theorem begin

Foundations 1
Class Test 2013

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Exercise 1

(a) Show using a calculation that
$\neg(P \Leftrightarrow Q) \stackrel{val}{==} (P \wedge \neg Q) \vee (\neg P \wedge Q)$.
State precisely at each step which equivalences you use.
You do not need to mention the steps of substitution or of Leibniz. [4]

(b) What can you deduce (using (a) above) about
$\neg(P \Leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$? [1]

Exercise 2

(a) Remove as many parenthesis as possible
from the following expression without changing its meaning: [3]
$(\lambda x.(\lambda y.(\lambda z.((((((xy)z)(\lambda x.x))(\lambda x.((xz)(yz))))(\lambda x.(((xx)y)y)))))))$

(b) Insert the full amount of parenthesis in the expression [3]
$(\lambda yz.(\lambda x.xx)yz)(\lambda x.x)x$

(c) β-reduce the following term until there are no more β-redexes showing all the reduction steps and all the possible reduction paths (note that there are four paths depending on the orders you choose for inside/outside redexes):
$(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.x)x$ [4]

Solutions to Exercise 1

1. (a) We show $\neg(P \Leftrightarrow Q) \stackrel{val}{==} (P \wedge \neg Q) \vee (\neg P \wedge Q)$ by the following calculation:

$$\begin{array}{ll} & \neg(P \Leftrightarrow Q) \\ \stackrel{val}{==} & \qquad \{\text{Bi-Implication}\} \\ & \neg((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ \stackrel{val}{==} & \qquad \{\text{Implication, twice}\} \\ & \neg((\neg P \vee Q) \wedge (\neg Q \vee P)) \\ \stackrel{val}{==} & \qquad \{\text{De Morgan}\} \\ & \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P) \\ \stackrel{val}{==} & \qquad \{\text{De Morgan, twice}\} \\ & (\neg\neg P \wedge \neg Q) \vee (\neg\neg Q \wedge \neg P) \\ \stackrel{val}{==} & \qquad \{\text{Double negation, twice}\} \\ & (P \wedge \neg Q) \vee (Q \wedge \neg P) \\ \stackrel{val}{==} & \qquad \{\text{commutativity}\} \\ & (P \wedge \neg Q) \vee (\neg P \wedge Q) \end{array}$$

(b) Using 2.(a) above, we can deduce that $\neg(P \Leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$ is a tautology.

Solution to Exercise 2(a)+(b)

(a) $\lambda xyz.xyz(\lambda x.x)(\lambda x.xz(yz))(\lambda x.xxyy)$.

(b) $(((\lambda y.(\lambda z.(((\lambda x.(xx))y)z)))(\lambda x.x))x)$

Solution to Exercise 2(c)

1. $(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.x)x \to_\beta (\lambda yz.(\lambda x.xx)yz)(\lambda x.x)x \to_\beta$ $(\lambda z.(\lambda x.xx)(\lambda x.x)z)x \to_\beta (\lambda z.(\lambda x.x)(\lambda x.x)z)x \to_\beta$ $(\lambda x.x)(\lambda x.x)x \to_\beta (\lambda x.x)x \to_\beta x$
2. $(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.x)x \to_\beta (\lambda yz.(\lambda x.xx)yz)(\lambda x.x)x \to_\beta$ $(\lambda z.(\lambda x.xx)(\lambda x.x)z)x \to_\beta (\lambda z.(\lambda x.x)(\lambda x.x)z)x \to_\beta$ $(\lambda z.(\lambda x.x)z)x \to_\beta (\lambda x.x)x \to_\beta x$
3. $(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.x)x \to_\beta (\lambda yz.(\lambda x.xx)yz)(\lambda x.x)x \to_\beta$ $(\lambda z.(\lambda x.xx)(\lambda x.x)z)x \to_\beta (\lambda z.(\lambda x.x)(\lambda x.x)z)x \to_\beta$ $(\lambda z.(\lambda x.x)z)x \to_\beta (\lambda z.z)x \to_\beta x$
4. $(\lambda xyz.xyz)(\lambda x.xx)(\lambda x.x)x \to_\beta (\lambda yz.(\lambda x.xx)yz)(\lambda x.x)x \to_\beta$ $(\lambda z.(\lambda x.xx)(\lambda x.x)z)x \to_\beta (\lambda x.xx)(\lambda x.x)x \to_\beta$ $(\lambda x.x)(\lambda x.x)x \to_\beta (\lambda x.x)x \to_\beta x$