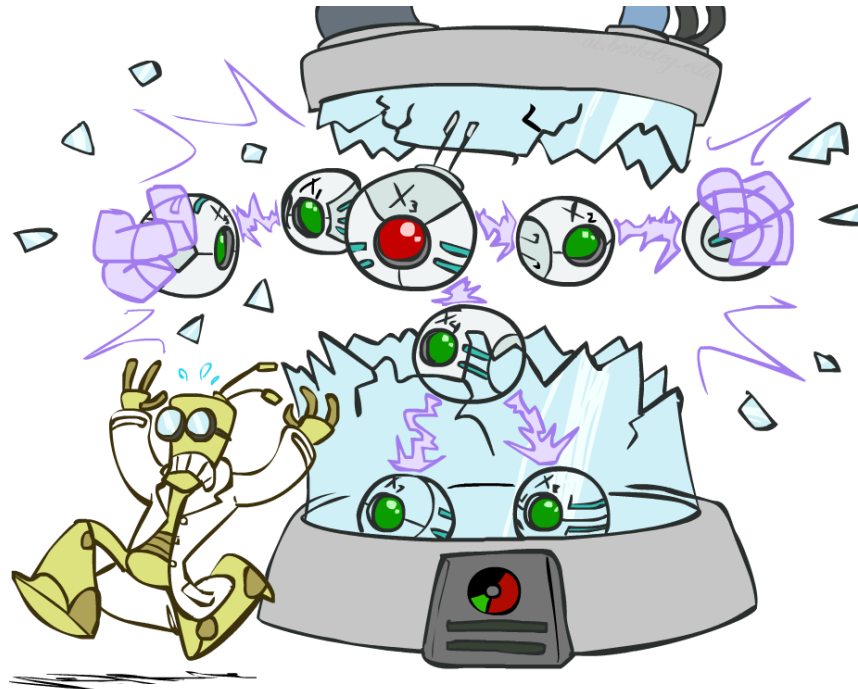


# Artificial Intelligence: Basics & Applications

## Bayes' Nets: Independence



Instructor: *Mahdi Javanmardi*

Amirkabir University of Technology

[slides adapted with permission from Dan Klein, Pieter Abbeel, Anca Dragan @ai.berkeley.edu]

# Probability Recap

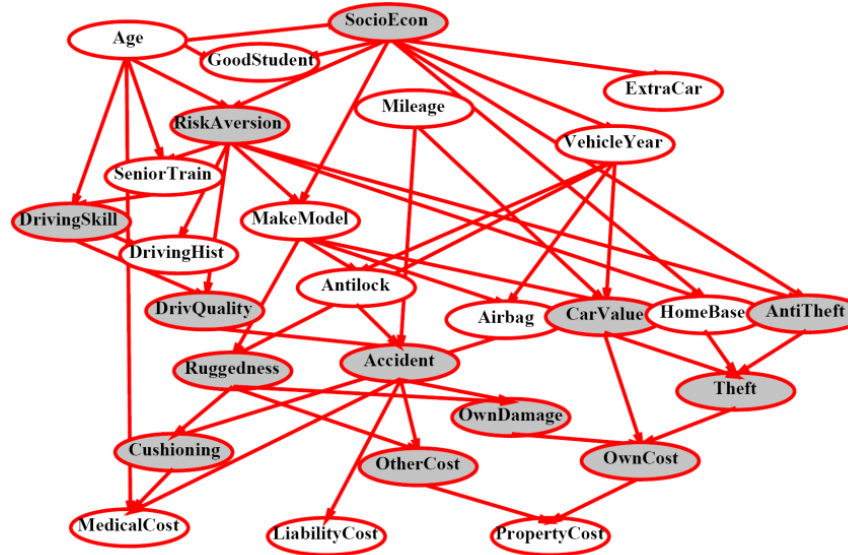
- Conditional probability  $P(x|y) = \frac{P(x, y)}{P(y)}$
- Product rule  $P(x, y) = P(x|y)P(y)$
- Chain rule 
$$\begin{aligned} P(X_1, X_2, \dots, X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots \\ &= \prod_{i=1}^n P(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if:  $\forall x, y : P(x, y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

$$X \perp\!\!\!\perp Y | Z$$

# Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is  $P(X \mid e)$ ?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

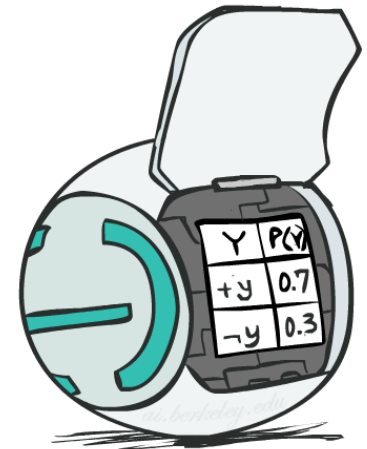
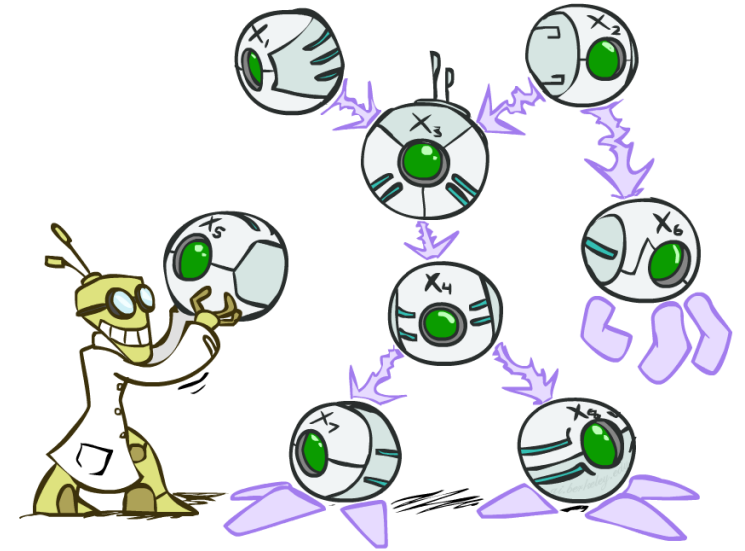
# Bayes' Net Semantics

- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over  $X$ , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$



