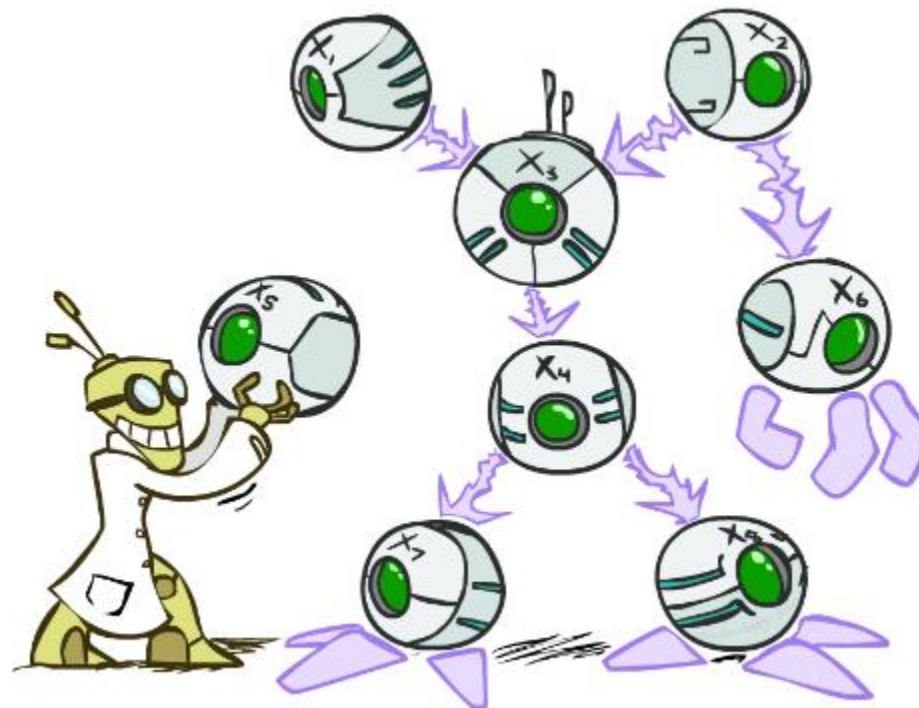


Artificial Intelligence: Basics & Applications

Bayes' Nets



Instructor: *Mahdi Javanmardi*

Amirkabir University of Technology

Probabilistic Models

- Models describe how (a portion of) the world works

- Models are always simplifications**

- May not account for every variable
- May not account for all interactions between variables
- “All models are wrong; but some are useful.”



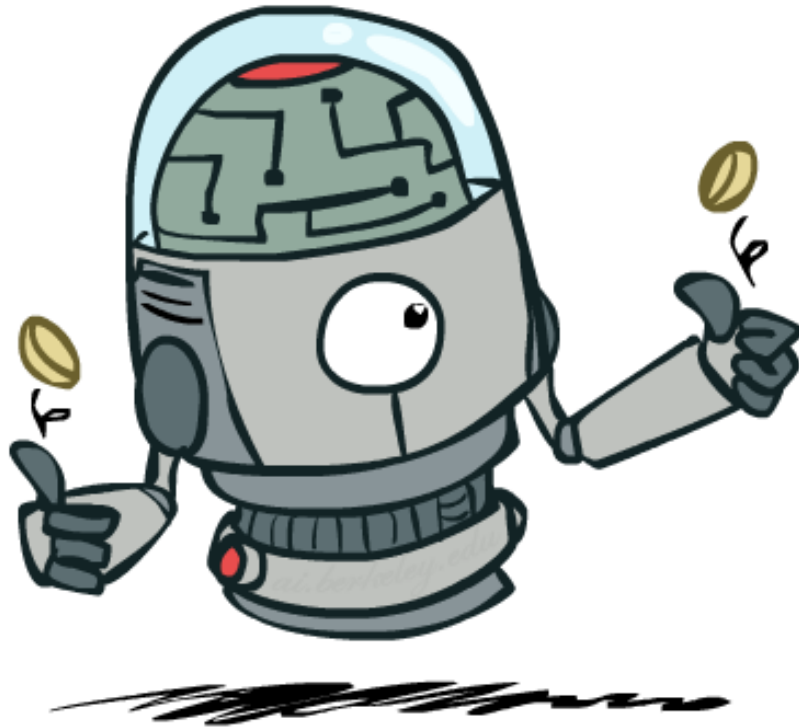
– George E. P. Box (1919 –2013) British statistician

- What do we do with probabilistic models?**

- We (or our agents) need to reason about unknown variables, given evidence
- Example: explanation (diagnostic reasoning)
- Example: prediction (causal reasoning)



Independence



Independence

- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

- This says that their joint distribution *factors* into a product two simpler distributions
- Another form:

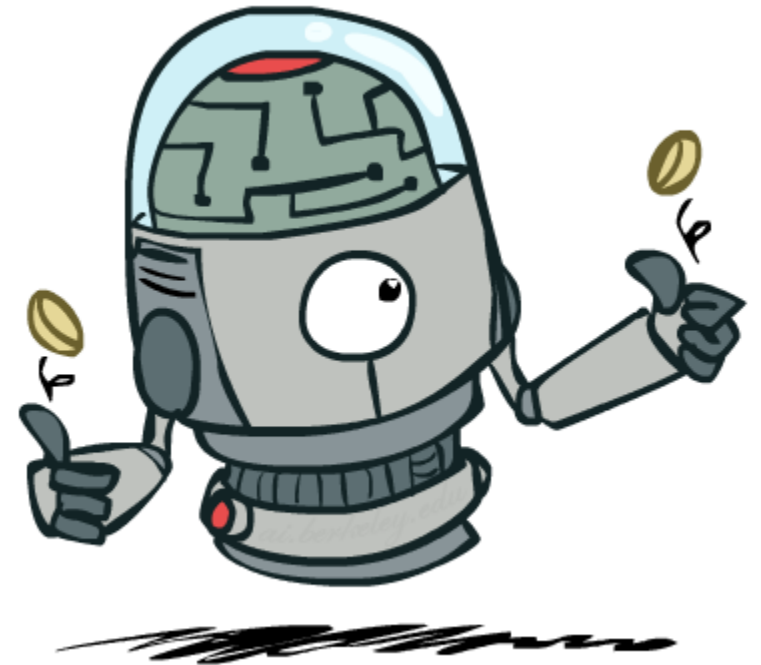
$$\forall x, y : P(x|y) = P(x)$$

- We write: $X \perp\!\!\!\perp Y$

- Independence is a simplifying *modeling assumption*

- Empirical* joint distributions: at best “close” to independent

- 4 What could we assume for {Weather, Traffic, Cavity, Toothache}?



Example: Independence?

