SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

+ SOLUTIONS

F17LP

Logic and Proof

Semester 1 – 2017/18

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

F17LP

Each question is worth 20 marks

(Throughout this exam paper,
wff is an abreviation for *well formed formula(e)*)

1. (a) Construct truth tables for each of the following wff.

    i. $p \oplus q$.

    ii. $p\,\mathbf{nand}\,q$.

    iii. $p \to q$.

    iv. $p \leftrightarrow q$.

    [1 mark each]

    (b) Construct the parse tree of $(p \leftrightarrow q) \to \neg r$. [2 marks]

    (c) Construct the truth table of $(p \leftrightarrow q) \to \neg r$. [3 marks]

    (d) Using the standard method, construct a wff in disjunctive normal form that has the following truth table. [4 marks]

p	q	r	A
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

    (e) Prove, **using truth tables**, that $\neg(p \vee (q \wedge r))$ is logically equivalent to $\neg p \wedge (\neg q \vee \neg r)$. [4 marks]

    (f) Define the binary connective **nor**. Show that $p \to q$ is logically equivalent to a wff constructed using only the connective **nor**. [3 marks]

2. (a) i. What is meant by *conjunctive normal form*? [1 mark]

ii. What is meant by a *Horn formula*? [1 mark]

iii. Write the following wff

$$(p \lor \neg q) \land (\neg c \lor \neg p \lor q) \land (\neg s \lor \neg r) \land d$$

in implicational form. [3 marks]

iv. Apply the algorithm described in the lectures to the wff in (iii) above to determine whether it is satisfiable or not. You should explain each step of the algorithm. [5 marks]

(b) This question **must** be answered using **only** truth trees.

i. Determine whether the following is a valid argument or not

$$p \to (q \lor r), \neg q \lor \neg r \models \neg p.$$

[5 marks]

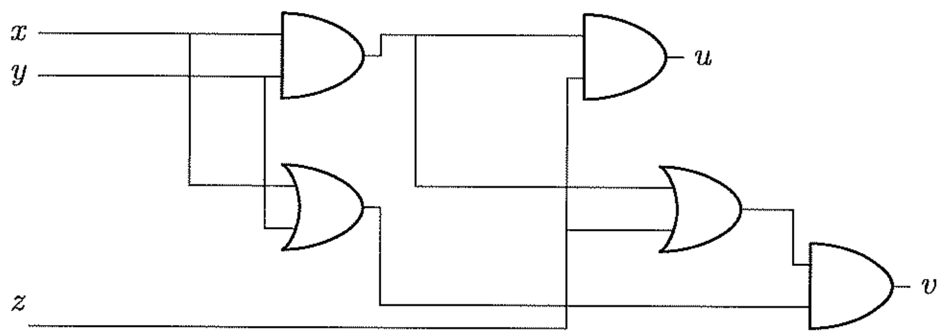
ii. Determine whether the following is a tautology or not.

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

[5 marks]

3. (a) In this question, you should use the Boolean algebra axioms listed at the end of this exam paper. You should also assume that $a^2 = a$ and $a+a = a$ for all elements a of a Boolean algebra. Prove $a0 = 0$. [5 marks]

(b) The following diagram shows a circuit with three inputs and two outputs. The symbols are recalled at the end of the exam paper. Describe each output u and v as a Boolean expression in terms of x, y and z. [5 marks]



(c) Draw a Venn diagram to represent the following Boolean expression

$$x\bar{y}z + xyz + \bar{x}yz$$

and then simplify this expression as much as possible using Boolean algebra properties. Any such properties used should be clearly stated. [5 marks]

(d) Describe the input/output behaviour of a transistor. Show that not-gates and or-gates can be constructed from transistors. Why does this imply that all combinational circuits can be constructed from transistors? [5 marks]

4. (a) Let $H(x)$ be interpreted as the 1-place predicate 'x is a Hobbit' and let $F(x)$ be interpreted as the 1-place predicate 'x has hairy feet'. Translate the following wff into colloquial English. [4 marks]

   i. $(\forall x)(H(x) \to F(x))$.

   ii. $(\forall x)(H(x) \to \neg F(x))$.

   iii. $(\exists x)(H(x) \wedge F(x))$.

   iv. $(\exists x)(H(x) \wedge \neg F(x))$.

