SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Computer Science

F29FA1

Foundations I

Class Test 2015/16

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13 October 2015
Duration: 0.5 Hours

Answer ALL questions

1. Let $M = ((\lambda y'.((z(\lambda x.(yx)))y'))z)$.

(a) Remove as many parenthesis as possible from the term M given above without changing its meaning. (1)

$M = (\lambda y'.z(\lambda x.yx)y')z$

Learning Objectives: Handling syntax

(b) Give the subterms of M. (1)

$(\lambda y'.z(\lambda x.yx)y')z$, $\lambda y'.z(\lambda x.yx)y'$, $z(\lambda x.yx)y'$, $z(\lambda x.yx)$, $(\lambda x.yx)$, yx, y', y, x, z,

Learning Objectives: Handling syntax

(c) Give the result of $M[z := y']$. (1)

$M[z := y'] \equiv (\lambda y''.y'(\lambda x.yx)y'')y'$

Learning Objectives: Handling syntax

(d) η-reduce M to an η-normal form M_1 and then give the β-normal form of M_1. (1)

$M_1 \equiv zyz$ and the β-nf is zyz.

Learning Objectives: Reduction in the lambda calculus

(e) β-reduce M to a β-normal form M_2 and then give the η-normal form of M_2. (1)

$M_2 \equiv z(\lambda x.yx)z$ and the η-nf is zyz.

Learning Objectives: Reduction in the lambda calculus

(f) Give the $\beta\eta$-normal form of M. (1)

zyz.

2. Suppose that A, B, and C are three lambda terms and the only things you know about A, B and C are the following three facts:
1) $A \twoheadrightarrow_\beta B$, 2) $A \twoheadrightarrow_\beta C$, 3) Both B and C are in β-normal forms.
For each of the following statements state whether it is false, true or possibly true or possibly false or both. In each case, justify your answer.

(a) $B \twoheadrightarrow_\beta A$. (1)

Possibly true and possibly false. An example of the first case is $A \equiv B$ (where we then get $B \twoheadrightarrow_\beta A$). An example of the second case is $A \equiv (\lambda x.x)y$ and $B \equiv y$ (where we then get $B \not\twoheadrightarrow_\beta A$).

Learning Objectives: Conversion in the lambda calculus.

(b) $A =_\alpha B$. (1)

Possibly true and possibly false. An example of the first case is $A \equiv B$ (where we then get $B =_\alpha A$). An example of the second case is $A \equiv (\lambda x.x)y$ and $B \equiv y$ (where we then get $B \neq_\alpha A$).

Learning Objectives: Conversion in the lambda calculus.

(c) $B \to_\beta C$. (1)

False because B is in β-normal form, hence there is no C such that $B \to_\beta C$.

Learning Objectives: Conversion in the lambda calculus.

(d) $B =_\beta C$. (1)

True. In fact even $B \equiv C$ by the Church Rosser Theorem.

Learning Objectives: Conversion in the lambda calculus.

(e) A is not weakly β-normalising. (1)

False. Since B is in normal form, then A is weakly β-normalising.

Learning Objectives: Conversion in the lambda calculus.

3. Let $A \equiv \lambda x.x\Omega\textbf{false}$ and $B \equiv \lambda xy.y\Omega$ where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ and $\textbf{false} \equiv \lambda xy.y$.

(a) State whether A is weakly β-normalising. Justify your answer. (1)

By normal order reduction, $\underline{\Omega} \to_\beta^{lmo} \underline{\Omega} \to_\beta^{lmo} \underline{\Omega} \to_\beta \ldots$ and hence by compatibility $\lambda x.x\underline{\Omega}\textbf{false} \to_\beta^{lmo} \lambda x.x\underline{\Omega}\textbf{false} \to_\beta^{lmo} \lambda x.x\underline{\Omega}\textbf{false} \ldots$. Hence, $A \to_\beta^{lmo} A \to_\beta^{lmo} A \ldots$ and by the normalisation theorem, the term A is not weakly normalising.

Learning Objectives: Weak versus Strong Termination and normalisation theorem.

(b) Define strong normalisation and show that A is not strongly β-normalising. (1)

SN is when the term always terminate no matter how we reduce it. A is not SN because it is not WN.

Learning Objectives: Weak versus Strong Termination and normalisation theorem.

(c) Is B weakly β-normalising? Justify your answer. (1)

By normal order reduction, $\underline{\Omega} \to_\beta^{lmo} \underline{\Omega} \to_\beta^{lmo} \underline{\Omega} \to_\beta^{lmo} \ldots$ and hence by compatibility $\lambda xy.y\underline{\Omega} \to_\beta^{lmo} \lambda xy.y\underline{\Omega} \to_\beta^{lmo} \lambda xy.y\underline{\Omega} \ldots$. Hence, $B \to_\beta^{lmo} B \to_\beta^{lmo} B \ldots$ and by the normalisation theorem, the term B is not weakly normalising.

Learning Objectives: Weak Termination and normalisation theorem.

(d) Is AB weakly β-normalising? Justify your answer. (1)

$AB \equiv \underline{(\lambda x.x\Omega\textbf{false})B} \to_\beta B\Omega\textbf{false} \equiv \underline{(\lambda xy.y\Omega)\Omega}\textbf{false} \to_\beta \underline{(\lambda y.y\Omega)\textbf{false}} \to_\beta \textbf{false}\Omega \equiv \underline{(\lambda xy.y)\Omega} \to_\beta (\lambda y.y)$. Hence AB is weakly β-normalising.

Learning Objectives: Weak versus Strong Termination.

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