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P_2(T, W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Example: Independence

- N fair, independent coin flips:

$P(X_1)$

H	0.5
T	0.5

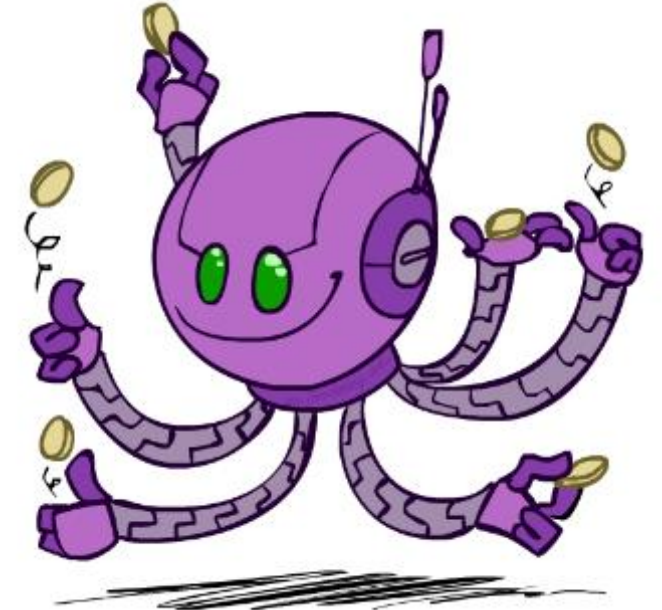
$P(X_2)$

H	0.5
T	0.5

...

$P(X_n)$

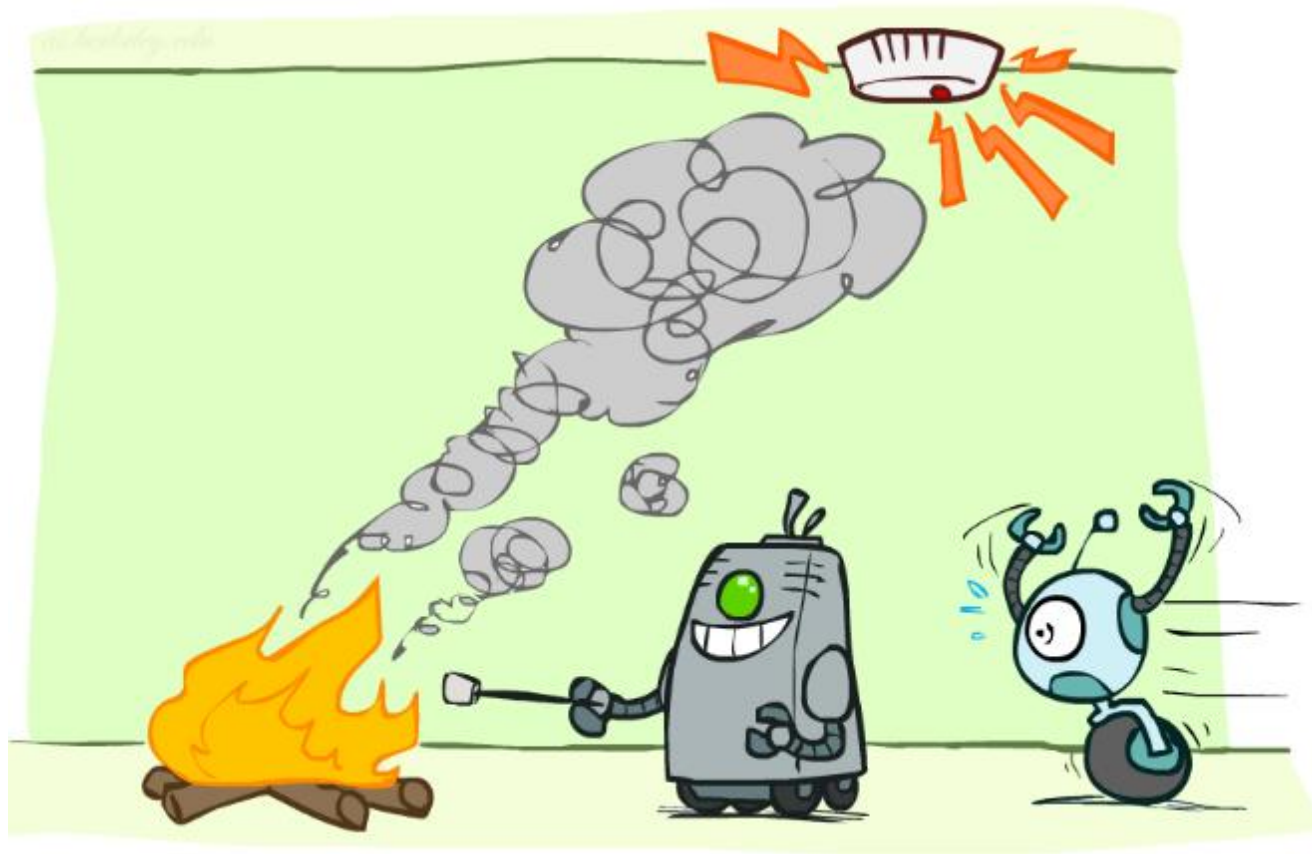
H	0.5
T	0.5



$P(X_1, X_2, \dots, X_n)$

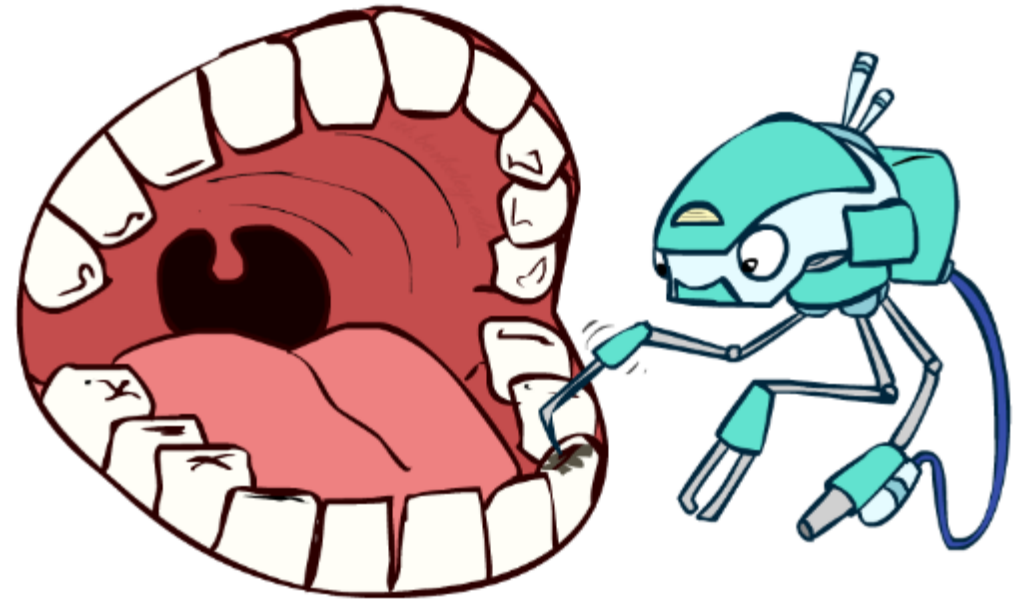


Conditional Independence



Conditional Independence

- $P(\text{Toothache}, \text{Cavity}, \text{Catch})$
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
 - $P(+\text{catch} \mid +\text{toothache}, +\text{cavity}) = P(+\text{catch} \mid +\text{cavity})$
- The same independence holds if I don't have a cavity:
 - $P(+\text{catch} \mid +\text{toothache}, -\text{cavity}) = P(+\text{catch} \mid -\text{cavity})$
- Catch is *conditionally independent* of Toothache given Cavity:
 - $P(\text{Catch} \mid \text{Toothache}, \text{Cavity}) = P(\text{Catch} \mid \text{Cavity})$
- Equivalent statements:
 - $P(\text{Toothache} \mid \text{Catch}, \text{Cavity}) = P(\text{Toothache} \mid \text{Cavity})$
 - $P(\text{Toothache}, \text{Catch} \mid \text{Cavity}) = P(\text{Toothache} \mid \text{Cavity}) P(\text{Catch} \mid \text{Cavity})$
 - One can be derived from the other easily



Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.

