1. MATRICES AND EIGEN VALUE PROBLEMS

- A rectangular array of elements (numeric or non-numeric) are called matrices.
- Matrices of each entry as zero is called zero matrix or null matrix
- A matrix of numbers of rows = to that of columns are square matrix
- Matrix having non-zero entities at all a_{ii} elements and all a_{ij}'s are zero, they are diagonal matrix
  - Diagonal matrix $D = diag(d_1, d_2, d_3 \cdots d_n)$ for all $i = \{1,2,3,\cdots n\}$.
  - if $d_i = d$, then the D is scalar matrix
  - if $d_i = 1$, then the D is unit matrix or identity matrix and it is denoted by I.
- A square matrix $A = [a_{ij}]$ is said to be a triangular matrix, where
  - if ($a_{ij} = 0$ for all $i > j$), then it is upper triangular matrix
  - if ($a_{ij} = 0$ for all $i < j$), then it is lower triangular matrix

Operations:

- Transpose of matrix: $A = [a_{ij}]$ for $m \times n$ is defined by $n \times m$ matrix $B\{b_{ij}\}$ for $1 < i < m(n)$.
- Addition: $c_{ij} = a_{ij} + b_{ij}$ for 2 matrix A & B to get result C.
- Multiplication:
  - Scalar multiplication: $kA = [ka_{ij}]$
- Ortogonal: If $AA^T = I$, then the matrix is ortogonal.

Rank of matrix :

- Determinant : Unique no. that is associated to the matrix
- Minor : The determinant of any submatrix of a square matrix order. Denoted by M_{ij}
- Rank : The value which is $<= n$ for $m \times n$ matrix, where one of the minors of matrix is non-zero and all orders $> n$ is zero.

$\boxed{\text{rank} \rightarrow \text{number} <= n \text{ (for } A_{m \times n}), \text{ where } M_r \neq 0, M_{n>r} = 0.}$

denoted by ρ.

$\boxed{\rho[A]_{m \times n} = r \mid M_r \neq 0, r <= \max(m,n) \quad M_{>r} = 0}$

Note: $\rho[A] = \rho[A^T]$.

$M_r \Rightarrow$ minor of square matrix of R&C as r

Method to find rank of matrix :

- Interchange two rows / columns, so as to get $a_{11} = 1$. a_{ij} is called pivot.
- (i) Multiply / add all elements on a R/C with any constant k
- form an upper triangular matrix by interchanging rows and columns.

This forms the matrix as echelon form.

~~Roch~~ é-~~Capelli~~ Th~~eorem~~

Rouch é-Capelli Theorem:

The system $Ax = B$ admits solution if and only if the rank (A) = rank (A|B)

A|B is called augumented matrix

- $\rightarrow \rho(A) = \rho(A|B) \rightarrow$ unique solution
- $\rightarrow \rho(A) = \rho(A|B) = n$ (rows (columns) of X) $\rightarrow$ unique solution
- $\rightarrow \rho(A) = \rho(A|B) < n$ (rows (columns) of X) $\rightarrow \infty$ many solutions
- $\rightarrow \rho(A) < \rho(A|B) \rightarrow$ no solution.

X $\Rightarrow$ matrix of unknowns

Eigen Value and Vector :

Let $A = [a_{ij}]_{n \times n}$,

if there exists a non-zero column vector X and

scalar λ || $AX = \lambda X$,

then X is Eigen vector and λ is Eigen value.

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$\boxed{(A - \lambda I)X = 0}$$ ← Charecteristic equation of matrix A.

For 2×2 matrix, characteristic equation is,

$$\lambda^2 - S_1\lambda + S_3 = 0$$

For 3×3 matrix, characteristic equation is,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

where

$S_1 = \sum_{(0,0)}^{(n,n)} a_{ij}, (i=j)$ ⟶ Sum of diagonals. ⟶ Trace of matrix

$S_2 = \sum_{(0,0)}^{(n,n)} M_{ij}, (i=j)$ ⟶ Sum of minor of diagonal

$S_3 = \sum_{1}^{n} a_{ij} A_{ij} \; [A_{ij} = (-1)^{i+j} \times M_{ij}]$ ⟶ Det (A).

Properties :

1. Eigen value of A = Eigen value of A^T

2. Σ (eigen value) = S_1 / Trace.

   Π (eigen value = $|A| / S_3$.

3. If λ_n is eigen value of A,
   - $k\lambda_n$ is eigen value of kA
   - λ_n^p is eigen value of A^p
   - $\frac{1}{\lambda_n}$ is the eigen value of A^{-1}.

4. If A is Δ matrix, then the diagonals elements are the eigen values.

Cayley - Hamilton Theorem :

Every square matrix satisfies its own characteristic equation

Usage :

→ for singular matrix, we can get A^{-1}

→ Higher positive integer power of A can be computed via the equation of degree less than the power by 1.

SELF ASSESSMENT QUESTIONS

Short Answer Questions

1) Given:

$$A = \begin{pmatrix} 1 & +2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{pmatrix}$$

To find: Sum & product of Eigen values.

Solution:

Characteristic eqn. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

Sum of Eigen value $= S_1 = \sum_1^3 a_{ii} = 1+0-3 = -2$

Product of Eigen value $= S_3 = |A| = 1\begin{vmatrix} 0 & 3 \\ -1 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ -2 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix}$

$= (0+3) - 2(-3+6) - 2(-1-0)$

$= 3-6+2 = 5-6 = -1$

2) Given:

$$A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

To find: Eigen values of $(A^2 \& A^{-1})$.

Solution:

Eigen value of A is $\{2, 3, 4\}$, since it is triangular mat

These eigen value of

$A^2 \Rightarrow \{2^2, 3^2, 4^2\} \Rightarrow \{4, 9, 16\}$

$A^{-1} = \{2^{-1}, 3^{-1}, 4^{-1}\} \Rightarrow \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$.

Given:

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

To prove: $AA^T = I$.

Proof:

$$A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Hence proved.

4) Given :

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

To Prove :

$(-3A^{-1})$ eigen value. = (A) eigen value.

Proof :

~~Eigen~~

characteristic equation must be $\lambda^2 - S_1\lambda + S_3 = 0$.

$S_1 = 1+1 = 2$ $\qquad$ $S_3 = 1-4 = -3$.

$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0$

$\therefore$ Eigen value of A $= \{+3 \ \& \ -1\}$

Eigen value of $-3A^{-1} = -3\left(\frac{1}{3}\right), -3\left(\frac{-1}{1}\right)$

$= \{-1, +3\}$

Hence proved.

5) Given :

$A = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & 8 \end{pmatrix}$

To do

To find :

λ_2, λ_3

$\lambda_1 = -9$

Solution :

Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 7-8+8 = 7$ $\qquad$ $S_2 = \begin{vmatrix} -8 & -1 \\ -1 & 8 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 7 & 4 \\ 4 & -8 \end{vmatrix}$

$= (-64-1) + (56+16) + (-56-16)$

$= -65 + 72 - 72 = -65$

$S_3 = 7$

Sum of Eigen value $= \lambda_1 + \lambda_2 + \lambda_3 = S_1$

$\lambda_1\lambda_2\lambda_3 = S_3$

$-9 + \lambda_2 + \lambda_3 = 7$

$(-9)\lambda_2\lambda_3 = -744$

$\lambda_2\lambda_3 = \frac{+744}{+9} =$

6. Given:

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

To find:

A^{-1} via Cayley Halminton.

Solution:

Charecteristic eqn $= \lambda^2 - S_1\lambda + S_3 = 0$

$S_1 = 1+4 = 5$ $\quad S_3 = 4-6 = -2$.

$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$

Acc. to Cayley Halminton theorem,

$A^2 - 5A - 2I = 0$. $\quad A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

$2I = A^2 - 5A$

$2I = \left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]^2 - 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Div. by A on both sides.

$2A^{-1} = A - 5I$

$A^{-1} = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

$A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

7. Applications of Cayley Halminton theorem:

→ If A is non-singular matrix then ~~the~~ we can get the inverse matrix of A, using this theorem

→ Highest ~~value~~ positive integer powers of A can be computed.

Long Answer Questions

8. Given:

$x_1 + x_2 + x_3 = 3$

$x_1 + 2x_2 + 3x_3 = 4$

$x_1 + 4x_2 + 9x_3 = 6.$

Solution:

Let $AX = B$ be the matrix equation.

by Rouch é-Capelli theorem,

$Ax = B$ provides unique solution if

$\rho(A) = \rho(A|B) = $ no. of unknowns.

where A/B is augmented matrix,

$$\begin{array}{ccc} A & X & = B \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$A|B = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

$R_2 = R_2 - R_1$; $R_3 = R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

$\cancel{R_3 = R_3 - 3R_1}$ $R_3 = R_3 - 3R_2$

$$\cancel{= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & -6 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

This ~~form~~ matrix is in echelon's form.

$\therefore$ ~~$\rho(A)$~~ $\rho(A|B) = 3$ (as there are 3 non zero rows)

$\rho(A) = 3$ (as there is 3 non-zero rows from result)

no. of unknowns $= 3$.

$\therefore$ It is unique solution

So,

$Ax = B$ is now,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 3$

$x_2 + 2x_3 = 1$

$2x_3 = 0$

$\boxed{x_3 = 0}$

$x_2 + 2x_3 = 1$

$\boxed{x_2 = 1}$

$x_1 + x_2 + x_3 = 3$

$x_1 + (1) + (0) = 3$

$x_1 + 1 = 3$

$\boxed{x_1 = 2}$

9) Given:

$x_1 - 4x_2 - x_3 = 3$

$3x_1 + x_2 - 2x_3 = 7$

$2x_1 + 3x_2 + x_3 = 10$

Solution

Let $Ax = B$ be the matrix equation.

by Rouch é Capeli theorem,

$Ax = B$ provides unique solution if

$\rho(A) = \rho(A|B) =$ no. of unknowns.