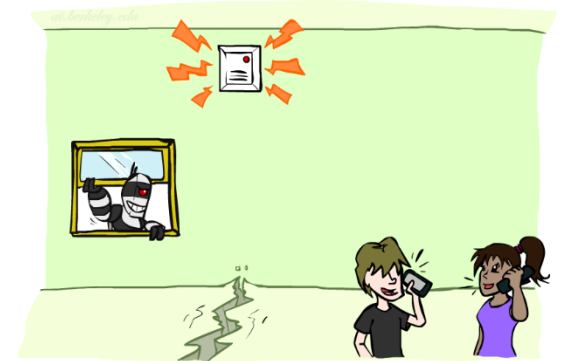
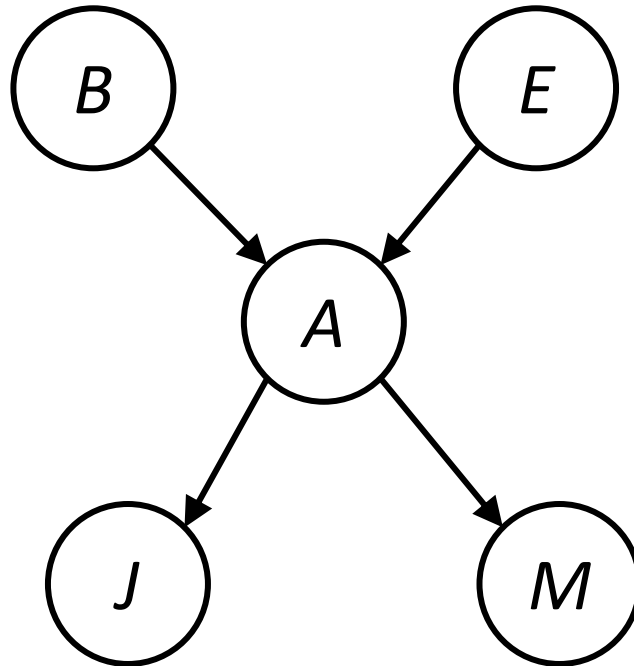
# Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999

E	P(E)
+e	0.002
-e	0.998

A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99



B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$P(+b, -e, +a, -j, +m) =$$

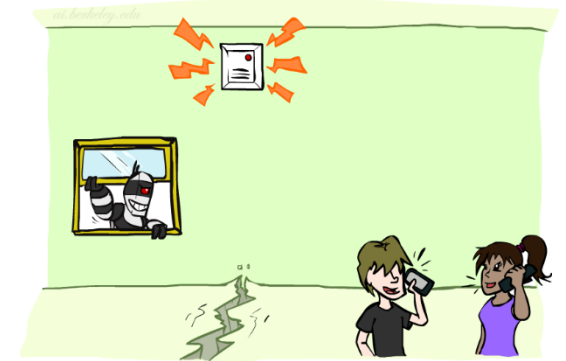
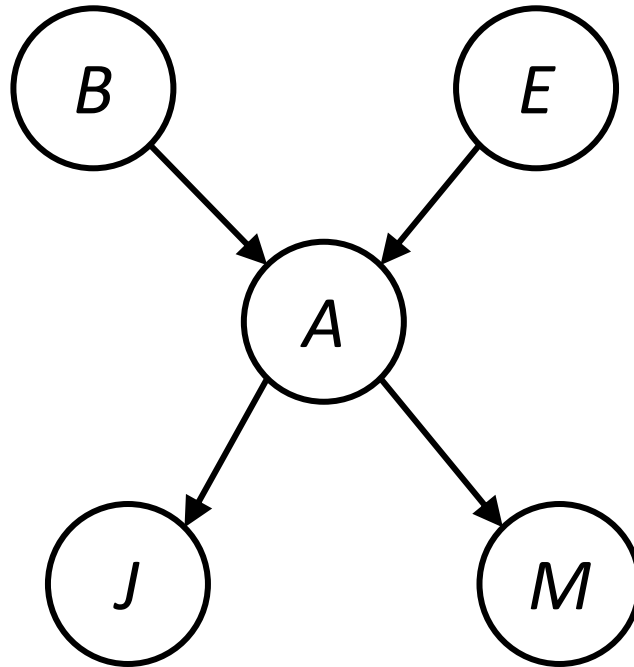
# Example: Alarm Network

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B	E	A	P(A B,E)
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-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$\begin{aligned}
 P(+b, -e, +a, -j, +m) &= \\
 P(+b)P(-e)P(+a|+b, -e)P(-j|+a)P(+m|+a) &= \\
 0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7
 \end{aligned}$$

# Size of a Bayes' Net

- How big is a joint distribution over  $N$  Boolean variables?

$$2^N$$

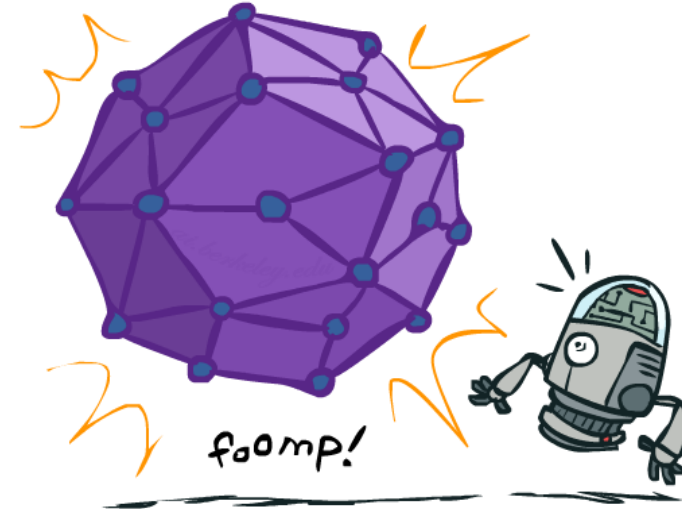
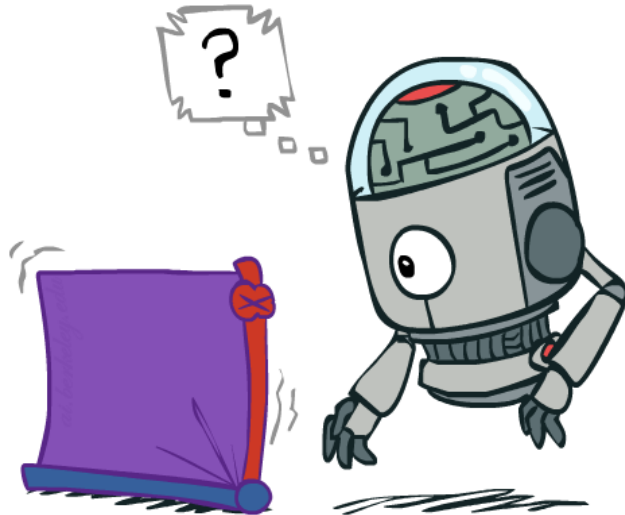
- How big is an  $N$ -node net if nodes have up to  $k$  parents?

$$O(N * 2^{k+1})$$

- Both give you the power to calculate

$$P(X_1, X_2, \dots, X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



# Bayes' Nets

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## Representation

- Conditional Independences
- Probabilistic Inference
- Learning Bayes' Nets from Data



# Conditional Independence

- X and Y are **independent** if

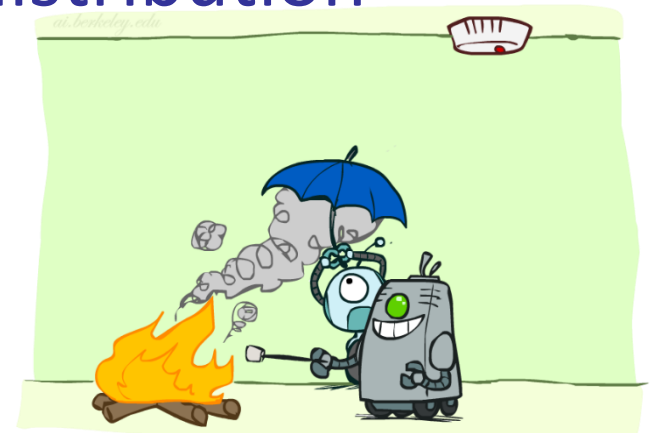
$$\forall x, y \quad P(x, y) = P(x)P(y) \quad \text{---} \rightarrow \quad X \perp Y$$