- X is conditionally independent of Y given Z

$$X \perp\!\!\!\perp Y \mid Z$$

if and only if:

$$\forall x, y, z : P(x, y | z) = P(x | z)P(y | z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x | z, y) = P(x | z)$$

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$$\forall x, y, z : P(x|z, y) = P(x|z)$$

$$\begin{aligned} P(x|z, y) &= \frac{P(x, z, y)}{P(z, y)} \\ &= \frac{P(x, y|z)P(z)}{P(y|z)P(z)} \\ &= \frac{P(x|z)P(y|z)P(z)}{P(y|z)P(z)} \end{aligned}$$

Conditional Independence

- What about this domain:

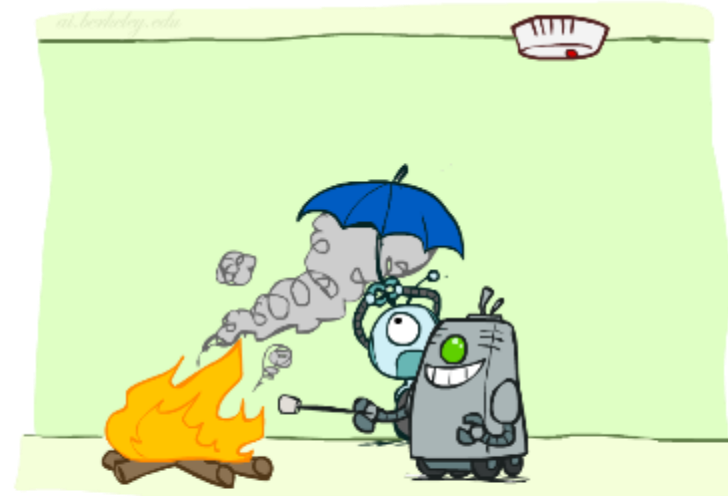
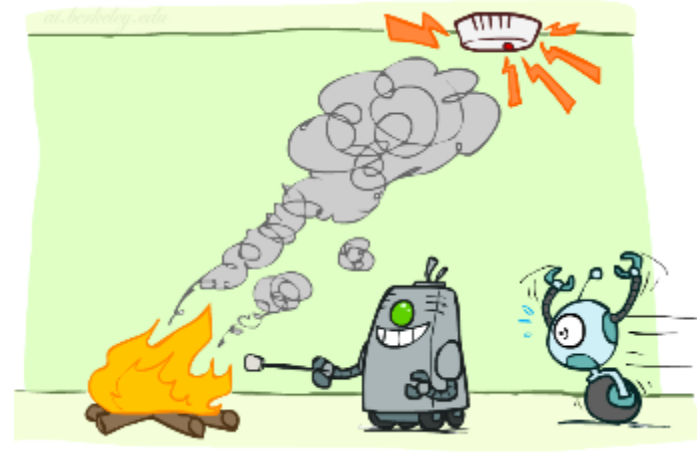
- Traffic
- Umbrella
- Raining



Conditional Independence

- What about this domain:

- Fire
- Smoke
- Alarm



Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes' nets / graphical models help us express conditional independence assumptions

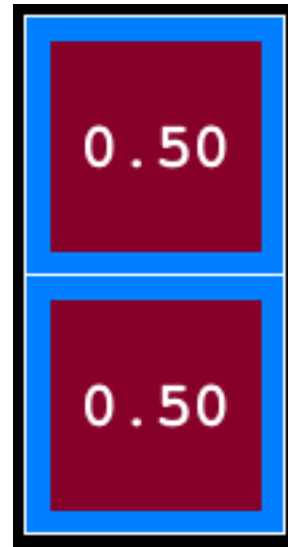


Ghostbusters Chain Rule

- Each sensor depends only on where the ghost is
- That means, the two sensors are conditionally independent, given the ghost position
- T: Top square is red
B: Bottom square is red
G: Ghost is in the top

$$P(T,B,G) = P(G) P(T|G) P(B|G)$$

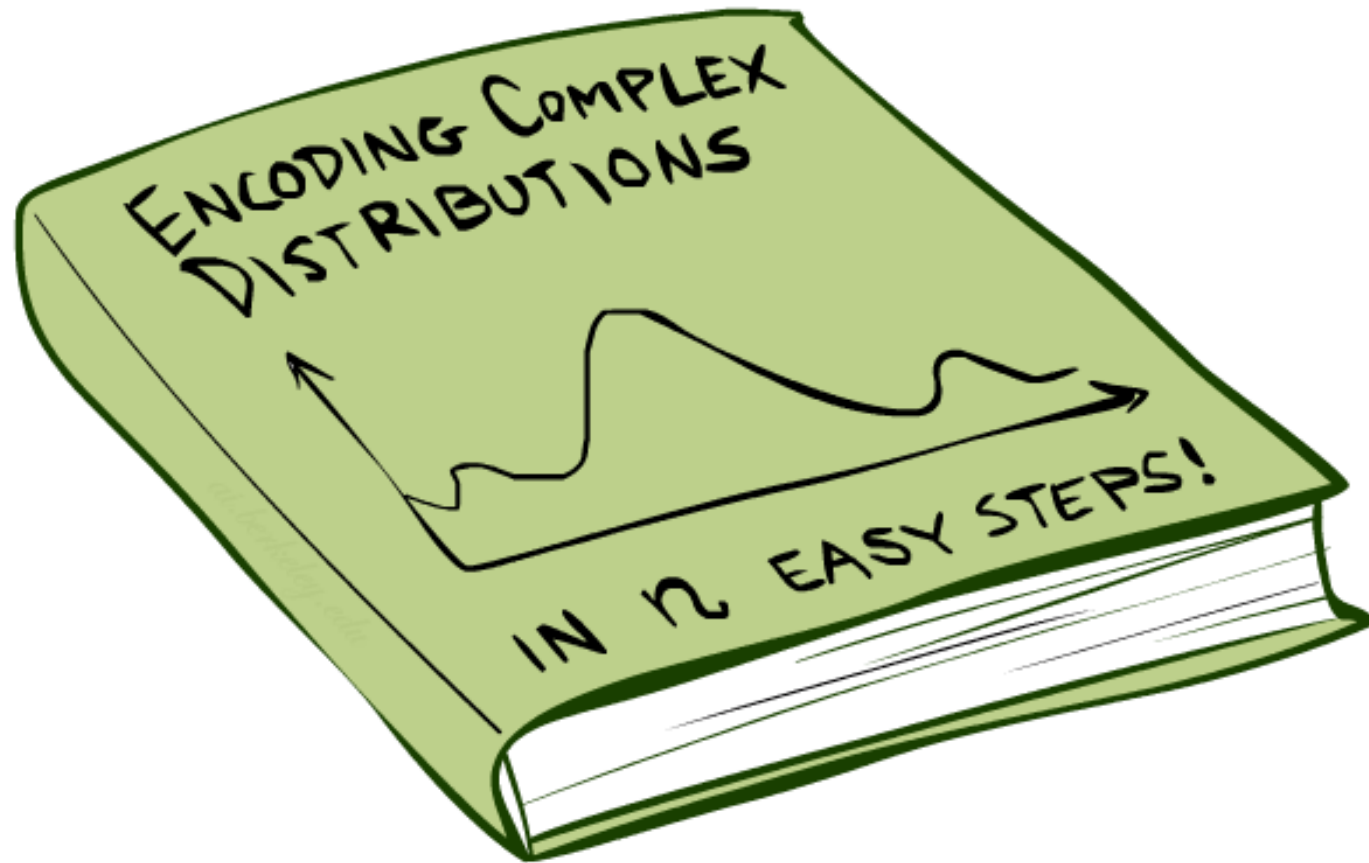
T	B	G	P(T,B,G)
+t	+b	+g	0.16
+t	+b	-g	0.16
+t	-b	+g	0.24
+t	-b	-g	0.04
-t	+b	+g	0.04
-t	+b	-g	0.24
-t	-b	+g	0.06
-t	-b	-g	0.06