In $Ax = B$,

$$A = \begin{bmatrix} 1 & -4 & -1 \\ 3 & 1 & -2 \\ 2 & -3 & +1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 1 & -4 & -1 & 3 \\ 3 & 1 & -2 & 7 \\ 2 & -3 & +1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -1 & 3 \\ 0 & 13 & 1 & -2 \\ 0 & 5 & 3 & 4. \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$10 - 2(3)$
$10 - 6$
$= 4.$

$-3 - 2(-4)$
$-3 + 8 = 5$

$1 - 2(-1)$
$1 + 2 = 3$

$7 - 3(3)$
$7 - 9$

$10 - 2(7)$
$10 - 14$

$1 - 2(-2)$

$-3 - 2(-4)$
$-3 + 8 = 5$

$1 - 3(-4)$
$1 + 12 = 13$

$-2 - 3(-1)$
$-2 + 3 = 1$

$7 - 3(3)$
$7 - 6$

$$= \begin{bmatrix} 1 & -4 & -1 & 3 \\ 0 & 13 & 1 & -2 \\ 0 & 5 & 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 & -1 & 3 \\ 0 & 13 & 1 & -2 \\ 0 & 0 & \frac{34}{13} & \frac{62}{13} \end{bmatrix} R_3 \to R_3 - \frac{5}{13} R_2$$

$5 - \frac{5}{13}(1)$

$3 - \frac{5}{13}$

$$\cancel{\begin{bmatrix} 1 & -4 & -1 & 3 \\ 0 & 1 & 1/13 & -2/13 \\ 0 & 0 & \frac{34}{13} & \frac{62}{13} \end{bmatrix}} \quad \cancel{R_2 \to R_2/13} \quad R_3 \to R_3 - 5R_2$$

$3 - \frac{5}{13} = \frac{39-5}{13} = \frac{34}{13}$

$4 + \frac{10}{13} = \frac{52+10}{13}$

It is in echelon form,

$\rho(A) = 3$

$\rho(A|B) = 3$

no. of unknowns $= 3$

Hence provides unique solution, such that

$x_1 - 4x_2 - x_3 = 3$ —(1) $\qquad 13x_2 + x_3 = -2$ —(2)

$\cancel{x_2 - \frac{1}{13}x_3 = \frac{-2}{13}}$ —(2) $\qquad \frac{34}{13}x_3 = \frac{62}{13}$ —(3).

$\cancel{\frac{34}{13}x_3 = \frac{62}{13}}$ —(3)

(3) $\to$ $x_3 = \frac{\cancel{62}^{31}}{\cancel{13}} \times \frac{\cancel{13}}{\cancel{34}_{17}} = \frac{31}{17}$ $\qquad \boxed{x_3 = \frac{31}{17}}$

(2) $\to$ $\cancel{x_2 - \frac{\frac{31}{17} \times 13}{13} = \frac{-2}{13}} \Rightarrow \cancel{\frac{13x_2 - \frac{31}{17}}{13} = \frac{-2}{13}}$

$\cancel{x_2 = \frac{-2}{13} + \frac{31 \times 13}{17}}$ $\qquad = \cancel{13x_2 = -2 + \frac{31}{17}}$

$\cancel{= \frac{-34 + 31 \times 13}{13 \times 17}}$ $\qquad x_2 = \frac{-34+31}{17 \times 13} = \frac{-3}{17 \times 13}$

$\cancel{(2) \to x_2 - \frac{1}{13}(x_3) = \frac{-2}{13}}$

$\cancel{x_2 - \frac{1}{13}\left(\frac{31}{17}\right) = \frac{-2}{13}}$

$\cancel{x_2 = \frac{-2}{13} + \frac{1}{13}\left(\frac{31}{17}\right)}$

$\cancel{= \frac{-34+31}{13(17)} = \frac{-3}{13 \times 17}}$

(2) $\Rightarrow$ $13x_2 + \frac{31}{17} = -2$

$$13x_2 = -2 - \frac{31}{17}$$

$$x_2 = \frac{-34 - 31}{17 \times 13}$$

$$= \frac{-65}{17 \times 13} = \frac{-5}{17}$$

$$\boxed{x_2 = \frac{-5}{17}}$$

(3). $x_1 - 4\left(\frac{-5}{17}\right) - \frac{31}{17} = 3$

$$x_1 + \frac{20}{17} - \frac{31}{17} = 3$$

$$\frac{17x_1 + 20 - 31}{17} = 3.$$

$$17x_1 - 11 = 51$$

$$17x_1 = 51 + 11$$

$$\boxed{x_1 = \frac{62}{17}}$$

$\therefore$ Solution is $\left(\frac{62}{17}, \frac{-5}{17}, \frac{31}{17}\right)$.

10. Given :

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

To find :

$A^4 - 4A^3 \mp 5A^2 + A + 2I$

Using ~~cramers~~ Cayley - Halmiton theorem.

Solution

~~Acc. to Cayley - Halmint~~

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

Charecteristic equation will be $\lambda^2 - S_1\lambda + S_3 = 0$.

$S_1 \Rightarrow \Sigma$ (diagonals)

$S_3 \Rightarrow |A|$

$S_1 = 1+3 = 4$ $\qquad S_3 = 3 - 8 = -5$.

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$.

By Cayley Halminton Theorem.

$A^2 - 4A - 5I = 0$.

$$\begin{array}{r|l} & A^2 \\ A^2 - 4A - 5I & A^4 - 4A^3 - 5A^2 + A + 2I \\ & A^4 - 4A^3 - 5A^2 \\ \hline & + A + 2I \\ \hline \end{array}$$

$\frac{A^4 A^2}{A^2}$

$\therefore A^4 - 4A^3 - 5A^2 + A + 2I = A^2(A^2 - 4A - 5I) + A + 2I$

$= 0 + A + 2I$.

$= A + 2I$

$= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+2 & 2+0 \\ 4+0 & 3+2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$

$$\boxed{A^4 - 4A^3 - 5A^2 + A + 2I = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}}$$

12. Given:

$$A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

to find:

Eigen value and vector.

Solution:

Charecteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = \Sigma(\text{diagonals}) = 7+6+5 = 18$

$S_2 = \Sigma(\text{Minors (diagonals)}) = \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$

$= (30-4) + (35) + (42-4)$

$= 26 + 35 + 38 = 99$

$S_3 = |A| = 7\begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + 2\begin{vmatrix} -2 & 0 \\ -2 & 5 \end{vmatrix} + 0.$

$= 7(30-4) + 2(-10)$

$= 7.(26) - 20 = 182 - 20 = 162.$

$\Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$

3	1	-18	99	-162
	0	3	-45	162
	1	-15	54	0

$\therefore \lambda^3 - 18\lambda^2 + 99\lambda - 162 = (\lambda-3)(\lambda^2 - 15\lambda + 54)$

$= (\lambda-3)(\lambda-6)(\lambda-9)$

$\therefore$ Eigen values are $\{3, 6, 9\}$.//

54

-6 -9

-15

Eigen Vector

Consider $[A - \lambda I]X = 0$.

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Case 1 : $\lambda = 3$.

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \Rightarrow \frac{x_1}{\begin{vmatrix} -2 & 0 \\ 3 & 2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 4 & 0 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix}}$$

$4x_1 - 2x_2 + 0x_3 = 0$ — (1)

$-2x_1 + 3x_2 - 2x_3 = 0$ — (2) $\Rightarrow \dfrac{x_1}{-4} = \dfrac{x_2}{-8} = \dfrac{x_3}{(12+4)}$

$0x_1 - 2x_2 + 2x_3 = 0$ — (3).

$x_1 \quad x_2 \quad x_3$

$-2 \quad 0 \quad 4 \quad -2$

$+3 \quad -2 \quad -2 \quad +3$

$\Rightarrow \dfrac{x}{-4} = \dfrac{x_2}{-8} = \dfrac{x_3}{16}$

$\boxed{x_1 = 1 ;\ x_2 = -2 ;\ x_3 = 4}$

$$\frac{x_1}{4+0} = \frac{x_2}{0-8} = \frac{x_3}{12+4}$$

$$\boxed{X_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{-8} = \frac{x_3}{16}$$

Case 2 : $\lambda = 6$.

$$\begin{bmatrix} 7-6 & -2 & 0 \\ 2 & 6-6 & -2 \\ 0 & -2 & 5-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}} = 0$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{-4} = 0. \qquad \boxed{X_2 = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}}.$$

Case 3: $\lambda = 9$.

$$\begin{bmatrix} 7-9 & -2 & 0 \\ -2 & 6-9 & -2 \\ 0 & -2 & 5-9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & 0 \\ -3 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -3 \end{vmatrix}} = 0$$

$$= \frac{x_1}{(4)} = \frac{x_2}{(4)} = \frac{x_3}{(6+4)} = 0.$$

$$= \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{10}$$

$$\boxed{X_3 = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}}$$

4. Verify ~~Hal~~ Cayley Halmintons's theorem, hence find A^{-1} if

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

The charecteristic eqn of A is: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where

~~$S_1 = 1+3+1 = 5$~~ $S_1 = \sum_{i=1}^{i=3} a_{ii}$

$S_1 = 1+3+1 = \boxed{S_1 = 5}$

$$S_2 = \sum_{(0,0)}^{(n,n)} M_{ii} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$= (3-0) + (1-0) + (3-2)$

$= 3+1+1 = 5 \quad \boxed{S_2 = 5}$

$$S_3 = |A| = 1\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix}$$

$= 1(3) - 1(2-4) = 3-(-2) = 5 \quad \boxed{S_3 = 5}$

$C.E = \lambda^3 - 5\lambda^2 + 5\lambda - 5 = 0.$

By Cayely Halmintos theorem,

$A^3 - 5A^2 + 5A - 5I = 0$

$$A^3 - 5A^2 + 5A - 5I = 0$$

$$\Rightarrow 5I = A^3 - 5A^2 + 5A$$

$$\div A \Rightarrow 5A^{-1} = A^2 - 5A + 5I$$

$$A^2 = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}$$

$$5A^{-1} = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -1-5+5 & 12-10+0 & -4+10+0 \\ -4-(-5)+0 & 7-15+5 & 2-0+0 \\ 2-0+0 & -8+10+0 & 1-5+5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 6 \\ 1 & -3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 2 & 6 \\ 1 & -3 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \qquad A^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

Revision carry forword:

15, 16, 17.

2. PROBABILITY AND RANDOM VARIABLES

PROBABILITY

TERMINOLOGIES & FORMULAE :

* Random experiment : A mechanism that produces definite outcome that cannot be predicted with certainity.
* Sample space : A set of all possible outcomes
* ~~Event~~ (event).
* Element : An event E on a particular trial of expn. where the outcome is observed is an element of set E.
* Probability : A value between 0 and 1 that measures the likelihood of which an event will occur on a single trial of random experiment. It is ~~It~~ is a sum of prob denoted by $P(A)$
* Conditional Probability : The conditional probability of event B is that the event B will occurs with the knowledge that event A has already occured, in 2 events. It is denoted (this case) by $P(B/A)$

If A & B are independant, $P(B/A) = P(B)$

else,

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

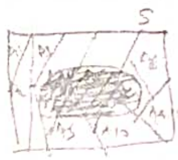
TOTAL PROBABILITY :

Let a sample space S is partitioned with collection of events $A_1, A_2, \ldots, A_n$, such that

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S \; ; \; A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \emptyset \; ; \; A_i \neq \emptyset \text{ for all } i.$$

Let B be the other event that occurs on sample space S $(B \in S)$,

$$B = (B \cap A_1) \cup (B \cap A_2) \ldots \cup (B \cap A_n)$$



→ set B

Then,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$$

$$\cancel{P(B)} = P(A_i) \; P(B_i | A_i) \quad \frac{P(B \cap A_i)}{P(A_i)} = P(B | A_i)$$

$$P(B \cap A_i) = P(B | A_i) \times P(A_i)$$

$$\therefore P(B) = \sum_{i=1}^{n} P(B | A_i) \times P(A_i).$$

~~Baye's Theor~~

BAYE'S THEOREM :

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B)}{\sum_{i=1}^{n} P(B|A_i)}$$

$$\because P(A\cap B) = P(A_i|B)\,P(B)$$

$$P(A_i|B) = \frac{P(A_i|B)\times P(B)}{\sum_{i=1}^{n} P(B|A_i)\times P(A_i)}$$

$$\boxed{P(A_i|B) = \frac{P(B|A_i)\times P(A_i)}{\sum_{i=1}^{n} P(B|A_i)\times P(A_i)}}$$

RANDOM VARIABLE

Random variable is a number generated by an random experiment. It is a result of an chance event that we can measure (via continuous random variable) or count (discreate random variable.