   (b) Consider the following wff

$$A = (\forall x)(\forall y)(\exists z)(R(x,y) \to R(x,z) \wedge R(z,y)).$$

   i. Draw the parse tree for A. [2 marks]

   ii. Interpret A in the structure whose domain is $\mathbb{Q}$, the set of all rational numbers (that is, positive and negative numbers that can be written as fractions) and $R(x,y)$ as $x < y$. Is A true or false in this interpretation? Explain. [2 marks]

   iii. Interpret A in the structure whose domain is $\mathbb{Z}$, the set of all positive or negative whole numbers numbers and $R(x,y)$ as $x < y$. Is A true or false in this interpretation? Explain. [2 marks]

   (c) Prove using truth trees that

$$(\exists x)(F(x) \wedge G(x)) \to (\exists x)F(x)$$

   is universally valid. [10 marks]

Boolean algebra axioms

(B1) $(x + y) + z = x + (y + z)$.

(B2) $x + y = y + x$.

(B3) $x + 0 = x$.

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$.

(B6) $x \cdot 1 = x$.

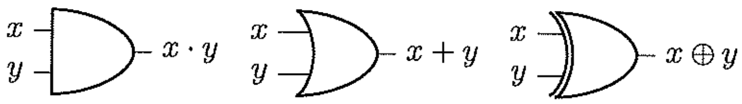
(B7) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(B8) $x + (y \cdot z) = (x + y) \cdot (x + z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

Circuit symbols



End of paper

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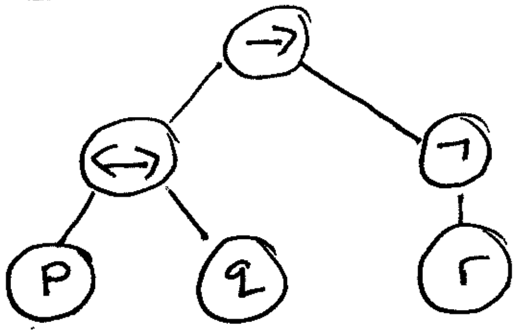
Solutions

1 (a)

p	q	p ⊕ q	p nand q	p → q	p ↔ q
T	T	F	F	T	T
T	F	T	T	F	F
F	T	T	T	T	F
F	F	F	T	T	T

(b)



(c)

p	q	r	p ↔ q	¬ r	(p ↔ q) → ¬ r
T	T	T	T	F	F *
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	F	F	T
F	T	F	F	T	T
F	F	T	T	F	F *
F	F	F	T	T	T

(d) $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

(e) Draw a truth table for $\neg(p \vee (q \wedge r))$ [1]
Draw a truth table for $\neg p \wedge (\neg q \vee \neg r)$. [1]
Observe that the truth tables are the same. [1]
Deduce that $\neg(p \vee (q \wedge r)) \equiv \neg p \wedge (\neg q \vee \neg r)$. [1]

(f) $p \underline{\text{nor}}\, q \overset{\text{def}}{=} \neg(p \vee q)$ [1 mark]

$$\begin{aligned}
p \to q &\equiv \neg p \vee q \\
&\equiv \neg\neg(\neg p \vee q) \\
&\equiv \neg[\neg(\neg p \vee q)] \\
&\equiv \neg[\neg p \ \underline{\text{nor}}\ q] \\
&\equiv \neg[(p \ \underline{\text{nor}}\ p) \ \underline{\text{nor}}\ q] \\
&\equiv [(p \ \underline{\text{nor}}\ p) \ \underline{\text{nor}}\ q] \ \underline{\text{nor}}\ [(p \ \underline{\text{nor}}\ p) \ \underline{\text{nor}}\ q]
\end{aligned}$$

[2 marks]

[using $x \ \underline{\text{nor}}\ x \equiv \neg x$]