- Givens:
 $P(+g) = 0.5$
 $P(-g) = 0.5$
 $P(+t | +g) = 0.8$
 $P(+t | -g) = 0.4$
 $P(+b | +g) = 0.4$
 $P(+b | -g) = 0.8$

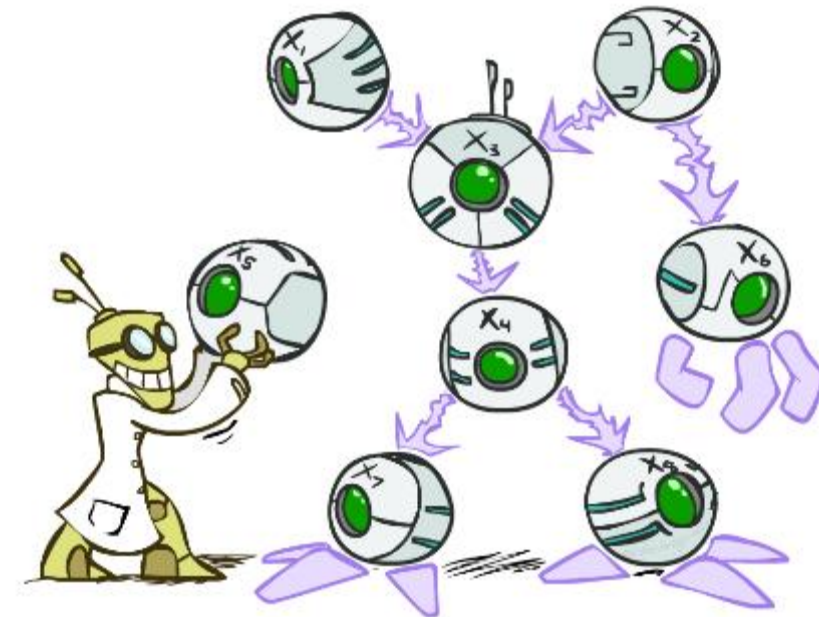


Bayes' Nets: Big Picture

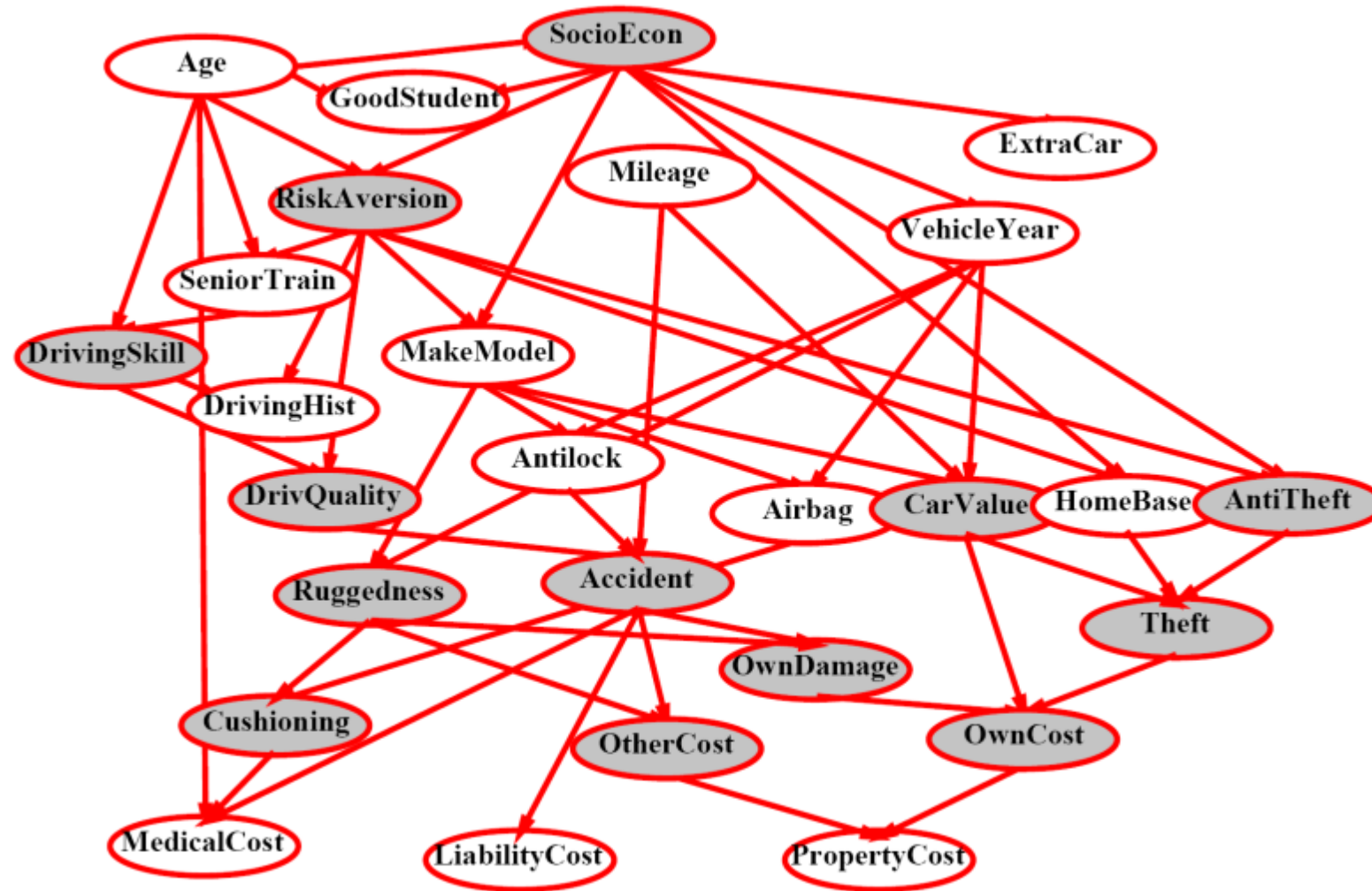


Bayes' Nets: Big Picture

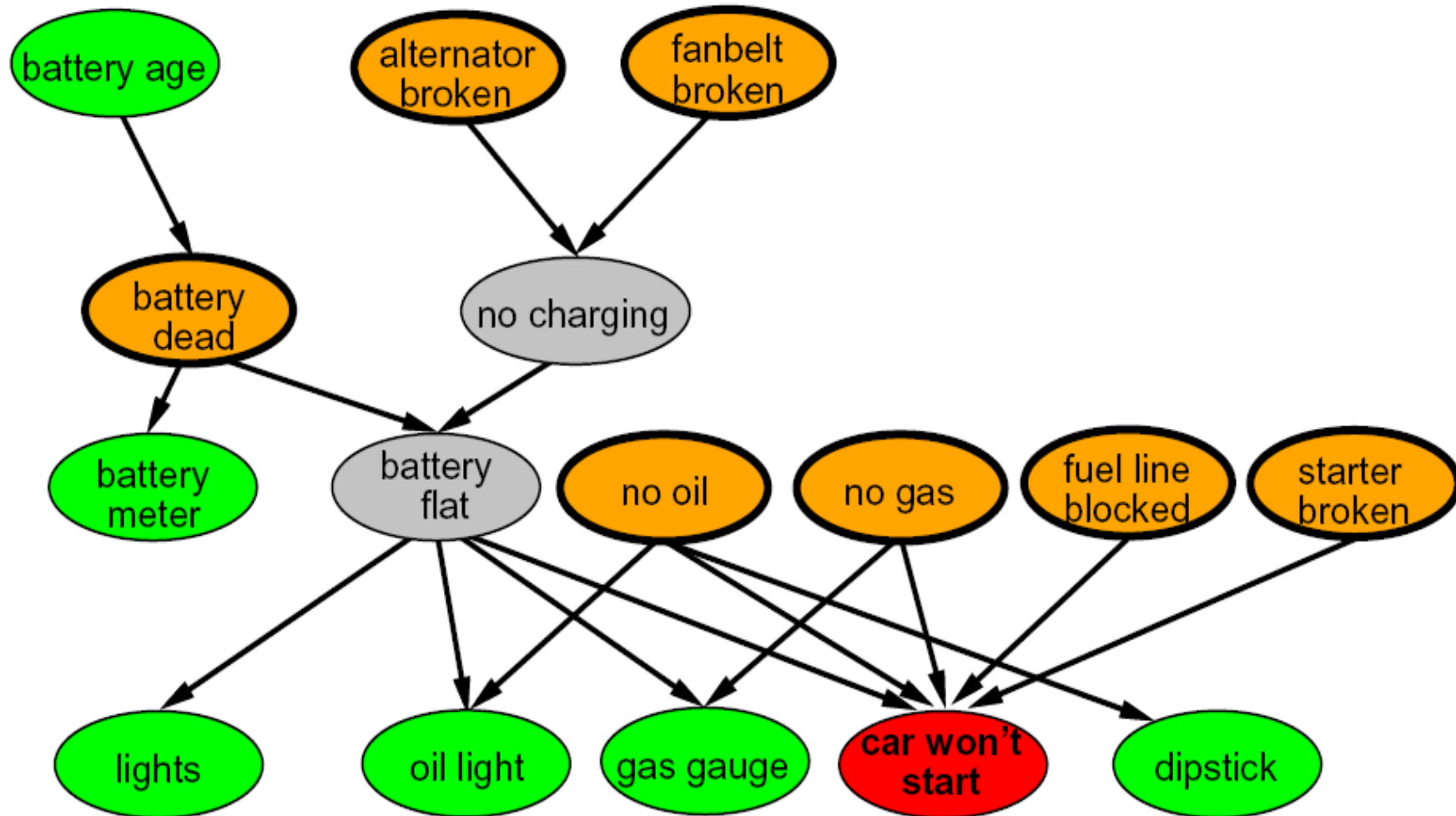
- Two problems with using **full joint distribution tables** as our **probabilistic models**:
 - Unless there are only a few variables, the joint is WAY too big to represent explicitly
