CS-2001
Data Structures
Spring 2022
Binary Tree and Tree ADT

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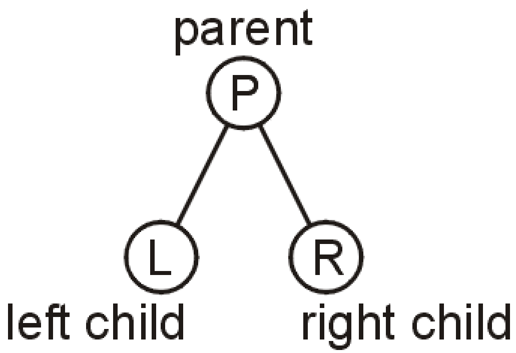
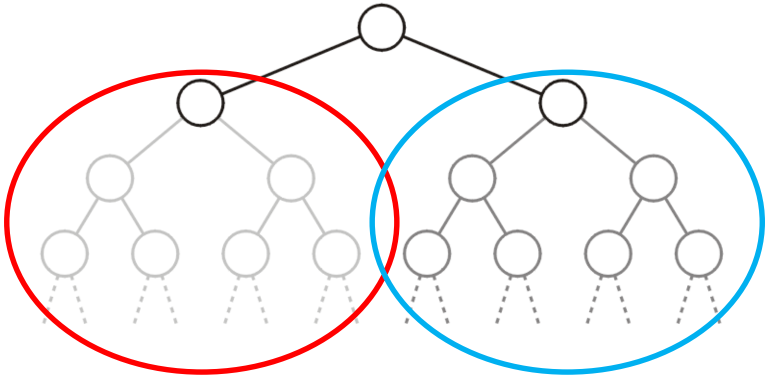
Binary Tree

- In a binary tree each node has at most two children
  - Allows to label the children as left and right

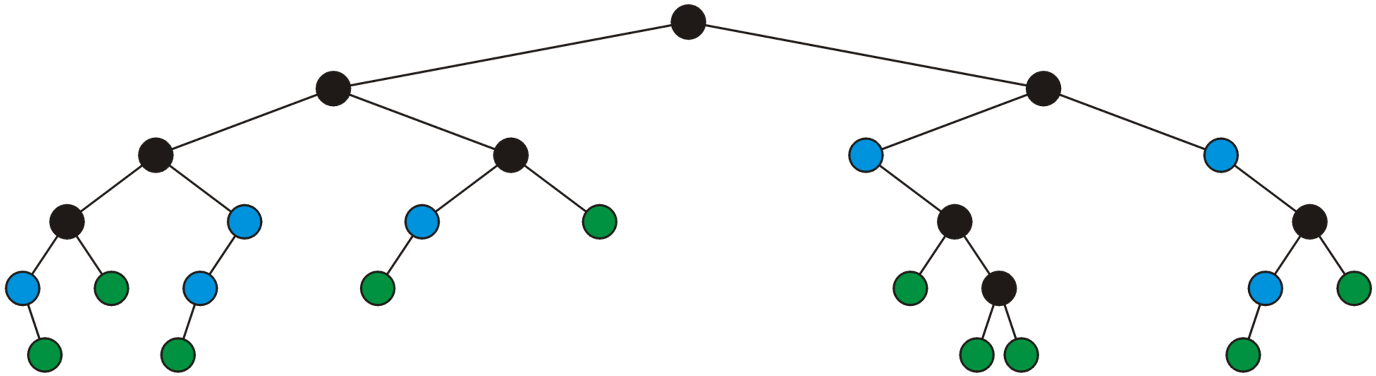


- Likewise, the two sub-trees are referred as
  - Left sub-tree
  - Right sub-tree



Binary Tree: Full **Node**

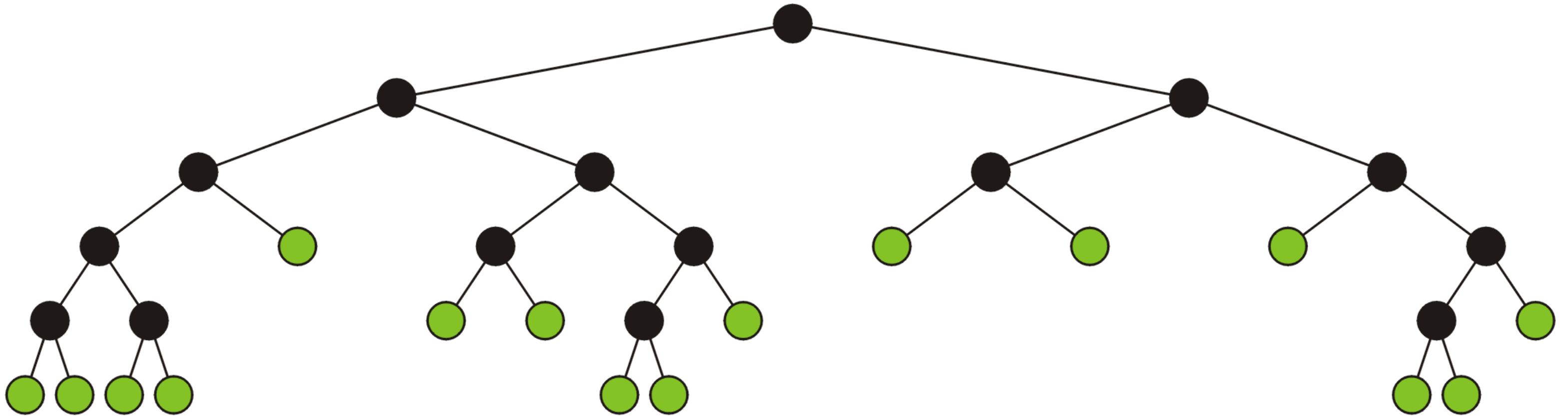
- A full node is a node where both the left and right sub-trees are non-empty trees
- (OR) if it has exactly two child nodes



full nodes neither leaf nodes

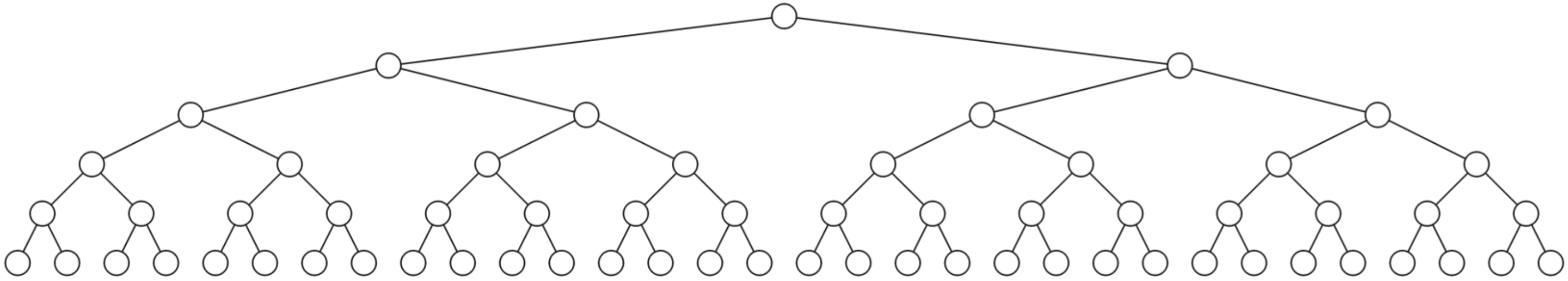
Full Binary Tree

- A full binary tree is where each node is:
  - A full node, or
  - A leaf node
- Full binary tree is also called **proper binary tree**, **strictly binary tree** or **2-tree**