2 (a)

(i) Wff is a conjunction of terms each of which is a disjunction of literals. [1]

(ii) A wff in CNF where each term (as above) contains <u>at most one positive literal</u>. [1]

(iii) $(q \to p) \wedge (c \wedge p \to q)$

$\wedge\ (s \wedge r \to \underline{f}) \wedge (\underline{t} \to d)$ [3]

(iv)

Marking from algorithm

$(q \to p) \wedge (c \wedge p \to q) \wedge (s \wedge r \to \underline{f}) \wedge (\underline{t} \to \dot{\underline{d}})$

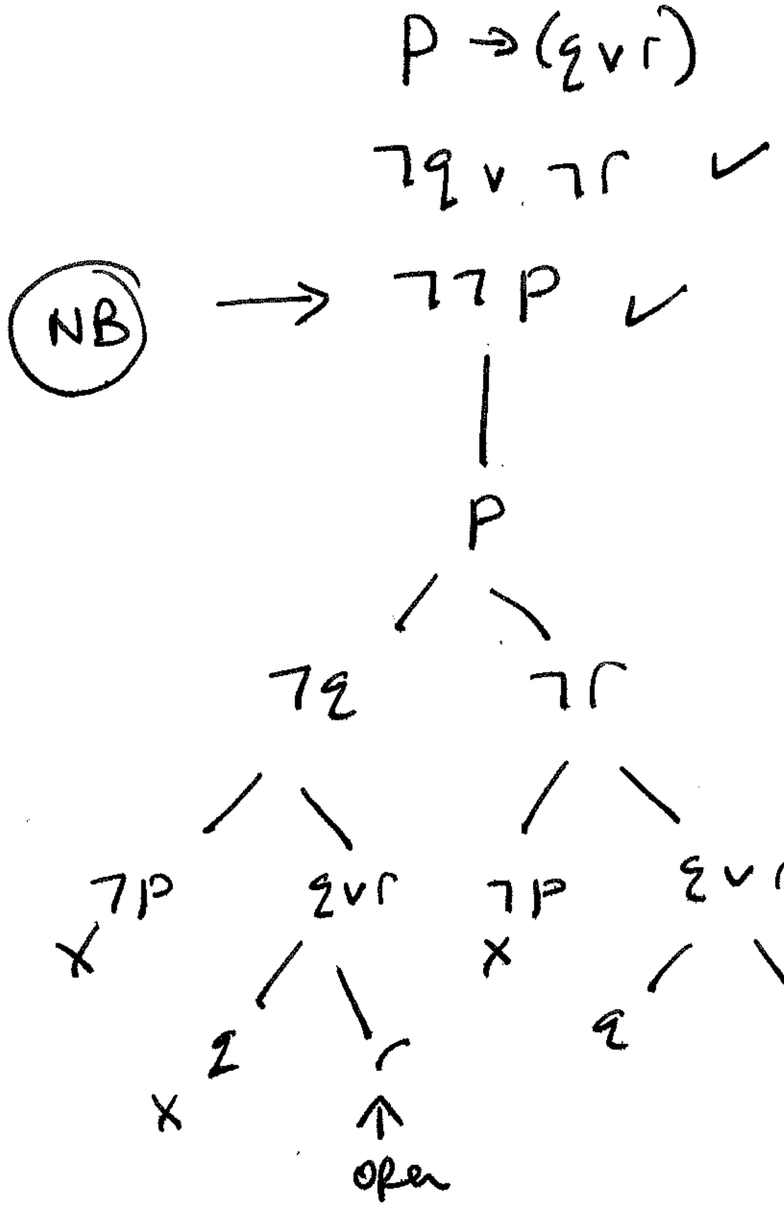
c	d	p	q	r	s
F	T	F	F	F	F

← This truth assignment is uniquely determined by the algorithm

? any partial credit

2 (b)

(i)



Thus this is <u>not</u> a valid argument.