 - Hard to learn (estimate) anything empirically about more than a few variables at a time (we saw similar problem in RL)
- **Bayes' nets**: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
 - More properly called **graphical models**
 - We describe how variables locally interact
 - Local interactions chain together to give global, indirect interactions
 - For about 10 min, we'll be vague about how these interactions are specified



Example Bayes' Net: Insurance

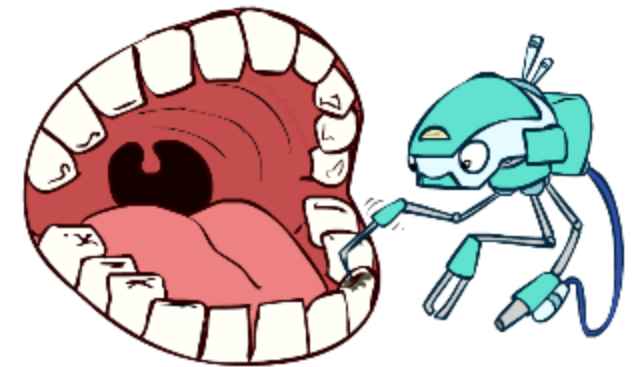
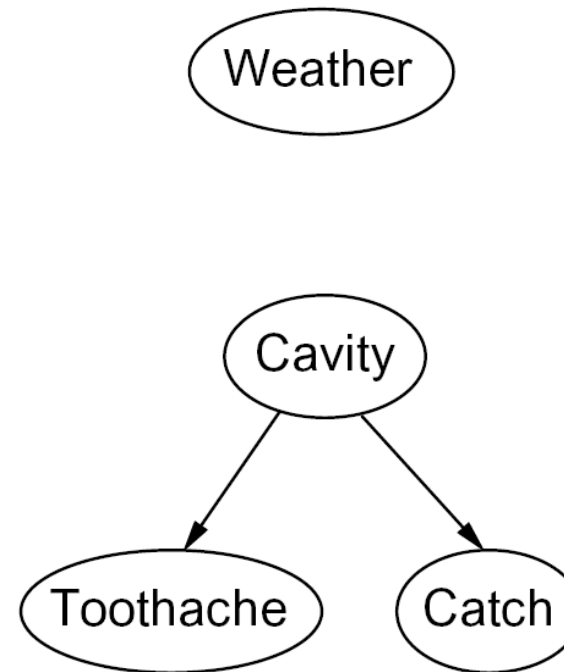


