Conway's Game of Life: Mathematical Problems & Solutions

Introduction

The Game of Life is a cellular automaton devised by John Horton Conway. It consists of a grid of cells which, based on a few mathematical rules, can live, die, or multiply.

The Rules (for a cell at state t):

1. **Underpopulation:** A live cell with < 2 live neighbors dies.
2. **Survival:** A live cell with 2 or 3 live neighbors lives.
3. **Overpopulation:** A live cell with > 3 live neighbors dies.
4. **Reproduction:** A dead cell with exactly 3 live neighbors becomes a live cell.

Problem 1: Fundamental Logic

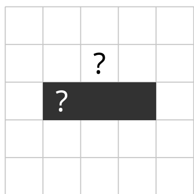
Question:
Consider a single live cell surrounded by 8 dead neighbors at time $t = 0$. 1. Apply the rules to determine the state of this cell at $t = 1$. 2. Apply the rules to determine if any neighbors become alive at $t = 1$. What is the total population at $t = 1$?

Solution:
1. **Center Cell:** It has 0 live neighbors. Rule 1 (Underpopulation) applies. The cell dies. 2. **Neighbors:** Each neighbor has exactly 1 live neighbor (the center cell). Rule 4 (Reproduction) requires exactly 3 live neighbors. No birth occurs.

Conclusion: The center cell dies, and no new cells are born. The total population at $t = 1$ is **0**.

Problem 2: The Blinker (Period 2 Oscillator)

Question:
Consider a row of three live cells (A "Blinker") at coordinates $(1, 2), (2, 2), (3, 2)$. Calculate the state of the cell at $(2, 3)$ (currently dead) and the cell at $(1, 2)$ (currently alive) for the next generation $t + 1$.



Solution:
Cell (2,3) [Dead]: Neighbors are at $(1, 2), (2, 2), (3, 2)$ (plus dead ones above). It has exactly **3** live neighbors. Rule 4 applies: **Birth**. The cell becomes Alive.

Cell (1,2) [Alive]: Neighbors are at $(2, 2)$ (Alive). All others are dead. It has exactly **1** live neighbor. Rule 1 applies: **Death** by underpopulation.

Note: This transformation rotates the line by 90 degrees.

Problem 3: The Block (Still Life)

Question:
A "Block" is a 2×2 square of live cells. Prove mathematically that this structure is a **Still Life** (it never changes state) by calculating the neighbor count for: 1. A live cell inside the block. 2. A dead cell immediately adjacent to the block.

Solution:
1. Live Cell Condition: Consider the top-left live cell. It shares edges/corners with the other 3

live cells in the block. Neighbor Count = 3. Rule 2 (Survival) applies. It stays **Alive**. (This holds for all 4 live cells).

2. Dead Cell Condition: Any dead cell adjacent to the block touches at most 2 live cells (sharing an edge with one and a corner with another). Neighbor Count ≤ 2. Rule 4 (Reproduction) requires exactly 3. No birth occurs.

Since no cells die and no cells are born, the structure is **stable**.

Problem 4: Glider Dynamics

Question:
A standard Glider repeats its shape every 4 generations, but displaced by 1 cell diagonally. If a Glider starts with its "head" at $(0, 0)$ at $t = 0$, calculate the position of the head at $t = 24$. Calculate the "speed" of the Glider in terms of c (where c is 1 cell per generation).

Solution:
Displacement: Every 4 generations, the glider shifts $(+1, +1)$. At $t = 24$, the number of cycles is $24/4 = 6$ cycles. Total displacement = $6 \times (1, 1) = (6, 6)$. Final Position: **(6, 6)**.

Speed: It travels $\sqrt{1^2 + 1^2} = \sqrt{2}$ distance in Euclidean space, or 1 unit in Chebyshev distance (grid logic), over 4 ticks. Usually, speed in Life is measured by grid displacement per tick. Speed = 1 cell / 4 generations = **c/4**.

Problem 5: Population Density

Question:
What is the maximum number of live neighbors a cell can have and still survive to the next generation? Based on this, can a solid 3×3 block of 9 live cells survive completely intact?

