE is based on an exponential ansatz for the transition probabilities and AD prices over a time horizon T and a series expansion of the deviation of its logarithm from that of a Gaussian process. Through this procedure, transition densities and AD prices are obtained

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as a power series in T that can be easily computed with a recursion involving only one-dimensional integrals. With several numerical examples, we have shown that the EE becomes asymptotically exact in the limit of small time to maturity but is remarkably accurate for time horizons up to several years. We have also shown how the range of application of the expansion can be systematically extended to large time horizons by means of a fast numerical convolution.

The accuracy and ease of computation of the exponent expansion makes it an effective practical alternative to PDE and MC for the computation of discount factors or survival probabilities in the IGBM model. This is especially useful when these building block are used in the context of time consuming multi-factor simulations like those necessary for the calculation of counterparty exposures or regulatory capital, particularly in presence of the so-called *wrong-way risk*; see, e.g., Gregory (2010).

5. Acknowledgments

Appendix A.

We here produce the explicit terms of the expansion, up to order four, of the function $W(x, x_0, T)$ given by Eq. (2.8) for both, the transition density and the associated Arrow-Debreu price.

A.1. *Transition densities*

The functions $W_n(x)$ in Eq. (2.8) are given, up to order $n = 4$, by

$$W_0(x) = \frac{ab\left(\mathrm{e}^{-x} - \mathrm{e}^{x_0}\right) + \left(a + \frac{\sigma^2}{2}\right)(x - x_0)}{\sigma^2},$$

$$W_1(x) = \frac{1}{8}\left(4a + \sigma^2 + \frac{4a^2}{\sigma^2} - \frac{2a^2b^2\left(\mathrm{e}^{-2x} - \mathrm{e}^{-2x_0}\right) + 8a^2b\left(\mathrm{e}^{-x} - \mathrm{e}^{-x_0}\right)}{\sigma^2(x - x_0)}\right.$$
$$\left. + \frac{8ab\left(\mathrm{e}^{-x} - \mathrm{e}^{-x_0}\right)}{x - x_0}\right),$$

$$W_2(x) = \frac{a^2b^2\mathrm{e}^{-2x}(x - x_0 + 1)}{4(x - x_0)^3} + \frac{a^2b^2\mathrm{e}^{-2x_0}(x - x_0 - 1)}{4(x - x_0)^3} - \frac{a^2b\mathrm{e}^{-x}(x - x_0 + 2)}{2(x - x_0)^3}$$
$$- \frac{ab\sigma^2\mathrm{e}^{-x}(x - x_0 + 2)}{2(x - x_0)^3} - \frac{ab\,\mathrm{e}^{-x_0}\left(a + \sigma^2\right)(x - x_0 - 2)}{2(x - x_0)^3},$$

$$W_3(x) = \frac{b^4\mathrm{e}^{-4x}(x - x_0 + 1)a^4}{32\sigma^2(x - x_0)^4} + \frac{b^2\mathrm{e}^{-2x}(x - x_0 + 2)a^4}{4\sigma^2(x - x_0)^4} - \frac{b^4\mathrm{e}^{-4x_0}(x - x_0 - 1)a^4}{32\sigma^2(x - x_0)^4}$$

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$$-\frac{b^3 e^{-3x}(2x-2x_0+3)a^4}{12\sigma^2(x-x_0)^4}+\frac{b^2 e^{-2x}(x-x_0+2)a^3}{2(x-x_0)^4}$$

$$+\frac{b^3 e^{-3x_0}\left(\sigma^2+a\right)(2x-2x_0-3)a^3}{12\sigma^2(x-x_0)^4}-\frac{b^2 e^{-2x}\sigma^2(x-x_0+3)a^2}{4(x-x_0)^5}$$

$$-\frac{b^3 e^{-3x}(2x-2x_0+3)a^3}{12(x-x_0)^4}+\frac{b e^{-x}\sigma^2\left(x^2-2x_0 x+6x+x_0{}^2-6x_0+12\right)a^2}{4(x-x_0)^5}$$

$$+\frac{b e^{-x}\sigma^4\left(x^2-2x_0 x+6x+x_0{}^2-6x_0+12\right)a}{4(x-x_0)^5}$$

$$+ e^{-2x_0}\left(\frac{b^3 e^{-x}a^4}{6\sigma^2(x-x_0)^3}-\frac{b^3 e^{-x}(2x-2x_0-3)a^4}{12\sigma^2(x-x_0)^4}-\frac{b^2(x-x_0-2)a^4}{4\sigma^2(x-x_0)^4}\right.$$

