Q1. Find the limit of the sequence if it converges otherwise indicate divergence

a) $a_n = \left\{\frac{\sin n}{n}\right\}$

d) $a_n = \left\{ \frac{\sin n}{n^2} \right\}_{n=1}^{\infty}$.

Recall that $-1 \le \sin n \le 1$ for all n

$-\frac{1}{n^2} \le \frac{\sin n}{n^2} \le \frac{1}{n^2}$, for all $n \geq 1$

$\lim_{n\to\infty} -\frac{1}{n^2} = 0 = \lim_{n\to\infty} \frac{1}{n^2}$

From the squeeze theorem we have $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$

$\therefore \{a_n\}$ converges.

b) $a_n = \left\{\frac{n+\ln n}{n^2}\right\}$

Apply Theorem 1, using L'Hôpital's Rule in the second step:

$$\lim_{n\to\infty} \frac{n+\ln n}{n^2} = \lim_{x\to\infty} \frac{x+\ln x}{x^2} = \lim_{x\to\infty} \frac{1+(1/x)}{2x} = 0$$

Q2- Find the sum of the telescoping series below.

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$$

10- Find the sum of the Telescoping series below.

a- $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$

Since $\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$, therefore

$$\begin{aligned} s_n &= \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots + \frac{1}{(3n-2)(3n+1)} \\ &= \frac{1}{3}\left[\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots \right. \\ &\quad \left. + \frac{1}{3n-5} - \frac{1}{3n-2} + \frac{1}{3n-2} - \frac{1}{3n+1}\right] \\ &= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) \rightarrow \frac{1}{3}. \end{aligned}$$

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Thus $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}$.

Q3- Evaluate the value of the following geometric series

$$\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^{n}$$

This is a geometric series with $r = -\frac{3}{4}$ and $c = 7$, starting at $n = 3$. By Eq. (5),

$$\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^{n} = \frac{7\left(-\frac{3}{4}\right)^{3}}{1-\left(-\frac{3}{4}\right)} = -\frac{27}{16} \qquad \blacksquare$$

Q4- Determine whether the following series is convergent or divergent.

$$i - \sum_{n=1}^{\infty} \frac{n}{n^4 - 2}$$

$\sum_{n=1}^{\infty} \frac{n}{n^4-2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ since

$$\lim \frac{\left(\frac{n}{n^4-2}\right)}{\left(\frac{1}{n^3}\right)} = 1, \quad \text{and} \quad 0 < 1 < \infty.$$

ii- $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

The function $f(x) = \frac{x}{(x^2+1)^2}$ is positive and continuous for $x \geq 1$. It is decreasing because $f'(x)$ is negative:

$$f'(x) = \frac{1-3x^2}{(x^2+1)^3} < 0 \qquad \text{for } x \geq 1$$

Therefore, the Integral Test applies. Using the substitution $u = x^2 + 1$, $du = 2x\,dx$, we have

$$\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_1^{R} \frac{x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \frac{1}{2}\int_2^{R} \frac{du}{u^2}$$

$$= \lim_{R\to\infty} \left.\frac{-1}{2u}\right|_2^{R} = \lim_{R\to\infty}\left(\frac{1}{4} - \frac{1}{2R}\right) = \frac{1}{4}$$

The integral converges. Therefore, $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ also converges. $\blacksquare$

$$\boldsymbol{iii-\sum_{n=2}^{\infty}\frac{\sqrt{n}}{3^n \ln n}}$$

$\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{3^n \ln n}$ converges by the ratio test, since

$$\rho = \lim \frac{\sqrt{n+1}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}}$$
$$= \frac{1}{3}\lim\sqrt{\frac{n+1}{n}} \cdot \lim\frac{\ln n}{\ln(n+1)} = \frac{1}{3} < 1.$$

5- Find the interval of convergency and radius and center of convergence.

$$\boldsymbol{\sum_{n=1}^{\infty}\frac{(-1)^n}{n4^n}(x-5)^5}$$

We compute ρ with $a_n = \dfrac{(-1)^n}{4^n n}(x-5)^n$:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{4^{n+1}(n+1)}\frac{4^n n}{(x-5)^n}\right|$$
$$= |x-5| \lim_{n\to\infty}\left|\frac{n}{4(n+1)}\right|$$
$$= \frac{1}{4}|x-5|$$

We find that

$$\rho < 1 \quad \text{if} \quad \frac{1}{4}|x-5| < 1, \quad \text{that is, if} \quad |x-5| < 4$$

Thus $F(x)$ converges absolutely on the open interval $(1, 9)$ of radius 4 with center $c = 5$. In other words, the radius of convergence is $R = 4$. Next, we check the endpoints:

$$x = 9: \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n}(9-5)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \qquad \text{converges (Leibniz Test)}$$

$$x = 1: \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n}(-4)^n = \sum_{n=1}^{\infty}\frac{1}{n} \qquad \text{diverges (harmonic series)}$$

We conclude that $F(x)$ converges for x in the half-open interval $(1, 9]$ shown in Figure 3.

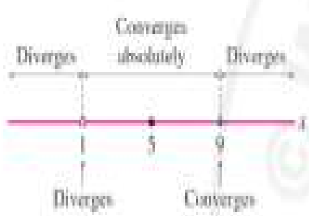

FIGURE 3 The power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n}(x-5)^n$