Perfect Binary Tree

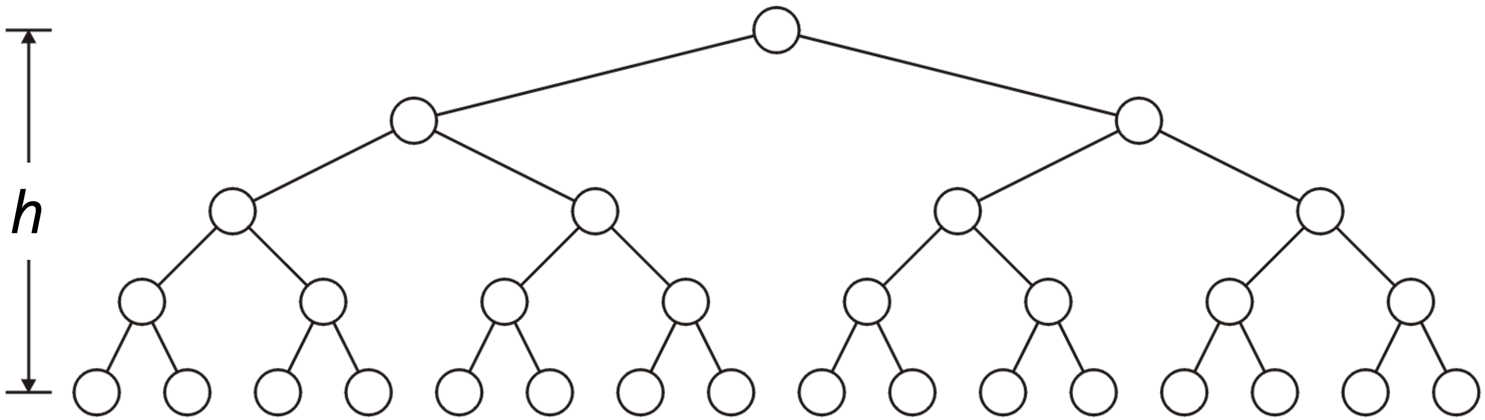
- A perfect binary tree of height h is a binary tree where
  - All leaf nodes have the same depth or level *L*
  - All other nodes are full-nodes



Binary Tree: Properties (3)

- A perfect binary tree with height h has 2^h leaf nodes

- A perfect binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$



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  - Total number of nodes: $2L-1 = 2^{h+1} - 1$

- A perfect binary tree with n nodes has **height** $\log_2(n + 1) - 1$

$$n = 2^{h+1} - 1$$

$$2^{h+1} = n + 1$$

$$h + 1 = \log_2(n + 1)$$

$$\Rightarrow h = \log_2(n + 1) - 1$$

Proof – Total Nodes of a Perfect Binary Tree

- Geometric Progression (finite)
  - $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}$
- Sum of Geometric Progression
  - $Sn = a + ar + ar^2 + ar^3 + ar^4 + \ldots. + ar^{n-1}$
  - $Sn = a\left[\frac{(r^n-1)}{r-1}\right]$ if r>1 and r!=1

Trees

- Total no. of nodes = sum of internal + external nodes
- Internal nodes = $2^h - 1$
- External nodes = 2^h
- Total nodes = $2^h - 1 + 2^h$

$= 2^h + 2^h - 1$; let say; a = 2^h

$= a + a - 1$

$= 2a + 1$

$= 2.2^{h+1} - 1$; replacing a with 2^h

$= 2^{h+1} - 1$

Proof – Total Nodes of a Perfect Binary Tree

- Sum of finite Geometric Progression
  - $Sn = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-1} = a\left[\frac{(r^n - 1)}{r-1}\right]$ -----------Eq. (1)
- Series we have in perfect binary trees
  - $Total\ nodes = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 \ldots\ldots + 2^h$
  - $Total\ nodes = 1.2^0 + 1.2^1 + 1.2^2 + 1.2^3 + 1.2^4 \ldots\ldots + 1.2^h$
  - a = 1, r = 2 and
  - n-1 = h OR n=h+1

  Putting above values in in the formula of Eq.(1)
  - $Total\ nodes = a\left[\frac{(r^n - 1)}{r-1}\right]$
  - $Total\ nodes = 1 . \left[\frac{(2^{h+1} - 1)}{2-1}\right]$
  - $Total\ nodes = 2^{h+1} - 1$

Binary Tree: Properties (4)

- A perfect binary tree with height h has 2^h leaf nodes

- A perfect binary tree of height h has $2^{h+1} - 1$ nodes
  - Number of leaf nodes: $L = 2^h$
  - Number of internal nodes: $2^h - 1$
  - Total number of nodes: $2L-1 = 2^{h+1} - 1$

- A perfect binary tree with n nodes has height $\log_2(n + 1) - 1$

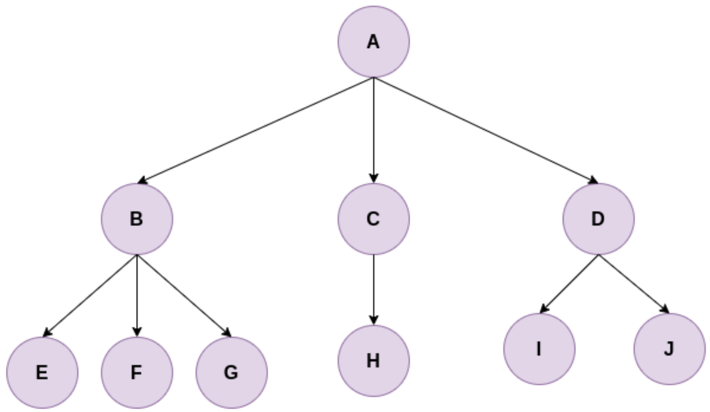
- **Number n of nodes in a binary tree of height h is at least h+1 and at most $2^{h+1} - 1$**

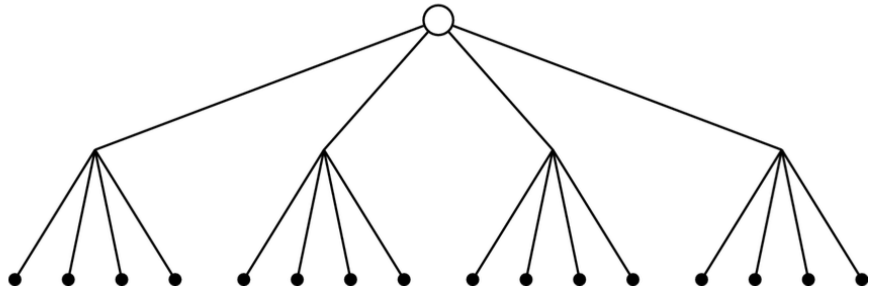
n-ary Trees

- What are n-ary trees?
- Strict n-ary trees
- Height vs Nodes?
- Internal vs External nodes

3-ary tree VS 4-ary trees

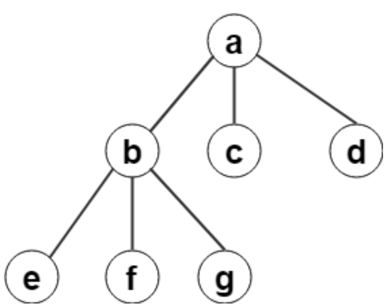
- 3-ary tree: {0,1,2,3}
- 4-ary tree: {0,1,2,3,4}

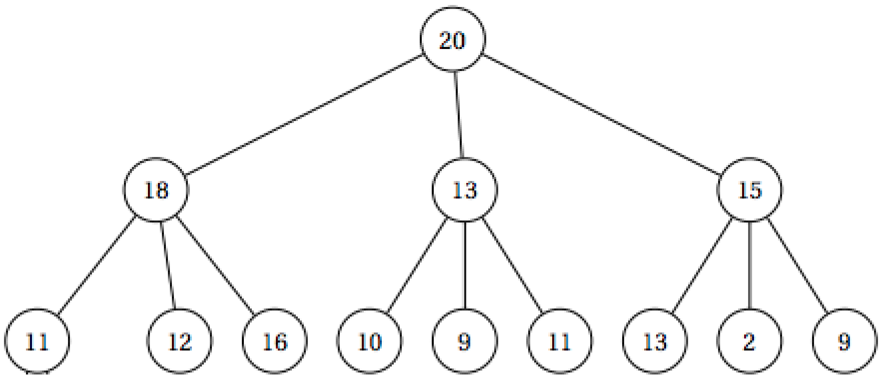




Strict/Full 3-ary Trees

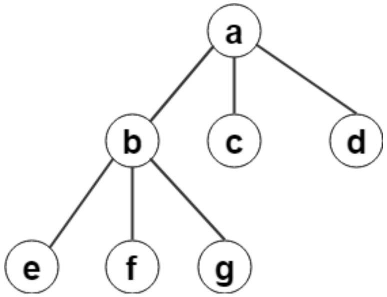
- Height given, what will be the no. of nodes?
- Min nodes: 3x3+1 = 7
- Min nodes: base*height+1 OR m*h+1
- Max nodes: 3^0+ 3^1+ 3^2+3^3……3n = $\frac{m^{h+1}-1}{m-1}$