$$-\frac{b^4 e^{-2x}a^4}{16\sigma^2(x-x_0)^4}-\frac{b^3 e^{-x}(2x-2x_0-3)a^3}{12(x-x_0)^4}-\frac{b^2(x-x_0-2)a^3}{2(x-x_0)^4}$$

$$\left.+\frac{b^3 e^{-x}a^3}{6(x-x_0)^3}-\frac{b^2\sigma^2(x-x_0-3)a^2}{4(x-x_0)^5}\right)+ e^{-x_0}\left(-\frac{b^2 e^{-x}(x-x_0+2)a^4}{4\sigma^2(x-x_0)^4}\right.$$

$$-\frac{b^3 e^{-2x}a^4}{6\sigma^2(x-x_0)^3}+\frac{b^3 e^{-2x}(2x-2x_0+3)a^4}{12\sigma^2(x-x_0)^4}+\frac{b^2 e^{-x}(x-x_0-2)a^4}{4\sigma^2(x-x_0)^4}$$

$$-\frac{b^2 e^{-x}(x-x_0+2)a^3}{2(x-x_0)^4}-\frac{b^3 e^{-2x}a^3}{6(x-x_0)^3}+\frac{b^3 e^{-2x}(2x-2x_0+3)a^3}{12(x-x_0)^4}$$

$$+\frac{b^2 e^{-x}(x-x_0-2)a^3}{2(x-x_0)^4}-\frac{b\sigma^2\left(x^2-2x_0 x-6x+x_0{}^2+6x_0+12\right)a^2}{4(x-x_0)^5}$$

$$\left.-\frac{b^2 e^{-x}\sigma^2 a^2}{(x-x_0)^4}-\frac{b\sigma^4\left(x^2-2x_0 x-6x+x_0{}^2+6x_0+12\right)a}{4(x-x_0)^5}\right),$$

$$W_4(x)= -\frac{a^4 e^{-4x}\left(2x^2-4x_0 x+5x+2x_0{}^2-5x_0+4\right)b^4}{16(x-x_0)^6}$$

$$+\frac{a^3 e^{-3x}\left(\sigma^2+a\right)\left(2x^2-4x_0 x+7x+2x_0{}^2-7x_0+8\right)b^3}{4(x-x_0)^6}$$

$$-\frac{a^2 b^2 e^{-2x}}{4(x-x_0)^7}\left(x^3\sigma^4-x_0{}^3\sigma^4+4x^2\sigma^4+3xx_0{}^2\sigma^4+4x_0{}^2\sigma^4+x\sigma^4\right.$$

$$-3x^2 x_0\sigma^4-8xx_0\sigma^4-x_0\sigma^4-15\sigma^4+4ax^3\sigma^2-4ax_0{}^3\sigma^2+20ax^2\sigma^2$$

$$+20ax_0{}^2\sigma^2+12axx_0{}^2\sigma^2+32ax\sigma^2-12ax^2 x_0\sigma^2-32ax_0\sigma^2-40axx_0\sigma^2$$

$$+2a^2 x^3-2a^2 x_0{}^3+10a^2 x^2+10a^2 x_0{}^2+6a^2 xx_0{}^2+16a^2 x-16a^2 x_0$$

$$\left.-6a^2 x^2 x_0-20a^2 xx_0\right)-\frac{abe^{-x}\sigma^4\left(\sigma^2+a\right)}{8(x-x_0)^7}\left(x^3-3x_0 x^2+12x^2+3x_0{}^2 x\right.$$

$$\left.-24x_0 x+60x-x_0{}^3+12x_0{}^2-60x_0+120\right)+ e^{-4x_0}\left(-\frac{a^4 b^4}{8(x-x_0)^4}\right.$$

$$\left.+\frac{5a^4 b^4}{16(x-x_0)^5}-\frac{a^4 b^4}{4(x-x_0)^6}\right)+ e^{-3x_0}\left(\frac{b^3 a^4}{2(x-x_0)^4}-\frac{7b^3 a^4}{4(x-x_0)^5}\right.$$