Notation:

- Let S be sample space,
- Then for random variable X, $X(S) = x$ is denoted by $X = x$. Here x is one of outcomes of S.

DISCRETE RANDOM VARIABLE

Probability mass function :

For a discrete random variable X that takes value $x_1, x_2, \ldots, x_i$. The probability $p_i = P(X = x_i)$. This is called probability mass function (p_i).

In simple terms,

The function $p(x)$ that satisfies below 2 condition

(i) $p_i \geq 0 \;\forall i$,

(ii) $\sum_{i=1}^{\infty} p_i = 1$

is the probability mass function or probability distribution of r.v. X.

Cumulative distributed function

- A.K.A Distribution function of a r.v X.
- For discrete r.v X,

$$\boxed{F(x) = \sum_{i=0}^{\infty} P(X = x_i) \ \forall \ x_i < x}$$

$$F(x) = P(X < x_n) = P(x=1) + P(x=2) + \cdots + P(x=n) = 1$$

- ~~Probability~~ Properties,

  - $P(a < X \leq b) = F(b) - F(a)$
  - $P(a \leq X \leq b) = P(X = a) + P(a < X \leq b)$
  - $P(a < X < b) = P(a < X \leq b) - P(X = b)$.

CONTINUOUS RANDOM VARIABLE

Probability density function: For a continuous ~~function~~ random variable X, The function that which x falls between 2 values, is the probability density function (pdf).

P.d.f, $P(a \leq x \leq b) = \int_a^b f(x)\,dx$.

Cumulative distributed function:

- For continuous R.V X,

$$\boxed{F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(x)\,dx}$$

~~Probability~~:

Properties:

- The probability at any point is always zero.
- $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$.

MATHEMATICAL EXPECTATIONS

→ ~~A.k.A~~ A.K.A $\begin{cases}\text{Expected value.}\\ \text{Mean value}\end{cases}$ of X

a, b are constant

$E(a) = a$

$E(ax \pm by) = aE(x) \pm bE(y)$

$Var(a) = 0$

$Var(ax \pm by) = a^2 V(x) \pm b^2 V(y)$

- Denoted by $\bar{X}$

$$\bar{X} = E(x) = \begin{cases}\sum x_i p_i & \text{(discreate)}\\ \int_{-\infty}^{\infty} x f(x)\, dx & \text{(continuous)}.\end{cases}$$

Variance: $Var(x) = E(x^2) - [E(x)]^2$.

$$E(x^2) = \begin{cases}\sum x_i^2 p_i \longrightarrow \text{discreate}\\ \int_{-\infty}^{\infty} x^2 f(x)\, dx \to \text{continuous.}\end{cases}$$

MOMENTS & MOMENT GENERIC FUNCTION

Moments: μ_r (when X is discrete) μ_r (when X is continuous).

Origin $\sum X^r P(X)$ $\int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$

Central $\sum (x-\bar{X})^r P(x)$ $\int_{-\infty}^{\infty} (x-\bar{x})^r f(x)\, dx$

Properties of Raw moments:

- $\mu_0 = \int_{-\infty}^{\infty} f(x)\,dx \,/\, \sum P(x) = 1$
- $\mu_1 = \int_{-\infty}^{\infty} x f(x)\, dx \,/\, \sum x P(x) = \bar{X}$ $\bar{X} \to$ mean.
- $\mu_2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx \,/\, \sum x^2 P(x) = (\overline{X^2})$

$$\boxed{Var(X) = \mu_2 - (\mu_1)^2}$$

for Moments about mean $\bar{X}$, $Var(x) = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\, dx$

Moment generic function:

- denoted by $M_x(t)$
- $M_x(t) = E(e^{tx})$ $t \to (-\infty, \infty)$

$$M_x(t) = \begin{cases}\int_{-\infty}^{\infty} e^{tx} f(x)\, dx\\ \sum_x e^{tx} p(x)\, dx.\end{cases}$$

Properties of Central moments:

- $\mu_0 = \sum (x-\bar{x})^0 P(x) \,/\, \int_{-\infty}^{\infty} (x-\bar{x})^0 \cdot f(x)\,dx$

$= \sum P(x) \,/\, \int_{-\infty}^{\infty} f(x)\,dx = 1.$

- $\mu_1 = \sum (x-\bar{x}) P(x) \,/\, \int_{-\infty}^{\infty} (x-\bar{x}) f(x)\,dx$

$= 0$

- $\mu_2 = \sum (x-\bar{x})^2 P(x) \,/\, \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx$

$= Var(x).$

Moment GENERIC FUNCTION :

- denoted by $M_x(t)$
- $M_x(t) = E(e^{tx})$, $t=(-\infty,\infty)$

$$M_x(t) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx. \\ \sum e^{tx} P(x) \end{cases}$$

$$M_x^{(r)} = \frac{d^r}{dx^r}(M_x(t)).$$

Property : $E(x^r) = \left[M_x^{(r)}(t)\right]_{t=0}.$

$(r) \rightarrow \frac{d^r}{dx^r}\left[E(x^r)\right]$

Prerequired formula:

nC_r (or) $C\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

$C\binom{3}{2} = \frac{3!}{2!(1)!} = \frac{3\times\cancel{2\times1}}{\cancel{2\times1\times1}} = 3.$

$C\binom{n}{n-1} = n$ $\quad C\binom{n}{n} = 1$

c

STANDARD DEVIATIONS

- Discreate distribution
  - binomial
  - poisson
  - geometric
- Continuous distribution
  - uniform
  - Exponential
  - Normal

DISCREATE DISTRIBUTION:

- Binomial discreate distribution:

In an experiment having many outcome that are categorized into two forms /results (success, failure). This is denoted by $B(n,p)$ $X \rightarrow B(n,p)$ follows

for p.m.f:
if $X \rightarrow B(n,p)$, where n, p are parameters.
then $\boxed{P(X=i) = C\binom{n}{x} . p^{x} . q^{n-x}}$, $x = [0, n]$

$n \rightarrow$ no. of trials $p \rightarrow$ probability of success $q \rightarrow$ probability of failure

- $p+q=1$ or $q = 1-p$
- Mean $= E(x) = np$
- $S.D = \sqrt{npq}$
- $Var(x) = npq$
- $E(x^2) = np(np-q)$
- Moment generic function $M_x(t) = (pe^t+q)^n$
- $M_{x+y}(t) = (q+pe^t)^{n_1+n_2}$, $n_1 \in x$, $n_2 \in y$

- Poisson Distribution

When the experiment having outcomes that are distributed exponentially, then the distribution follows poison distribution $P(\lambda)$.

for p.m.f, if $x \rightarrow P(\lambda)$, then

$$\boxed{P(X=i) = \frac{e^{-\lambda} . \lambda^{x}}{x!}}, \quad x = [0, \lambda]$$

• Mean $E(x) = \lambda = Var(x)$

• $M_x(t) = e^{\lambda(e^t - 1)}$

• $X_1 + X_2 \rightarrow \lambda_1 + \lambda_2$.

Poission distribution is limited of binomial under the following

* Number of trials should be ∞
* probability of succes should be small
* $np = \lambda$

$B(n, p) = P(\lambda)$ if

• Geometric Distribution

Geometric distribution offers to the distribution when situation turns that the final (predicted) attempt would be succes where as the previous attempts are failure.

~~The distribution G( ) follow~~

for The ~~pdf~~/pmf, x follows $G(\lambda)$ when

$$\boxed{P(X=x) = pq^x}, \quad x = [0, \infty)$$

• Mean $E(x) = q/p$ • $var(x) = q/p^2$ • $M_x(t) = \frac{p}{(1-qe^t)}$

• Memoryless property.

$$P(X > s+t \mid X > s) = P(X > t)$$

CONTINUOUS DISTRIBUTION:

• Uniform or Rectangular distribution.

A random variable x is set to follow uniform distributed if it has the pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , \text{otherwise.} \end{cases}$$

$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$ | Mean $= \frac{a+b}{2}$ | $var(x) = \frac{(b-a)^2}{12}$

• Exercise.
• Question paper
• Example

• Exponential distribution

A continuous random variable 'x' is said to follow an exponential distribution with $\lambda > 0$ if

pdf $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

• Mean $= \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{\lambda}$

• Variable $= \frac{1}{\lambda^2}$

• $E(x^2) = \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$

• Gaussian Distributions

A random variable x is said to follow normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma)$

If pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty)$
$\mu \in (-\infty, \infty)$
$\sigma \in [0, \infty)$

• $M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

• Mean $= \mu$ • Variance $= \sigma^2$

• $\int e^x\, dx = e^x + c$

• $\int e^{-ax}\, dx = \frac{-1}{a} e^{-ax} + c$

• Normal distribution variable $z = \frac{x-\mu}{\sigma}$

$\frac{1 + 0.67 + 2 + 1.34 + 3 + 2.01 \simeq 10.00}{10} = 1$

$\frac{3 \times 0.67^2}{01}$

$1 + k + 2 + 2k + 3 + 3k = 10$

$6 + 6k = 10$

$6(1+1k) = 10$

$\frac{10}{6} = 1 + 1k$

$1.66 = 1 + 1k$

$1k = 0.66$

$1 + k + 2 + 2k + 3 + 3k = 10$

$6 + 6k = 10$

$6(1+1k) = \frac{10}{6}$

$1 + 1k = 1.67$

$1k = 1.67 - 1$

$= 0.67$

SEMESTER EXAM QUESTIONS

February 2022

2 MARKS:

1. If $P(A) = 0.4$ $P(B) = 0.67$ and $P(A \cap B) = 0.15$. Find $P(A' \cap B')$

$P(A) = 0.4$ $P(B) = 0.67$ $P(A \cap B) = 0.15$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.67 - 0.15$

$= 1.07 - 0.15$

$= 0.92$

$P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - 0.92$

$= 0.08$

2. The Moment generic function of a random variable X is given by $M_x(t) = e^{3(e^t - 1)}$. What is $P(x=0)$?

~~$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda}$~~ If $M_x(t) = e^{\lambda(e^t - 1)}$, $\boxed{\therefore \lambda = 3}$

It follows poisson distribution. Hence $P(x=i) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$P(x=0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-3}$ $\boxed{P(x=0) = -3}$

15 Marks:

12) a) A random variable X has the following probability distributions.

