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Answer 1

a) To test if it is surjective $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $f(x) = y$ must be true but since $x = \pm\sqrt{y}$ $y \geq 0$ so negative values for y are not covered. It is not surjective.
To test if it is injective suppose $f(a) = f(b)$ for $a, b \in \mathbb{R}$
$a^2 = b^2$
$|a| = |b|$
$a = \pm b$
So it is not injective.

b) To test if it is surjective $\forall y \in \mathbb{R}, \exists x \in \overline{\mathbb{R}}^+$ such that $f(x) = y$ must be true but since $x = \pm\sqrt{y}$ $y \geq 0$ so negative values for y are not covered. It is not surjective.
To test if it is injective suppose $f(a) = f(b)$ for $a, b \in \overline{\mathbb{R}}^+$
$a^2 = b^2$
$|a| = |b|$
$a = b$
So it is injective.

c) To test if it is surjective $\forall y \in \overline{\mathbb{R}}^+, \exists x \in \mathbb{R}$ such that $f(x) = y$ must be true but since $x = \pm\sqrt{y}$, $y \geq 0$ so it holds for all values of y it is surjective.
To test if it is injective suppose $f(a) = f(b)$ for $a, b \in \mathbb{R}$
$a^2 = b^2$
$|a| = |b|$
$a = \pm b$
So it is not injective.

d) To test if it is surjective $\forall y \in \overline{\mathbb{R}}^+, \exists x \in \overline{\mathbb{R}}^+$ such that $f(x) = y$ must be true but since $x = \pm\sqrt{y}$, $y \geq 0$ so it holds for all values of y it is surjective.
To test if it is injective suppose $f(a) = f(b)$ for $a, b \in \overline{\mathbb{R}}^+$
$a^2 = b^2$
$|a| = |b|$
$a = b$
So it is injective.

Answer 2

a) Since $x \in A \subset \mathbb{Z}$ then if $x \neq x_0$, $||x - x_0|| \geq 1$
So if we assume $\delta = 0.9$ the left side of the expression $(||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon)$ is always 0 when $x \neq x_0$ and hence the expression is always true.
If $x = x_0$ the left side becomes 1 and right side also always 1 because $||f(x) - f(x_0)|| < \epsilon, \forall \epsilon \in \overline{\mathbb{R}}^+$ since $||f(x) - f(x_0)|| = 0$ and the expression will continue to be true.
So we can say every function $f : A \subset \mathbb{Z} - > \mathbb{R}$ is continuous.

b) Assume that f is not a constant function. We can find $\exists x, x_0 \in \overline{\mathbb{R}}^+$ such that $f(x) \neq f(x_0)$. Since $f(x), f(x_0) \in \mathbb{Z}$, $||f(x) - f(x_0)|| \geq 1$ which makes the right side of the expression $(||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon)$ false for $\forall \epsilon \in \overline{\mathbb{R}}^+$ and since $||x - x_0||$ is finite $\exists \delta \in \overline{\mathbb{R}}^+$ such that $(||x - x_0|| < \delta)$ so the left side is true. Which makes the expression false hence f is not continuous.

Assume that f is a constant function. Since $\forall x, x_0, f(x) = f(x_0)$ the right side of the expression $(||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon)$ is true $\forall \epsilon \in \overline{\mathbb{R}}^+$ because $f(x) - f(x_0) = 0$. Which makes the expression true since $false \implies true$ and $true \implies true$ are both true.

We learned that if f is not a constant fucntion it is not continuous. By using contraposition we can derive "If f is continuous it is a constant function." And also we proved that if f is a constant function it is continuous. So we can say that it is necessary and sufficient for a function of the form $f : A \subset \mathbb{R} - > \mathbb{Z}$ to be constant for it to be continuous.

Answer 3

a)

$$X = A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) | a_i \in A_i \text{ for } i = 1, 2, ..., n\}$$

Let's denote the elements of sets with a_{ij} where $a_i \in A_i$ and j is the order of the element in the set.

X will be of the form:

$\{(a_{11}, a_{21}, ..., a_{n1}),$
$(a_{12}, a_{21}, ..., a_{n1}),$
$(a_{11}, a_{22}, ..., a_{n1}),$
$...$
$(a_{11}, a_{21}, ..., a_{n2}),$
$...\}$

Since we can write all tuples contiuously according to the sum of their j values which will go from n to some countable number which is the sum of the number of elements of each of the sets of the form A_i. And make a 1-1 correspondance with the set of natural numbers.

Hence we can say that this set is countable.

b) The infinite product will look like:

$$B = X \times X \times ... \times \ X \times ... = \{(a_1, a_2, ..., a_n, ...) | a_i \in \{0, 1\}, n, i \in \mathbb{N}\}$$

it will be the infinite combination of these tuples.

