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Forward and Backward Euler Methods

Let's denote the time at the nth time-step by t_n and the computed solution at the nth time-step by y_n, i.e., $y_n \equiv y(t = t_n)$. The step size h (assumed to be constant for the sake of simplicity) is then given by $h = t_n - t_{n-1}$. Given (t_n, y_n), the forward Euler method (FE) computes y_{n+1} as

$$y_{n+1} = y_n + hf(y_n, t_n), \quad \text{(Explicit) Forward Euler Method.} \tag{6}$$

The forward Euler method is based on a truncated Taylor series expansion, i.e., if we expand y in the neighborhood of t=t_n, we get

$$y(t_n + h) \equiv y_{n+1} = y(t_n) + h\frac{dy}{dt}|_{t_n} + \mathrm{O}(h^2) = y_n + hf(y_n, t_n) + \mathrm{O}(h^2). \tag{7}$$

From (8), it is evident that an error is induced at every time-step due to the truncation of the Taylor series, this is referred to as the *local truncation error* (LTE) of the method. For the forward Euler method, the LTE is O(h^2). Hence, the method is referred to as a *first order* technique. In general, a method with O(h^{k+1}) LTE is said to be of kth order. Evidently, higher order techniques provide lower LTE for the same step size. The truncation error is different from the *global* error g_n, which is defined as the absolute value of the difference between the true solution and the computed solution, i.e., $g_n = |y^e(t_n)-y_n|$. In most cases, we do not know the exact solution and hence the global error is not possible to be evaluated. However, if we neglect roundoff errors, it is reasonable to assume that the global error at the nth time step is n times the LTE, since n is proportional to $1/h$, g_n should be proportional to LTE/h. This implies that for a kth order method, the global error scales as h^k.

A *convergent* numerical method is the one where the numerically computed solution approaches the exact solution as the step size approaches 0. Once again, if the true solution is not known a priori, we can choose, depending on the precision required, the solution obtained with a sufficiently small time step as the 'exact' solution to study the convergence characteristics.

Another important observation regarding the forward Euler method is that it is an *explicit* method, i.e., y_{n+1} is given explicitly in terms of known quantities such as y_n and $f(y_n,t_n)$. Explicit methods are very easy to implement, however, the drawback arises from the limitations on the time step size to ensure *numerical stability*. In order to see this better, let's examine a linear IVP, given by $dy/dt = -ay$, $y(0)=1$ with $a>0$. As we know, the exact solution $y^e = \exp(-at)$, which is a stable and a very smooth solution with $y^e(0) = 1$ and $y^e(\infty) = 0$. Now, what is the discrete equation obtained by applying the forward Euler method to this IVP? Using Eq. 7, we get

$$y_{n+1} = y_n - ah\, y_n = (1-ah)\, y_n = (1-ah)^2\, y_{n-1} = \ldots = (1-ah)^n\, y_1 = (1-ah)^{n+1}\, y_0. \tag{8}$$

Eq. 9 implies that in order to prevent the amplification of the errors in the iteration process, we require $|1-ah| < 1$ or for stability of the forward Euler method, we should have $h<2/a$.

These results can be better perceived from Figures 1 and 2. The test problem is the IVP given by $dy/dt = -10y$, $y(0)=1$ with the exact solution $y = \exp(-10t)$. The stability criterion for the forward Euler method requires the step size h to be less than 0.2. In Figure 1, we have shown the computed solution for h=0.001, 0.01 and 0.05 along with the exact solution[1]. As seen from there, the method is numerically stable for these values of h and becomes more accurate as h decreases. However, based on the stability analysis given above, the forward Euler method is stable only for $h < 0.2$ for our test problem. The numerical instability which occurs for $h \geq 0.2$ is shown in Figure 2. For h =0.2, the instability is oscillatory between $y = \pm 1$, whereas for h>0.2, the amplitude of the oscillation grows in time without bound, leading to an explosive numerical instability.