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$$
\begin{aligned}
&+ \frac{2b^3a^4}{(x-x_0)^6} + \frac{b^3\sigma^2a^3}{2(x-x_0)^4} - \frac{7b^3\sigma^2a^3}{4(x-x_0)^5} + \frac{2b^3\sigma^2a^3}{(x-x_0)^6} \Bigg) \\
&+ \mathrm{e}^{-2x_0}\Bigg(-\frac{b^2a^4}{2(x-x_0)^4} + \frac{5b^2a^4}{2(x-x_0)^5} + \frac{b^4\mathrm{e}^{-2x}a^4}{2(x-x_0)^6} - \frac{4b^2a^4}{(x-x_0)^6} - \frac{b^2\sigma^2a^3}{(x-x_0)^4} \\
&+ \frac{5b^2\sigma^2a^3}{(x-x_0)^5} - \frac{8b^2\sigma^2a^3}{(x-x_0)^6} + \frac{b^3\mathrm{e}^{-x}\left(\sigma^2+a\right)(x-x_0-8)a^3}{4(x-x_0)^6} - \frac{b^2\sigma^4a^2}{4(x-x_0)^4} \\
&+ \frac{b^2\sigma^4a^2}{(x-x_0)^5} - \frac{b^2\sigma^4a^2}{4(x-x_0)^6} - \frac{15b^2\sigma^4a^2}{4(x-x_0)^7}\Bigg) + \mathrm{e}^{-x_0}\Bigg(-\frac{ab\sigma^6}{8(x-x_0)^4} \\
&+ \frac{3ab\sigma^6}{2(x-x_0)^5} - \frac{15ab\sigma^6}{2(x-x_0)^6} + \frac{15ab\sigma^6}{(x-x_0)^7} - \frac{a^2b\sigma^4}{8(x-x_0)^4} + \frac{3a^2b\sigma^4}{2(x-x_0)^5} \\
&- \frac{15a^2b\sigma^4}{2(x-x_0)^6} + \frac{15a^2b\sigma^4}{(x-x_0)^7} - \frac{a^3b^3\mathrm{e}^{-2x}\left(\sigma^2+a\right)(x-x_0+8)}{4(x-x_0)^6} \\
&+ \frac{8a^2b^2\mathrm{e}^{-x}\left(\sigma^2+a\right)^2}{(x-x_0)^6}\Bigg).
\end{aligned}
$$

A.2. *Arrow-Debreu price*

Considering Eq. (2.9) and this time the recursion (2.16), the functions $W_n(x)$ in Eq. (2.8) are given, up to order $n=4$, by

$$
\begin{aligned}
W_0(x) =& \frac{ab\left(\mathrm{e}^{-x}-\mathrm{e}^{x_0}\right)+\left(a+\frac{\sigma^2}{2}\right)(x-x_0)}{\sigma^2}, \\
W_1(x) =& \frac{a^2\left(-2b^2\mathrm{e}^{-2x}+2b^2\mathrm{e}^{-2x_0}+8b\mathrm{e}^{-x}-8b\mathrm{e}^{-x_0}+4x-4x_0\right)}{8\sigma^2(x-x_0)} \\
&+ \frac{4a\sigma^2\left(2b\mathrm{e}^{-x}-2b\mathrm{e}^{-x_0}+x-x_0\right)}{8\sigma^2(x-x_0)} + \frac{\sigma^2\left(\sigma^2(x-x_0)+8\mathrm{e}^{x}-8\mathrm{e}^{x_0}\right)}{8\sigma^2(x-x_0)}, \\
W_2(x) =& \frac{a^2b^2\mathrm{e}^{-2x}(x-x_0+1)}{4(x-x_0)^3} + \frac{a^2b^2\mathrm{e}^{-2x_0}(x-x_0-1)}{4(x-x_0)^3} - \frac{a^2b\mathrm{e}^{-x}(x-x_0+2)}{2(x-x_0)^3} \\
&- \frac{ab\sigma^2\mathrm{e}^{-x}(x-x_0+2)}{2(x-x_0)^3} - \frac{ab\mathrm{e}^{-x_0}\left(a+\sigma^2\right)(x-x_0-2)}{2(x-x_0)^3} \\
&+ \frac{\sigma^2\mathrm{e}^{x_0}(x-x_0+2)}{2(x-x_0)^3} + \frac{\sigma^2\mathrm{e}^{x}(x-x_0-2)}{2(x-x_0)^3}, \\
W_3(x) =& \frac{b^4\mathrm{e}^{-4x}(x-x_0+1)a^4}{32\sigma^2(x-x_0)^4} + \frac{b^2\mathrm{e}^{-2x}(x-x_0+2)a^4}{4\sigma^2(x-x_0)^4} - \frac{b^3\mathrm{e}^{-3x}(2x-2x_0+3)a^4}{12\sigma^2(x-x_0)^4} \\
&- \frac{b^4\mathrm{e}^{-4x_0}(x-x_0-1)a^4}{32\sigma^2(x-x_0)^4} + \frac{b^2\mathrm{e}^{-2x}(x-x_0+2)a^3}{2(x-x_0)^4} - \frac{b^3\mathrm{e}^{-3x}(2x-2x_0+3)a^3}{12(x-x_0)^4} \\
&- \frac{b^2\mathrm{e}^{-2x}\sigma^2(x-x_0+3)a^2}{4(x-x_0)^5} + \frac{b\left(x^2-2x_0x+x_0^2+2\right)a^2}{(x-x_0)^4} - \frac{\mathrm{e}^{2x}\sigma^2(x-x_0-2)}{4(x-x_0)^4} \\
&+ \frac{b\mathrm{e}^{-x}\sigma^2\left(x^2-2x_0x+6x+x_0^2-6x_0+12\right)a^2}{4(x-x_0)^5} + \frac{b^2\mathrm{e}^{-x}(2x-2x_0-1)a^2}{4(x-x_0)^4}
\end{aligned}
$$