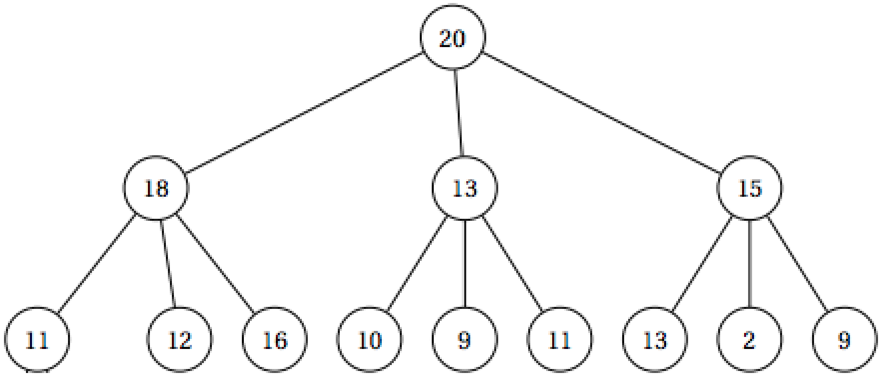




Strict/Full 3-ary Trees

- Nodes given, what will be the height?
- Max height:
  - $n = m*h+1 \Rightarrow h = (n-1)/m$
- Min height:
  - $n = \frac{m^{h+1}-1}{m-1}$
  - Find h??

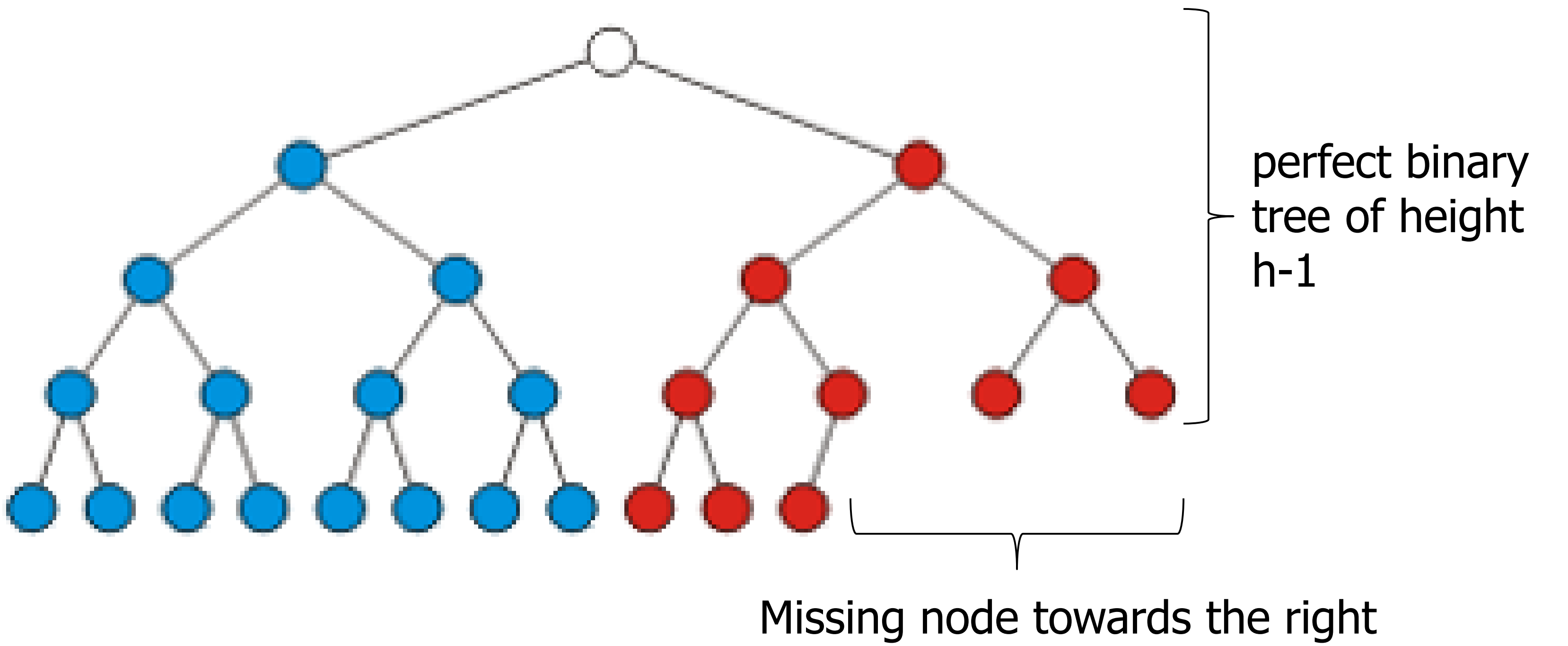




[Almost] Complete Binary tree

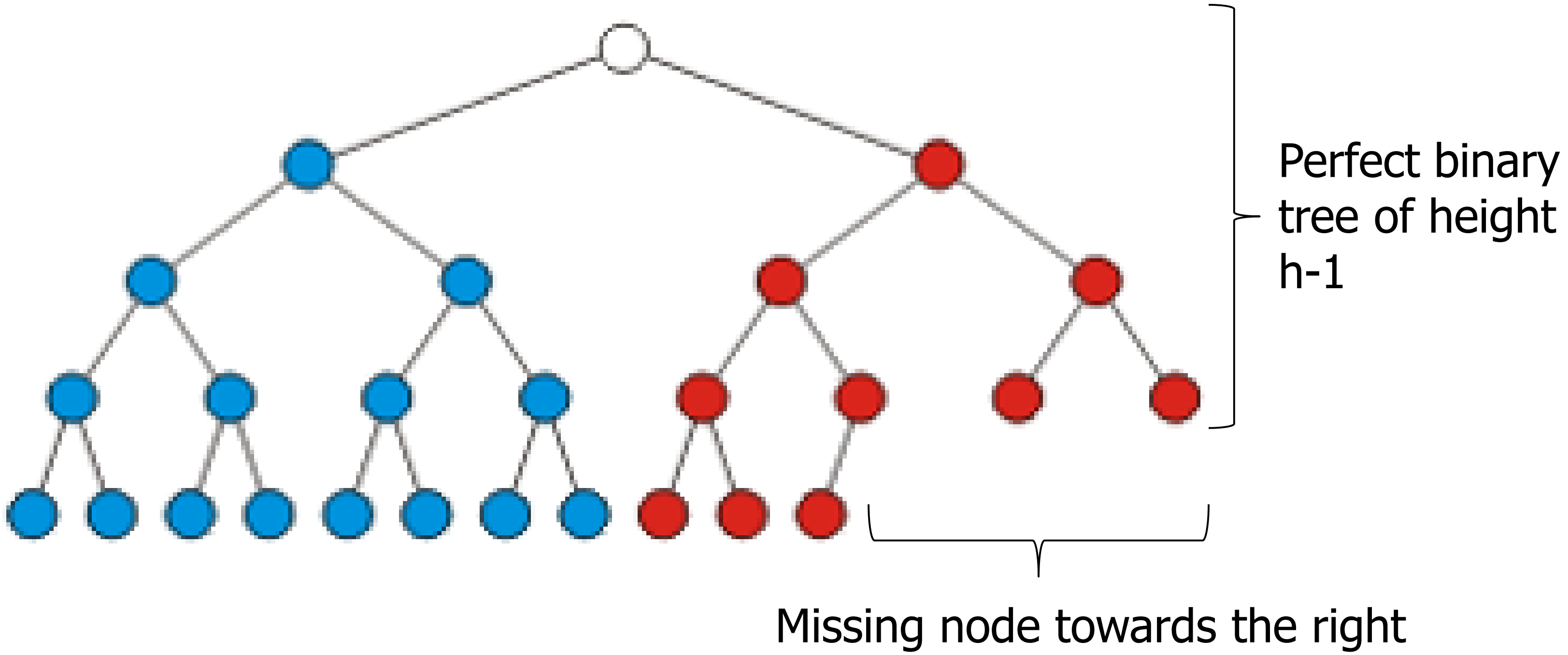
Almost (or Nearly) Complete Binary Tree

- Almost complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth `d` for `d = 1,2,...,h−1`
     - Each leaf in the tree is either at level `h` or at level `h - 1`
  2. The nodes at depth `h` are as far left as possible