Figure 1: Convergence of the FE method: dt = 0.05, 0.01 & 0.001

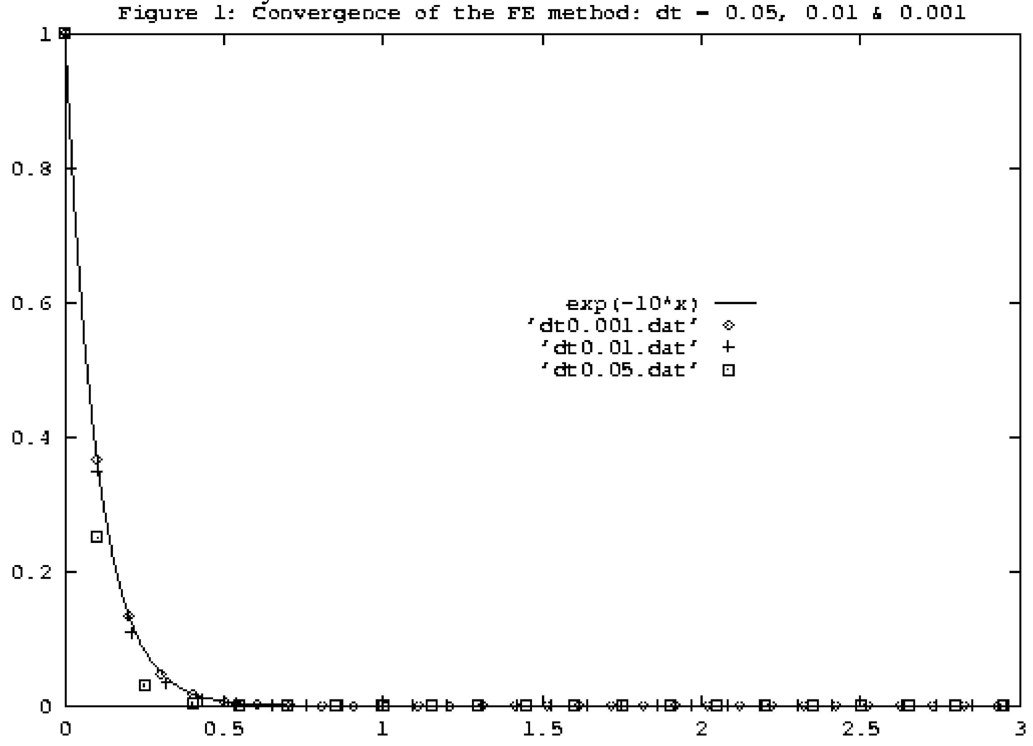
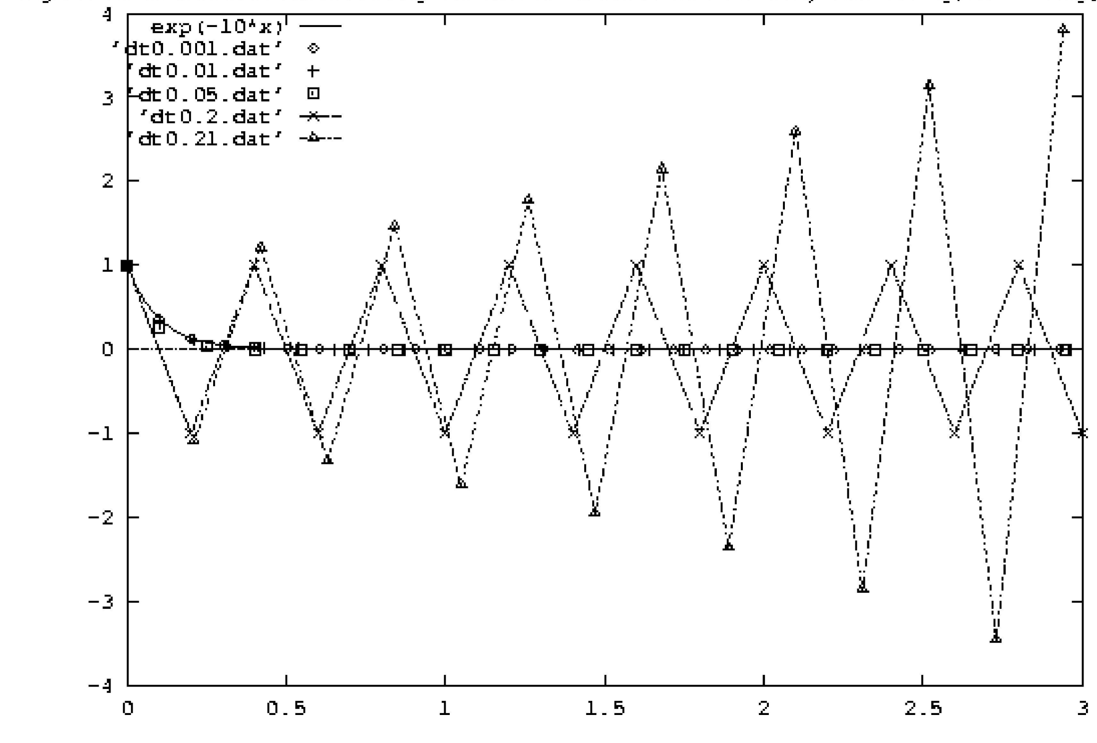
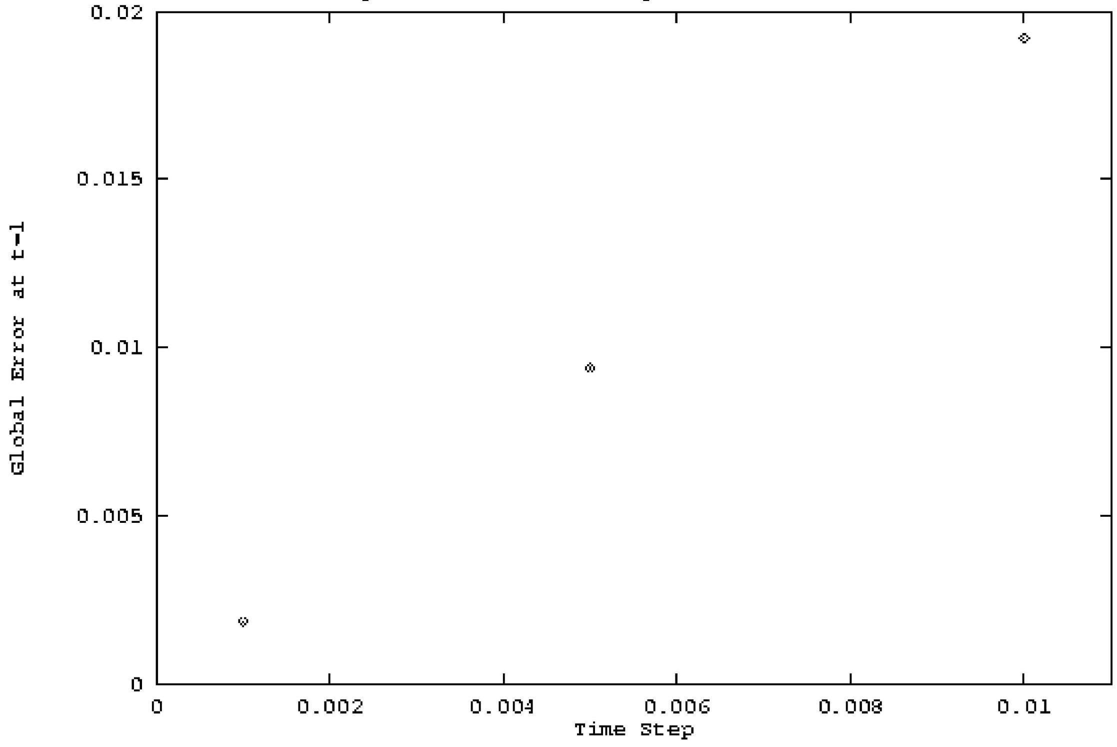