- X and Y are **conditionally independent** given Z

$$\forall x, y, z \quad P(x, y|z) = P(x|z)P(y|z) \quad \text{---} \rightarrow \quad X \perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example:  $Alarm \perp Fire|Smoke$



# Bayes Nets: Assumptions

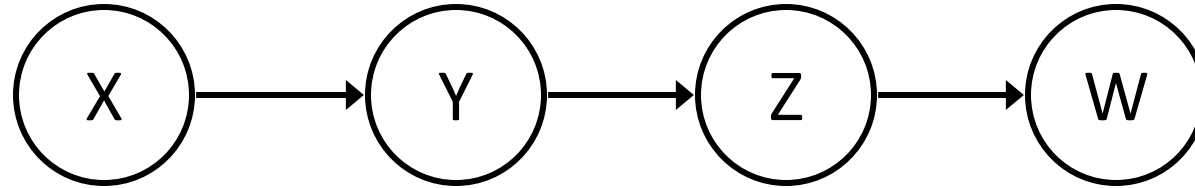
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



# Example



- Conditional independence assumptions directly from simplifications in chain rule:

$$P(x, y, z, w) = P(x) P(y | x) P(z | x, y) P(w | x, y, z)$$

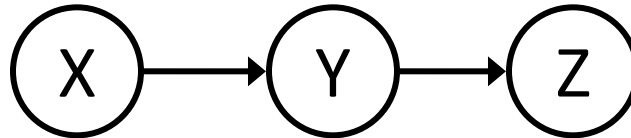
$$P(x, y, z, w) = P(x) P(y | x) P(z | y) P(w | z) \quad z \perp\!\!\!\perp x \mid y \quad w \perp\!\!\!\perp x, y \mid z$$

- Additional implied conditional independence assumptions?

$$w \perp\!\!\!\perp x \mid y$$

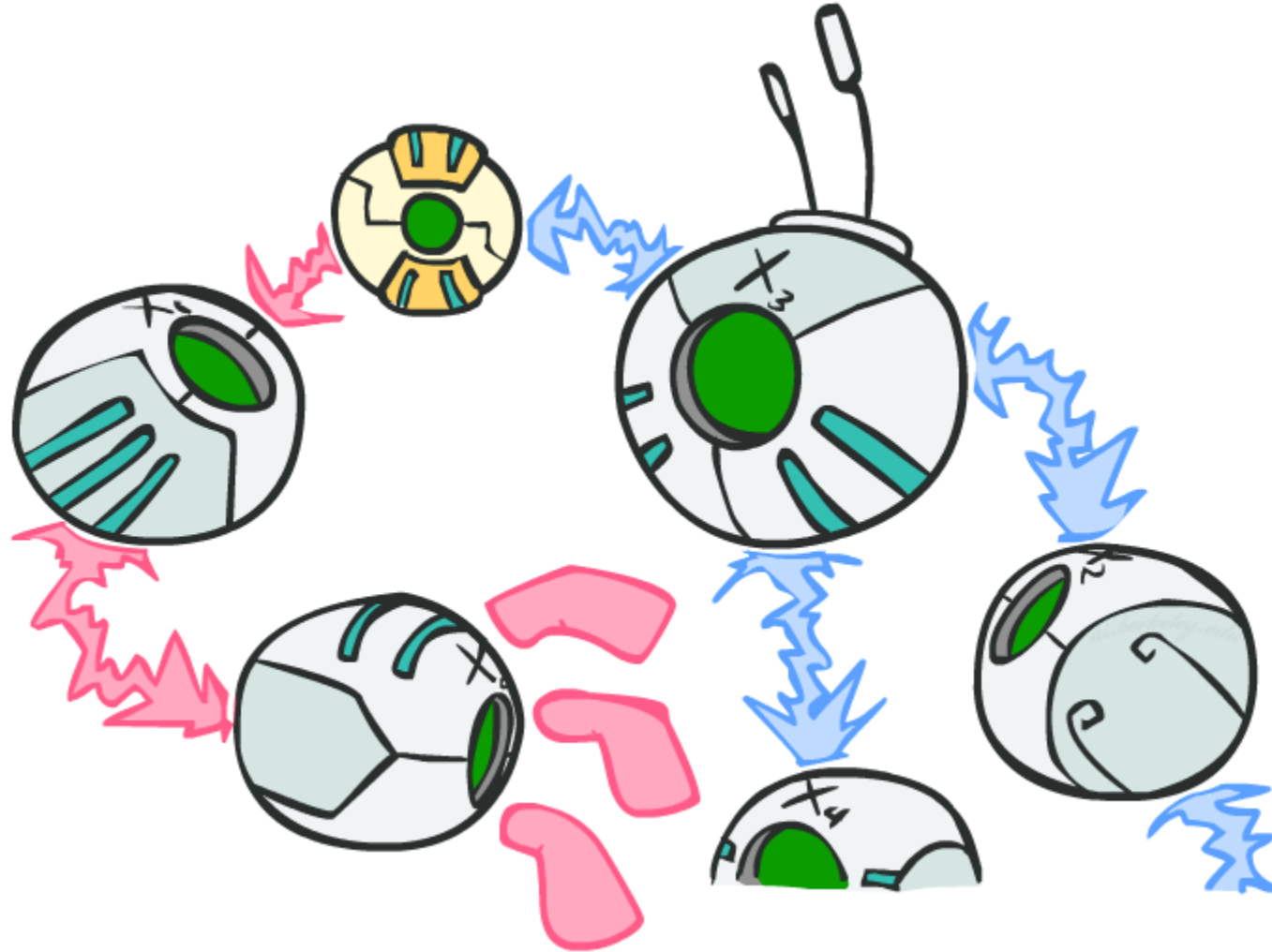
# Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



- Question: are X and Z necessarily independent?
  - Answer: no. Example: low pressure causes rain, which causes traffic.
  - X can influence Z, Z can influence X (via Y)
  - Addendum: they *could* be independent: how?