Complete Binary Tree

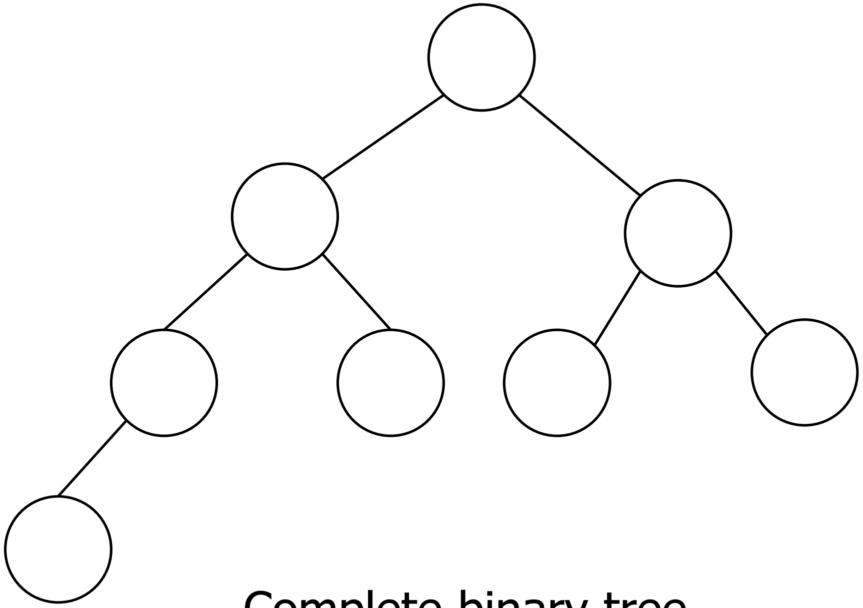
- Complete binary tree of height h is a binary tree in which
  1. There are 2^d nodes at depth `d` for `d = 1,2,...,h−1`
     - Each leaf in the tree is either at level `h` or at level `h - 1`
  2. **The nodes at depth h are as far left as possible (Formal ?)**



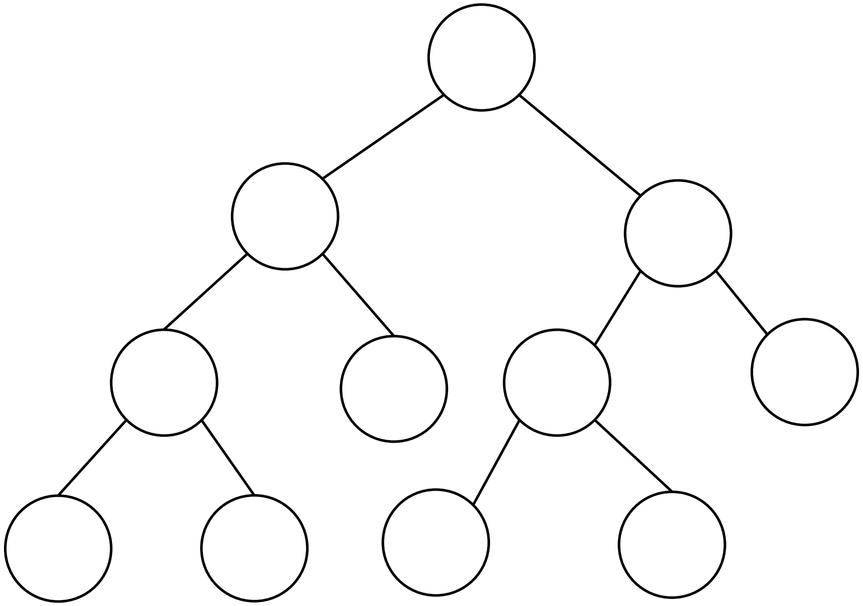
Complete Binary Tree

Condition 2: The nodes at depth h are as far left as possible

- If a node p at depth h−1 has a left child
  - Every node at depth h−1 to the left of *p* has 2 children
- If a node at depth h−1 has a right child
  - It also has a left child