Solution:
Max Neighbors: Rule 3 states a cell dies if neighbors > 3. Therefore, a cell survives if neighbors are 2 or 3. The maximum is **3**.

3x3 Block Analysis: Consider the center cell of a 3×3 block. It is surrounded by 8 live cells. $8 > 3$, so Overpopulation applies. The center cell **dies**. Therefore, a solid 3×3 block cannot survive intact; it will become a ring (or evolve further).

Problem 6: Reverse Engineering (Predecessors)

Question:
A specific configuration consists of exactly one live cell at $t = 1$. Is it possible to find a configuration at $t = 0$ that produces this result? *Hint: Consider the reproduction rule.*

Solution:
No, it is **impossible**. 1. For a live cell to exist at $t = 1$, it must have either survived (was alive at $t = 0$) or been born (was dead at $t = 0$). 2. If it survived, it must have had 2 or 3 neighbors at $t = 0$. 3. If it was born, it must have had exactly 3 neighbors at $t = 0$.

In all cases, a configuration at $t = 0$ requires multiple cells to sustain or create a cell at $t = 1$. However, these "parent" cells would also interact with each other. It is a known property of Life that a population can vanish, but a single cell cannot spontaneously appear without "parents" that would also leave traces or die. More formally, a single cell is unstable and dies immediately. To result in *only* one cell is contradictory because the neighbors required to make it would typically create other siblings or survive themselves.

Problem 7: The "Garden of Eden"

Question:
In the context of Cellular Automata, define what a "Garden of Eden" pattern is. Do such patterns exist in the Game of Life?

Solution:
Definition: A Garden of Eden is a configuration that has **no predecessor**. It cannot arise from the evolution of any previous state; it can only exist if it is manually set as the initial state ($t = 0$).

Existence: Yes, they exist. It was proven by Roger Banks (and others) that the Game of Life is not surjective (not all states are reachable). There are finite patterns that cannot be formed by any previous arrangement of living cells.

Problem 8: Geometric Expansion

Question:
If a pattern grows at the maximum possible speed (speed of light, c) in all directions, what shape would the boundary of the living cells approximate after a large number of generations? *Note: In GoL, influence propagates to the 8 neighbors.*

Solution:
In the Game of Life (and Moore neighborhood CA), influence propagates to all 8 surrounding cells in 1 tick. Information travels horizontally, vertically, and diagonally at 1 cell per tick. Therefore, the "event horizon" or maximum boundary of expansion forms a **Square** (Chebyshev ball), not a circle.

Problem 9: The Toad (Period 2 Oscillator)

Question:
The "Toad" consists of two horizontal rows of 3 cells, offset by 1. $t = 0$: Live cells at $(1, 1), (2, 1), (3, 1)$

and $(2,0),(3,0),(4,0)$. Calculate the neighbor count for the "gap" cells at $(1,0)$ and $(4,1)$ to determine if they are born.

Solution:
Cell (1,0) [Dead]: Neighbors: 1. $(1,1)$ - Alive 2. $(2,1)$ - Alive 3. $(2,0)$ - Alive Total = 3. **Birth**.

Cell (4,1) [Dead]: Neighbors: 1. $(3,1)$ - Alive 2. $(3,0)$ - Alive 3. $(4,0)$ - Alive Total = 3. **Birth**.

(The Toad works by "inflating" in the middle while the outer corners die off, flipping between two shapes).

Problem 10: Infinite Growth

Question:
Is it possible for a finite initial pattern (finite number of live cells) to grow into an infinitely large population over infinite time? What is the name of the first such pattern discovered?

Solution:
Possibility: Yes. While the grid is infinite, the number of initial cells is finite. It was originally conjectured that all patterns stabilize or die, but this was disproved.

The Pattern: The first discovery was the **Gosper Glider Gun** (found by Bill Gosper in 1970). It is a finite stationary pattern that periodically emits Gliders. Since Gliders travel forever and add population to the grid as they travel away, the total population grows indefinitely (linear growth). Later, "Spacefillers" were found which grow quadratically.