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$$+\frac{b\sigma^2\left(x^2-2x_0 x+x_0{}^2+2\right)a}{(x-x_0)^4}+\frac{be^{-x}\sigma^4\left(x^2-2x_0 x+6x+x_0{}^2-6x_0+12\right)a}{4(x-x_0)^5}$$

$$+e^{-2x_0}\left(-\frac{b^2 a^4}{4\sigma^2(x-x_0)^3}-\frac{b^4 e^{-2x}a^4}{16\sigma^2(x-x_0)^4}+\frac{b^3 e^{-x}a^4}{4\sigma^2(x-x_0)^4}+\frac{b^2 a^4}{2\sigma^2(x-x_0)^4}\right.$$

$$\left.-\frac{b^2 a^3}{2(x-x_0)^3}+\frac{b^3 e^{-x}a^3}{4(x-x_0)^4}+\frac{b^2 a^3}{(x-x_0)^4}+\frac{b^2 e^x a^2}{4(x-x_0)^4}-\frac{b^2\sigma^2(x-x_0-3)a^2}{4(x-x_0)^5}\right)$$

$$+e^{-3x_0}\left(\frac{b^3 a^4}{6\sigma^2(x-x_0)^3}-\frac{b^3 a^4}{4\sigma^2(x-x_0)^4}+\frac{b^3 a^3}{6(x-x_0)^3}-\frac{b^3 a^3}{4(x-x_0)^4}\right)$$

$$+e^{-x_0}\left(\frac{b^3 e^{-2x}a^4}{4\sigma^2(x-x_0)^4}-\frac{b^2 e^{-x}a^4}{\sigma^2(x-x_0)^4}+\frac{b^3 e^{-2x}a^3}{4(x-x_0)^4}-\frac{2b^2 e^{-x}a^3}{(x-x_0)^4}\right.$$

$$-\frac{b\left(\sigma^2+2b\right)a^2}{4(x-x_0)^3}-\frac{be^x a^2}{(x-x_0)^4}-\frac{b^2 a^2}{4(x-x_0)^4}-\frac{b^2 e^{-x}\sigma^2 a^2}{(x-x_0)^4}+\frac{3a^2 b\sigma^2(x-x_0-2)}{2(x-x_0)^5}$$

$$\left.-\frac{b\sigma^4 a}{4(x-x_0)^3}-\frac{be^x\sigma^2 a}{(x-x_0)^4}+\frac{3b\sigma^4(x-x_0-2)a}{2(x-x_0)^5}\right)+e^{x_0}\left(-\frac{3(x-x_0+2)\sigma^4}{2(x-x_0)^5}\right.$$

$$\left.-\frac{\sigma^4}{4(x-x_0)^3}-\frac{abe^{-x}\sigma^2}{(x-x_0)^4}-\frac{e^x\sigma^2}{(x-x_0)^4}+\frac{a^2 b^2 e^{-2x}}{4(x-x_0)^4}-\frac{a^2 be^{-x}}{(x-x_0)^4}\right)$$

$$+\frac{e^{2x_0}\sigma^2(x-x_0+2)}{4(x-x_0)^4}+\frac{e^x\sigma^4\left(x^2-2x_0 x-6x+x_0{}^2+6x_0+12\right)}{4(x-x_0)^5},$$

$$W_4(x)=\frac{e^x\left(x^3-3x_0 x^2-12x^2+3x_0{}^2 x+24x_0 x+60x-x_0{}^3-12x_0{}^2-60x_0-120\right)\sigma^6}{8(x-x_0)^7}$$

$$+\frac{16abx_0\sigma^4}{(x-x_0)^7}\quad\frac{e^{2x}\left(x^2-2x_0 x-5x+x_0{}^2+5x_0+8\right)\sigma^4}{2(x-x_0)^6}-\frac{abe^{-x}\left(\sigma^2+a\right)\sigma^4}{8(x-x_0)^4}$$

