Horn Clauses and Satisfiability

review

Horn Formula Definition

- $H = (p \rightarrow q) \wedge (t \wedge r \rightarrow T) \wedge (p \wedge r \wedge s \rightarrow \perp)$
- Horn formulas are <u>conjunctions</u> of **Horn clauses**
- Horn clause is an implication whose assumption A is a conjunction of proposition of type P and whose conclusion is also of type P (P::= $\perp$ | T | atom).

Horn Formula Definition

- $H = (p \rightarrow q) \wedge (t \wedge r \rightarrow T) \wedge (p \wedge r \wedge s \rightarrow \perp)$
- In other words:
  - 1 or more clauses separated by AND
  - Each clause must have:
    - NO negations
    - One implication
    - The left hand side of implication:
      - can be **one or more** of ($\perp$ | T | atom) separated by an AND
    - The right hand side of implication:
      - must be **one** of ($\perp$ | T | atom)

Horn Formula – other definition

- A formula is a Horn formula if it is in **CNF** and every disjunction contains at most one positive literal.
- Horn clauses are clauses, which contain **at most** one positive literal.
- H=(p V ~q) ^ (~c V ~p V q) ^ (~t V ~r) ^ d
- H =(q $\rightarrow$ p) ^ (c^p $\rightarrow$ q) ^ (t^r $\rightarrow$ $\perp$) ^ (T $\rightarrow$d)

Horn Formula – cont.

- Horn formula allows to efficiently compute satisfiability.
- If a set of formulas is not satisfiable
  - There is a contradiction / inconsistency in the rules
- Useful to build a knowledge base

The Algorithm to test for Satisfiability

```
Function HORN (ϕ)
//precondition: ϕ is a Horn formula
//postcondition: HORN (ϕ) decides the satisfiability for ϕ
{
    mark all occurrences of T in ϕ
    while there is a conjunct  p1 ^ p2 ^ … pn → P of ϕ
                such that all pj are marked but P is not
            mark P
    end while
    if ⊥ is marked
            return 'unsatisfiable'
    else
            return 'satisfiable'
    end if
}
```

Exercises

- $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
- Mark: q, s, r through $(T \rightarrow q), (T \rightarrow s), (T \rightarrow r)$
- $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
- Mark: p through $(r \rightarrow p)$
- $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
- Mark: v through $(p \wedge q \wedge s \rightarrow v)$
- $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
- Return?
  - Satisfiable

Exercises

- $(p \wedge q \wedge s \rightarrow p) \wedge (q \wedge r \rightarrow p) \wedge (p \wedge s \rightarrow s)$
- No occurrences of T in ϕ
- Nothing is marked
- Returning: Satisfiable

Exercises

- $(p \wedge q \wedge s \rightarrow p) \wedge (q \wedge r \rightarrow p) \wedge (p \wedge s \rightarrow s)$
- Remember that a formula ϕ is satisfiable if there is an interpretation that makes the formula ϕ true.
- Let p, q, r, s be false, then
  - $(p \wedge q \wedge s \rightarrow p)$ is True (false implying anything is True),
  - $(q \wedge r \rightarrow p)$ is True
  - $(p \wedge s \rightarrow s)$ is True
  - The formula is True

Exercises

- $(p \wedge q \wedge s \rightarrow \perp) \wedge (q \wedge r \rightarrow \perp) \wedge (s \rightarrow \perp)$
- No occurrences of T in ϕ
- Nothing is marked
- Returning: Satisfiable
- When p, q, r, s are false, the formula is True