HERIOT WATT UNIVERSITY

SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Computer Science

F29FA1

Foundations I

Class Test 2014/15

13 October 2014
Duration: 0.5 Hour

Answer ALL questions

1. (a) Remove as many parenthesis as possible from the following expression without changing its meaning: $(\lambda x.(\lambda y.(\lambda z.((((((xy)z)(\lambda x.x))(\lambda x.((xz)(yz))))(\lambda x.(((xx)y)y))))))$. (1)

$\lambda xyz.xyz(\lambda x.x)(\lambda x.xz(yz))(\lambda x.xxyy)$.

Learning Objectives: Syntax and semantics of the λ-calculus.

(b) Give the subterms of $\lambda xyz.xyz(\lambda x.x)$. (1)

$\{\lambda xyz.xyz(\lambda x.x), \lambda yz.xyz(\lambda x.x), \lambda z.xyz(\lambda x.x), x, y, xy, z, (xy)z, \lambda x.x, xyz(\lambda x.x)\}$.

Learning Objectives: Syntax and semantics of the λ-calculus.

(c) Find λ-terms A, B and C such that: A, B and C are all in β-normal form, BC does not have a β-normal form and $A(BC)$ has a β-normal form. (2)

Let $A = \lambda x.z$, and $B \equiv C \equiv \lambda x.xx$. Each of A, B and C is in β-normal form, but BC does not have a β-normal form and $A(BC)$ has the β-normal form z.

Learning Objectives: Reduction and normalisation in the λ-calculus.

(d) If for all A, B, we have $A =_\beta B$, would the lambda calculus be allowed to be a model of computation? (1)

No. In any model of computation we need to have at least two different elements. If for all A, B, we have $A =_\beta B$ then any two programs have the same meaning.

Learning Objectives: The λ-calculus is a sound model of computation.

(e) Do you know of a computable function which cannot be represented in the λ-calculus? If so give it. Otherwise say why it is not the case. (1)

I expect them to say that anything computable can be represented in the λ-calculus.

Learning Objectives: The λ-calculus is the language of computation.

(f) Let $K = \lambda xy.x$. Show that $KAB =_\beta A$.
Hint: you can assume that $y \notin FV(A)$ and hence $A[y := B] \equiv A$. (2)

$KAB \equiv (\lambda xy.x)AB \to_\beta (\lambda y.x)[x := A]B \equiv (\lambda y.A)B \to_\beta A[y := B] \equiv A$.
Since $KAB \twoheadrightarrow_\beta A$, then $KAB =_\beta A$.

Learning Objectives: Calculation in the λ-calculus.

(g) Assume $v \notin FV(A)$ and $v \notin FV(B)$. Show that if $Av =_{\beta\eta} Bv$ then $A =_{\beta\eta} B$. (3)

Assume $v \notin FV(A)$, $v \notin FV(B)$ and $Av =_{\beta\eta} Bv$.

- By compatibility, $\lambda v.Av =_{\beta\eta} \lambda v.Bv$.
- $\lambda v.Av =_{\beta\eta} A$ by (η), since $v \notin FV(A)$
- $\lambda v.Bv =_{\beta\eta} B$ by (η), since $v \notin FV(B)$
- Hence, $A =_{\beta\eta} B$, since $=_{\beta\eta}$ is an equivalence relation.

Learning Objectives: Understanding extensionality and the use of free variables.

(h) Is $(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.xx)x(\lambda x.xx)(\lambda x.xx)$ β-normalising? If yes, give the β-normal form showing all the reduction steps you used to reach it. If not, give a detailed proof why it is not. (3)

$\underline{(\lambda xyz.xyz)(\lambda x.xx)}(\lambda x.xx)x(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$
$\underline{(\lambda yz.(\lambda x.xx)yz)(\lambda x.xx)}x(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$

$\underline{(\lambda z.(\lambda x.xx)(\lambda x.xx)z)x}(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$
$(\underline{(\lambda x.xx)(\lambda x.xx)}x)(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$
$(\underline{(\lambda x.xx)(\lambda x.xx)}x)(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$
$(\underline{(\lambda x.xx)(\lambda x.xx)}x)(\lambda x.xx)(\lambda x.xx) \to_\beta^{lmo}$
...

Since the leftmost reduction path does not terminate, by the normalisation theorem $(\lambda yz.(\lambda x.xx)yz)(\lambda x.xx)x(\lambda x.xx)(\lambda x.xx)$ is not β-normalising.

Learning Objectives: normalisation theorem.

(i) Give the $\beta\eta$-normal form of $\lambda z.(z(\lambda y.((\lambda x.x)(\lambda x.xx)(\lambda y.y))y))z$. (1)

$\lambda z.z(\lambda y.y)z$.

Learning Objectives: reduction.

END OF PAPER