$$-\frac{a^2 b^2 e^{-2x}\left(x^3-3x_0 x^2+4x^2+3x_0{}^2 x-8x_0 x+x-x_0{}^3+4x_0{}^2-x_0-15\right)\sigma^4}{4(x-x_0)^7}$$

$$-\frac{4ab\sigma^4}{(x-x_0)^4}-\frac{3abe^{-x}\left(\sigma^2+a\right)\sigma^4}{2(x-x_0)^5}-\frac{15abe^{-x}\left(\sigma^2+a\right)\sigma^4}{2(x-x_0)^6}-\frac{15abe^{-x}\left(\sigma^2+a\right)\sigma^4}{(x-x_0)^7}$$

$$-\frac{16abx_0\sigma^4}{(x-x_0)^7}+\frac{16a^2 bx_0\sigma^2}{(x-x_0)^7}-\frac{a^3 b^2 e^{-2x}\left(x^2-2x_0 x+5x+x_0{}^2-5x_0+8\right)\sigma^2}{(x-x_0)^6}$$

$$-\frac{a^2 b^2 e^{-x}\left(2x^2-4x_0 x+11x+2x_0{}^2-11x_0-8\right)\sigma^2}{4(x-x_0)^6}-\frac{4a^2 b\sigma^2}{(x-x_0)^4}-\frac{16a^2 bx\sigma^2}{(x-x_0)^7}$$

$$+e^{-4x_0}\left(-\frac{a^4 b^4}{8(x-x_0)^4}+\frac{5a^4 b^4}{16(x-x_0)^5}-\frac{a^4 b^4}{4(x-x_0)^6}\right)$$

$$\frac{a^4 b^2 e^{-2x}\left(x^2-2x_0 x+5x+x_0{}^2-5x_0+8\right)}{2(x-x_0)^6}$$

$$+\frac{a^3 b^3 e^{-3x}\left(\sigma^2+a\right)\left(2x^2-4x_0 x+7x+2x_0{}^2-7x_0+8\right)}{4(x-x_0)^6}$$

$$-\frac{a^4 b^4 e^{-4x}\left(2x^2-4x_0 x+5x+2x_0{}^2-5x_0+4\right)}{16(x-x_0)^6}$$

