Student Information

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Answer 1

Let $G(x)$ be the generating function for the sequence $\{a_n\}$, that is, $G(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$
$x \cdot G(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} \cdot x^n$
$x^2 \cdot G(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} \cdot x^n$

$G(x) - 3x \cdot G(x) - 4x^2 \cdot G(x) = \sum_{n=0}^{\infty} a_n \cdot x^n - 3 \cdot \sum_{n=1}^{\infty} a_{n-1} \cdot x^n - 4 \cdot \sum_{n=2}^{\infty} a_{n-2} \cdot x^n$
$(1 - 3x - 4x^2) \cdot G(x) = a_0 + a_1 \cdot x - 3x \cdot a_0 + \sum_{n=2}^{\infty}(a_n - 3a_{n-1} - 4a_{n-2}) \cdot x^n$
By using the recurrence relation $a_n - 3a_{n-1} - 4a_{n-2} = 0$ and $a_0, a_1 = 1$:
$(1 - 3x - 4x^2) \cdot G(x) = 1 - 2x$
$G(x) = \frac{1-2x}{1-3x-4x^2}$
By using partial fractions:
$G(x) = \frac{A}{1-4x} + \frac{B}{x+1}$
$A - 4B = -2$
$A + B = 1$
$A = 2/5$
$B = 3/5$
$G(x) = \frac{2/5}{1-4x} + \frac{3/5}{1+x}$
Using the identity: $\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n \cdot x^n$
$G(x) = \frac{2}{5} \cdot \sum_{n=0}^{\infty} 4^n \cdot x^n + \frac{3}{5} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot x^n$
So we have:

$$a_n = \frac{2}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n$$

Answer 2

a)

Let's say that $f(x)$ is the generating function.
It's of the form:

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + ...$$

For the given sequence:

$$f(x) = 2x^0 + 5x^1 + 11x^2 + 29x^3 + ...$$

We know that

$$\frac{2}{1-x} = 2 \cdot \sum_{n=0}^{\infty} x^n$$

So substracting $\frac{2}{1-x}$ we get:

$$f(x) - \frac{2}{1-x} = 0x^0 + 3x^1 + 9x^2 + 27x^3 + ...$$

$$f(x) - \frac{2}{1-x} + 1 = x^0 + 3x^1 + 9x^2 + 27x^3 + ...$$

$$f(x) - \frac{2}{1-x} + 1 = x^0 + 3^1x^1 + 3^2x^2 + 3^3x^3 + ...$$

Since

$$x^0 + 3^1x^1 + 3^2x^2 + 3^3x^3 + ... = \sum_{n=0}^{\infty} 3^n \cdot x^n = \frac{1}{1-3x}$$

$$f(x) - \frac{2}{1-x} + 1 = \frac{1}{1-3x}$$

$$f(x) = \frac{1}{1-3x} + \frac{2}{1-x} - 1$$

$$f(x) = \frac{-3x^2 - 3x + 2}{3x^2 - 4x + 1}$$

b)

By using partial fractions:

$$G(x) = \frac{7-9x}{1-3x+2x^2} = \frac{A}{2x-1} + \frac{B}{x-1}$$

$A + 2B = -9$
$-A - B = 7$
$A = -5$
$B = -2$
We know that

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n \cdot x^n$$

$$G(x) = \frac{5}{1-2x} + \frac{2}{1-x} = 5\sum_{n=0}^{\infty} 2^n \cdot x^n + 2 \cdot \sum_{n=0}^{\infty} x^n$$

So we have:

$$a_n = 2 + 5 \cdot 2^n$$

Thus, the sequence corresponding to the generating function:

$$< 7, 12, 22, 42, 82, 162, ... >$$

Answer 3

a)

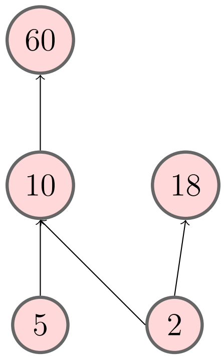
Let's check if R is reflexive:
aRa means $(a^2 + a^2 = n^2) \vee (a^2 + n^2 = a^2)$
For first condition since n and a are integers n can't be equal to $a\sqrt{2}$ which is not an integer.
For second condition for n to be an edge it must be a positive integer and can't be zero.
So it is not reflexive and hence, not an equivalence relation.

b)

Because $2x_1 + y_1 = 2x_1 + y_1$ for all real numbers x_1, y_1. Hence $(x_1, y_1)R(x_1, y_1)$ for all real numbers x_1, y_1. So R is reflexive.
Suppose $(x_1, y_1)R(x_2, y_2)$. Then $2x_1 + y_1 = 2x_2 + y_2$, so $2x_2 + y_2 = 2x_1 + y_1$ is also true. Hence $(x_2, y_2)R(x_1, y_1)$, it follows that R is symmetric.
If $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$, $2x_1 + y_1 = 2x_2 + y_2 = 2x_3 + y_3$. Hence $(x_1, y_1)R(x_3, y_3)$. Thus, R is transitive.
Consequently R is an equivalence relation.
Let's assume that (x, y) is the equivalence class of $(1, -2)$.
So, $2x + y = 2 \cdot 1 - 2 = 0$.
$x = -\frac{y}{2}$
Let's say y is equal to some constant k then the equivalence class is of the form: $(-\frac{k}{2}, k)$
And it represents the line $y = -2x$ in the cartesian coordinate system.

Answer 4

a)



b)

Columns and rows are 2, 5, 10, 18, 60

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

$$(x, y) = \{(10, 2), (18, 2), (60, 2), (10, 5), (60, 5), (60, 10)\}$$

Columns and rows are 2, 5, 10, 18, 60

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

d)

If we are allowed to replace only a single element it is not possible because we must change either 2 or 5 and also remove 18 since its not related to anything but itself and this is not possible with a single change.
But if we are allowed to remove 2 and add 1 element. It is possible, for example we can remove 2 and 18 and add 1 instead we get:
$A = \{1, 5, 10, 60\}$ And we would get a total ordering in this way.