# D-separation: Outline



# D-separation: Outline

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- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

# Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic

- In numbers:

$$P(+y \mid +x) = 1, P(-y \mid -x) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

# Causal Chains

- This configuration is a “causal chain”

- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

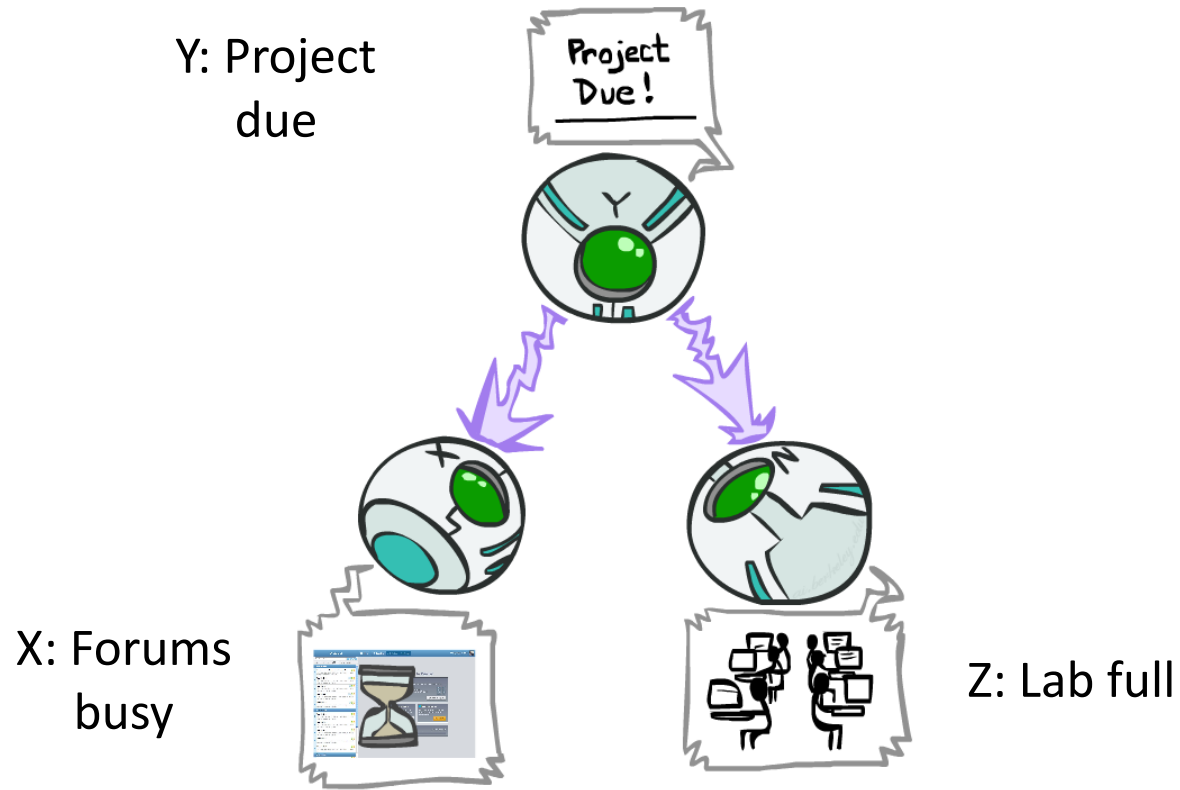
*Yes!*

- Evidence along the chain “blocks” the influence



# Common Cause

- This configuration is a “common cause”



- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Project due causes both forums busy and lab full

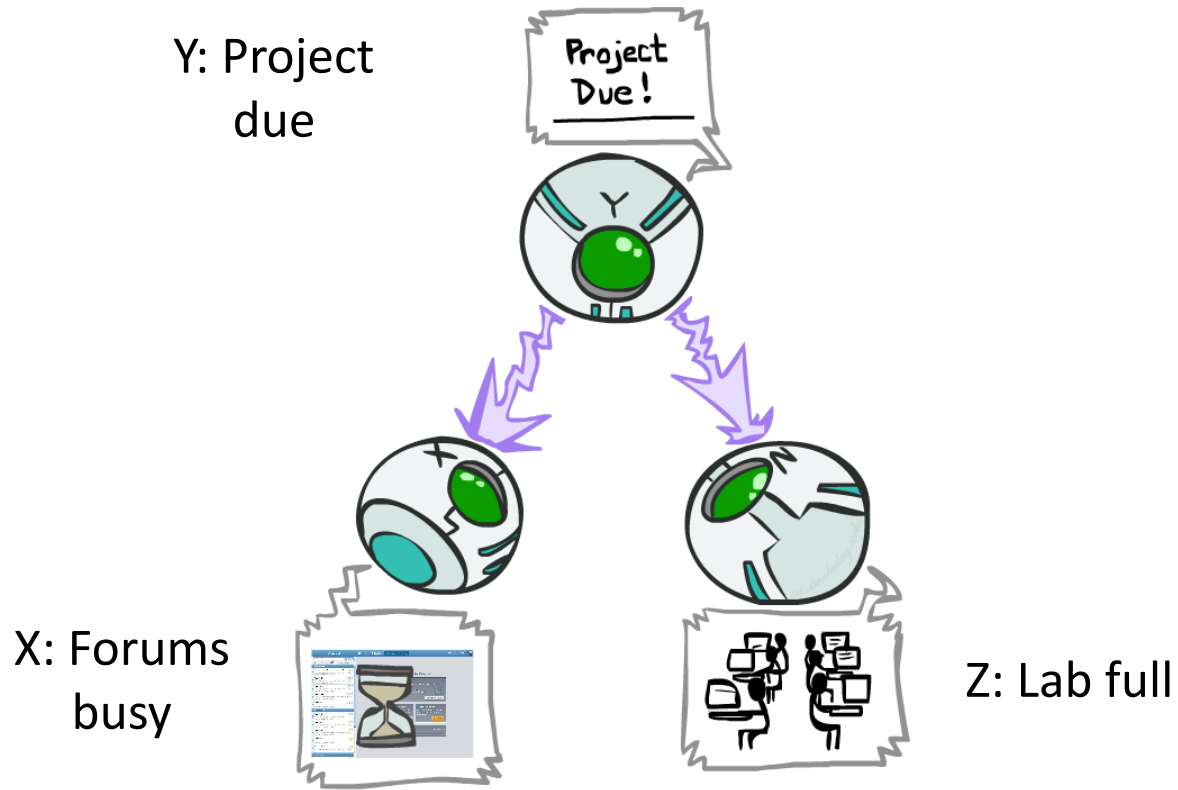
- In numbers:

$$P(+x \mid +y) = 1, P(-x \mid -y) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

# Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

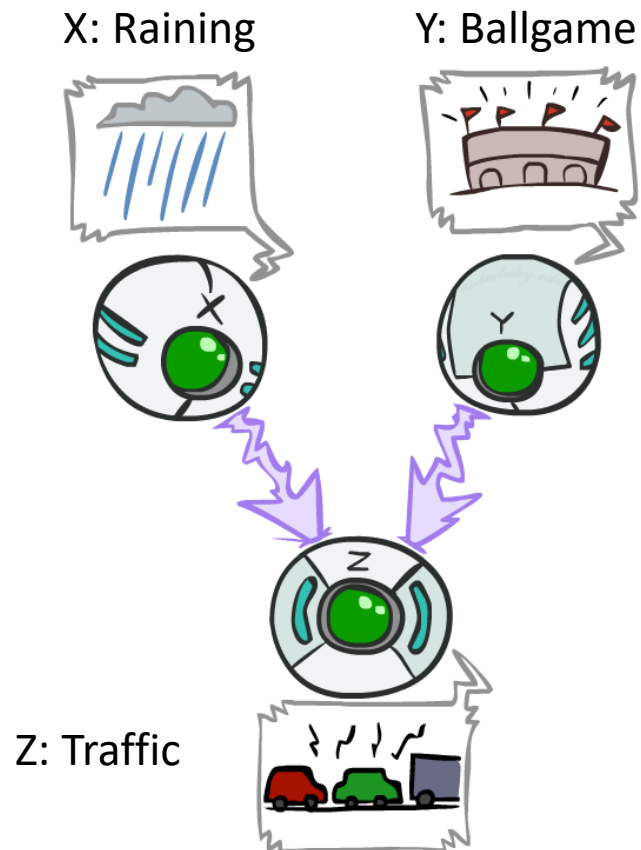
**Yes!**

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Observing the cause blocks influence between effects.