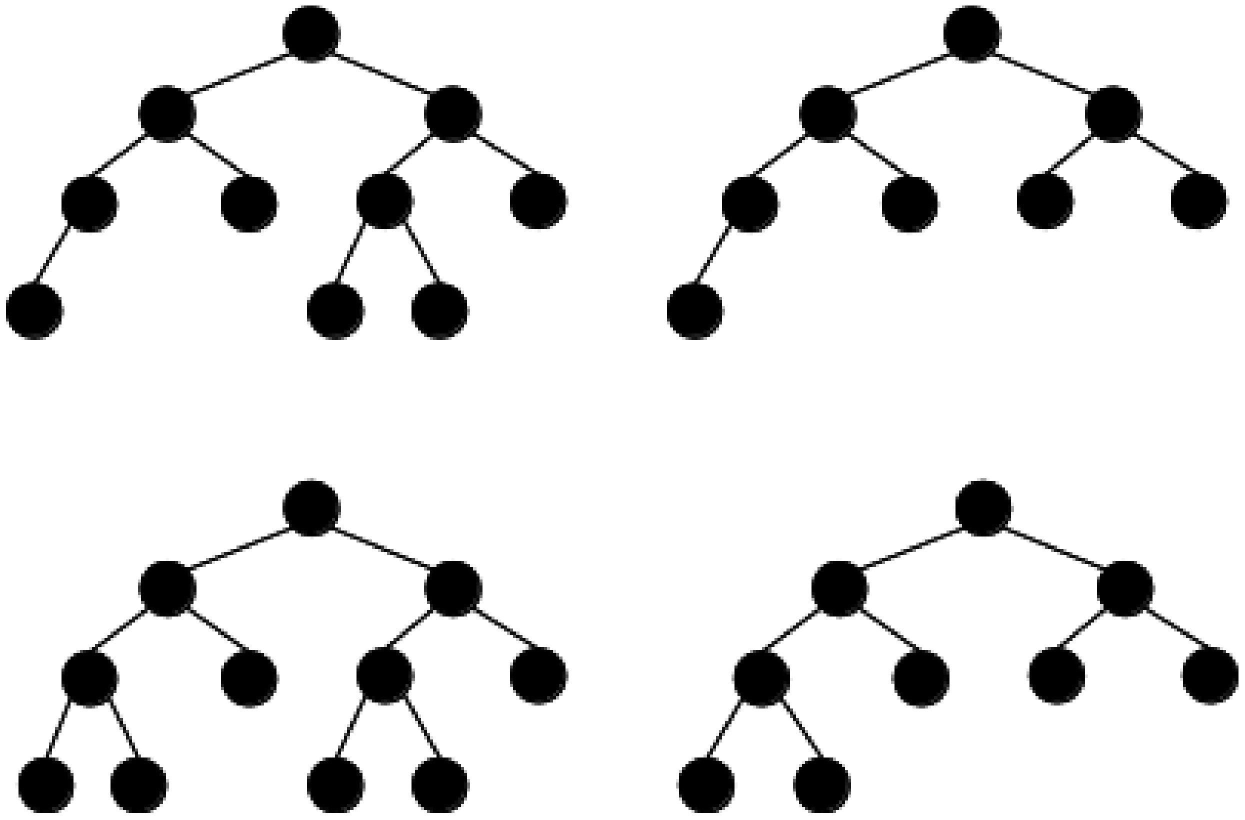


Complete binary tree

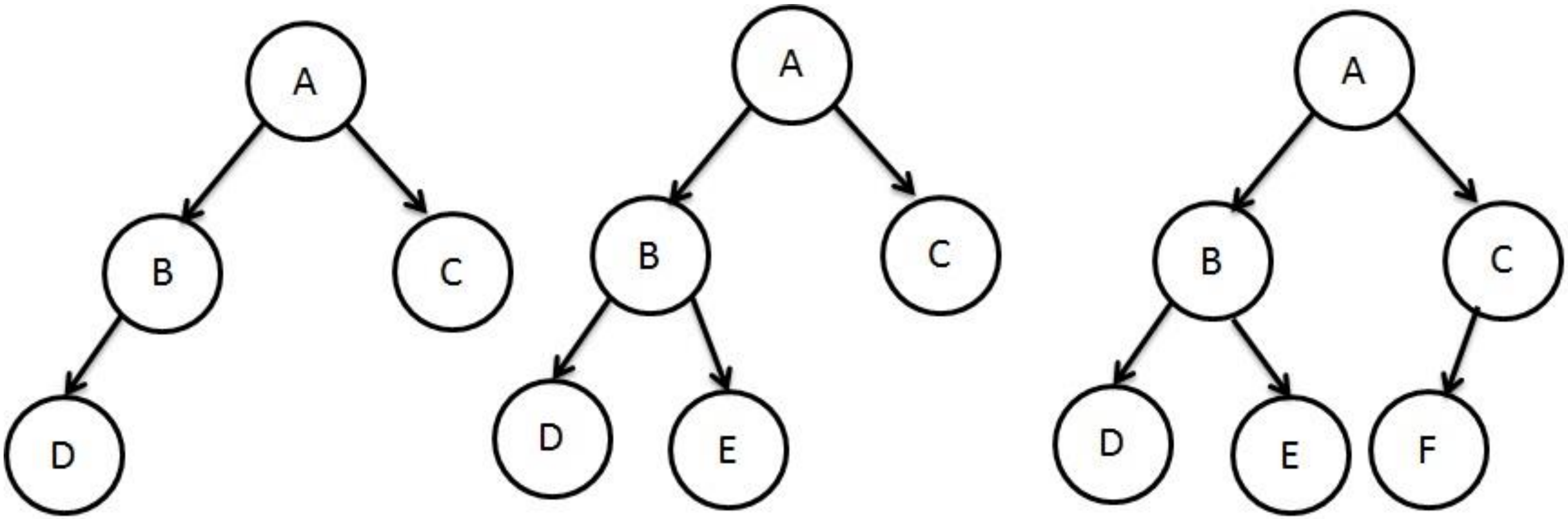


Not a Complete binary tree
(condition 2 violated)

Full vs. Complete Binary Tree



Complete Binary Trees…



What is the height and number of nodes for each tree?

Complete Binary Tree: Properties

- Total number of nodes `n` are between
  - **At least:** perfect binary tree of height `h-1 + 1 (i.e., 1 node in the next level)` → $2^h - 1 + 1 = 2^h$ nodes
  - **At most:** perfect binary tree of height `h`, i.e., $2^{h+1} - 1$ nodes

- Height `h` is equal to $\lfloor Log_2(n) \rfloor$

Balanced Binary Tree

- Balanced binary tree
  - For each node, the difference in height of the right and left sub-trees is no more than one
  - Both Perfect binary trees and complete binary trees are balanced as well

- Completely balance binary tree
  - Left and right sub-trees of every node have the same height
  - A perfect binary tree is completely balanced

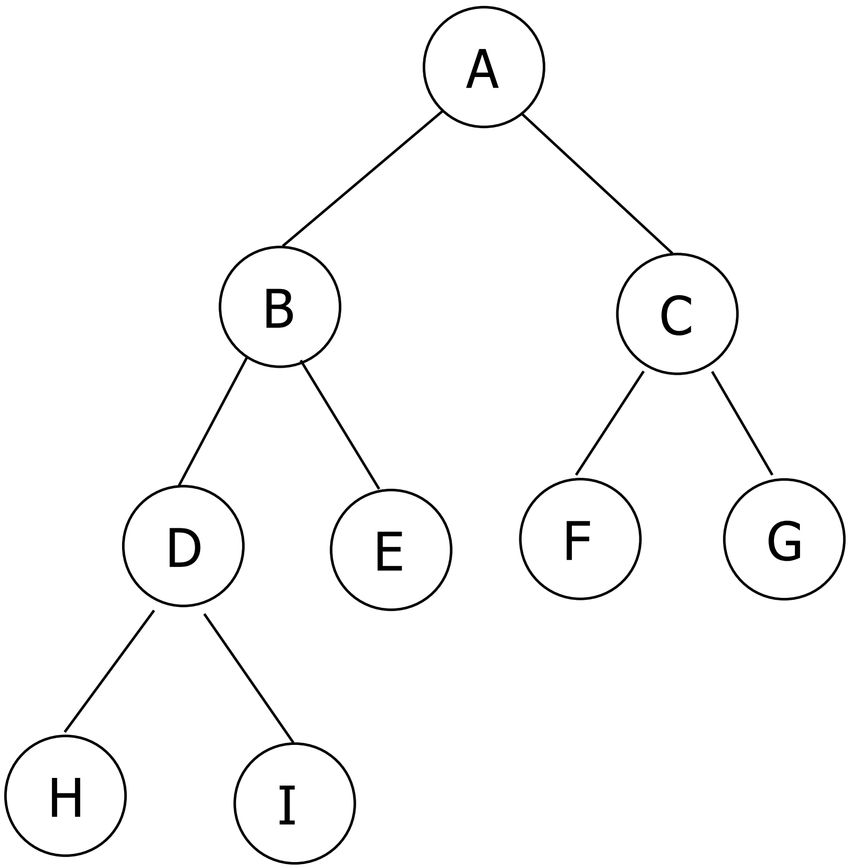
Tree ADT

Binary Tree Storage

- Contiguous storage
- Linked-list based storage

Contiguous Storage

Array Storage Example (1)



[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Array Storage Example (1)

Element	Index	Left-Child	Right-Child
A	1	2	3
B	2	2*2	2*2+1
C	3	2*3	2*3+1
.			
.			
.			
Node	i	2*i	2*i+1

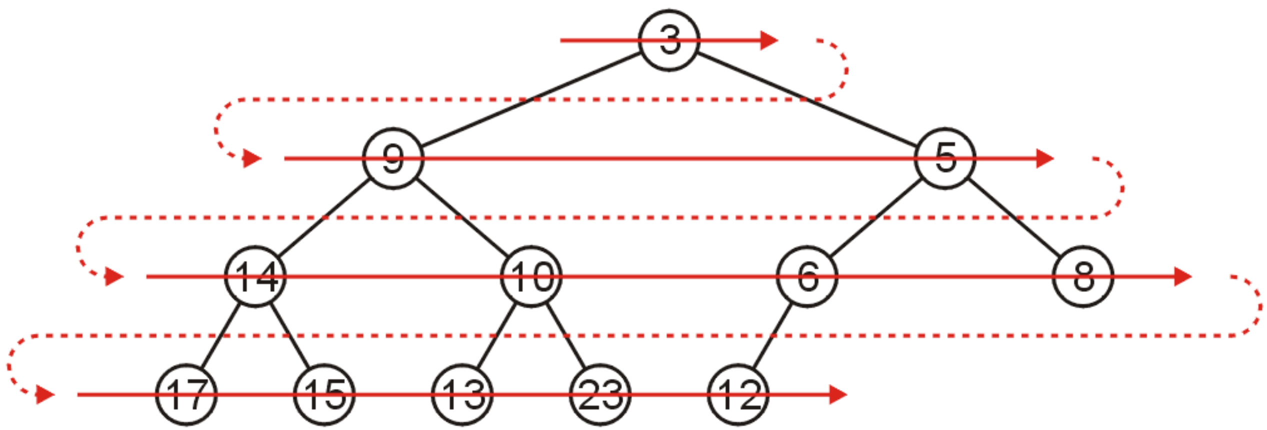
Parent $= \frac{n}{2}$

[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Array Storage (1)