$X = i$	-2	-1	0	1	2	3
$P(x=i)$	0.1	k	0.2	2k	0.3	3k

(i) Value of $k = ?$

Total probability $= \sum_{i=0}^{\infty} P(x=i) = 1$

$= 0.1 + k + 0.2 + 2k + 0.3 + 3k = 6k + 0.6 = 1$

$6k = 0.4$

$k = \frac{0.4}{6}$ ~~0.06~~ $= \frac{0.2}{3} \simeq 0.066$

$\therefore P(X=-1) = 0.06$ $P(X=1) = 0.12$ $P(X=3) = 0.18$

$0.1 + 0.06 + 0.2 + 0.12 + 0.3 + 0.18$

$0.16 + 0.2 + 0.42 + 0.18$

(ii) $P(X<2)$, $P(-2<X<2)$

$$\cancel{P(X<2) = P(X=0)+P(X=1)}$$

$$\cancel{= 0.2+0.16}$$

$$\cancel{P(X<2) = 0.36}$$

$$\cancel{P(-2<X<2) = P_{-1}+P_0+P_1}$$

$$\cancel{= 0.08+0.2+0.16}$$

$$\cancel{P(X\in[-2,2]) = 0.44}$$

$$P(X<2) = P_{-2}+P_{-1}+P_0+P_1+P_2$$

$$= 0.88$$

$$P(-2<X<2) = F(2)-F(-2)-P(X=2)$$

$$= 0.88-0.1-0.3$$

$$= 0.48$$

(iii) CF (x)

$$F(x) = \sum_{i=0}^{x} P(X=x_i), \forall\ x_i<X$$

X	-2	-1	0	1	2	3
P	0.1	0.08	0.2	0.12	0.3	0.18
F	0.1	0.18	0.38	0.50	0.88	1.00

$$0.1+k+0.2+2k+0.3+3k=1$$

$$0.6+6k=1$$

$$6k = 1-0.6$$

$$6k = 0.4$$

$$k = \frac{0.4}{6}$$

$$= 0.0667$$

(iv) Evaluate Mean (x).

$$\bar{X} = E(X) = \sum_{i=0}^{\infty} x_i P_i$$

$$= (-2)(0.1)+(-1)(0.06)+(0)(0.2)+(1)(0.12)+(2)(0.3)+(3)(0.18)$$

$$= 1$$

b) The time is required to repair a machine is exponential distribution with parameter $\lambda = 1/2$

(i) What is the probability that the repair time exceeds 2hrs?

(ii) What is the probability that a repair takes atleast 10hrs given that the duration exceeds 9hrs?

From the given,

$\lambda = 1/2$. $X \to$ time required to repair the machine

To find:

(i) $P(X>2)$

(ii) $P(X\geq 10 | X>9)$

We know that,

$$P(X = x_i) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, \quad x \in [0, \lambda]$$

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

~~In order to~~

As the X is exponentially distributed.

$$\lambda = \frac{1}{2}.$$

$$\therefore f(x) = \frac{1}{2} e^{-\frac{1}{2}x}$$

(i) $P(X > 2) = \int_2^\infty f(x)\,dx$

$$= \frac{1}{2}\int_2^\infty e^{-x/2}\,dx.$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^\infty$$

$$= -\frac{1}{2}\left[\frac{e^{-\infty}}{1/2}\right] - \left[-\frac{1}{2}\left[\frac{e^{-1}}{1/2}\right]\right]$$

$$= -0 - (-e^{-1}) = -0 + e^{-1} = e^{-1} = 0.3679$$

(ii) $P(X \geq 10 \mid X > 9)$

~~$X \geq 10 \to A$~~ ~~$X > 9 \to B$.~~ ~~Ans $\to X \geq 10 \cap X > 9$~~ ~~10...~~ ~~10.~~

~~$P(A|B) = \frac{P(A \cap B)}{P(B)} =$~~

$$. P(X > s+t \mid X > s) = P(X > t),$$

$$P(X > 9+1 \mid X > 9) = P(X > 1)$$

$$= \int_1^\infty \frac{1}{2} e^{-x/2}\,dx.$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1} \times 2\right]_1^\infty$$

$$= \frac{1}{2}\left[-2e^{-\infty} - (-2e^{-1/2})\right]$$

$$= \frac{1}{\cancel{2}}\left[(-2)\left[0 - e^{-1/2}\right]\right] = -e^{-1/2} =$$

12. b) (i) The mark obtained a number of students in a certain subject are assumed to be normally distributed with mean 65 and standard deviation 5. If 3 students are selected at random from this set what is the probability that exactly 2 of them will have marks over 70?

Mean = 65

S.D = 5

If X represents normally distributing random variable then

$P[\text{marks over } 70] = P[X>70]$

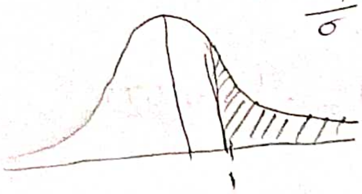
$= P[70 < X < \infty]$

$= P\left[\frac{70-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \infty\right]$

standardization of X

$= \frac{X-\mu}{\sigma}$

$= P\left[\frac{5}{5} < z < \infty\right]$



$= P[1 < z < \infty]$ $= ~~1-P[0 \le z \le 1]~~$

$= P[-\infty < z < \infty] - P[-\infty < z < 1]$ $= 1 - 0.5$

$= 1 - 0.5398$

$= 0.4602$

$P[\text{marks} > 70] = 0.4602$

$P(2 < z < \infty)$

Then $q = 1-p = 0.5398$ Given $n = 3$

$= P(-\infty < z < \infty) - P(0 < z < 2)$

Since P is same for all students the number Y follows Binomial distribution

$= 1 - 0.9772$

$= 0.0228$ //.

$P(\text{atleast } 2 \text{ scores} > 70) = P(Y \geq 2)$

$= 1 - P(Y < 2)$

$= 1 - [P(Y=0) + P(Y=1) + P(Y=2)]$

$= 1 - \left[C\binom{3}{0} p^0 q^{3-0} + C\binom{3}{1} \cdot p^1 q^2 + C\binom{3}{2} \cdot p^2 q^1\right]$

$= 1 - [$

(ii) A car firm has 2 cars which it hires out day by day. The number of demands of cars on each day follows a Poission distribution with mean 1.5. Calculate the proportion of days on which (i) Neither car is used. (ii) Some demand is not fulfilled.

Mean $= \lambda = 1.5$

Let x denote no. of cars which hired day by day.

For poisson distribution,

$$P(X=r) = \frac{e^{-\lambda} \times \lambda^{r}}{r!} = \frac{e^{-1.5} \times 1.5^{r}}{r!}$$

(i) Neither the car is used.

$$P(X=0) = e^{-1.5}$$

(ii) Some demands is not fulfilled.

= Demand is more than 2 cars per days.

$$P(x>2) = 1 - P(x \leq 2)$$

$$= 1 - P(x=0) - P(x=1) - P(x=2)$$

$$= 1 - e^{-1.5} - 1.5e^{-1.5} - \frac{1.5^2 \cdot e^{-1.5}}{2}$$

$$= 1 - \frac{1}{e^{1.5}} - \frac{1.5}{e^{1.5}} - \frac{1.5^2}{2e^{1.5}}$$

$$= 1 - \frac{1}{e^{1.5}}\left[1 - 1.5 - \frac{2.25}{2}\right]$$

$$= 1 - 0.2231\left[\overset{-1.625}{\cancel{1.125}}\right]$$

$$= 1 + 0.3625 = \cancel{0}.3625$$

March 2022

2 Marks'

(1) 1. The number of monthly breakdown of a computer is a random variable having the poission distribution of mean equal to 1.8. Find the probability that this computer will function for a month with atleast one breakdown.

(2) 2. The length of time a person speaks over phone follows exponential distribution with mean $1/6$. what is the probability that the person will talk for more than 8 minutes?

13 Marks: Given Same as Assignments.

September 2020

2 Marks

(3) 1. The time (in hours) requires to repair a machine is exponentially distributed with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 2hrs?

(4) 2. A pair of dice is thrown 4 times. If getting a doublet is contains a successes. Find the probability of 2 successes.

13 Marks

(5) 12. a) Same as Assignment

b) If 10% of the screws produced by a automic machine are defective, find the probability that out of 20 screws selected at random there are (i) exactly 2 defective; (ii) at most 3 defective (iii) at least 2 defectives (iv) between 1 and 3 defectives (inclusive).

16 Mark.

~~16. a) Fit a binomial distribution for the following data and also test the goodness of fit~~

~~x: 0~~

~~f:~~

1) The number of monthly breakdown of a computer is a random variable having the ~~position~~ poisson distribution of mean equal to 1.8. Find the probability that the computer will function for a month with atleast 1 breakdown.

Given: Mean = 1.8

$X \to$ poisson distribution.

To find: $P(X \geq 1)$.

Solution:

For X to follow poisson distribution,

$$P(X=x_i) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$\boxed{Mean = \lambda = 1.8}$

$$P(X \geq 1) = 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - \left[\frac{e^{-1.8} \cdot 1.8^{0}}{0}\right] = 1 - \frac{1}{e^{1.8}} = \frac{e^{1.8}-1}{e^{1.8}} = \frac{5.0496}{6.0496}$$

$\boxed{P(X \geq 1) = 0.834.}$

2. The length of time a person speaks over phone follows exponential distribution with mean 1/6. What is the probability that the person will talk for more than 8 minutes?

$X \to$ ~~Poisson~~ Exponential distribution.

Mean = 1/6.

$P(X>8)$

w.k.t X follows exponential distribution

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{6} e^{-x/6}$$

$$P(X>8) = \int_{8}^{\infty} f(x)\, dx = \frac{1}{6} \int_{8}^{\infty} e^{-x/6}\, dx$$

$$= \frac{1}{6} \times -6 \left[e^{-\frac{x}{6}}\right]_{8}^{\infty}$$

$$= -1 \left[0 - e^{-4/3}\right] = -1\left[-6.11 \times 10^{-3}\right]$$

$$= -1[-0.006]$$

$\boxed{P(X>8) = 0.006}$

3) The time (in hrs) required to repair the machine is exponentially distributed with mean = 1/2. What is the probability that exceeds 2 hrs?