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$$
\begin{aligned}
&+ \mathrm{e}^{2x_0}\left(-\frac{5x_0{}^2\sigma^4}{2(x-x_0)^7} + \frac{5xx_0\sigma^4}{(x-x_0)^7} + \frac{4x_0\sigma^4}{(x-x_0)^7} - \frac{\sigma^4}{2(x-x_0)^4} - \frac{5x^2\sigma^4}{2(x-x_0)^7} - \frac{4x\sigma^4}{(x-x_0)^7}\right) \\
&+ \mathrm{e}^{-2x_0}\left(\frac{5b^2x_0{}^2a^4}{2(x-x_0)^7} + \frac{4b^2x_0a^4}{(x-x_0)^7} - \frac{5b^2xx_0a^4}{(x-x_0)^7} - \frac{b^2a^4}{2(x-x_0)^4} + \frac{b^4\mathrm{e}^{-2x}a^4}{2(x-x_0)^6}\right. \\
&+ \frac{5b^2x^2a^4}{2(x-x_0)^7} - \frac{4b^2xa^4}{(x-x_0)^7} + \frac{5b^2\sigma^2x_0{}^2a^3}{(x-x_0)^7} + \frac{8b^2\sigma^2x_0a^3}{(x-x_0)^7} - \frac{10b^2\sigma^2xx_0a^3}{(x-x_0)^7} - \frac{b^2\sigma^2a^3}{(x-x_0)^4} \\
&+ \frac{b^3\mathrm{e}^{-x}\left(\sigma^2+a\right)(x-x_0-8)a^3}{4(x-x_0)^6} + \frac{5b^2\sigma^2x^2a^3}{(x-x_0)^7} - \frac{8b^2\sigma^2xa^3}{(x-x_0)^7} + \frac{b^2\sigma^4x_0{}^2a^2}{(x-x_0)^7} + \frac{b^2\sigma^4x_0a^2}{4(x-x_0)^7} \\
&- \frac{2b^2\sigma^4xx_0a^2}{(x-x_0)^7} - \frac{b^2\sigma^4a^2}{4(x-x_0)^4} + \frac{b^2\mathrm{e}^{x}\sigma^2(3x-3x_0-8)a^2}{4(x-x_0)^6} - \frac{15b^2\sigma^4a^2}{4(x-x_0)^7} + \frac{b^2\sigma^4x^2a^2}{(x-x_0)^7} \\
&\left.- \frac{b^2\sigma^4xa^2}{4(x-x_0)^7}\right) + \mathrm{e}^{x_0}\left(\frac{3x_0{}^2\sigma^6}{2(x-x_0)^7} - \frac{3xx_0\sigma^6}{(x-x_0)^7} - \frac{15x_0\sigma^6}{2(x-x_0)^7} + \frac{\sigma^6}{8(x-x_0)^4} + \frac{3x^2\sigma^6}{2(x-x_0)^7}\right. \\
&+ \frac{15x\sigma^6}{2(x-x_0)^7} + \frac{15\sigma^6}{(x-x_0)^7} + \frac{8\mathrm{e}^{x}\sigma^4}{(x-x_0)^6} + \frac{2ab\mathrm{e}^{-x}\left(\sigma^2+a\right)\sigma^2}{(x-x_0)^5} + \frac{8ab\mathrm{e}^{-x}\left(\sigma^2+a\right)\sigma^2}{(x-x_0)^6} \\
&\left.- \frac{a^2b^2\mathrm{e}^{-2x}(3x-3x_0+8)\sigma^2}{4(x-x_0)^6}\right) + \mathrm{e}^{-x_0}\left(\frac{3abx_0{}^2\sigma^6}{2(x-x_0)^7} + \frac{15abx_0\sigma^6}{2(x-x_0)^7} - \frac{3abxx_0\sigma^6}{(x-x_0)^7}\right. \\
&- \frac{ab\sigma^6}{8(x-x_0)^4} + \frac{3abx^2\sigma^6}{2(x-x_0)^7} + \frac{15ab\sigma^6}{(x-x_0)^7} - \frac{15abx\sigma^6}{2(x-x_0)^7} + \frac{3a^2bx_0{}^2\sigma^4}{2(x-x_0)^7} + \frac{15a^2bx_0\sigma^4}{2(x-x_0)^7} \\
&- \frac{3a^2bxx_0\sigma^4}{(x-x_0)^7} - \frac{a^2b\sigma^4}{8(x-x_0)^4} + \frac{8a^2b^2\mathrm{e}^{-x}\sigma^4}{(x-x_0)^6} + \frac{3a^2bx^2\sigma^4}{2(x-x_0)^7} + \frac{15a^2b\sigma^4}{(x-x_0)^7} - \frac{15a^2bx\sigma^4}{2(x-x_0)^7} \\
&+ \frac{11a^2b^2x_0{}^2\sigma^2}{4(x-x_0)^7} - \frac{a^3b^3\mathrm{e}^{-2x}(x-x_0+8)\sigma^2}{4(x-x_0)^6} - \frac{2a^2b^2x_0\sigma^2}{(x-x_0)^7} - \frac{11a^2b^2xx_0\sigma^2}{2(x-x_0)^7} - \frac{a^2b^2\sigma^2}{2(x-x_0)^4}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{16a^3b^2\mathrm{e}^{-x}\sigma^2}{(x-x_0)^6} - \frac{2ab\mathrm{e}^{x}\left(\sigma^2+a\right)(x-x_0-4)\sigma^2}{(x-x_0)^6} + \frac{11a^2b^2x^2\sigma^2}{4(x-x_0)^7} + \frac{2a^2b^2x\sigma^2}{(x-x_0)^7} \\
&- \frac{a^4b^3\mathrm{e}^{-2x}(x-x_0+8)}{4(x-x_0)^6} + \frac{8a^4b^2\mathrm{e}^{-x}}{(x-x_0)^6}\Bigg) + \mathrm{e}^{-3x_0}\left(-\frac{7b^3x_0{}^2a^4}{4(x-x_0)^7} - \frac{2b^3x_0a^4}{(x-x_0)^7}\right. \\
&+ \frac{7b^3xx_0a^4}{2(x-x_0)^7} + \frac{b^3x_0{}^3\mathrm{Ei}(x_0-x)a^4}{4(x-x_0)^7} - \frac{3b^3xx_0{}^2\mathrm{Ei}(x_0-x)a^4}{4(x-x_0)^7} + \frac{3b^3x^2x_0\mathrm{Ei}(x_0-x)a^4}{4(x-x_0)^7} \\
&- \frac{b^3x^3\mathrm{Ei}(x_0-x)a^4}{4(x-x_0)^7} + \frac{b^3a^4}{2(x-x_0)^4} - \frac{7b^3x^2a^4}{4(x-x_0)^7} + \frac{2b^3xa^4}{(x-x_0)^7} - \frac{7b^3\sigma^2x_0{}^2a^3}{4(x-x_0)^7} \\
&- \frac{2b^3\sigma^2x_0a^3}{(x-x_0)^7} + \frac{7b^3\sigma^2xx_0a^3}{2(x-x_0)^7} + \frac{b^3\sigma^2x_0{}^3\mathrm{Ei}(x_0-x)a^3}{4(x-x_0)^7} - \frac{3b^3\sigma^2xx_0{}^2\mathrm{Ei}(x_0-x)a^3}{4(x-x_0)^7} \\
&+ \frac{3b^3\sigma^2x^2x_0\mathrm{Ei}(x_0-x)a^3}{4(x-x_0)^7} + \frac{b^3\left(\sigma^2+a\right)\mathrm{Ei}(x_0-x)a^3}{4(x-x_0)^4} - \frac{b^3\sigma^2x^3\mathrm{Ei}(x_0-x)a^3}{4(x-x_0)^7} \\
&\left.+ \frac{b^3\sigma^2a^3}{2(x-x_0)^4} - \frac{7b^3\sigma^2x^2a^3}{4(x-x_0)^7} + \frac{2b^3\sigma^2xa^3}{(x-x_0)^7}\right).
\end{aligned}
$$

