National University of Computer and Emerging Sciences

Lahore Campus

Discrete Structures (CS1005)

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Final Exam (Solution Key)

Total Time: 3 Hours

Total Marks: 100

Total Questions: 05

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Roll No Section Student Signature

CLO #1: Express statements in terms of predicates, quantifiers and logical connectives. Apply formal logic proofs, logical reasoning to practical problems related to offered program.

Q. No 1: **[5+10+5=20]**

a) Use quantifiers and predicates with more than one variable to express these statements.

i) Every CS student has used at least one lab this semester.

$\forall \boldsymbol{x}(\boldsymbol{S}(\boldsymbol{x}) \rightarrow \exists \boldsymbol{y}(\boldsymbol{L}(\boldsymbol{y}) \wedge \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y})))$

ii) There exists a student who has not used every lab on campus.

$\exists \boldsymbol{x}(\boldsymbol{S}(\boldsymbol{x}) \wedge \neg \forall \boldsymbol{y}(\boldsymbol{L}(\boldsymbol{y}) \rightarrow \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y})))$

$\boldsymbol{Or}\ \exists \boldsymbol{x}(\boldsymbol{S}(\boldsymbol{x}) \wedge \exists \boldsymbol{y}(\boldsymbol{L}(\boldsymbol{y}) \wedge \neg \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y})))$.

b) **Solution:** Use these propositional variables:

- BC: Lord Hazelton was killed by a blow on the head with a brass candlestick.
- C: The cook was in the kitchen at the time of the murder.
- L: Lady Hazelton was in the dining room.
- S: Sara (the maid) was in the dining room.
- B: the butler killed Lord Hazelton with a fatal dose of strychnine.
- Ch: the chauffeur killed Lord Hazelton.
- W: the wine steward killed Lord Hazelton.

Premises:

1. $BC \quad (Fact\ I)$
2. $L \vee S \quad (Fact\ II)$
3. $C \rightarrow B \quad (Fact\ III)$
4. $L \rightarrow Ch\ (Fact\ IV)$
5. $\neg C \rightarrow \neg S\ (Fact\ V)$
6. $S \rightarrow W \quad (Fact\ VI)$

Additional formalization from "there was only one cause of death": Since III explicitly says the butler's method (strychnine) is different from the brass-candlestick cause, we encode:

1. B→¬BC (Fact VII)

Derivation:

1. C→B — premise (3).
2. B→¬BC. — premise (7).
3. C→¬BC — from (1) and (2) by **Hypothetical Syllogism**
4. BC — premise (1).
5. ¬C— from (3) and (4) by **Modus Tollens**
6. ¬C→¬S — premise (5).
7. ¬S — from (5) and (6) by **Modus Ponens**.
8. L∨S — premise (2).
9. L — from (7) and (8) by **Disjunctive Syllogism**
10. L→Ch — premise (4).
11. Ch . — from (9) and (10) by **Modus Ponens**.

Thus Ch (the chauffeur killed Lord Hazelton) is derived.

c) Prove, using proof by cases, that for any integer n, the integer $n^2 + 2n$ has the same parity as n.
Proof ($By\ Cases\ on\ the\ Parity\ of\ n$)
We consider two exhaustive cases:
$\boldsymbol{Case\ 1: n\ is\ even}$
$If\ n\ is\ even, then\ there\ exists\ an\ integer\ k\ such\ that$
$\boldsymbol{n = 2k}$
$Compute\ n^2 + 2n$:
$n^2 + 2n = (2k)^2 + 2(2k) = 4k^2 + 4k = 4(k^2 + k)$

$Since\ 4(k^2 + k)\ is\ a\ multiple\ of\ 4, it\ is\ even.$
$Thus, when\ n\ is\ even, n^2 + 2n\ is\ also\ even.$
$So\ they\ have\ the\ same\ parity.$

$\boldsymbol{Case\ 2: n\ is\ odd}$
$If\ n\ is\ odd, then\ there\ exists\ an\ integer\ k\ such\ that$
$\boldsymbol{n = 2k + 1}$
$Compute\ n^2 + 2n$:
$n^2 + 2n = (2k + 1)^2 + 2(2k + 1)$
$= (4k^2 + 4k + 1) + (4k + 2)$
$= 4k^2 + 8k + 3$
$= 2(2k^2 + 4k + 1) + 1.$

$The\ expression\ is\ of\ the\ form\ 2m + 1, which\ is\ odd.$
$Thus, when\ n\ is\ odd, n^2 + 2n\ is\ also\ odd.$

$So\ they\ have\ the\ same\ parity.$

CLO #2: Apply fundamental concepts of number theory, such as divisibility, greatest common divisors, modular arithmetic, prime numbers, and congruences and cardinality of sets.