$\lambda = \frac{1}{2}$

$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{1}{2}x}$

~~$P(X \geq 2) = 1 - P(X \leq 2)$ $[P_0 + P_1 + P_2]$~~

$P(X>2) = \int_2^\infty f(x)dx = \frac{1}{2}\left[\int_2^\infty e^{-x/2}dx\right]_2^\infty = \frac{1}{2}\left[-2e^{-x/2}\right]_2^\infty$

$= -[0 - e^{-1}]$

$= -[0 - 0.3679]$

$= 0.3679.$

4) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of 2 successes.

From the above problem, the X which describes the R.V of dice trials follows binomial distribution where

$n \to 4$ times.

$p = P(\text{getting doublet}) = \frac{n(A)}{n(S)} = \frac{6}{36}$ $A \Rightarrow \{(1,1),(2,2),\dots(6,6)\}$

$q = 1-p = 1-\frac{1}{6} = \frac{5}{6}.$

$P(x) = nC_x . p^x . q^{n-x}$

$= 4C_x p^x . q^{4-x}$

$P(\text{getting 2 successes}) = 4C_2 . p^2 . q^2$

$= 6.\left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^2 = 6.\frac{1}{6\times6}.\frac{25}{6\times6} = \frac{25}{6^3} = 0.1157$

$\boxed{\therefore P(\text{getting 2 success}) = 0.1157}$

5) If 10% of the screws produced by an automic machine are defective, find the probability that out of 20 screws selected at random that are

(i) exactly 2 defective
(ii) at most 3 defective
(iii) atleast 2 defective
(iv) between 1 and 3 defective (inclusive).

Number of screws $n = 20$

$p(\text{defective screws}) = 10\% = \frac{1}{10}$

$p = \frac{1}{10}$

$q = \frac{9}{10}$

As the event follows binomial distribution.

$p(x=x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$,

$p(x) = {}^{20}C_{x} \cdot \left(\frac{1}{10}\right)^{x} \cdot \left(\frac{9}{10}\right)^{20-x}$

i) Exactly 2 defective

$p(x=2) = {}^{20}C_{2} \cdot \left(\frac{1}{10}\right)^{2} \cdot \left(\frac{9}{10}\right)^{18} = 0.2851$

$\boxed{p(x=2) = 0.2851}$

(ii) Atmost 3 defective.

$$p(x \le 3) = \sum_{x=0}^{3} p(x=x)$$

$$= \left[{}^{20}C_{3} \cdot \left(\frac{1}{10}\right)^{3} \cdot \left(\frac{9}{10}\right)^{17}\right] + \left[{}^{20}C_{2} \cdot \left(\frac{1}{10}\right)^{2} \cdot \left(\frac{9}{10}\right)^{18}\right]$$

$$+ \left[{}^{20}C_{1} \cdot \left(\frac{1}{10}\right)^{1} \cdot \left(\frac{9}{10}\right)^{19}\right] + \left[{}^{20}C_{0} \cdot \left(\frac{1}{10}\right)^{0} \cdot \left(\frac{9}{10}\right)^{20}\right]$$

$= 0.1901 + 0.2851 + 0.2701 + 0.1215$

$= 0.8668$.

(iii) Atleast 2 defective

$p(x \ge 2) = 1 - p(x<2) = 1 - [p(x=0) + p(x=1)]$

$= 1 - (0.1216 + 0.2701) = 0.6083$.

(iv) Between 1 & 3 defective (inclusive)

$p(1<x<3) = p(x=2) = 0.2851$

3. Two Dimentional Random Variables

Let S be the sample space.

$X \Rightarrow X(s)$. $Y \Rightarrow Y(s)$. X, Y $\in R$.

$\therefore (X, Y) \rightarrow$ 2D random variable.

Joint distribution:

→ Like discrete random variable (1D), we define the PMF as $P(x) = P(X = x_i)$.

→ Similarly, for 2 random variable X & Y, we define the JPMP as $P(X, Y) = P(X = x_i, Y = y_j)$.

DISCREATE RANDOM 2D VARIABLE

Joint probability mass function:

→ for 2 variables X and Y, Joint PMF is denoted by

$$P(X = x_i, Y = y_j) = P_{ij}$$

$$P_{ij} \geq 0 \ \forall \ i, j \neq 0.$$

→ Total probability: $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(x_i y_j) = 1.$

Marginal Probability Function:

Marginal probability function of 2DRV (X,Y) is given by,

$$P_{i*} = \sum_{i=\infty}^{\infty} P_{ij} \quad ; \quad P_{*j} = \sum_{j=\infty}^{\infty} P_{*j}$$

Joint Cumulative Discreate Function (JCMF).

Joint cumulative distributed function of 2DRV is denoted by,

$$F(X, Y) = P(X \leq x, Y \leq y) = P((X \leq x) \cap (Y \leq y))$$

nth form

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} P(x, y), \quad n, m \in R.$$

Condition probability function:

$$P(X = i \mid Y = j) = \frac{P(X = i \cap Y = j)}{P(Y = j)} = \frac{P_{ij}}{P_{*j}}$$

Vise versa for $P(Y/X)$.

Properties:

~~$F(-\infty, \infty) = F$~~

- $F(-\infty, y) = F(x, -\infty) = 0$
- $F(-\infty, \infty) = 1$
- $P(a < X < b, Y \leq y) = F(b, y) - F(a, y)$
- $P(X \leq x, c < X < d) = F(x, d) - F(x, c)$
- $P(a < x < b, c < y < d) = (F_{bd} - F_{ad}) - (F_{bc} - F_{ac})$
- $P(X+Y) = \sum_{i=0}^{i} \sum_{j=0}^{j} P_{ij}, \quad i+j = z, \; z \in (X+Y).$

CONTINUOUS RANDOM
2D VARIABLE

Joint Probability density function

If (X, Y) are 2DRV in an continuous environment,
JPDF is denoted by $f(x, y)\, dx dy$.

- $f(x, y) \geq 0 \;\forall\; x, y \in R.$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$ (total probability).

Marginal Probability Function:

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Condition Probability Function:

$$f_{\frac{x}{y}} = \frac{f(x, y)}{f(y)}$$

Independant Random variable

$f(x,y) = f(x).f(y)$ if (x,y) are independant

———x———x———x———

COVARIANCE AND CORRELATION

COVARIENCE

$$\mathrm{cov}(X,Y) = E(X,Y) - E(X)E(Y)$$

where
$$E(a_i) = \begin{cases} \sum a_i p_i \\ \int_{-\infty}^{\infty} a f(a)\, da \end{cases}$$

Properties:

- $\mathrm{Cov}(X,Y) = 0$ if x,y are independant
- $\mathrm{cov}(X,X) = \mathrm{var}(X)$
- $\mathrm{cov}(X,Y) = \mathrm{cov}(Y,X)$
- $\mathrm{cov}(aX,Y) = a\,\mathrm{cov}(X,Y)$
- $\mathrm{cov}(X+c,Y) = \mathrm{cov}(X,Y)$
- $\mathrm{cov}(X+Y,Z) = \mathrm{cov}(X,Y) + \mathrm{cov}(Y,Z)$

On the whole ;

$$\mathrm{cov}\left(\sum_{i=1}^{m} a_i X_i,\ \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \,\mathrm{cov}(X_i, Y_j)$$

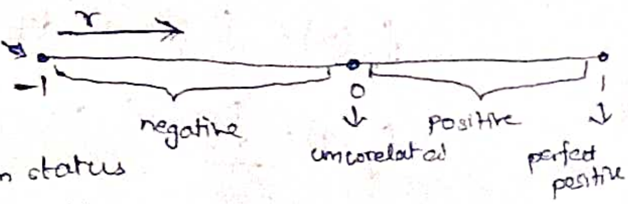
CORRELATION

Correlation of X,Y is denoted by

$$r(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_y}$$ {Karl Person's correlation coefficient}

properties:

- $r \in [-1,1]$



~~r = 1 then r is perfect~~ corr

condition	correlation status
$r=1$	perfect positive
$r\in[0,1]$	positive
$r\in[-1,0]$	negative
$r=0$	uncorrelated.

REGRESSION

Regression → study of relationship between two variables.

Line (Regression, $y \to x$)

$$y - \bar{y} = b_{yx}(x - \bar{x}) \quad , \quad b_{yx} = \frac{\sigma_y}{\sigma_x} \times r$$

Line (Regression, $x \to y$)

$$x - \bar{x} = b_{xy}(y - \bar{y}) \quad , \quad b_{xy} = \frac{\sigma_x}{\sigma_y} \times r$$

~~where~~

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

Angle between two lines of regression.

$$\theta = \tan^{-1}\left[\frac{1 - r^2}{r}\left(\frac{\sigma_y \sigma_x}{\sigma_x^2 + \sigma_y^2}\right)\right]$$

—— x —— x —— x ——

TRANSFORMATION OF 2DRV

* $f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$
* $f_v(v) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, du$
* $f_{uv}(u,v) = f_{XY}(x,y) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

—— x —— x —— x ——

CENTRAL

CENTRAL LIMIT THEOREM

"under certain condition, the sum of a large number of random variable is approximately normal".

normal in the same normal distribution.

If $X_1, X_2, X_3 \ldots X_n$ are RV of random samples

Given a large number enough sample of random sample we will see the distribution of these variable coverage to a normal distribution.

$$\bar{X}_n = \frac{\sum_{i=1}^{n} X_n}{n} \quad ; \quad E(\bar{X}) = \mu \quad ; \quad Var(\bar{X}) = \frac{\sigma^2}{n}.$$

Normalized random variable.

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Coverage in distribution to standard normal random variable as

$$\lim_{n \to \infty} P(Z_n \leq x) = \phi(x).$$

$\phi_x \rightarrow$ standard normal CDF.

Worked Problems

1, Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes no. of red ball drawn, find Joint Probability distribution of (X,Y).

As X → white balls drawn

Y → red balls drawn.

$P(X_0, Y_0) = P(\text{drawing 3 balls other than white \& Red})$

$= P(\text{drawing 4 balls of black}) = \frac{4C_3}{9C_3} = \frac{1}{21}$

$P(X_0, Y_1) = P(\text{drawing 0 white, 1 red, 2 black}) = \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$

$\quad\quad 3C_1 \quad 4C_2.$

$P(X_0, Y_2) = P(\text{drawing 0 white, 2 red, 1 black}) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{1}{7}$

$P(X_0, Y_3) = P(\text{drawing 0 w, 3 red}) = \frac{3C_3}{9C_3} = \frac{1}{84}$

$P(X_1, Y=0) = P(1W, 0R) = \frac{2C_1 \times 4C_2}{9C_3} = \frac{1}{7}$

$P(X=1, Y=1) = P(1W, 1R) = \frac{2C_1 \times 3C_1 \times 4C_1}{9C_3} = \frac{2}{7}$

$P(X=1, Y=2) = P(1W, 2R) = \frac{2C_1 \times 3C_2}{9C_3} = \frac{1}{14}$

$P(X=1, Y=3) = 0$ (3 balls drawn).