- We can store a binary tree as an array

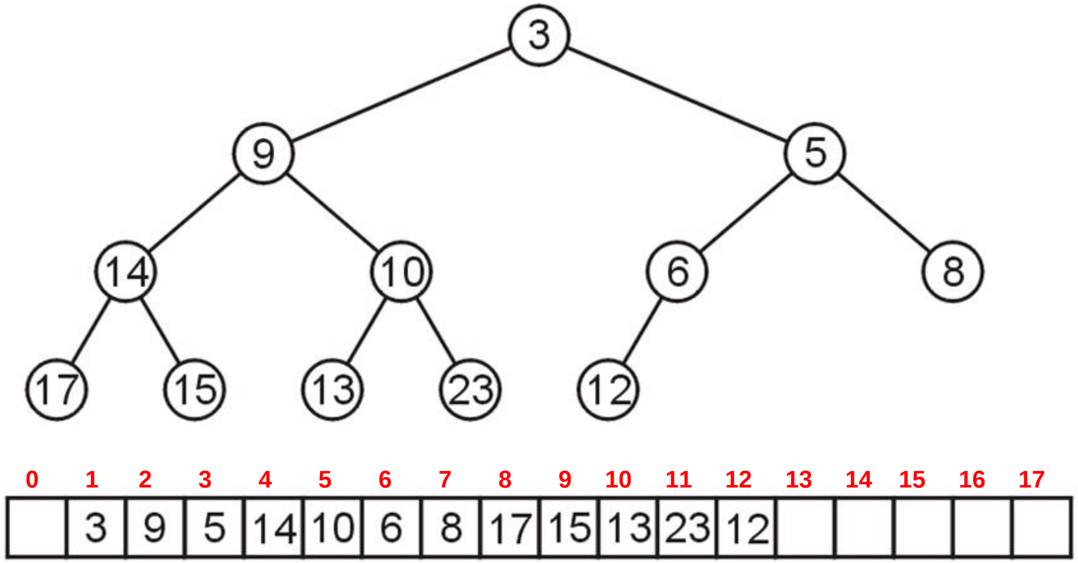
- Traverse tree in breadth-first order, placing the entries into array
  - Storage of elements (i.e., objects/data) starts from root node
  - Nodes at each level of the tree are stored left to right



	3	9	5	14	10	6	8	17	15	13	23	12				

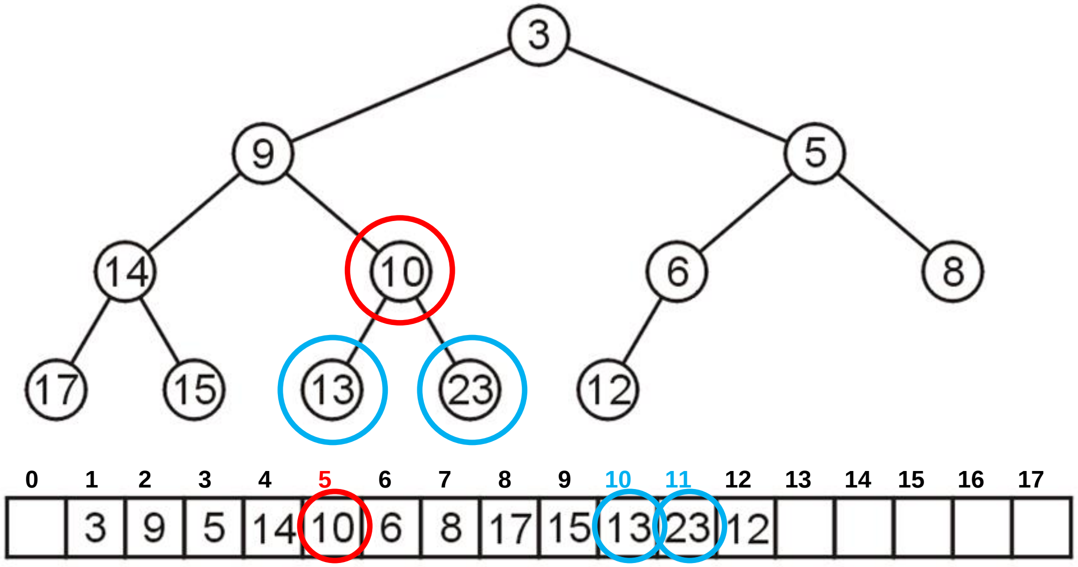
Array Storage (2)

- The children of the node with index `k` are in `2k` and `2k + 1`
- The parent of node with index `k` is in `k ÷ 2`



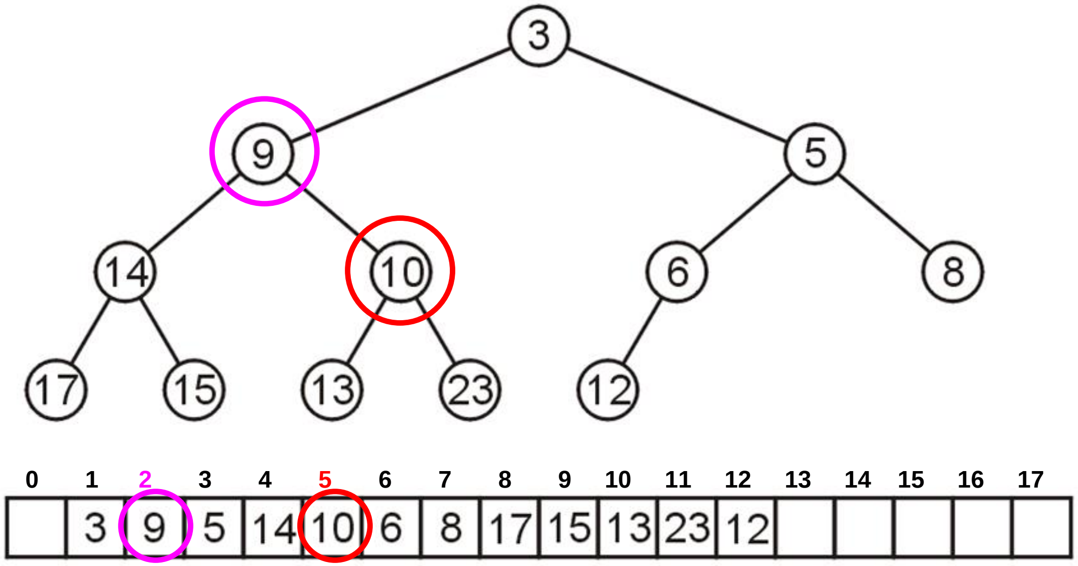
Array Storage Example (3)

- Node 10 has index 5
  - Its children 13 and 23 have indices 10 and 11, respectively



Array Storage Example (4)