Q. No. 2: **[5+5+10=20]**

a) Determine whether the sets of integers divisible by 5 but not by 3 is countable or uncountable. If countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

b) Use extended Euclidean algorithm to express $gcd(521,345)$ as linear combination of 521 and 345.
Solution:
We perform the Euclidean algorithm:
$521 = 1 \cdot 345 + 176$
$345 = 1 \cdot 176 + 169$
$176 = 1 \cdot 169 + 7$
$169 = 24 \cdot 7 + 1$
$7 = 7 \cdot 1 + 0$

$$So:\ gcd(521,345) = 1.$$

Back-substitution
$1 = 169 - 24 \cdot 71$
$1 = 169 - 24(176 - 169)$
$= 169 - 24.176 + 24.169$
$= 25.169 - 24.176$
$= 25\,(345 - 176) - 24\,.176$
$= 25.345 - 25\,.176 - 24.176$
$= 25\,.345 - 49\,.176$
$= 25\,.345 - 49\,(521 - 345)$
$= 25.345 - 49.521 + 49\,.345$
$= 74.345 - 49\,.521$

Final Answer

$$1 = 74 \cdot 345 - 49 \cdot 521$$

c) Use Chinese remainder theorem to find all solutions to the system of congruences

$$x \equiv 2\ (mod\ 3)\ , x \equiv 1\ (mod\ 4), and\ x \equiv 3\ (mod\ 5)$$

Solution:
Step 1 — Compute the modulus

$$M = 3 \cdot 4 \cdot 5 = 60.$$

Step 2 — Compute each partial modulus
$M_1 = \frac{M}{3} = 20, M_2 = \frac{M}{4} = 15, M_3 = \frac{M}{5} = 12$
Step 3 — Find inverses
We must find y_i such that:
- $20y_1 \equiv 1(mod3)$
- $20(2) \equiv 1(mod3)$
  $y_1 = 2$
- $15y_2 \equiv 1(mod4)$
- $15(3) \equiv 1(mod4)$
  $y_2 = 3$

- $12y_3 \equiv 1(mod5)$
- $12(3) \equiv 1(mod5)$
  $y_3 = 3$

Step 4 — Apply CRT formula

$$x \equiv a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3(mod60)$$
$$x \equiv (80 + 45 + 108)(mod\ 60)$$
$$= 233\ mod\ 60 = 53.$$

So, $x \equiv 53(mod60)$
$\boldsymbol{x = 53 + 60k, k \in Z.}$

CLO #3: Apply mathematical induction to prove properties of sequences, Learn about recursive relations, relations and their properties.

Q. No. 3: **[5+10+5=20]**

a) Find a recurrence relation for the number of bit strings of length *n* that does not contain a pair of consecutive 0s. What are the initial conditions? How many bit strings of length seven contain no consecutive 0s?

Solution: The bit strings of length *n* ending with 1 that do not have two consecutive 0s are precisely the bit strings of length *n* − 1 with no two consecutive 0s with a 1 added at the end. Consequently, there are a_{n-1} such bit strings. Bit strings of length *n* ending with a 0 that do not have two consecutive 0s must have 1 as their (*n* − 1)st bit; otherwise, they would end with a pair of 0s. Hence, the bit strings of length *n* ending with a 0 that have no two consecutive 0s are precisely the bit strings of length *n* − 2 with no two consecutive 0s with 10 added at the end. Consequently, there are a_{n-2} such bit strings.

$$a_n = a_{n-1} + a_{n-2}\ for\ n \geq 3.$$

2. Initial Conditions

We compute small values directly:
The initial conditions are $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11.

