SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Computer Science

F29FA

FOUNDATIONS I

MOCK TEST — 2020/21

Duration: 0.5 Hour

ANSWER ALL QUESTIONS

1. Let $M \equiv (\lambda xyz.xz(yz))(\lambda xy.x)((\lambda z.zz)(\lambda z.zz))(\lambda x.x)x$.
$K \equiv \lambda xy.x$.
$S \equiv \lambda xyz.xz(yz)$.
$I \equiv \lambda x.x$.
$B \equiv \lambda xy.y(xxy)$.
$C \equiv BB$.
$\Omega \equiv (\lambda z.zz)(\lambda z.zz)$.
$F \equiv \lambda x.yzx$.

(a) Give $F[y := xI]$ showing all the substitution steps. (2.5)

$F[y := xI] \equiv \underline{(\lambda x.yzx)[y : xI]} \equiv^6$
$\lambda x'.\underline{(yzx)[x := x']}[y := xI] \equiv^3$
$\lambda x'.(\underline{(yz)[x := x']}x[x := x'])[y := xI] \equiv^3$
$\lambda x'.(\underline{y[x := x']}z[x := x']x[x := x'])[y := xI] \equiv^2$
$\lambda x'.(y\underline{z[x := x']}x[x := x'])[y := xI] \equiv^2$
$\lambda x'.(yz\underline{x[x := x']})[y := xI] \equiv^1$
$\lambda x'.\underline{(yzx')[y := xI]} \equiv^3$
$\lambda x'.\underline{(yz)[y := xI]}x'[y := xI] \equiv^3$
$\lambda x'.\underline{y[y := xI]}z[y := xI]x'[y := xI] \equiv^1$
$\lambda x'.xI\underline{z[y := xI]}x'[y := xI] \equiv^2$
$\lambda x'.xIz\underline{x'[y := xI]} \equiv^2$
$\lambda x'.xIzx'$.

(b) Give the meaning of the following terms:

- K. (0.5)
- S. (1)

- K is the function which takes two arguments, ignores the second and returns the first.
- S is the function that takes 3 arguments, applies the first to the third and then applies their result to the application of the second to the third.

(c) Insert the full parenthesis in $S(KI)$. Note here that you should write S, K and I in full. (1.5)

$S({}_1KI)_1 \equiv ({}_1({}_2\lambda x.({}_3\lambda y.({}_4\lambda z.({}_5({}_6xz)_6({}_6yz)_6)_5)_4)_3)_2({}_2({}_3\lambda x.({}_4\lambda y.x)_4)_3({}_3\lambda x.x)_3)_2)_1$.
Note that you do not need to give me the numbering on the parenthesis. I just did them for you.

(d) Write M using the *minimum* number of symbols you need from K, S, I, B, C, Ω and x (you may not need all these symbols and your answer should have the minimum number of symbols needed, also you cannot use anything else in M except these symbols; for example C is written using only the B symbol). (1)

$M \equiv SK\Omega Ix$.

(e) Which of the terms Ω, SK and $K\Omega$ is a subterm of M. Explain why. (1.5)

Since $M \equiv ((((SK)\Omega)I)x)$ then only Ω and SK are subterms, and the others do not appear between parenthesis.

(f) Give all the subterms of SK. (1.5)

$\{SK, S, K, \lambda yz.xz(yz), \lambda z.xz(yz), \lambda y.x, xz(yz), xz, yz, x, y, z\}$.

(g) Is M strongly β-normalising? Prove your answer always underlying the redex you are working on. (1.5)

$M \equiv SK\underline{\Omega}Ix \to_\beta$
$M \equiv SK\underline{\Omega}Ix \to_\beta$
$M \equiv SK\underline{\Omega}Ix \to_\beta$
$M \equiv SK\underline{\Omega}Ix \to_\beta$
$M \equiv SK\underline{\Omega}Ix \to_\beta$
....
Since we have given an infinite β-reduction path, M is not strongly β-normalising.

(h) Is M β-normalising? If yes, β-reduce M until there are no β-redexes left, showing all the β-reduction steps, underlining at each stage the redex you are contracting, and always keeping the term as compact as possible. If the term is not β-normalising, give a detailed proof why it is not. (2)

$M \equiv SK\Omega Ix \equiv$
$\underline{(\lambda xyz.xz(yz))K}\Omega Ix \to_\beta$
$\underline{(\lambda yz.Kz(yz))\Omega}Ix \to_\beta$
$\underline{(\lambda z.Kz(\Omega z))I}x \to_\beta$
$KI(\Omega I)x \equiv$
$\underline{(\lambda xy.x)I}(\Omega I)x \to_\beta$
$\underline{(\lambda y.I)(\Omega I)}x \to_\beta$
$Ix \equiv$
$\underline{(\lambda x.x)x} \to_\beta$
x.
Hence the term is β-normalising.

(i) Does M have a β-normal form? If yes, give the β-normal form. If not, say why not. (0.5)

Yes since by above, $M \twoheadrightarrow_\beta x$ and hence $M =_\beta x$ where x is in β-normal form. So, the β-normal form of M is x.

(j) Show that for any term A we have $CA \twoheadrightarrow_\beta A(CA)$. (1.5)

$CA \equiv BBA \equiv \underline{(\lambda xy.y(xxy))B}A \to_\beta \underline{(\lambda y.y(BBy))A} \to_\beta A(BBA) \equiv A(CA)$.
Hence, $CA \twoheadrightarrow_\beta A(CA)$.

END OF PAPER