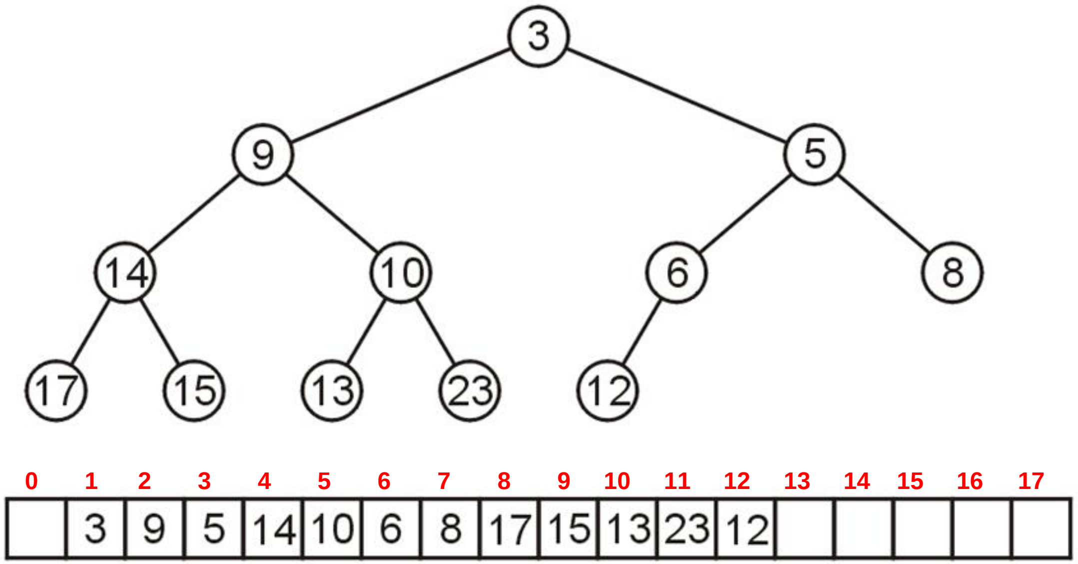
- Node 10 has index 5
  - Its children 13 and 23 have indices 10 and 11, respectively
  - Its parent is node 9 with index 5/2 = 2



Array Storage (3)

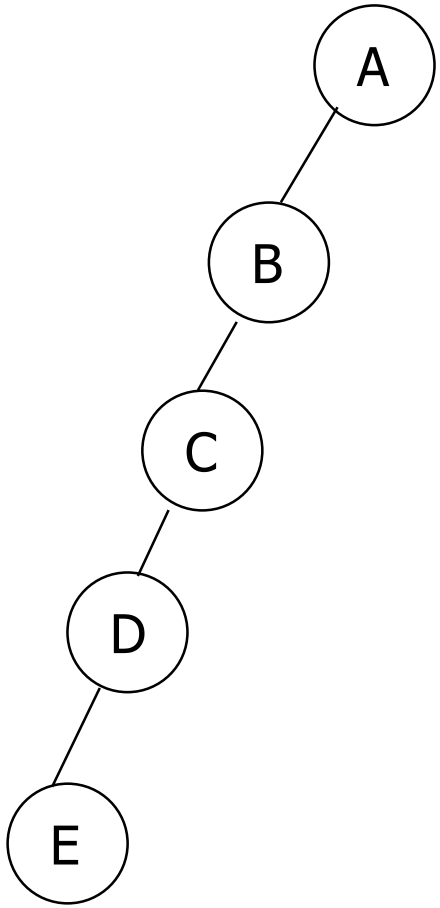
- Why array index is not started from 0
  - In C++, this simplifies the calculations

```
parent = k >> 1;
left_child = k << 1;
right_child = left_child | 1;
```



Array Storage Example (2)

- Unused nodes in tree represented by a predefined bit pattern

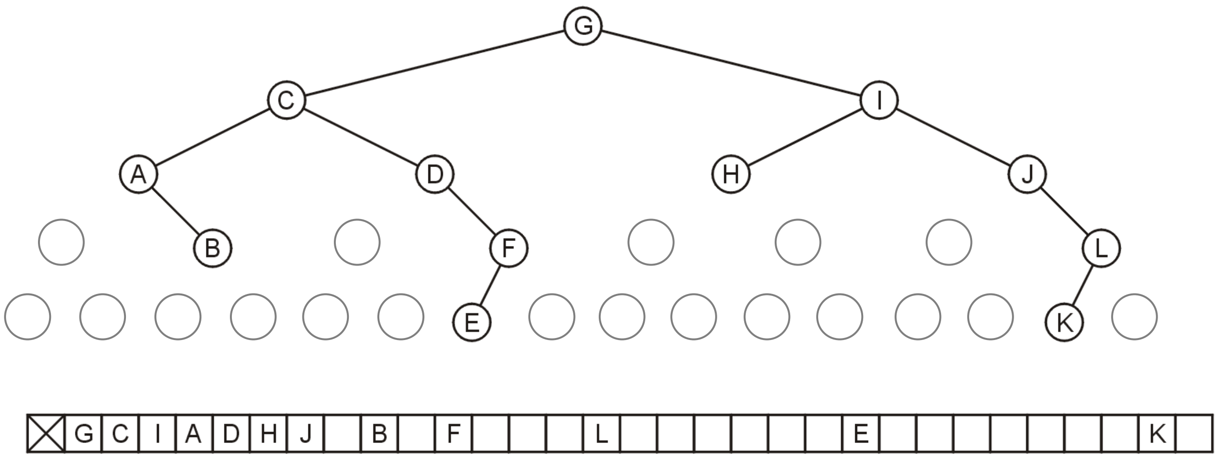


[1]	A
[2]	B
[3]	-
[4]	C
[5]	-
[6]	-
[7]	-
[8]	D
[9]	-
…	…
[16]	E

Array Storage: Disadvantage

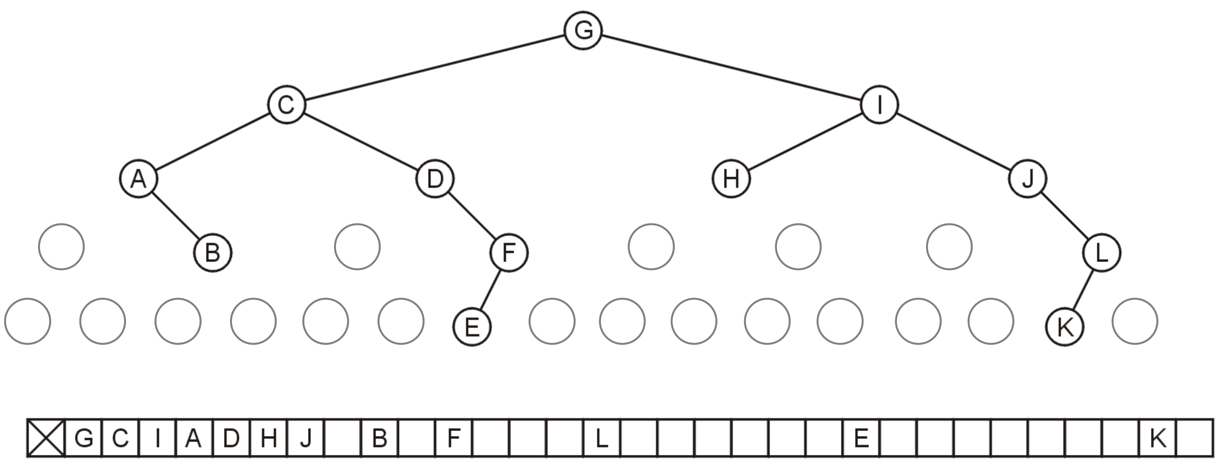
- Why not store any tree as an **array** using breadth-first traversals?
  - Because there is a significant potential for a lot of wasted memory

- Consider the following tree with 12 nodes
  - What is the required size of array?