Example Bayes' Net: Car



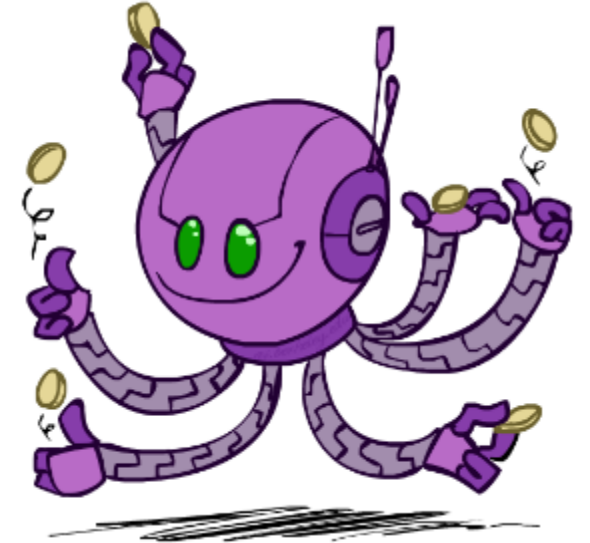
Graphical Model Notation

- **Nodes: variables (with domains)**
 - Can be assigned (observed) or unassigned (unobserved)
- **Arcs: interactions**
 - Similar to CSP constraints
 - Indicate “direct influence” between variables
 - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don't!)



Example: Coin Flips

- N independent coin flips



- No interactions between variables: **absolute independence**

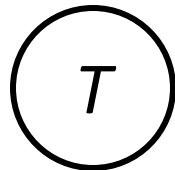
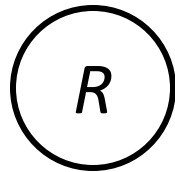
Example: Traffic

- Variables:

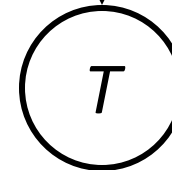
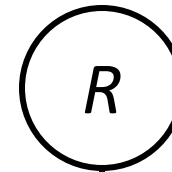
- R: It rains
- T: There is traffic



- Model 1: independence



- Model 2: rain causes traffic

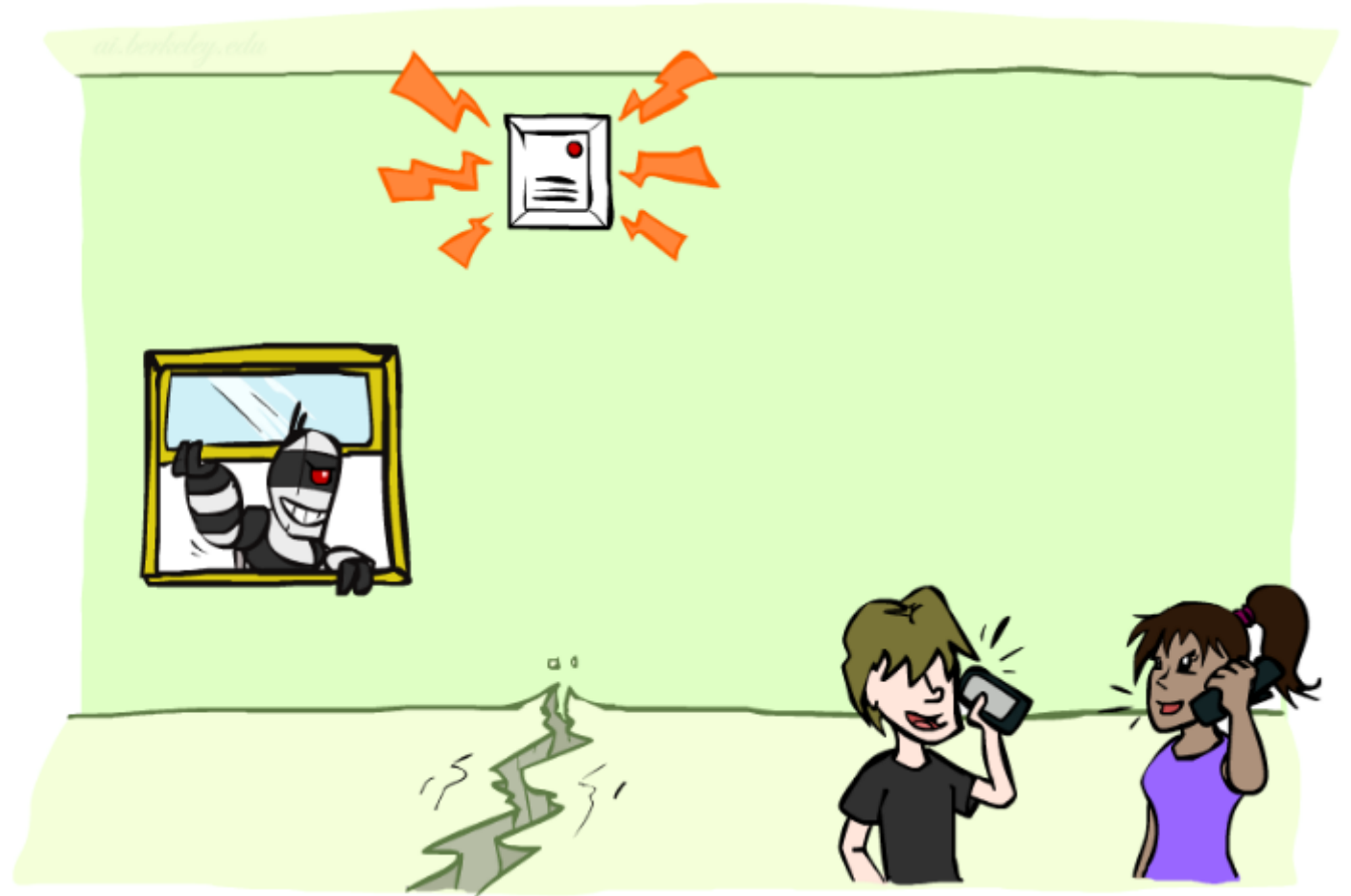


- Why is an agent using model 2 better?