$P(X=2, Y=0) = P(2W, 0R, 1B) = \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$

$P(X=2, Y=1) = P(2W, 1R) = \frac{2C_2 \times 3C_1}{9C_3} = \frac{1}{28}$

$P(X=2, Y=2) = 0$

$P(X=2, Y=3) = 0$

The Joint Probability distribution are defined as below,

X \ Y	0	1	2	3
0	1/21	3/14	1/7	1/84
1	1/7	2/7	1/14	0
2	1/21	1/28	0	0

2) From the following joint distribution of X and Y, find the marginal distributions.

Y \ X	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

Y \ X	0	1	2	$\sum_{j=0}^{\infty} P(y_i) = P_{*j}$
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	6/14
2	1/28	0	0	1/28
$P_{j*} = \sum_{i=0}^{\infty} P(x_i)$	10/28	15/28	3/28	1

P_{i*} → Marginal Probability of X over Y.

P_{*j} → Marginal Probability of Y over X.

Note: After the above, we need to represent each ($P_x(X=0)$ to $P_x(X=2)$, and $P_y(Y=0)$ to $P_y(Y=2)$) in below format.

$$P_{xy}(y) = \begin{cases} 15/28, & y=0 \\ 6/14, & y=1 \\ 1/28, & y=2. \end{cases}$$

$$P_x(x) = \begin{cases} 10/28, & x=0 \\ 15/28, & x=1 \\ 3/28, & x=2. \end{cases}$$

3. The joint probability mass function of (X, Y) is given by $P(X,Y) = k(2x+3y)$, $x = \{0, 1, 2\}$, $y = \{1, 2, 3\}$. Find,

(i) Marginal probability
(ii) Conditional Probability
(iii) Probability distribution of $X+Y$.

The Joint Probability function is represented in below table,

X \ Y	1	2	3	P_{i*}
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k.	30k.
P_{j*}	15k	24k	33k	72k

$k(2x+3y)$
0 + 3
2 + 3
4 + 3

From the above table.

$$\sum_{i=0}^{2}\sum_{j=1}^{3} P(x_i, y_j) = 72k = 1$$

$$\boxed{\therefore k = \frac{1}{72}}$$

(i) Marginal Probability.

$$P_x(x) = \begin{cases} 18k = 18/72 = 1/4, & x=0 \\ 24k = 24/72 = 1/3, & x=1 \\ 30k = 30/72 = 5/12, & x=2. \end{cases}$$

$$P_Y(y) = \begin{cases} 15k = 15/72, & y = 1 \\ 24k = 1/3, & y = 2 \\ 33k = 11/24, & y = 3 \end{cases}$$

(ii) Condition probability

Let $i \in \{0,1,2\}$.

$$P(X=i \mid Y=1) = \begin{cases} \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{3k}{15k} = \dfrac{1}{5}, & X=0 \\ \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{5k}{15k} = \dfrac{1}{3}, & X=1 \\ \dfrac{P(X=2, Y=1)}{P(Y=1)} = \dfrac{7k}{15k} = \dfrac{7}{15}, & X=2. \end{cases}$$

$$P(X=i \mid Y=2) = \begin{cases} \dfrac{P_{02}}{P_{*2}} = \dfrac{6}{24} = \dfrac{1}{4}, & X=0 \\ \dfrac{P_{12}}{P_{*2}} = \dfrac{8}{24} = \dfrac{1}{3}, & X=1 \\ \dfrac{P_{22}}{P_{*2}} = \dfrac{10}{24} = \dfrac{5}{12}, & X=2. \end{cases}$$

$$P(X=i \mid Y=3) = \begin{cases} \dfrac{P_{03}}{P_{*3}} = \dfrac{9}{33} = \dfrac{3}{11}, & X=0 \\ \dfrac{P_{13}}{P_{*3}} = \dfrac{11}{33}, & X=1 \\ \dfrac{P_{23}}{P_{*3}} = \dfrac{13}{33}, & X=2. \end{cases}$$

~~Let~~

Let $j \in \{1,2,3\}$

$$P(Y=j|X=0) = \begin{cases} \frac{P_{01}}{P_{0*}} = \frac{1}{6} \\ \frac{P_{02}}{P_{0*}} = \frac{1}{3} \\ \frac{P_{03}}{P_{0*}} = \frac{1}{2} \end{cases}$$

$$P(Y=j|X=1) = \begin{cases} \frac{P_{11}}{P_{1*}} = \frac{5}{24} \\ \frac{P_{12}}{P_{1*}} = \frac{1}{3} \\ \frac{P_{13}}{P_{1*}} = \frac{11}{24}. \end{cases}$$

$$P(Y=j|X=2) = \begin{cases} \frac{P_{21}}{P_{2*}} = \frac{7}{30} \\ \frac{P_{22}}{P_{2*}} = \frac{1}{3} \\ \frac{P_{23}}{P_{2*}} = \frac{13}{30}. \end{cases}$$

(ii) Probability distribution of $X+Y$.

$$P(X+Y)_z = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{ij} \quad , \quad m \in X, n \in Y, \; m+n \in (X+Y).$$

$X+Y$	$P(X+Y)$ eq.	soln.
1	P_{01}	$3/72$
2	$P_{02} + P_{11}$	$11/72$
3	$P_{03} + P_{12} + P_{21}$	$24/72$
4	$P_{13} + P_{22}$	$21/72$
5	P_{23}	$13/72$.

1) The JDF of the 2DRV (X, Y) is given by

$$f(x,y) = \begin{cases} 8xy & , 0 < x < 1, \ 0 < y < x \\ 0 & , \text{otherwise} \end{cases}$$

Find (i) $f_x(x)$ (ii) $f_y(y)$ (iii) $f(y/x)$.

(i) $f_x(X)$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

$$= \int_0^{\cancel{1}x} 8xy\,dy.$$

$$= 8x \int_0^x y\,dy$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x = 8x\left[\frac{0}{2} - \frac{x^2}{2}\right] = {}^4\cancel{8}x\left(-\frac{x^2}{\cancel{2}}\right)$$

$$= -4x^3, \ x \in (0,1)$$

(ii) $f_y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$$= \int_0^1 8xy\,dx = 8y\int_0^1 x\,dx = 8y\left[\frac{x^2}{2}\right]_0^1 = 4y.$$

(iii) $f(y|x)$

$$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f_x(x)} = \frac{8xy}{-4x^3} = -2yx^{-2},$$

$$\boxed{f(y|x) = -2yx^{-2}}$$

Ex-11

(5) If $f(x,y) = \begin{cases} 2-x-y & , 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$

is the JDF of 2DRV (X,Y), find the coefficient of correlation of X and Y.

i)

We know that,

$$\text{Correlation Coeffient} = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y}$$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^1 (2-x-y)\, dy$$

$$= \int_0^1 2\,dy - \int_0^1 x\,dy - \int_0^1 y\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= \left[2 - x - \frac{1}{2}\right] - [0]$$

$$\boxed{f_x(x) = \frac{3-2x}{2}}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \int_0^1 2\,dx - \int_0^1 x\,dx - \int_0^1 y\,dx$$

$$= \left[2x - \frac{x^2}{2} - yx\right]_0^1$$

$$= \left[2 - \frac{1}{2} - y\right]_0^1 - [0]$$

$$\boxed{f_y(y) = \frac{3-2y}{2}}$$

$$E(X) = \int_0^1 x\, f(x)\, dx.$$

$$= \int_0^1 x \frac{(3-2x)}{2}\, dx.$$

$$= \frac{1}{2}\int_0^1 (3x - 2x^2)\, dx.$$

$$= \frac{1}{2}\left[\int_0^1 3x\,dx - \int_0^1 2x^2\,dx\right]$$

$$= \frac{1}{2}\left[3\frac{x^2}{2} - 2\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{3}{2} - \frac{2}{3}\right) - (0)\right]$$

$$= \frac{1}{2}\left[\frac{9-4}{6}\right]$$

$$= \frac{1}{2}\left[\frac{5}{6}\right] = \frac{5}{12}$$

$$E(Y) = \int_0^1 y\, f(y)\, dy$$

$$= \int_0^1 \frac{y(3-2y)}{2}\, dy$$

$$= \frac{1}{2}\left[\frac{3y^2}{2} - \frac{2y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{3}{2} - \frac{2}{3}\right) - 0\right]_0$$

$$= \frac{1}{2}\left[\frac{5}{6}\right] = \frac{5}{12}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

$$= \int_0^1 x^2 \left(\frac{3}{2} - \frac{2x}{2}\right) dx.$$

$$= \frac{1}{2}\int_0^1 \left[3x^2 - 2x^3\right] dx.$$

$$\cancel{\frac{1}{2}\left\{(3-2)(x)\right\}}$$

$$= \frac{1}{2}\left[\cancel{3}\frac{x^3}{\cancel{3}} - \cancel{2}\frac{x^4}{\cancel{4}2}\right]_0^1$$

$$= \frac{1}{2}\left[x^3 - \frac{x^4}{2}\right]_0^1$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) - 0\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\right] = \frac{1}{4}$$

$$\boxed{E(x^2) = \frac{1}{4}}$$

$$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy$$

$$= \int_0^1 y^2 \left(\frac{3-2y}{2}\right) dy$$

$$= \frac{1}{2}\int_0^1 y^2(3-2y)\,dy$$

$$= \frac{1}{2}\int_0^1 (3y^2 - 2y^3)\,dy$$

$$= \frac{1}{2}\left[\int_0^1 3y^2\,dy - \int_0^1 2y^3\,dy\right]$$

$$= \frac{1}{2}\left[3\left(\frac{y^3}{3}\right)_0^1 - \cancel{2}\left(\frac{y^4}{4}\right)_0^1\right]$$

$$= \frac{1}{2}\left[\frac{3}{3} - \frac{\cancel{2}1}{\cancel{4}2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\right] = \frac{1}{4}$$

$$\boxed{E(y^2) = \frac{1}{4}}$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy\left[2-x-y\right] dx\,dy.$$

$$= \int_0^1\left[\int_0^1 (2xy - 2x^2y - 2xy^2)\,dx\right] dy$$

$$= \int_0^1 \left[\cancel{x}\, 2y \cdot \frac{x^2}{2} - 2y\cdot\frac{x^3}{3} - 2y^2\cdot\frac{x^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{2y}{2} - \frac{2y}{3} - \frac{2y^2}{2}\right] dy = 2\int_0\left(\frac{y}{2} - \frac{y}{3} - \frac{y^2}{2}\right) dy.$$

$$= 2\left[\frac{1}{2}\cdot\frac{y^2}{2} - \frac{1}{3}\cdot\frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{4} - \frac{1}{6} - \frac{1}{6}\right] = 2\left[\frac{1}{4} - \frac{1}{3}\right]$$

$$= 2\left[\frac{3-4}{12}\right]$$

$$\cancel{= \frac{1}{6}}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

$$\sigma_x^2 = E(x^2) - (E(x))^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\sigma_x = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12} \qquad \boxed{\sigma_x = \frac{\sqrt{11}}{12}}$$

$$\sigma_y^2 = E(y^2) - (E(y))^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{\sqrt{11}}{12} \qquad \boxed{\sigma_y^2 = \frac{\sqrt{11}}{12}}$$

$$\begin{array}{r} 24 \\ 6\overline{)144} \\ 12 \\ \hline 24 \\ 24 \\ \hline \end{array}$$

