Logic and Proof Revision Manual

In this revision manual, you will be asked to refer to Prof. Lawson's material that is posted online.

First Order Logic

1- The World of Propositional Logic (A brief summary)

Logical constants:	true (T), false (F)
Atomic sentences:	p, q, r are typically used to represent these
Literal:	an atomic sentence or a negated atomic sentence
Composite sentences:	use connectives, brackets and literals to build new structures
	$( )$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

p: it is sunny
q: it is hot
r: I need a hat

$(p \wedge q) \rightarrow r$

There are limitations to what can be represented using propositional logic. You cannot represent, for example: Every rabbit eats grass or there is one rabbit that does not eat grass.

2- First Order Logic

This is more expressive, and models the world using
subjects: individual identities, for example rabbit or lion.
predicates: properties of subjects for example 'eats Grass', and a variable, say E, may be used to represent them.

For example E(x)= 'x eats grass'. But we cannot determine whether this is true or false
A sentence then can consist of

E(rabbit) or,
E(lion)

with the first sentence being true, and the second one being false (do lions eat grass maybe?)
In the above example there was only one predicate. We may use two predicates, or more.
For example, A(x, y)= 'x is the ancestor of y'

3- Quantifiers
Quantifiers can be either universal or existential.

Universal quantification

Let $G(x)$= 'x eats grass'; $D(x)$='x eats dandelion'; $M(x)$='x eats meat';
$(\forall x)G(x)$ means that G holds for all values of variable x in the domain associated with x
$(\forall x)D(x)$ means that D holds for all values of variable x in the domain associated with x
$(\forall x)M(x)$ means that M holds for all values of variable x in the domain associated with x
What does the following then represent?

$(\forall x)\ G(x) \rightarrow D(x)$

$(\forall x)\ G(x) \rightarrow \neg\ M(x)$

Existential quantification

$(\exists\ x)G(x)$ means that G holds for at least one value of x (maybe more) in the domain associated variable x.
$(\exists\ x)M(x)$ means that G holds for at least one value of x (maybe more) in the domain associated variable x.
What does the following then represent

$(\exists\ x)\ (G(x) \wedge M(x))$

Free variables are the ones that are not bound by a universal or an existential quantifier.

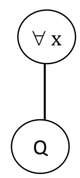
$(\forall y)\ A(x,y)$ has x bound as a universally quantified variable, but y is free.

Here y will be the any human being that ever lived, but is not bound by the universal quantifier: x is a free variable and can be either 'Adam' or 'Eve'.

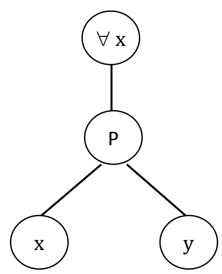
Using more than one quantifier

$(\forall\ x)(\forall\ y)P(x,y)$ and $(\forall\ y)(\forall\ x)P(x,y)$ are equivalent
and
$(\exists\ x)(\ \exists\ y)P(x,y)$ and $(\exists\ y)(\ \exists\ x)P(x,y)$ are equivalent
but
$(\forall\ x)(\ \exists\ y)P(x,y)$ and $(\exists\ y)(\ \forall\ x)P(x,y)$ are NOT equivalent
To illustrate why these are not equivalent using the example $A(x, y)$= 'x is the ancestor of y'
$(\forall\ x)(\ \exists\ y)A(x,y)$ means that every x will have at least one y for which x is the ancestor (so saying: x has a child , or grandchild, or great-grandchild)
But, $(\exists\ y)(\ \forall\ x)A(x,y)$means there is at least one y which has every x as an ancestor (so saying: any x is an ancestor of that particular y)

Quantifier representation:
Trees are used to represent, for example $(\forall\ x)Q(x)$



You can also represent (∀ x)P(x,y), note how you can 'see' that x is a bound variable while y is free.



But in this example, both x and y are bound