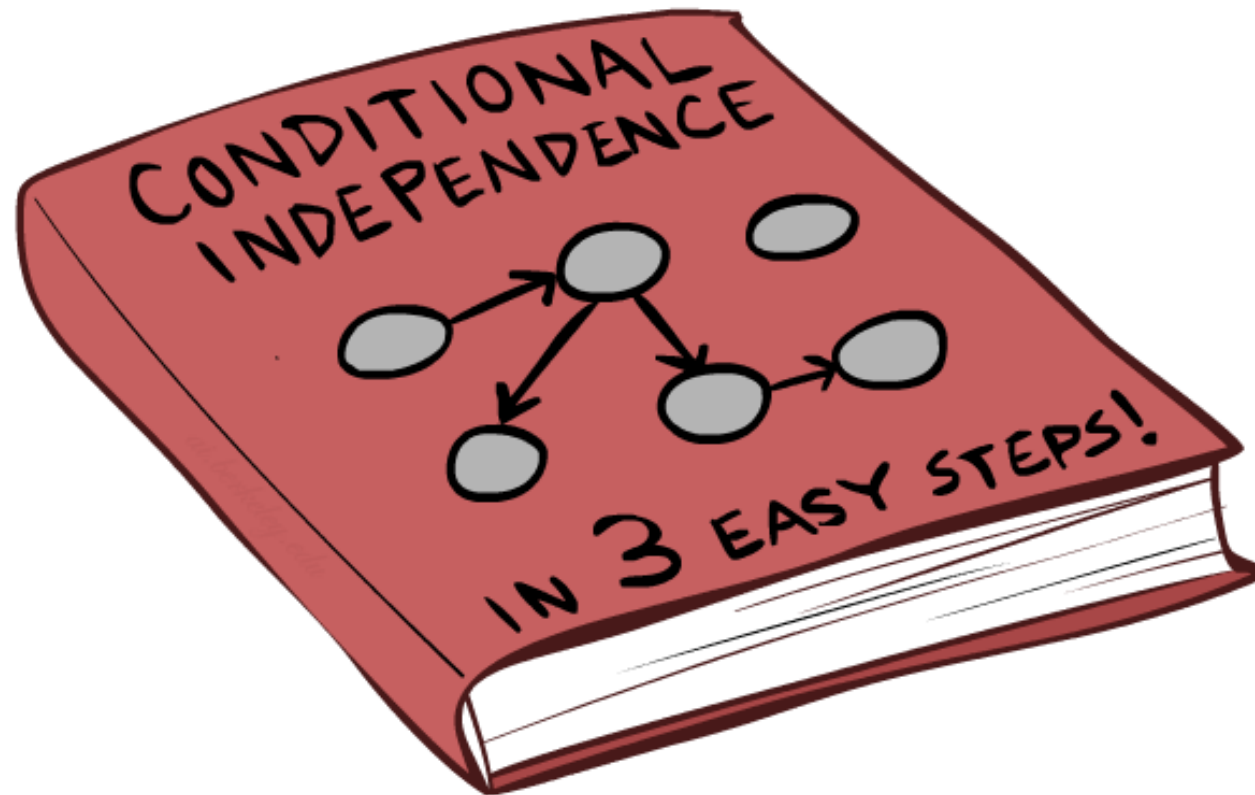
# Common Effect

- Last configuration: two causes of one effect (v-structures)



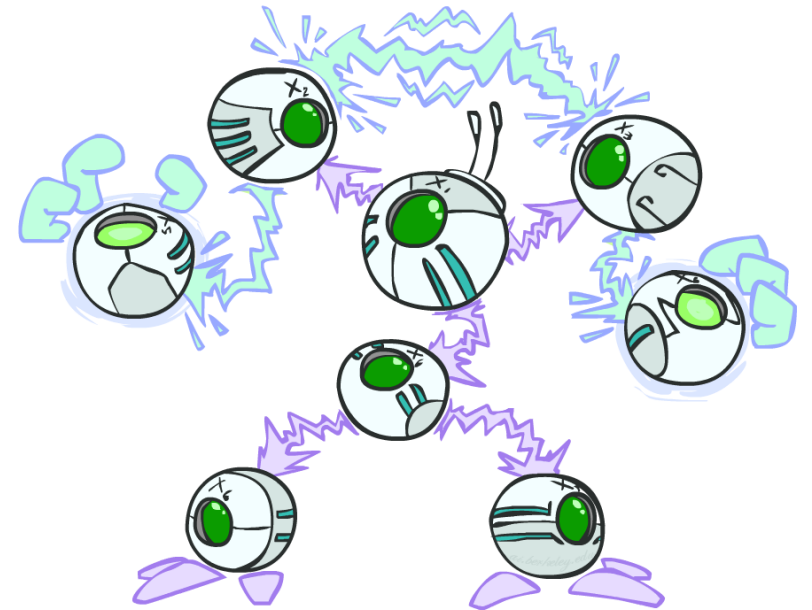
- Are X and Y independent?
  - **Yes**: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - **No**: seeing traffic puts the rain and the ballgame in competition as explanation.
- **This is backwards from the other cases**
  - Observing an effect **activates** influence between possible causes.