Example: Alarm Network

- Variables

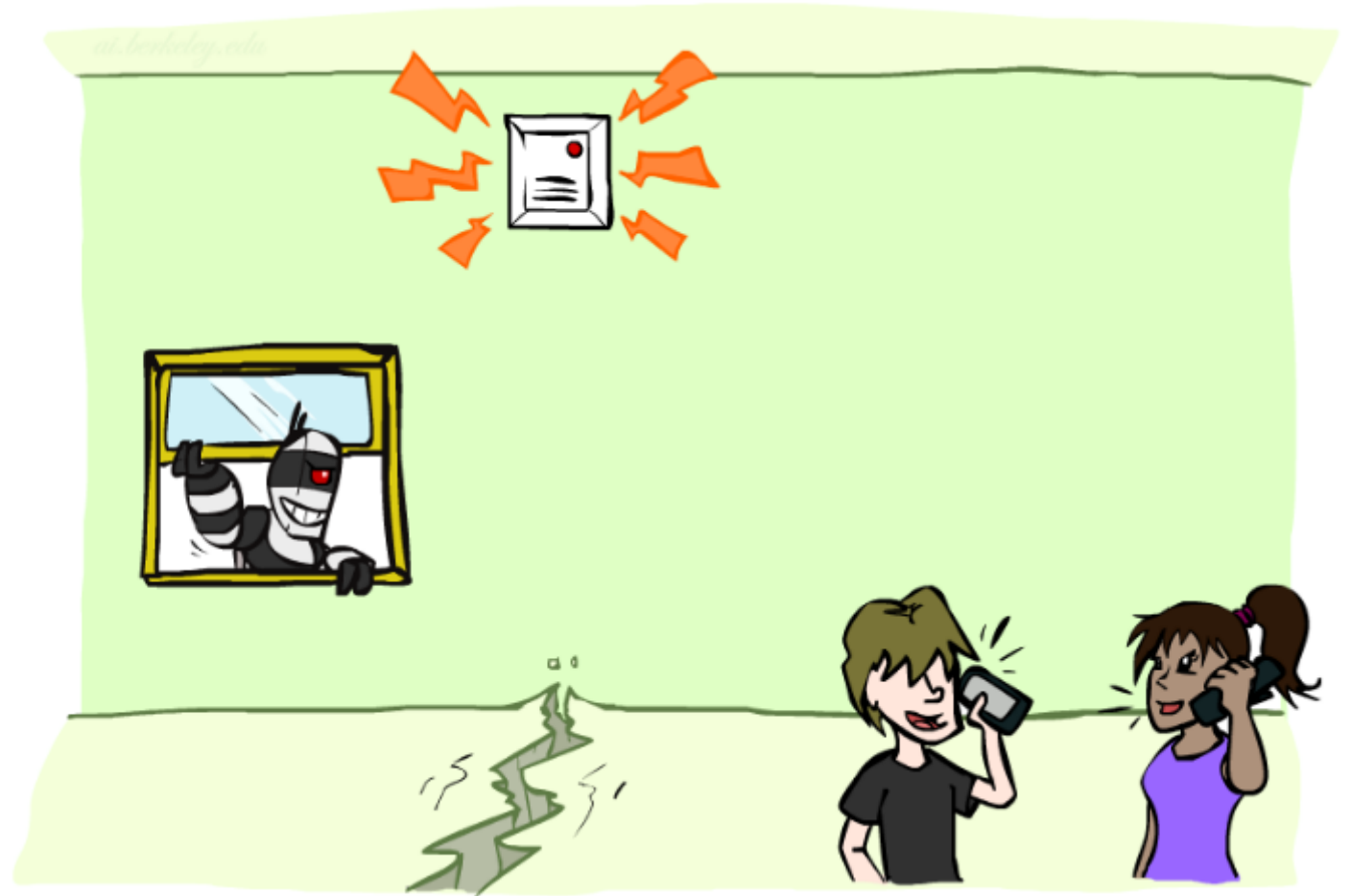
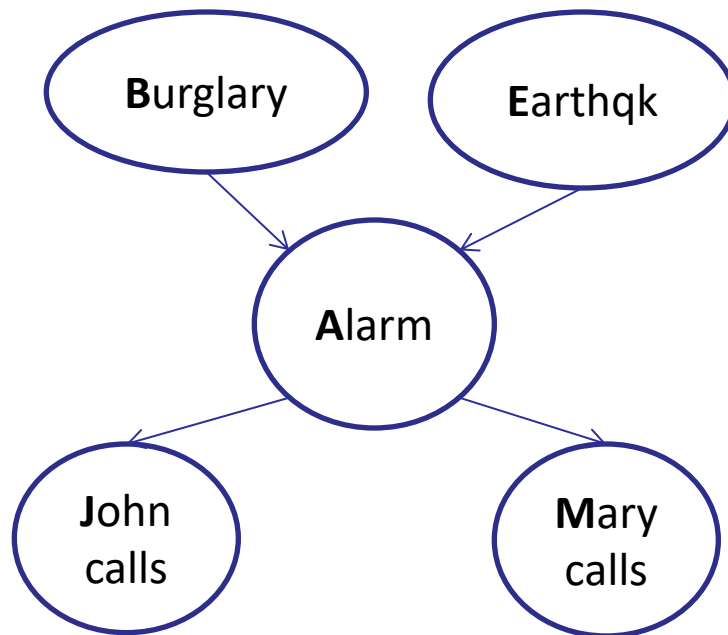
- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



Example: Alarm Network

■ Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



Example: Traffic II

- Variables

- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- B: Ballgame
- C: Cavity



Bayes' Net Semantics



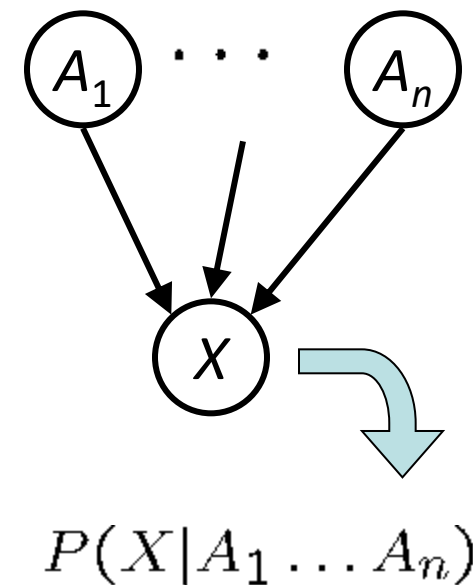
Bayes' Net Semantics



- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
 - A collection of distributions over X , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- CPT: conditional probability table
- Description of a noisy “causal” process



A Bayes net = Topology (graph) + Local Conditional Probabilities

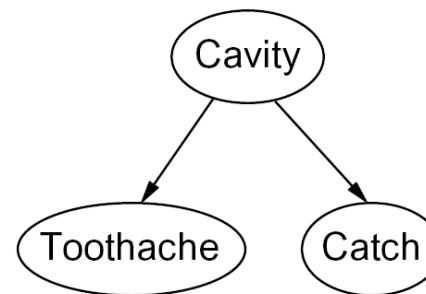
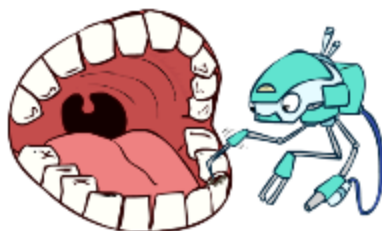
Probabilities in BNs