Figure 2: Numerical instability in the FE method for dt>0.2; IVP is dy/dx = -10y, y(0



The convergence of the solution can be analyzed quantitatively. Let's look at the global error $g_n = |y^e(t_n) - y(t_n)|$ for our test problem at t=1. We know that the local truncation error (LTE) at any given step for the Euler method scales with h^2. Hence, the global error g_n is expected to scale with nh^2. However, for the integration within a fixed time interval, n is proportional to $1/h$. So the global error g_n at the nth Euler step is proportional to h. This result is confirmed by the computational results presented in Figure 3, where the global error at t=1 is plotted against the time step size h.

Figure 3: Linear Convergence of the FE method



The *conditional stability*, i.e., the existence of a critical time step size beyond which numerical instabilities manifest, is typical of *explicit* methods such as the forward Euler technique. *Implicit* methods can be used to replace explicit ones in cases where the stability requirements of the latter impose stringent conditions on the time step size. However, implicit methods are more expensive to be implemented for non-linear problems since y_{n+1} is given only in terms of an implicit equation. The implicit analogue of the explicit FE method is the backward Euler (BE) method. This is based on the following Taylor series expansion

$$y_n \equiv y(t_{n+1} - h) = y(t_{n+1}) - h\frac{dy}{dt}|_{t_{n+1}} + \mathrm{O}(h^2), \tag{9}$$

which gives

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1}), \quad \text{(Implicit) Backward Euler Method.} \tag{10}$$