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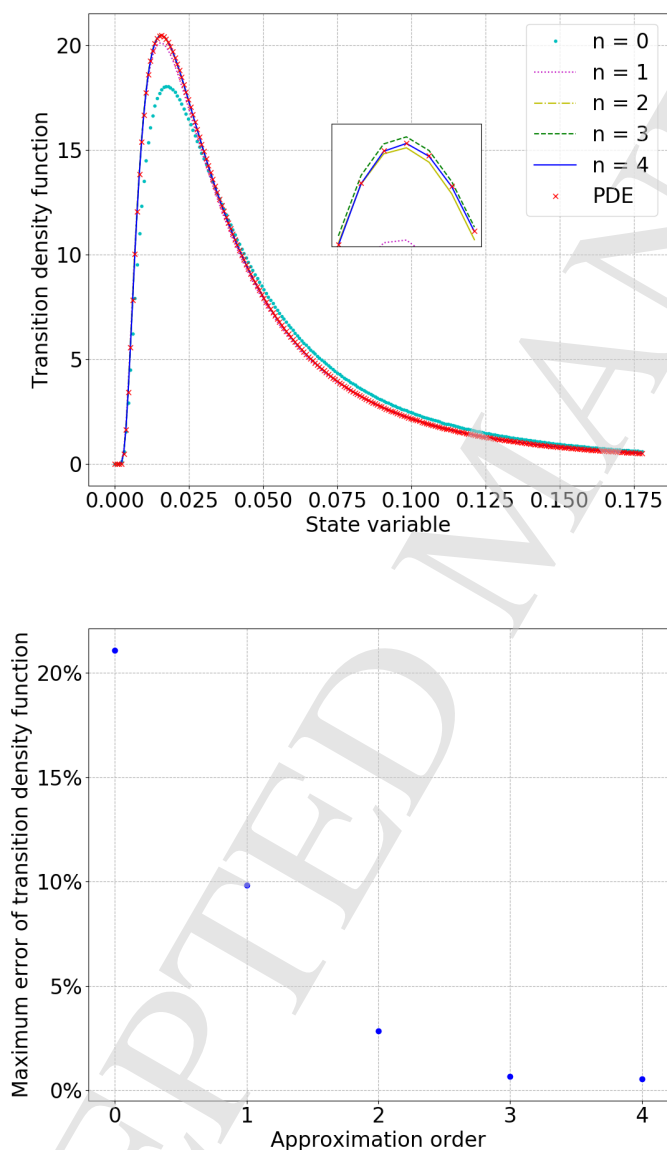
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Fig. 1. Accuracy of the exponent expansion for the transition density of the IGBM model (1.2), for $a = 0.1$, $b = 0.04$, $\sigma = 0.6$, $y_0 = 0.06$ and $T = 3$. Bottom: maximum relative error. Top: transition density function for $n = 0$ (dotted), $n = 1$ (long dashed), $n = 2$ (short dashed), $n = 3$ (continuous), $n = 4$, (dot-long dashed), exact (crosses). The inset is an enlargement of the region of the maximum.