# The General Case



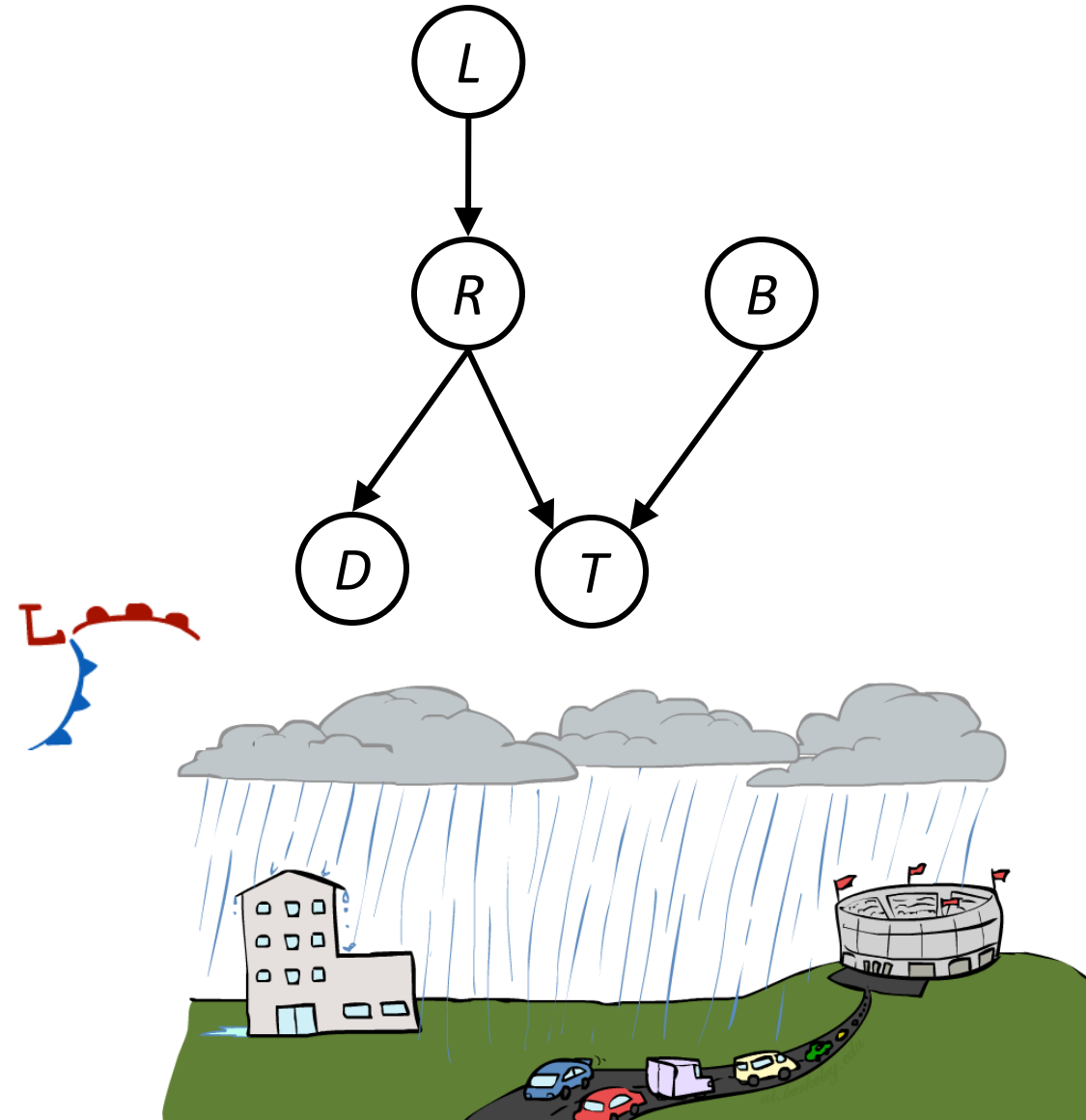
# The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



# Reachability

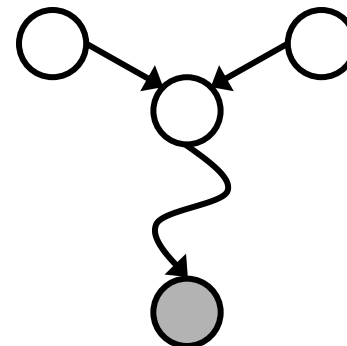
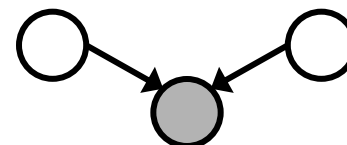
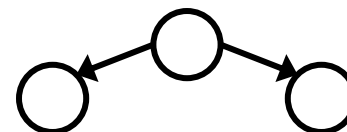
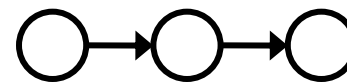
- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



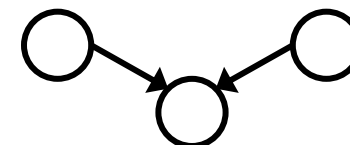
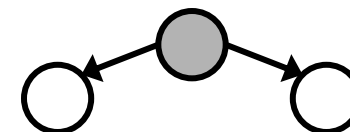
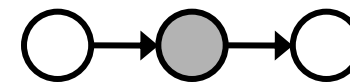
# Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y “d-separated” by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Causal chain  $A \rightarrow B \rightarrow C$  where B is unobserved (either direction)
  - Common cause  $A \leftarrow B \rightarrow C$  where B is unobserved
  - Common effect (aka v-structure)  
 $A \rightarrow B \leftarrow C$  where B or one of its descendants is observed
- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



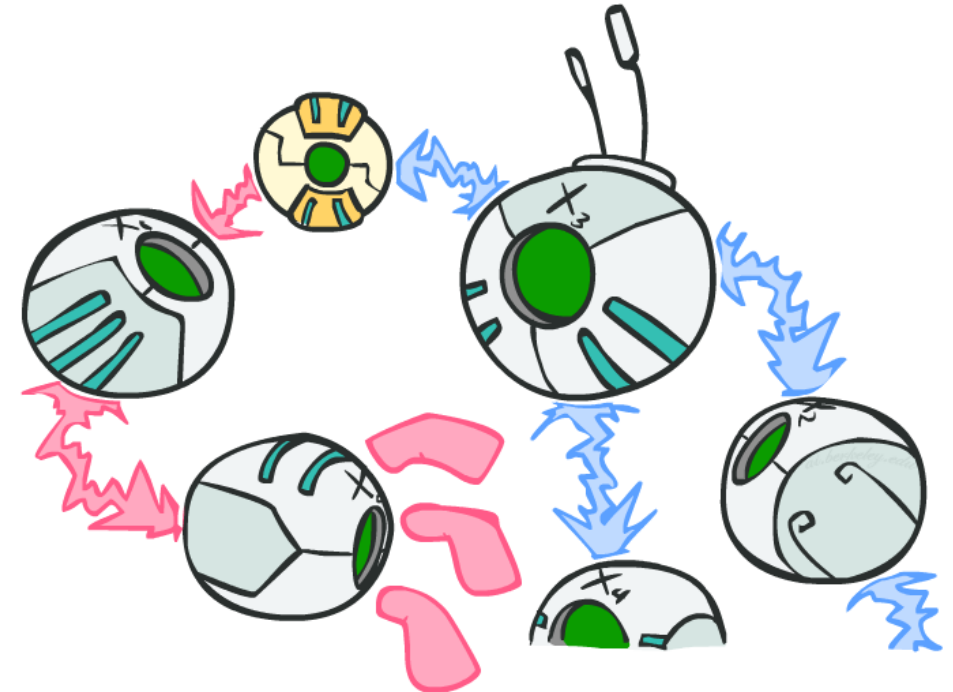
# D-Separation

- Query:  $X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\} \text{ ?}$
- Check all (undirected!) paths between  $X_i$  and  $X_j$ 
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$

- Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$





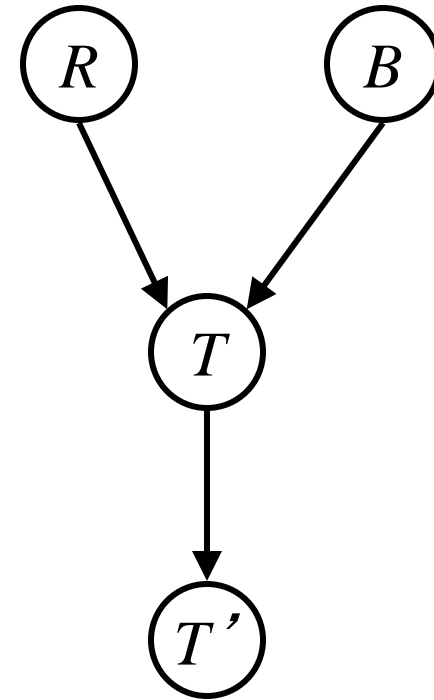
# Example

$$R \perp\!\!\!\perp B$$

*Yes*

$$R \perp\!\!\!\perp B | T$$

$$R \perp\!\!\!\perp B | T'$$



# Example

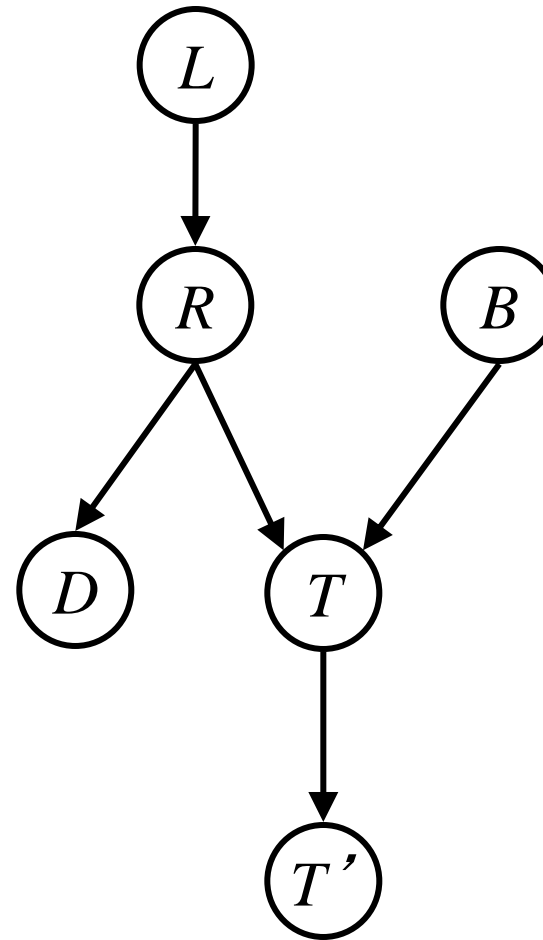
$L \perp\!\!\!\perp T' | T$  *Yes*

$L \perp\!\!\!\perp B$  *Yes*

$L \perp\!\!\!\perp B | T$

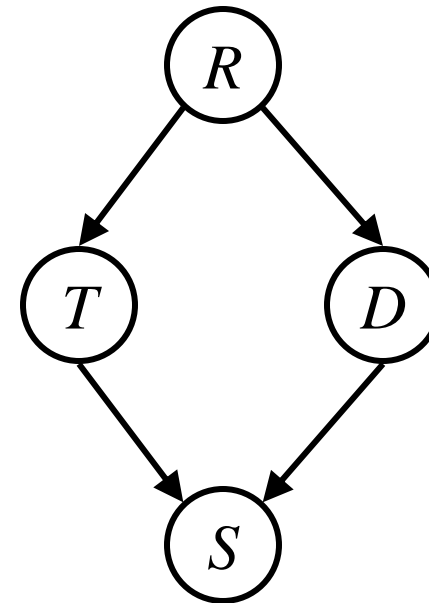
$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$  *Yes*



# Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:



$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R \quad \text{Yes}$$

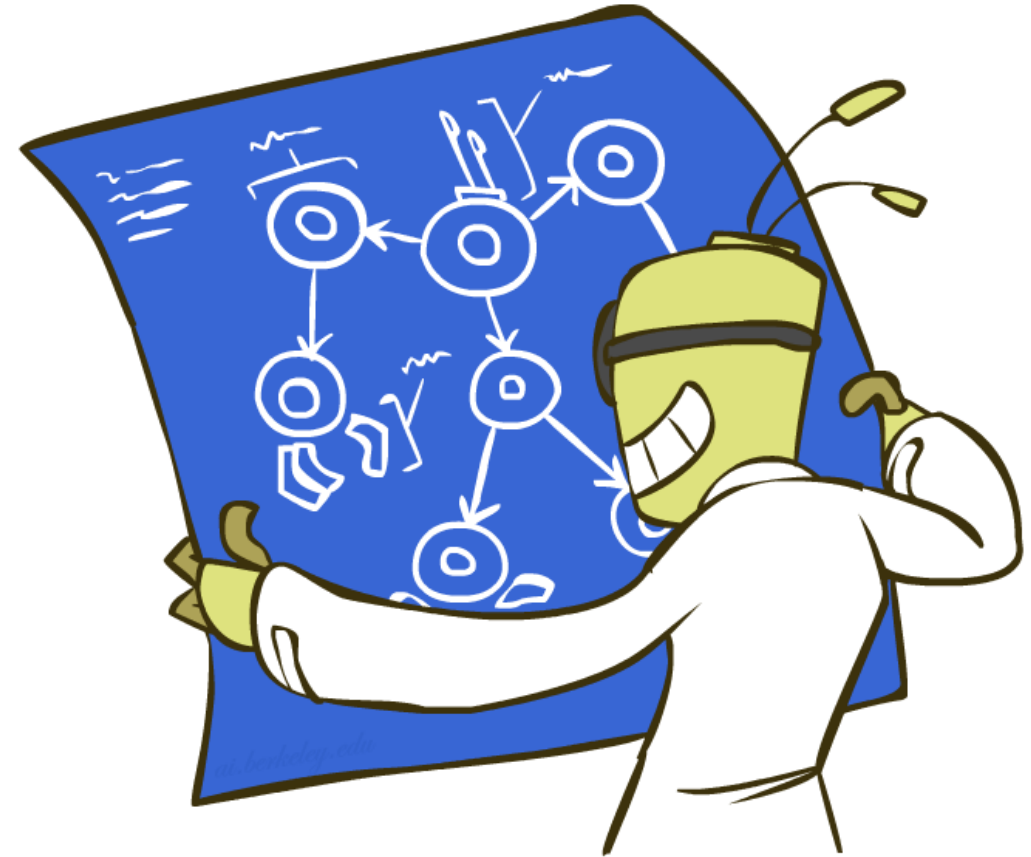
$$T \perp\!\!\!\perp D | R, S$$

# Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

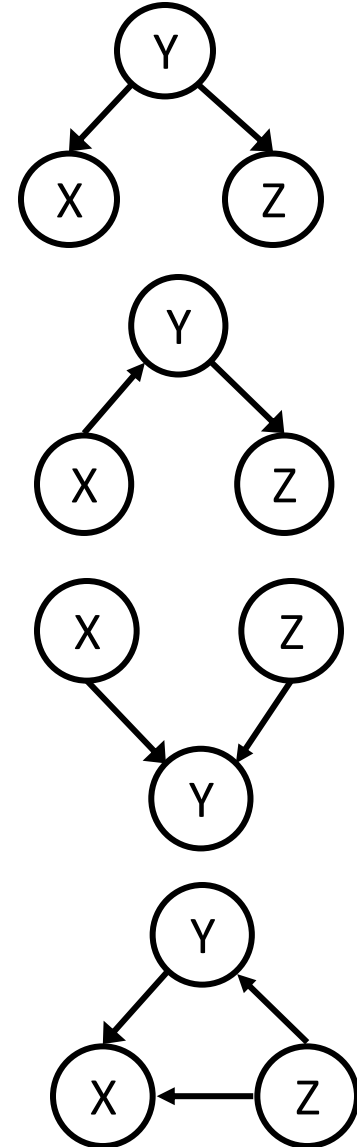
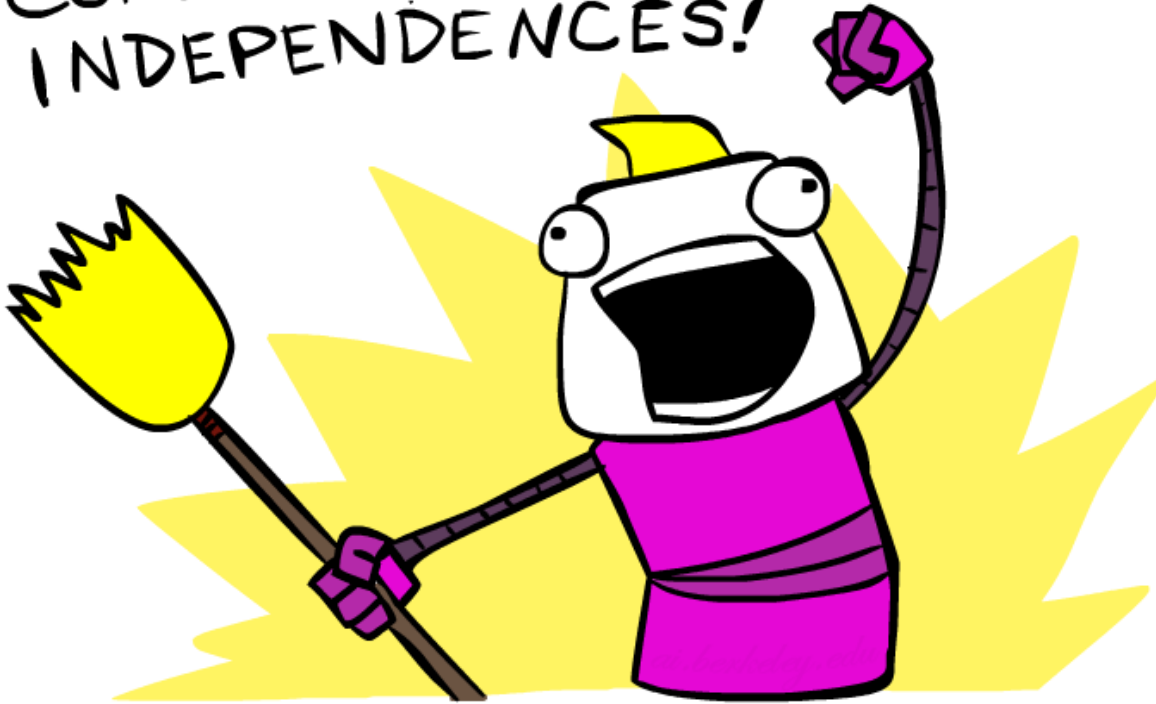
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



# Computing All Independences

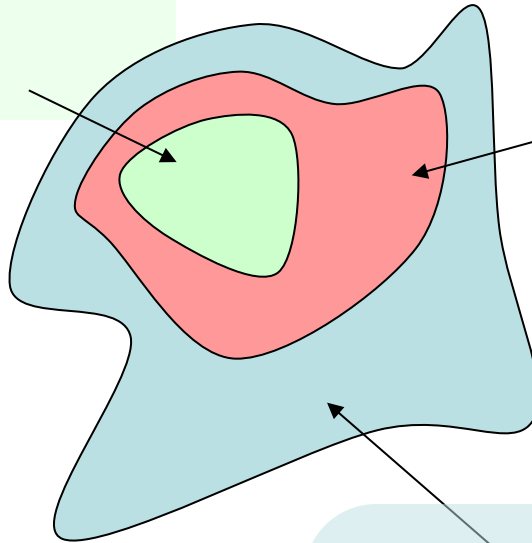
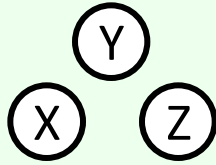
COMPUTE ALL THE  
INDEPENDENCES!



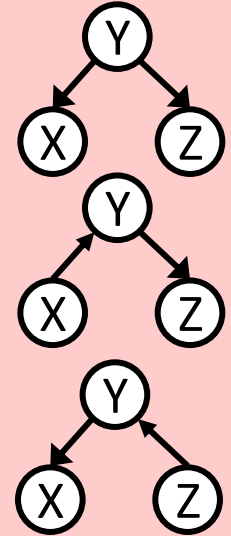
# Topology Limits Distributions

- Given some graph topology  $G$ , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

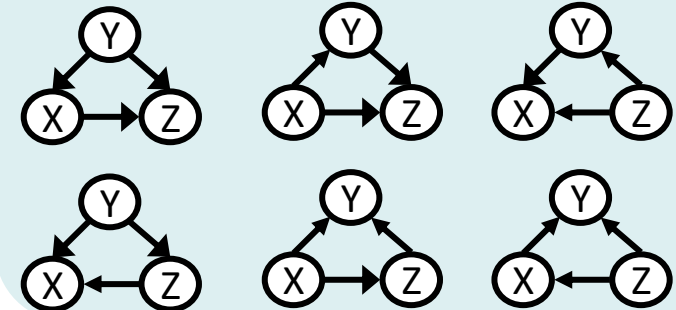
$$\{X \perp\!\!\!\perp Y, X \perp\!\!\!\perp Z, Y \perp\!\!\!\perp Z, \\ X \perp\!\!\!\perp Z \mid Y, X \perp\!\!\!\perp Y \mid Z, Y \perp\!\!\!\perp Z \mid X\}$$



$$\{X \perp\!\!\!\perp Z \mid Y\}$$



$$\{\}$$



# Bayes Nets Representation Summary

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- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

# Bayes' Nets

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Representation



Conditional Independences

- Probabilistic Inference

- Enumeration (exact, exponential complexity)
- Variable elimination (exact, worst-case exponential complexity, often better)
- Probabilistic inference is NP-complete
- Sampling (approximate)

- Learning Bayes' Nets from Data