Let's say a_{ij} is the jth element of the ith tuple. When we write all combinations of these tuples we can still find an element such that $(a_1, a_2, ..., a_n)$ which is not in the set by using an algorithm like $a_n = 0$ if $a_{nn} = 1$ and $a_{nn} = 1$ if $aii = 0$

So it is uncountably infinite.

Answer 4

$$(n!)^2, 5^n, 2^n, n^{51} + n^{49}, n^{50}, \sqrt{n} \cdot \log(n), (log(n))^2$$

a)

$$\lim_{n\to\infty} \frac{(n!)^2}{5^n}$$

$$\lim_{n\to\infty} \frac{n^2 \cdot n - 1^2 \cdot n - 3^2 ...}{5 \cdot 5 \cdot 5 \cdot ...} \geq \lim_{n\to\infty} \left(\frac{9}{5}\right)^{n-2} \cdot \frac{4}{25} = \infty$$

since right side's limit is ∞ by comparison test left side is also ∞

We can say $(n!)^2$ grows faster than 5^n

Hence by limit asymptotic theorem 5^n is O($(n!)^2$)

b)

$$\lim_{n\to\infty} \left(\frac{5}{2}\right)^n = \infty$$

We can say 5^n grows faster than 2^n

Hence by limit asymptotic theorem 2^n is O(5^n)

c)

$$\lim_{n\to\infty} \frac{2^n}{n^{51} + n^{49}}$$

if we apply L'Hospital rule 51 times:

$$\lim_{n\to\infty} \frac{2^n \cdot (\ln 2)^{51}}{51!} = \infty$$

We can say 2^n grows faster than $n^{51} + n^{49}$

Hence by limit asymptotic theorem $n^{51} + n^{49}$ is O(2^n)

d)

$$\lim_{n\to\infty} \frac{n^{51} + n^{49}}{n^{50}} = lim_{n\to\infty}(\frac{n + \frac{1}{n}}{1}) = \infty$$

We can say $n^{51} + n^{49}$ grows faster than n^{50}

Hence by limit asymptotic theorem n^{50} is O($n^{51} + n^{49}$)

e)

$$\lim_{n\to\infty} \frac{n^{50}}{\sqrt{n}\cdot\log(n)} = \lim_{n\to\infty} \frac{n^{\frac{99}{2}}}{\log(n)}$$

Apply L'Hospital rule:

$$\lim_{n\to\infty} \frac{99}{2}\cdot n^{\frac{99}{2}} = \infty$$

We can say n^{50} grows faster than $\sqrt{n}\cdot\log(n)$
Hence by limit asymptotic theorem $\sqrt{n}\cdot\log(n)$ is $\mathrm{O}(n^{50})$

f)

$$\lim_{n\to\infty} \frac{\sqrt{n}\cdot\log(n)}{(log(n))^2}$$

Apply L'Hospital rule:

$$\lim_{n\to\infty} \frac{1}{2}\cdot\sqrt{n} = \infty$$

We can say $\sqrt{n} * \log(n)$ grows faster than $(log(n))^2$
Hence by limit asymptotic theorem $(log(n))^2$ is $\mathrm{O}(\sqrt{n}\cdot\log(n))$

Answer 5

a)

$$\gcd(94,134) = \gcd(134, 94 \bmod 134) = \gcd(94, 134 \bmod 94) = \gcd(40, 94 \bmod 40)$$

$$= \gcd(14, 40 \bmod 14) = \gcd(12, 14 \bmod 12) = \gcd(2, 12 \bmod 2) = \gcd(2,0)$$

Since the second argument became 0 the function returns the first argument 2.
gcd(94,134)=2

b) Goldbach's conjecture:

$$A = \{x|\text{x is a prime number}\}$$

$$\exists a, b \in A$$

$$2\cdot k = a + b \text{ for } \forall k > 1 \text{ and } \forall k \in \mathbb{Z}$$

To find even integers that are greater than 5 as a sum of three primes we can add 2 to both sides of the above equation:
$\exists a, b \in A$
$2\cdot k + 2 = a + b + 2$ for $k > 1$ and $\forall k \in \mathbb{Z}$

Since 2 is a prime number and $2 \cdot k + 2$ represents all even numbers greater than 5 this shows that we can write all even numbers greater than 5 using 3 prime numbers.

To find odd numbers we can do the same thing using 3 this time:

$\exists a, b \in A$

$2 \cdot k + 3 = a + b + 3$ for $k > 1$ and $\forall k \in \mathbb{Z}$

In this equation since 3 is also a prime number and $2 \cdot k + 3$ represents all odd numbers greater than 5. So we can write all odd numbers greater than 5 using 3 prime numbers.

We have showed that we can calculate both even and odd numbers that are greater than 5 using three prime numbers. Which means we can write all numbers greater than 5 using 3 prime numbers.