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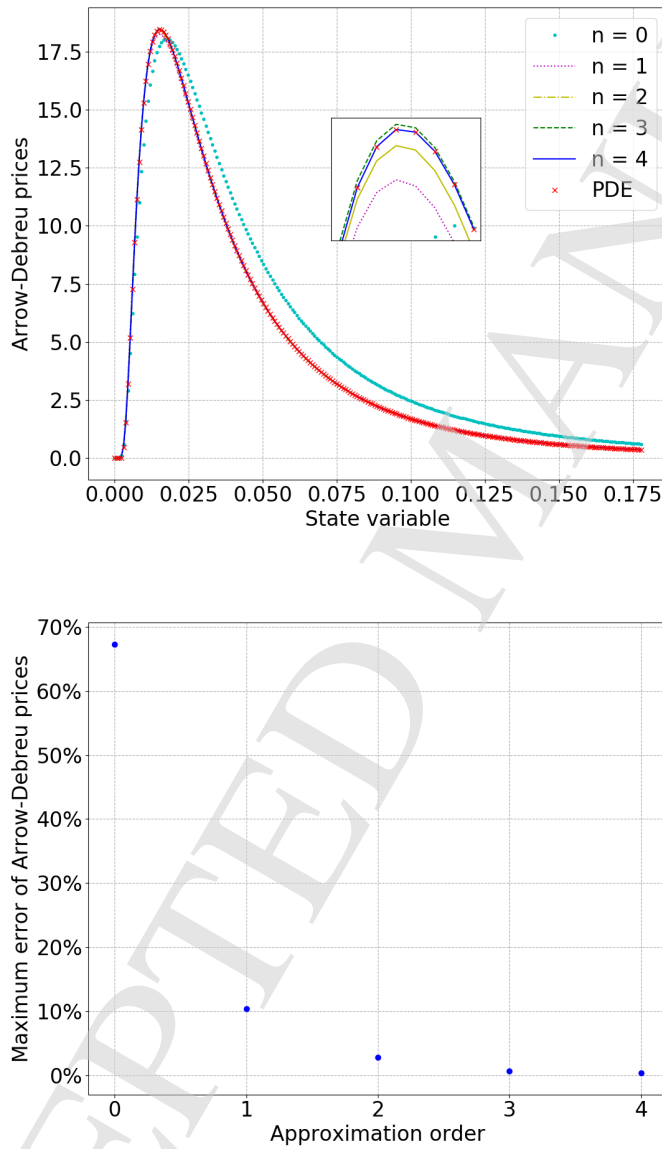

Fig. 2. Accuracy of the exponent expansion for the AD prices of the IGBM model (1.2), for $a = 0.1$, $b = 0.04$, $\sigma = 0.6$, $y_0 = 0.06$ and $T = 3$. Bottom: maximum relative error. Top: Arrow-Debreu prices for $n = 0$ (dotted), $n = 1$ (long dashed), $n = 2$ (short dashed), $n = 3$ (continuous), $n = 4$, (dot-long dashed), exact (crosses). The inset is an enlargement of the region of the maximum.

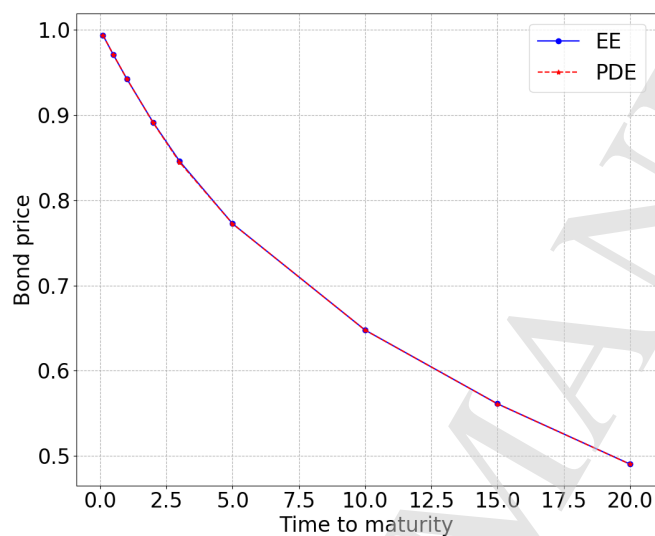16   *Luca Capriotti, Yupeng Jiang, and Gaukhar Shaimerdenova*

Fig. 3. Zero-coupon bond price as a function of time to maturity for the IGBM model Equation (1.2), with mean reversion speed $a = 0.1$, level $b = 0.04$, volatility $\sigma = 0.6$ and initial rate $y_0 = 0.06$: 4th order exponent expansion (continuous line), PDE (dashed line).