2 (b) (ii)

NB $\rightarrow$ $\neg\left[(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))\right]$ ✓

$p \rightarrow (q \rightarrow r)$

$\neg\left[(p \rightarrow q) \rightarrow (p \rightarrow r)\right]$ ✓

$p \rightarrow q$ ✓

$\neg(p \rightarrow r)$ ✓

p

$\neg r$

$\neg p$ ✗	q

$\neg p$ ✗	$q \rightarrow r$

$\neg q$ ✗	r ✗

Truth tree closes so it is a tautology.

6

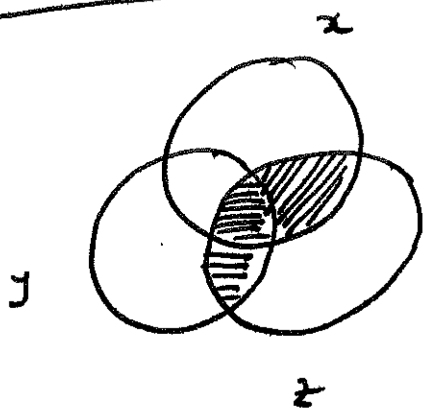
3 (a)

$$a0 = a(a \cdot \overline{a}) \quad \text{by (B10)}$$
$$= (aa)\overline{a} \quad \text{by (B4)}$$
$$= a^2\overline{a}$$
$$= a\overline{a} \quad \text{Given that } a^2 = a$$
$$= 0 \quad \text{by (B10)}$$

(b) $u = (x \cdot y) \cdot z$. [2]

$v = (x \cdot y + z) \cdot (x + y)$. [3]

(c)



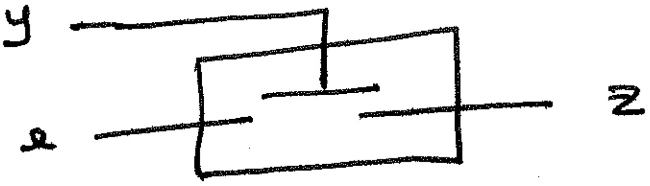
$x\overline{y}z + xyz + \overline{x}yz$

Looks like $(x+y) \cdot z$

$$x\overline{y}z + xyz + \overline{x}yz = xz(\overline{y} + y) + \overline{x}yz$$
$$= xz + \overline{x}yz$$
$$= (x + \overline{x}y)z$$
$$= \underline{(x+y)z}. \quad \text{absorption}$$

~~4~~ 7

(d)



when $y = 0$ then $z = x$

when $y = 1$ then $z = 0$

x	y	z
1	1	0
1	0	1
0	1	0
0	0	0

$z = x\bar{y} = x \square y$ (say)

[1]

- $1 \square y = 1\bar{y} = \bar{y}$. Not-gate [1]
- $x + y = \overline{(\bar{x} \cdot \bar{y})}$

  $= \bar{x} \square \bar{y}$

  $= (1 \square x) \square (1 \square y)$. Or-gate [2]

Every combinational circuit can be constructed from not-gates and or-gates only. Thus every combinational circuit can be constructed from transistors. [1]

4 (a)