- Bayes' nets **implicitly** encode joint distributions
 - As a product of local conditional distributions
 - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- Example:



$$P(+cavity, +catch, -toothache)$$

$$= P(-toothache | +cavity) P(+catch | +cavity) P(+cavity)$$

Probabilities in BNs



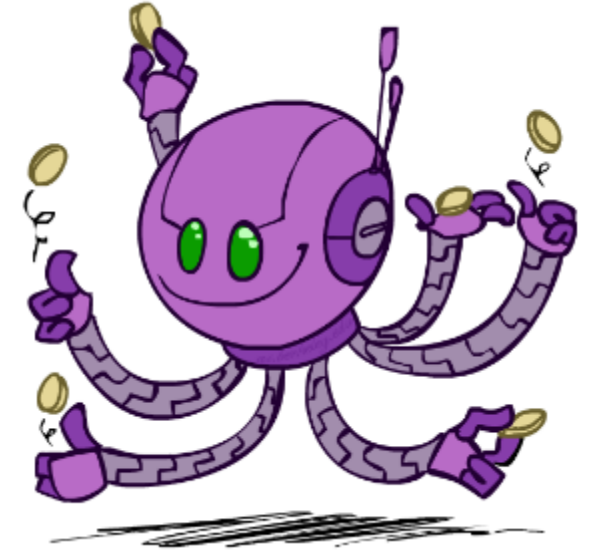
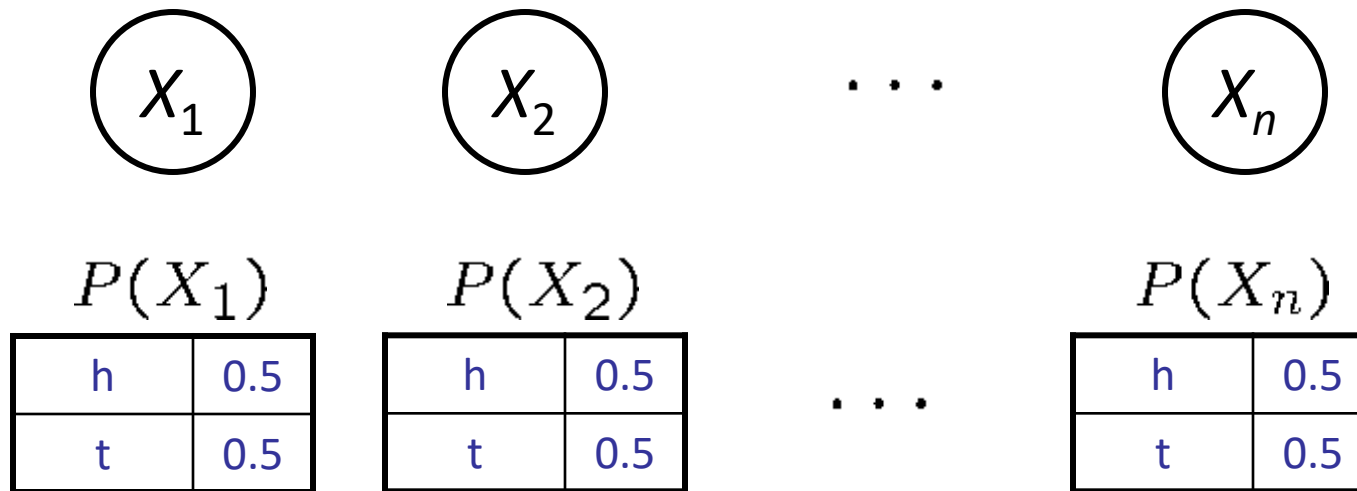
- Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- Assume conditional independences: $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$
 - Consequence: $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$
- Not every BN can represent every joint distribution

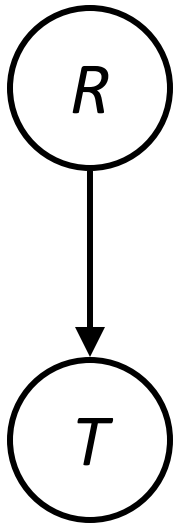
Example: Coin Flips



$$P(h, h, t, h) = P(h)P(h)P(t)P(h)$$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



$P(R)$	
+r	1/4
-r	3/4

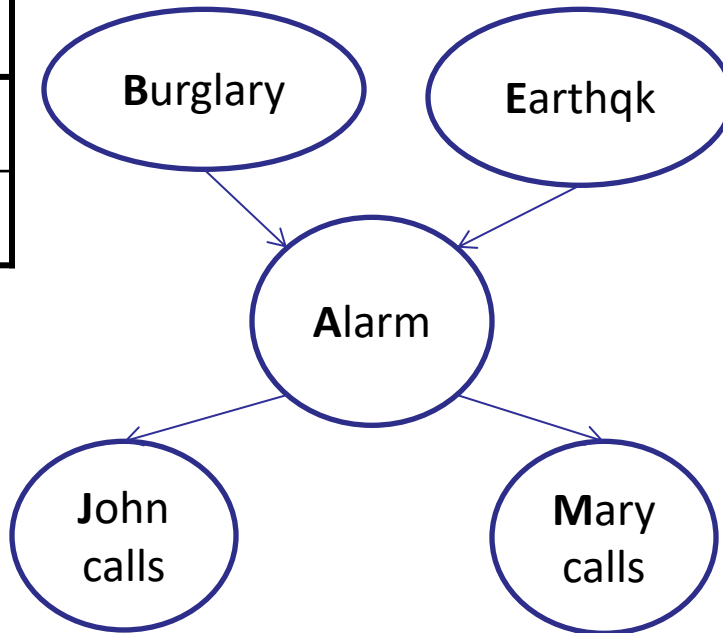
$P(T R)$					
+r	<table><tr><td>+t</td><td>3/4</td></tr><tr><td>-t</td><td>1/4</td></tr></table>	+t	3/4	-t	1/4
+t	3/4				
-t	1/4				
-r	<table><tr><td>+t</td><td>1/2</td></tr><tr><td>-t</td><td>1/2</td></tr></table>	+t	1/2	-t	1/2
+t	1/2				
-t	1/2				

$$P(+r, -t) = P(+r)P(-t|+r) = 1/4 * 1/4$$



Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

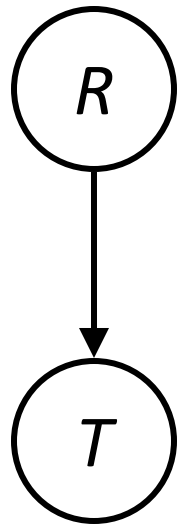
A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$\begin{aligned}
 &P(M|A)P(J|A) \\
 &P(A|B,E)P(E) \\
 &P(B)
 \end{aligned}$$

Example: Traffic

- Causal direction



$P(R)$

+r	1/4
-r	3/4

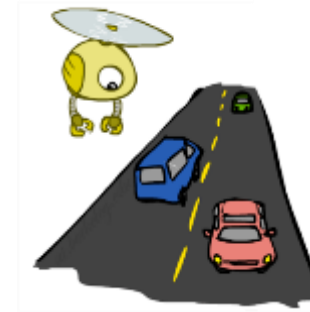
$P(T|R)$

+r	+t	3/4
	-t	1/4

-r	+t	1/2
	-t	1/2

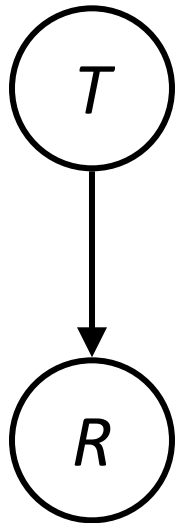
$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16



Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7



$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Causality?

- When Bayes' nets reflect the true causal patterns:
 - Often simpler (nodes have fewer parents)
 - Often easier to think about
 - Often easier to elicit from experts
 - BNs need not actually be causal
 - Sometimes no causal net exists over the domain (especially if variables are missing)
 - E.g. consider the variables *Traffic* and *Drips*
 - End up with arrows that reflect correlation, not causation
 - What do the arrows really mean?
 - Topology may happen to encode causal structure
 - **Topology really encodes conditional independence**
- $$P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$$