3. Number of bit strings of length 7 with no consecutive 0s

Use the recurrence:
$a_1 = 2, \quad a_2 = 3$
$a_3 = a2 + a1 = 3 + 2 = 5$
$a_4 = a3 + a2 = 5 + 3 = 8$
$a_5 = a4 + a3 = 8 + 5 = 13$
$a_6 = a5 + a4 = 13 + 8 = 21$
$a_7 = a6 + a5 = 21 + 13 = 34$

b) Solve these recurrence relations together with the initial conditions given.

$$a_n = -4a_{n-1} + 5a_{n-2}\ ,\quad for\ n \geq 3, \qquad a_0 = 2\ \ ,\ a_1 = 8$$

Solution:
The Characteristic Equation is

$$r^2 + 4r - 5 = 0$$
$$(r + 5)(r - 1) = 0$$

$$r_1 = -5, \qquad r_2 = 1$$

General Solution:

$$a_n = A\,(1)^n + B\,(-5)^n$$

Use Initial Conditions:

$$For\ n = 0\ ;\ a_n = A(1)^0 + B(-5)^0$$
$$= A + B = 2$$
$$For\ n = 1;\ a_n = A(1)^1 + B(-5)^1$$
$$= A - 5B = 8$$

Solving the System

$$A + B = 2$$
$$A - 5B = 8$$
$$(A - 5B) - (A + B) = 8 - 2$$
$$\Longrightarrow -6B = 6$$
$$\Longrightarrow B = -1$$
$$A + (-1) = 2$$
$$\Longrightarrow A = 3$$

The solution is

$$a_n = 3\,(1)^n - 1\,(-5)^n$$

Or

$$a_n = 3 - 1\,(-5)^n$$

c) Determine whether the relation R on the set $A = \{0,1,2,3,4\}$ is reflexive, symmetric, antisymmetric, and/or transitive $where\ R = \{(\boldsymbol{a}, \boldsymbol{b})\ \in\ \boldsymbol{AxA}\ if\ and\ only\ if\ \boldsymbol{ab}\ \geq\ \boldsymbol{1}\}$.

Also, represent the matrix of the given relation.

Step 1: Determine the pairs in R:

$R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

Step 2: Check properties

1. Reflexive:

A relation is reflexive $(a, a) \in R, for\ all\ a\ \in A$.

- $(0,0) \notin R\ \rightarrow$ **not reflexive**

2. Symmetric:

A relation is symmetric if $(a, b) \in R \Longrightarrow (b, a) \in R$.

$If\ if\ (a, b) \in R$

- $ab \geq 1\ \leftrightarrow\ ba\ \geq 1$
- $\Longrightarrow (b, a) \in R$
- So **symmetric** ☑

3. Antisymmetric:

A relation is antisymmetric $if\ (a, b) \in R\ and\ (b, a) \in R\ \rightarrow\ a = b$

- $Counterexample$: $(1,2) \in R, and\ (2,1) \in R, but\ 1 \neq 2 \rightarrow$ **not antisymmetric**

4. Transitive:

A relation is transitive $if\ (a, b) \in R\ and\ (b, c) \in R\ \rightarrow\ (a, c) \in R$

$if\ (a, b) \in R\ and\ (b, c) \in R$

- $ab \geq 1\ and\ bc \geq 1$ for all $a, b, c \geq 1$

- $As\ a \geq 1, b \geq 1, c \geq 1$
- $\rightarrow\ ac \geq 1$
- $\Longrightarrow (a, c) \in R$
- So **transitive** ☑

Step 3: Represent the matrix of R

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

CLO #4: Apply fundamental counting principles to solve combinatorial problems.

Q. No. 4: **[5+5+10=20]**

a) 8 processes are assigned to 3 CPU registers. Find the least number of processes that some register must get? How many ways to assign processes to registers in order?

Solution:
You have 8 processes and 3 registers.

By the pigeonhole principle:

$\left\lceil \frac{8}{3} \right\rceil$ **= 3** So at least one register must receive at least 3 processes.

Ways to assign processes = $\mathbf{3^8 = 6561}$

b) A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes contain at most three tails?