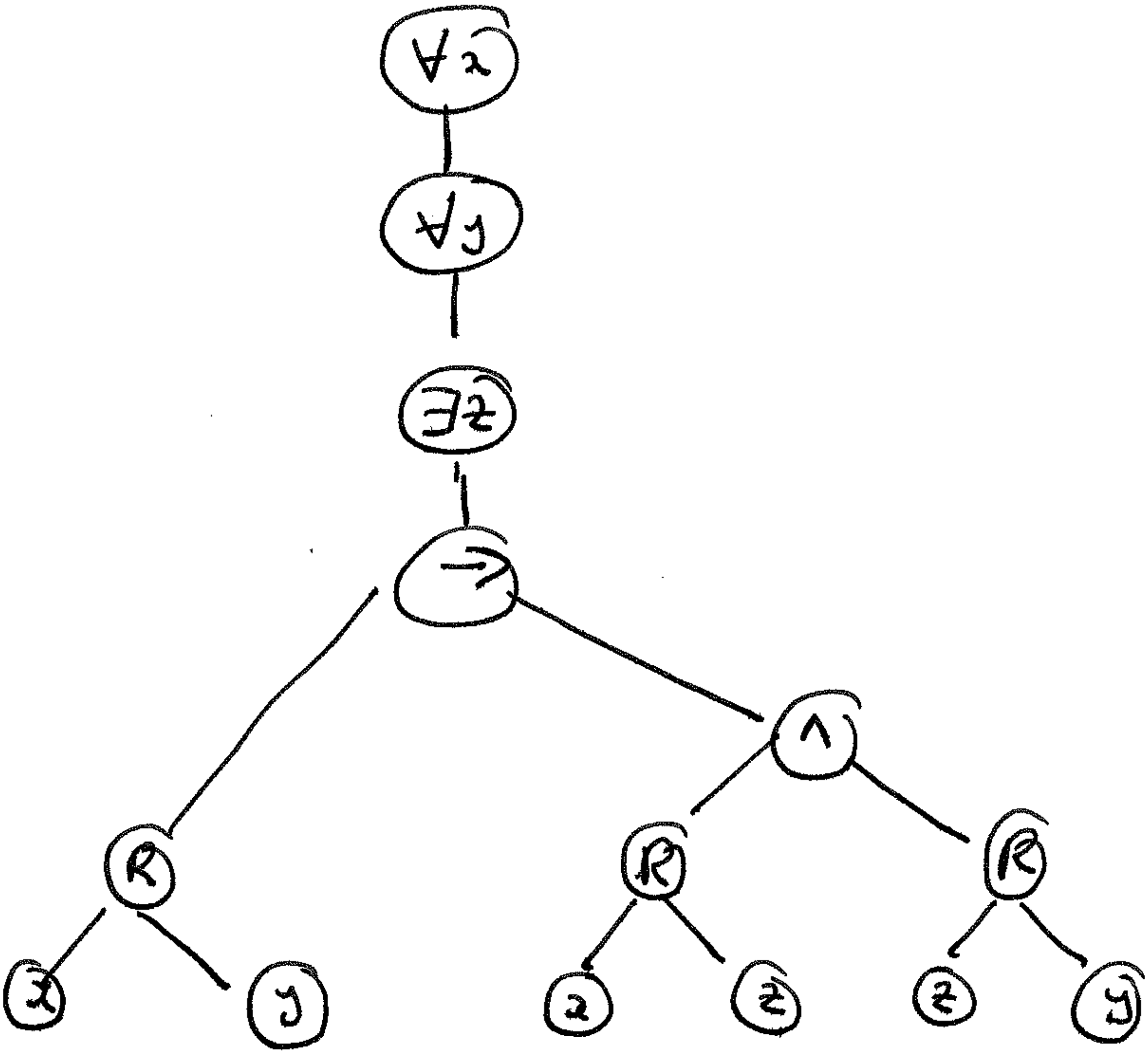
(i) All Hobbits have hairy feet.

(ii) No Hobbits have hairy feet.

(iii) Some Hobbits have hairy feet.

(iv) Some Hobbits don't have hairy feet.

(b) (i)



(b) (ii) If $x, y \in \mathbb{Q}$ and $x < y$ then there is $z \in \mathbb{Q}$ such that $x < z < y$. That is, between any two rational numbers there is a third.

This is true.

(iii) If $x, y \in \mathbb{Z}$ and $x < y$ then there is $z \in \mathbb{Z}$ such that $x < z < y$. This is false because $0 < 1$ but there is no integer between them.

4 (c) NB

$$\neg\left[(\exists x)(F(x) \wedge G(x)) \rightarrow (\exists x)F(x)\right] \checkmark$$

$$(\exists x)(F(x) \wedge G(x)) \checkmark$$

$$\neg(\exists x)F(x) \checkmark$$

$$(\forall x)\neg F(x) \quad (*)$$

$$F(a) \wedge G(a)$$ New constant a

$$F(a)$$

$$G(a)$$

Instantiate (*) with x→a

$$\neg F(a)$$

$$\times$$

Thus the wff is universally valid.