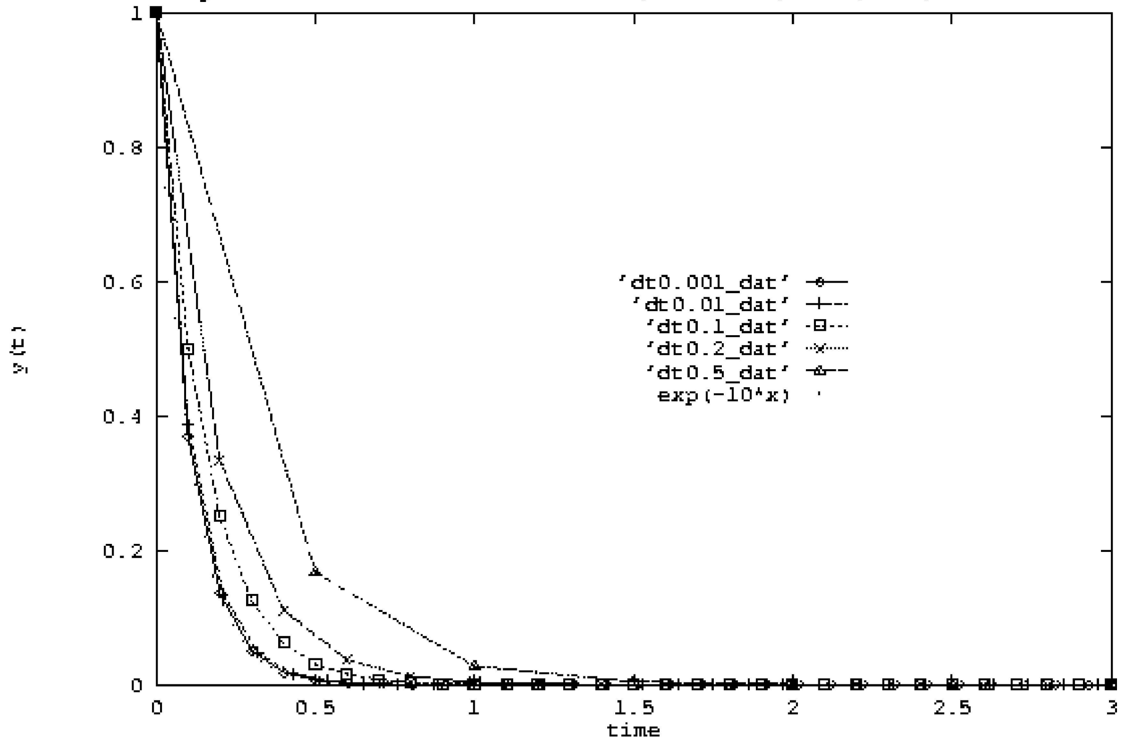
Once again, note that in Eq. 11, $f(y_{n+1}, t_{n+1})$ is not known, hence it gives us an implicit equation for the computation of y_{n+1} (Compare Eqs. 7 and 11). For instance, let $f(y,t) \equiv p(y) = y\cos(y)$. This means that to obtain y_{n+1}, we need to solve the non-linear equation $y_{n+1} - hy_{n+1}\cos(y_{n+1}) = y_n$ at any given time step n. A suitable root finding technique such as the Newton-Raphson method can be used for this purpose. This is evidently much more time consuming than the explicit FE method where, for the problem above, we have $y_{n+1} = y_n + hy_n\cos(y_n)$.

Well, why do we resort to implicit methods despite their high computational cost? The reason is that implicit techniques are stable. Let's examine this for the same linear test problem we considered in the context of the FE method: $dy/dt = -10\,y$, $y(0) = 1$. In the case of linear problems, using BE is as easy as using FE, applying Eq. 11, we have

$$y_{n+1} = \frac{y_n}{1+10h}, \tag{11}$$

which gives a numerical scheme stable for all h>0. In Figure 4, I have plotted the solutions computed using the BE method for h=0.001, 0.01, 0.1, 0.2 and 0.5 along with the exact solution. Note that there is no numerical instability in this case. The accuracy of the computed solution deteriorates as h is increased, and we expect the global error to scale linearly with h.

Figure 4: The backward Euler method, dt=0.001, 0.01, 0.1, 0.2 and 0.5



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