Correlation of coefficient

$$r = \frac{E(xy) - E(x)E(y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{\frac{1}{6} - \left(\frac{5}{12}\right)^2}{\left(\frac{\sqrt{11}}{12}\right)^2} = \frac{\frac{1}{6} - \frac{25}{144}}{\frac{11}{144}} = \frac{\frac{24-25}{144}}{\frac{11}{144}}$$

$$\boxed{r = \frac{-1}{11}}$$

(5). If $y = 2x-3$ and $y = 5x-7$ are two regression lines, find the mean values of x and y. Find the correlation coefficient b/w x & y. Find the estimate of x when $y = 1$.

Let $y = 2x - 3$ — ①

$y = 5x - 7$ — ②

∵ Two lines ① & ② can pass through their mean $\bar{x}$ and $\bar{y}$,

$\bar{y} = 2\bar{x} - 3$ — ③

$\bar{y} = 5\bar{x} - 7$ — ④

Consider ① = ②,

$2\bar{x} - 3 = 5\bar{x} + 7$

$2\bar{x} - 5\bar{x} = +7 + 3$

$-3\bar{x} = 10$

$\cancel{\bar{x}} = -10 \quad \boxed{\bar{x} = \frac{-10}{3}}$

$\therefore \bar{y} = 2\left(\frac{-10}{3}\right) - 3$

$= \frac{-20-9}{3} = \frac{-29}{3}$

$\boxed{\bar{y} = \frac{-29}{3}}$

Correlation Coefficient

$$r = \pm\sqrt{b_{xy} \cdot b_{yx}}$$

consider ① $\Rightarrow$ $y = 2x - 3$

comparing with $y - \bar{y} = b_{yx}\, x - \bar{x}$

$$\boxed{b_{yx} = 2}$$

lll^{ly} considering ② $\rightarrow$ $y = 5x + 7$.

$$5x = y - 7$$

$$x = \frac{y-7}{5}$$

$$\boxed{b_{xy} = \frac{1}{5}}$$

$$r = \pm\sqrt{2 \cdot \frac{1}{5}} = \sqrt{\frac{2}{5}} = \pm 0.63.$$

as $b_{yx}, b_{xy} > 0, r > 0$, hence $\boxed{r = 0.63}$

Substitutin $y = 1$,

$$5x = 1 - 7 = 6$$

$$\boxed{x = 6/5}.$$

4. Testing of Hypothesis

— Points to Remember —

* Sample : A portion of population that is studied to learn about characteristics of population.

* Population: Collection of individual units which can be perform or characteristics.

* Random Sampling: One in which each item of a population has an equal chance of selected for testing or experiment

* Sample size: no. of items selected in a sample for test.

  Based on sample size, tests can be classified to,

  - large sample test : Test with sample size $n \geq 30$
  - small sample test : Test with sample size $n < 30$.

* Parameters : Measurement of population's ~~sample's~~ mean, S.D, var, etc.,
* Statistics : Measurement of population as an estimate.
* Sampling distribution: If we group the mean value taken from each sample (size n), according to their frequency ; The distribution formed is sampling distribution

* Notation :

  $\mu \rightarrow$ population mean

  $\sigma \rightarrow$ population standard deviation

  $\bar{x} \rightarrow$ sample mean.

  $s \rightarrow$ sample standard deviation.

  $\gamma \rightarrow$ Degrees of freedom.

* Level of significance : It is the maximum number of the probability of rejecting H_0, when H_0 is true and is usually decided before testing an hypothesis. Usualy the probability of rejection would be refered as LOS on 5% and 1%..

* Type of error :

Based on type of hypothesis testing, there are two types :

(i) Type 1 Error : Occurs when H_0 is true but we reject H_0. This is a.k.a false positive.

(ii) Type 2 Error : Occurs when H_0 is false but we accept the H_0. This is a.k.a false negative.

~~* Types of Hypothesis testing (statistical)~~ :

* Types of Hypothesis (statistical) :

(i) Null Hypothesis (H_0) : when there is no significant difference b/w parameter and ~~value~~ statistical value.

(ii) Alternative Hypothesis (H_1) : when there is some significant difference between parameter and statistical value.

LARGE SAMPLE (SIZE) TEST :

I) z - test :

1) population - sample ($\bar{x}_1, \mu, \sigma$) :

$$z = \begin{cases} \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} & \text{, when } \sigma \text{ exists} \\ \dfrac{\bar{x} - \mu}{s/\sqrt{n}} & \text{, when } \sigma \text{ doesn't exists.} \end{cases}$$

2) 2 samples ($\bar{x}_n, \sigma_1$) :

$$z = \begin{cases} \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} & , \sigma \text{ doesn't exists} \\ \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} & , \sigma \text{ exists.} \end{cases}$$

3) Proportions :

$P \rightarrow$ ~~population~~ proportion

$p \rightarrow$ population

$q \rightarrow (p-1)$

$$p = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2} \text{ (estimate)}$$

$$z = \begin{cases} \dfrac{p - P}{\sqrt{\dfrac{pq}{n}}} \\ \dfrac{P_1 - P_2}{\sqrt{pq\left[\dfrac{1}{n_1} + \dfrac{1}{n_2}\right]}} \end{cases}$$

4. 2 standard deviations :

$$z = \begin{cases} \dfrac{s_1 - s_2}{\sigma\sqrt{\frac{1}{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \\ \dfrac{s_1 - s_2}{\frac{1}{2}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}} \end{cases}$$

5. Population - Sample (S.D) : $Z = \dfrac{s-\sigma}{\sigma/\sqrt{2n}}$.

Exact sampling distribution or

Small sample test

(I) t-Test : $(n<30)$.

1). 1 population 1 sample :

$$t = \begin{cases} \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}, & \gamma = n-1 \\ \dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}, & \gamma = n-1 \end{cases}$$

Feducial limits :

$$\bar{x} \pm t_{0.05}\frac{S}{\sqrt{n-1}}$$

2) 2 sample :

$$t = \begin{cases} \dfrac{\bar{x}_1-\bar{x}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}, & \gamma = n_1+n_2-2 \\ \dfrac{\bar{x}_1-\bar{x}_2}{S\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}, & S = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}, \ \gamma = n_1+n_2-2. \\ \dfrac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{s_1^2+s_2^2}{n-1}}}, & \gamma = 2n-2 \end{cases}$$

(II) F-test : (Test of comparison on variance (σ^2)).

$$F_0 = \frac{\sigma_1^2}{\sigma_2^2} \quad , \quad \gamma_1, \gamma_2 \quad \boxed{\gamma_x = n-1}$$

Remember : $\sigma_1 > \sigma_2$ (always)

$$s_1^2 = \frac{\Sigma(x_1 - \bar{x})^2}{n_1} \quad ; \quad s_2^2 = \frac{\Sigma(x_1 - \bar{x})^2}{n_2}$$

$$\sigma_1^2 = \left(\frac{n_1}{n_1 - 1}\right) s_1^2 \quad ; \quad \sigma_2^2 = \left(\frac{n_2}{n_2 - 1}\right) s_2^2$$

Tips :

- To test whether 2 independant sample have the drawn from same population, we have to test the below.,
  (i) Equality of population means, using t-test or z-test based on sample size.
  (ii) Equality of population variances using f-test.

(III) Chi Test (χ^2 test)

(i) Goodness of fit :

$$\chi_0^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i	E_i	$O_i - E_i$	Sq $(O_i - E_i)$	sq/E_i
				χ_0^2

$\gamma = n-1$

O → observation frequency.

E → Expected frequency.

(ii) Independant of Attributes : For 2×2 contigeous table with the cell frequencies a, b, c, d, then χ^2 value will be.

$$\chi^2 = \frac{N\left[\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right]^2}{(a+c)(b+d)(a+b)(c+d)} \quad , \quad \gamma = (m-1)(n-1).$$

m, n → row, column.

How to frame expectation table if,

step① $O[i\,j] =$

	M	N	
X	a_1	b_1	A_1
Y	a_2	b_2	B_1
	A_2	B_2	N.

$A_1 = (a+b)$
$A_2 = (a+c)$
$B_1 = (a_1+b_2)$ $B_2 = (b+d)$.
$N = a+b+c+d$.

step②

$E[i\,j] =$

	M	N	
X	A_1B_1	A_1B_2	E_1
Y	A_2B_1	A_2B_2	F_1
	E_2	F_2	

$$\boxed{A_iB_j = \frac{(a_i)(b_j)}{N}}$$

for 3 variables,

$A_3B_3 = E_n - \sum A_iB_j$

$i,j \in \{1,2\}$.

step③

Tabulate ~~Formulate~~ the result observation.

O	E	(O−E)	$(O-E)^2$	$(O-E)^2/E_1$

⋮

$$\boxed{\chi_0^2 = \sum \frac{(O-E)^2}{E_1}}$$

7,8, 14 a, b, 16a

Question paper code BA1401

2 marks

7) list any two properties of x^2 distribution?

8) Example the various steps involved in testing of hypothesis?

5 marks

14)

a) Two independent samples are chosen from two schools A and B, a common test is given in a subject. The scores at the of the students as follows:

School A	76	68	70	43	94	68	33	
School B	40	48	92	85	70	76	68	22

Can we conclude that students of school A performed better than students of School B

(or)

b) A group of 10 rats fed on diet A and anther group of 8 rats fed on diet B recorded the following increase in weight

Diet A	5	6	8	1	12	4	3	9	6	10
Diet B	2	3	6	8	10	1	2	8		

Find if the variances are significantly different

Question paper code BA1401

2 marks

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8) Example the various steps involved in testing of hypothesis?

