SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Computer Science

F29FA1

Foundations I

Class Test 2016/17

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11 October 2016
Duration: 0.5 Hour

Answer ALL questions

1. Let $A \equiv x(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y)$.

(a) Insert as many parenthesis as possible in A without changing its meaning. (1)

$((x(\lambda x.x))(((\lambda x.(\lambda y.(xy)))(\lambda z.(zz)))y))$.

Learning Objectives: Syntax and semantics of the λ-calculus.

(b) Give the subterms of A, each subterm on a separate line. (2)

$\{$
$x(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y)$
$((\lambda xy.xy)(\lambda z.zz)y)$
$(\lambda xy.xy)(\lambda z.zz)$
$x(\lambda x.x)$
$\lambda xy.xy$
$\lambda y.xy$
$\lambda z.zz$
$\lambda x.x$
xy
zz
x
y
z
$\}$.

Learning Objectives: Syntax and semantics of the λ-calculus.

(c) Is A β-normalising? If yes, β-reduce A until there are no β-redexes left, showing all the β-reduction steps. If not, explain why not. (1)

$x(\lambda x.x)(\underline{(\lambda xy.xy)(\lambda z.zz)}y) \to_\beta$
$x(\lambda x.x)(\underline{(\lambda y.(\lambda z.zz)y)y}) \to_\beta$
$x(\lambda x.x)(\underline{(\lambda z.zz)y}) \to_\beta$
$x(\lambda x.x)(yy)$.
Hence A is β-normalising since there are no more β-redexes.

Learning Objectives: Reduction and normalisation in the λ-calculus.

(d) Give the $\beta\eta$-normal form of A is it exists, otherwise, say why it does not exist. (1)
$x(\lambda x.x)(yy)$

Learning Objectives: Reduction and normalisation in the λ-calculus.

(e) Give the term $A[x := (\lambda x.xx)(\lambda x.xx)]$. (1)

$(\lambda x.xx)(\lambda x.xx)(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y)$.

Learning Objectives:Substitution in the λ-calculus.

(f) Is $A[x := (\lambda x.xx)(\lambda x.xx)]$ β-normalising? If yes, give the β-normal form showing all the reduction steps you used to reach it. If not, give a detailed proof why it is not. (2)

$A[x := (\lambda x.xx)(\lambda x.xx)] \equiv$
$\underline{(\lambda x.xx)(\lambda x.xx)}(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y) \to_\beta^{lmo}$
$\underline{(\lambda x.xx)(\lambda x.xx)}(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y) \to_\beta^{lmo}$
$\ldots$

Since the leftmost reduction path does not terminate, by the normalisation theorem $A[x := (\lambda x.xx)(\lambda x.xx)]$ is not β-normalising.

Learning Objectives: normalisation theorem.

(g) Give the term $A[x := \lambda x.x]$. (1)

$A[x := \lambda x.x] \equiv (\lambda x.x)(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y)$.

Learning Objectives:Substitution in the λ-calculus.

(h) Is $A[x := \lambda x.x]$ β-normalising? If yes, give the β-normal form showing all the reduction steps you used to reach it. If not, give a detailed proof why it is not. (2)

$A[x := \lambda x.x] \equiv$
$\underline{(\lambda x.x)(\lambda x.x)}((\lambda xy.xy)(\lambda z.zz)y) \to_\beta^{lmo}$
$\underline{(\lambda x.x)((\lambda xy.xy)(\lambda z.zz)y)} \to_\beta^{lmo}$
$\underline{(\lambda xy.xy)(\lambda z.zz)}y \to_\beta^{lmo}$
$\underline{(\lambda y.(\lambda z.zz)y)y} \to_\beta^{lmo}$
$\underline{(\lambda z.zz)y} \to_\beta^{lmo}$
yy.
Hence $A[x := \lambda x.x]$ is β-normalising since there are no more β-redexes. The β-normal form is yy.

Learning Objectives: normalisation theorem.

(i) Give three terms A, B and C such that $B \to_\beta C$ and $A[x := B] \to_\beta^3 A[x := C]$ (i.e., B β-reduces to C in one step whereas $A[x := B]$ β-reduces to $A[x := C]$ in three steps. (2)

Let $A \equiv xxx$, $B \equiv (\lambda y.y)z$ and $C \equiv z$. Then, $B \to_\beta C$ and $A[x := B] \to_\beta^3 A[x := C]$

Learning Objectives: interaction of substitutions and reduction.

(j) For each of the reduction paths below, state whether it is standard or not and also whether it follows the leftmost β-reduction strategy or not. In each case, justify your answer. (2)

1. $(\lambda z.\underline{(\lambda x.(\lambda y.xy)z)(\lambda z.z)})x \to_\beta (\lambda z.\underline{(\lambda y.(\lambda z.z)y)z})x \to_\beta (\lambda z.\underline{(\lambda z.z)z})x \to_\beta (\lambda z.z)x$.
2. $(\lambda z.(\lambda x.\underline{(\lambda y.xy)z})(\lambda z.z))x \to_\beta (\lambda z.\underline{(\lambda x.xz)(\lambda z.z)})x \to_\beta (\lambda z.\underline{(\lambda z.z)z})x \to_\beta (\lambda z.z)x$.

1. $(\lambda z.\underline{(\lambda x.(\lambda y.xy)z)(\lambda z.z)})x \overset{R_0}{\to}_\beta (\lambda z.\underline{(\lambda y.(\lambda z.z)y)z})x \overset{R_1}{\to}_\beta (\lambda z.\underline{(\lambda z.z)z})x \overset{R_2}{\to}_\beta (\lambda z.z)x$ is standard because for any pair (R_i, R_{i+1}) where $0 \leq i \leq 1$, the λ of the redex R_{i+1} comes from a λ in A_i which is to the right of the λ of R_i in A_i (here we take $A_0 \equiv (\lambda z.\underline{(\lambda x.(\lambda y.xy)z)(\lambda z.z)})x$ and $A_1 \equiv (\lambda z.\underline{(\lambda y.(\lambda z.z)y)z})x$ and $A_2 \equiv (\lambda z.\underline{(\lambda z.z)z})x$.
This path however is not leftmost because we are not reducing the leftmost redex (the one starting with the leftmost λz).
2. $(\lambda z.(\lambda x.\underline{(\lambda y.xy)z})(\lambda z.z))x \to_\beta (\lambda z.\underline{(\lambda x.xz)(\lambda z.z)})x \to_\beta (\lambda z.\underline{(\lambda z.z)z})x \to_\beta (\lambda z.z)x$ is not standard because for $A_0 \equiv (\lambda z.(\lambda x.(\lambda y.xy)z)(\lambda z.z))x$, $R_0 \equiv \underline{(\lambda y.xy)z}$ and $R_1 \equiv \underline{(\lambda x.xz)(\lambda z.z)}$, the λ of the redex R_1 does not come from a λ in A_0 which is to the right of the λ of R_0 in A_0. Moreover, this path is not leftmost because we are not reducing the leftmost redex (the one starting with the leftmost λz).

Learning Objectives: interaction of substitutions and reduction.

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