Possible Outcomes = $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}$
$= 1 + 10 + 45 + 120$
$= 176.$

c) Use the binomial theorem to expand $(5x^2 + 2y^3)^4$

$$(5x^2 + 2y^3)^4 = \sum_{k=0}^{4} \binom{4}{k} (5x^2)^{4-k} (2y^3)^k$$

$$= \binom{4}{0}(5x^2)^4 + \binom{4}{1}(5x^2)^3(2y^3) + \binom{4}{2}(5x^2)^2(2y^3)^2 + \binom{4}{3}(5x^2)(2y^3)^3 + \binom{4}{4}(2y^3)^4$$

$$= 625x^8 + 1000x^6y^3 + 600x^4y^6 + 160x^2y^9 + 16y^{12}.$$

✔ **Final answer:**

$$\boxed{625x^8 + 1000x^6y^3 + 600x^4y^6 + 160x^2y^9 + 16y^{12}}$$

CLO #5: Express statements in terms of predicates, quantifiers and logical connectives. Apply formal logic proofs, logical reasoning to practical problems related to offered program.

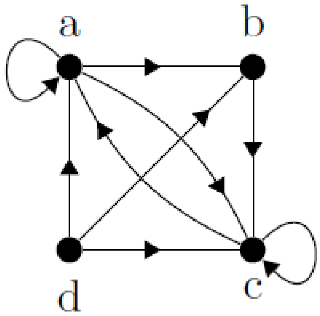
Q. No. 5: **[5+5+10=20]**

a) Draw a graph with the given adjacency matrix. Also, write the name of the graph.

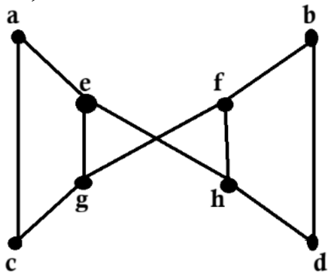
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution:

Graph: Directed Multigraph



b) Define the term Bipartite Graph. Also, determine whether the given graph is bipartite.



Solution:

Bipartite Graph:

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

Graph: Graph is Bipartite as

$\boldsymbol{Set\ 1}$: $V_1 = \{a, g, h, b\}$

$\boldsymbol{Set\ 2}$: $V_2 = \{e, c, f, d\}$

c) Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

Solution:

- Vertices of G and H = 5
- Edges of G and H= 7
- Both G and H have the same degree sequence: (4,3,3,2,2)

Structure:

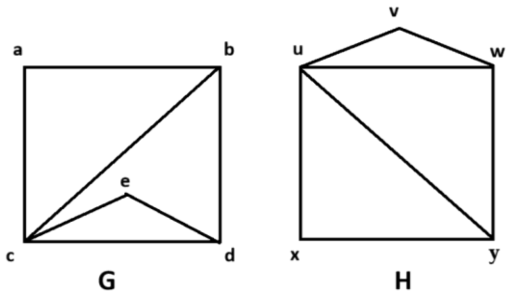
In G

- Vertex **c** has degree **4** (connected to a, d, b, e).
  → Only vertex of degree 4.
- Vertex **u** in H also has degree **4** (connected to v, w, x, y).
  → Only vertex of degree 4.

So: $\boldsymbol{f(c) = u}$

We must match vertices preserving triangle structures.

Notice triangles:



In **G**:

- Triangle: c−b−d−c
- Triangle: c−e−d−c

In **H**:

- Triangle: u−y−x−u
- Triangle: u−v−w−u

Thus:

Triangle 1 in G ↔ Triangle 1 in H (c−b−d−c ↔ u−y−x−u)

$$So \quad \boldsymbol{f(b) = y}$$

$$\boldsymbol{f(d) = w}$$

Triangle 2 in G ↔ Triangle 2 in H (c −e−d−c ↔ u−v−w−u)

$$Thus: \quad \boldsymbol{f(e) = v}$$

The remaining vertex

- In G: a (a adjacent to c and b)
- In H: x (x adjacent to u and y)

This matches perfectly. $\boldsymbol{f(a) = x}$

Mapping

$$\boldsymbol{c} \leftrightarrow \boldsymbol{u}$$
$$\boldsymbol{b} \leftrightarrow \boldsymbol{y}$$
$$\boldsymbol{d} \leftrightarrow \boldsymbol{w}$$
$$\boldsymbol{e} \leftrightarrow \boldsymbol{v}$$
$$\boldsymbol{a} \leftrightarrow \boldsymbol{x}$$