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Can we conclude that students of school A performed better than students of school B

(or)

b) A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded the following increase in weight

Diet A	5	6	8	1	12	4	3	9	6	10
Diet B	2	3	6	8	10	1	2	8		

Find if the variances are significantly different

16 a) 1000 students at college level were graded According to their I.Q. and their economic conditions. What conclusion can you draw from the following data?

Economic conditions	I.Q level	
	High	Low
Rich	460	140
Poor	240	160

5. Design of Experiments

ANALYSIS OF VARIENCE :

- ANOVA developed by R.A. Fisher in 1920s
- This technique is used when multiple sample cases are exists.
- For other tests, we have 2 sample cases to compare to figure out the significant differences b/w them.
- Assumptions :
  - All observ. are independant of one another and randomly selected
  - Population @ each factor lvl is approx. normal.
  - Variances of each factor lvl are approx. equal.
- Basic principles to apply : Randomization, replication, and Local control.
- We can test based on below classes.

  (i) One way (ii) two way (iii) three factor (~~ii~~)
  - RBD / LSD

(i) One way classification :

One way classification on Anova is to analyse data @ just one independant variable.

steps :

① Find N, T, so as to find correct factor $CF = T^2/N$ $\boxed{T = \sum_{i=1}^{N} x_i}$

② Find SST, $SST = \sum_{j=1}^{N} \sum_{i=1}^{j} x_i^2 - C.F$

③ Find SSC, $SSC = \sum_{j=1}^{r} \left[\frac{\left(\sum_{i=1}^{j} x_i\right)^2}{n_j} \right] - \frac{T^2}{N}$ $df_c = N - r$, $df_r = r - 1$

④ $SSE = SST - SSC$

⑤ Tabulate ANOVA as below structure and find the value of

$MSC = SSC/df_r$; $MSE = SSE/df_c$

⑥ Refer to f-table and get f_e, $f_{0.05}(df_r, df_c)$ @ 5% LOS

⑦ Find $F_o = \left(\frac{MSC}{MSE}\right)$ and compare f_o with f_e.

(ii) Two way classification :

This is to analyse and conclude data ~~f~~ that has multiple independant variables ~~that~~ so as to for 2D of data representation.

steps :

① Find N, T and $CF = T^2/N$

② Find sum square formula,

$$SST = \sum_j \sum_i x_{ij}^2 - CF$$

$$SSC = \sum_j \left[\frac{(\sum_j x_i)^2}{n_j} \right] - CF$$

$$df_c = c-1$$
$$df_r = r-1$$

$$SSR = \sum_j \left[\frac{(\sum_i y_i)^2}{n_j} \right] - CF$$

$$SSE = SST - (SSC + SSR)$$

③ Find Mean square formula, (~~formula~~ Tabulating Anova)

$$MSC = \frac{SSC}{df_c} \; ; \; MSR = \frac{SSR}{df_r} \; ; \; MSE = \frac{SSE}{(c-1)(r-1)}$$

④ Find F_o,

$$F_r = \frac{MSR}{MSE} \; ; \; F_c = \frac{MSC}{MSE}$$

⑤ Find F_e from F-table @ $F(c-1, r-1)$ @ 5% of LOS

⑥ Draw condution,

For each part of dimention (Row, col in this case),

If $F_o < F_e$, we accept H_o.

(iii) Three factor classification or Latin Square Design.

A Latin square is a square table of m row and n col. such that each symbols appear once and only once at each row and column.

It reduces the number of treatement contribution and controls two source of variation.

COMPARISON:

Latin Square design.	Randomized Block design
(i) $n(r) = n(c)$, ∴ replication = no. of treatements	no replication
(ii) LSD is suitable when no. of treatements $\in [5,12]$	Can hand any no. of treatment
(iii) It removes variation b/w rows and columns from that within rows resulting in reduction of experiment error to large extend	Does not follow that advantage. Hence prefered to use LSD over RBD.
(iv) Equally on rectangular plots	more/less a square field is required.

STEPS:

(1) Find N, T, $CF = T^2/N$

(2) Find $SST = \sum_j^p \sum_i^j (x_i^2)$

(3) Find SSC, SSR, SSE, SSK.

To find SSK, we need another table to transform from 3D to 2 way classification model.

Field \ Yield	1	2	3	4	total	total^2	total$^2/4$
A							
B							
C							
D							
					$\boxed{\Sigma}$		

$\boxed{S8'}$

− CF

SSK

④ Formulate the ANOVA table and find

· MSC, MSR, MSK, MSE

⑤ Find F_c, F_r, $F_T\left(= \frac{MSK}{MSE}\right)$.

⑥ Find $F_e(df_c, df_r)$ @ 5% LOS.

⑦ Draw conclusion

w.r.t r, c, t

ANOVA TABLE

Source of Variable	Square Sum	df	Mean Square = $\frac{\text{Sum Square}}{df}$	Variance ratio
Sample (for 1c) { C^*	SSC	df_c	$MSC = SSC/df_c$	$F_c = MSC/MSE$
R^*	SSR^*	df_r	$MSR = SSR/df_r$	$F_r = MSR/MSE$
T^{**}	SSTK	$(p-1)^{**}$	$MSK^{**} = SSK/p-1$	$F_t = MSK/MSE$
E^{**}	SSE	$df_c \cdot df_r$	$MSE = SSE/df_c \cdot df_r$	
	SST	$(p^2-1)^{**}$		

$*$ -> 2way classification

$**$ -> 3way classification.

EVERGREEN FORMULAE

~~wed~~ Let ~~yo~~ $\frac{d}{dx}(f(x)) = f'(x)$,

$f'(x^n) = nx^{n-1}$

$f'(a) = 0$

$f'(a f(x)) = a f'(x)$

$f(g \pm h) = g' \pm h'$

$f(gh) = gh' \pm hg'$

$f(g/h) = \frac{hg' - gh'}{g^2}$

$f(e^x) = e^x$ $\qquad$ $f(e^{ax}) = ae^{ax}$

$\int dx = x + c$

$\int a\,dx = ax + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\int a^x\,dx = \frac{a^x}{\log a} + c$

$\int \log x\,dx = x(\log x - 1) + c$

$\int 1/x\,dx = \log(x) + c$

$\int (\log x)^2\,dx = x(\log x)^2 - 2x\log + 2x$

$\int e^x\,dx = e^x + c$

$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

$\int e^{-ax}\,dx = \frac{-e^{-ax}}{a} + c$

Reg. No. : 6 7 2 2 2 2 0 0 0 7 6

Question Paper Code : AK2501

M.C.A. (2 Years) DEGREE EXAMINATION, FEBRUARY/MARCH 2023

First Semester

DMC 6101 — MATRICES PROBABILITY AND STATISTICS

(Regulations 2018)

Time : Three hours Maximum : 100 marks

(Use of statistical tables may be permitted)

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Two eigen values of the main $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ are 1 and 2. Find third eigen value and the value of $|A|$.

2. Find the eigen values of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

3. A bag contains 3 red and 4 white balls. Two draws are made without replacement. What is the probability that both the balls are red?

4. Use Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.

5. Find the conditional density function of Y given $X = x$ if $f(x, y) = 2, 0 < x < y < 1$.

6. Find the value of k, if the joint probability density function of random variable x and Y is given by $f(x, y) = kxye^{-(x^2+y^2)}, x > 0, y > 0$.

7. List any two properties of Chi-square distribution.

8. Define Type I and Type II errors in testing of hypothesis.

9. What are the uses of analysis of variance?

10. Write down the ANOVA table with the usual notation for two-way classification.

PART B — (5 × 13 = 65 marks)

11. (a) State any two properties of eigen values and eigen vectors. Calculate the eigenvales and eigenvectors of the matrix (13)

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Or

(b) State Cayley – Hamilton theorem. Make use of the theorem to find the value of the matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1$ (13)

12. (a) The density function of a random variable X is given by $f(x) = kx(2-x), 0 \le x \le 2$. Find k, mean, variance and r^{th} moment. (13)

Or

(b) The members of a consulting firm rent cars from rental agencies A, B and C as 60 percent, 30 percent and 10 percent respectively. If 9, 20 and 6 percent of cars from A, B and C agencies need turn and if a rental car delivered to the firm does not need turn up, what is the probability that it came from B agency. (13)

13. (a) A study of prices at Chennai and Madurai gave the following data: (13)

	Chennai	Madurai
Mean	19.5	17.75
S.D.	1.75	2.5

Also the coefficient of correlation between the two is 0.8. Estimate the most likely price (i) at Chennai corresponding to the price of 18 at Madurai and (ii) at Madurai corresponding to the price of 17 at Chennai.

Or

(b) If the joint probability density function of (X,Y) is given by $f_{xy}(x, y) = x + y, 0 \le x \le 1, 0 \le y \le 1$. Find the probability density function of $U = XY$. (13)

14. (a) (i) A certain injection administered to each of 12 patients resulted in the following difference of blood pressure: 5,2,8,−1,3,0,6,−2,1,5,0,4. Can it be concluded that the injection will be, in general, accompanied by an increase in B.P.? (6)

(ii) Theory predicts that the proportion of beans in four groups A,B,C, D should be 9:3:3:1. In an experiment among 1600 beans, the numbers in the four groups were 882,313,287 and 118. Does the experiment support the theory? (7)

Or

(b) Fit a Poisson distribution for the following distribution and also test the goodness of fit: (13)

x :	0	1	2	3	4	5	Total
f :	142	156	69	27	5	1	400

15. (a) The following table shows the lifetime in hours from three different types of television tables manufactured by the company. Analyse the data using one way classification at 5% level of significance.

Sample 1	407	411	409		
Sample 2	404	406	408	405	402
Sample 3	410	408	406	408	

Or

(b) A experiment was conducted to study the performance of 4 different detergents for cleaning .. injectors. The following "Cleanliness"' readings were obtained with specially designed equipments.

		Engine		
		I	II	III
	A	45	43	51
Detergent	B	47	46	52
	C	48	50	55
	D	42	37	49

Test at 1% significance whether these are differences in the detergents.

PART C — (1 × 15 = 15 marks)

16. (a) Two random variables X and Y have the following density function (15)

$f(x,y)=\begin{cases} 2-x-y, & 0\le x\le 1, 0\le y\le 1 \\ 0, & otherwise \end{cases}$. Find $Var(X), Var(Y)$, $Cov(X,Y)$ and also the correlation between X and Y.

Or

(b) Analyze the variance in the Latin square of yields paddy, where P, Q, R and S denote the different methods of cultivation. Examine whether the different methods of cultivation have given significantly different yields. (15)

S 122	P 121	P 123	Q 122
Q 124	R 123	P 122	S 125
P 120	Q 199	S 120	R 121
R 122	S 123	Q 121	P 121