Array Storage: Disadvantage

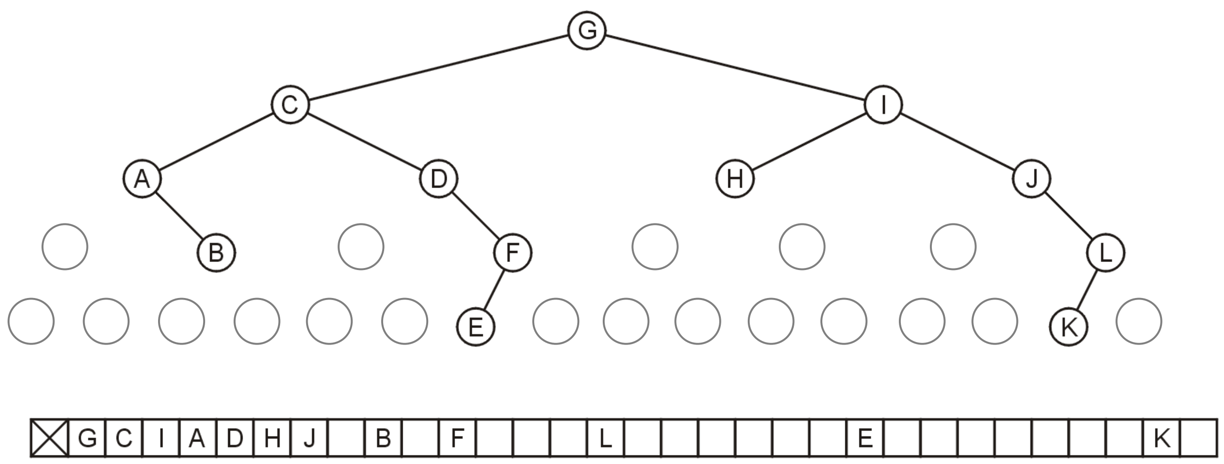
- Why not store any tree as an array using breadth-first traversals?
  - There is a significant potential for a lot of wasted memory
- Consider the following tree with 12 nodes
  - What is the required size of array? **32**
  - What will be the array size if a child is added to node K?



Array Storage: Disadvantage

- Why not store any tree as an array using breadth-first traversals?
  - There is a significant potential for a lot of wasted memory

- Consider the following tree with 12 nodes
  - What is the required size of array? **32**
  - What will be the array size if a child is added to node K? **double**



Linked List Storage

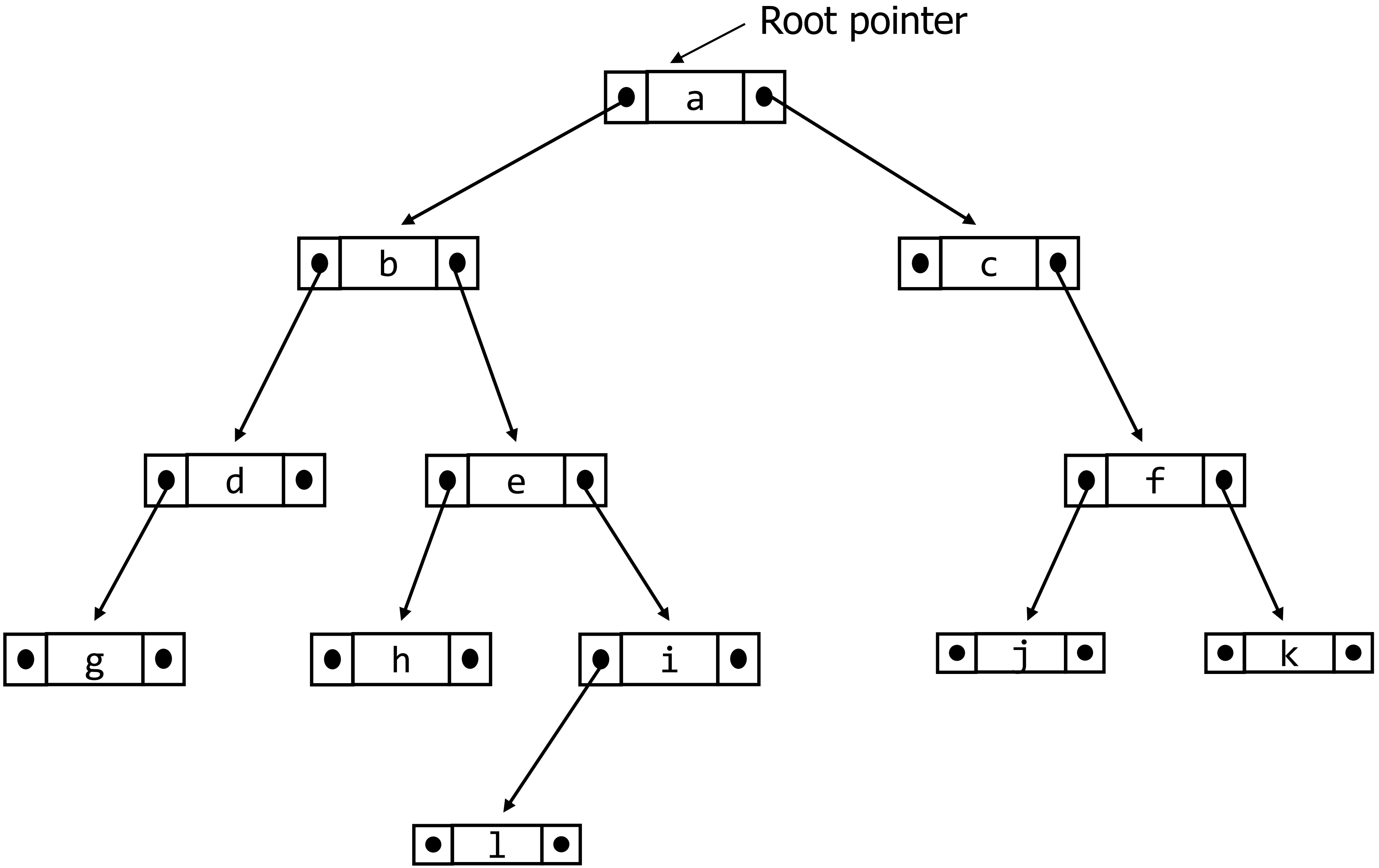
As Linked List Structure (1)

- We can implement a binary tree by using a struct which stores:
  - An element
  - A left child pointer (pointer to first child)
  - A right child pointer (pointer to second child)

```
struct Node{
   Type value;
   Node *LeftChild,*RightChild;
}*root;
```

- The root pointer points to the root node
  - Follow pointers to find every other element in the tree
- Leaf nodes have `LeftChild` and `RightChild` pointers set to `NULL`

As Linked List Structure: Example



Tree Traversal

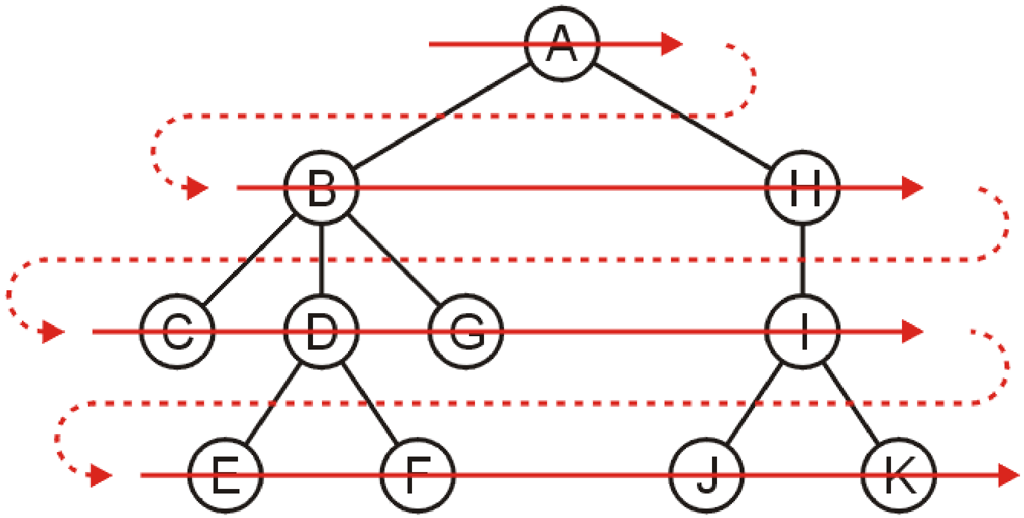
Tree Traversal

- To traverse (or walk) the tree is to visit (printing or manipulating) each node in the tree exactly once
  - Traversal must start at the root node
    - There is a pointer to the root node of the binary tree

- Two types of traversals
  - Breadth-First Traversal
  - Depth-First Traversal

Breadth-First Traversal (For Arbitrary Trees)

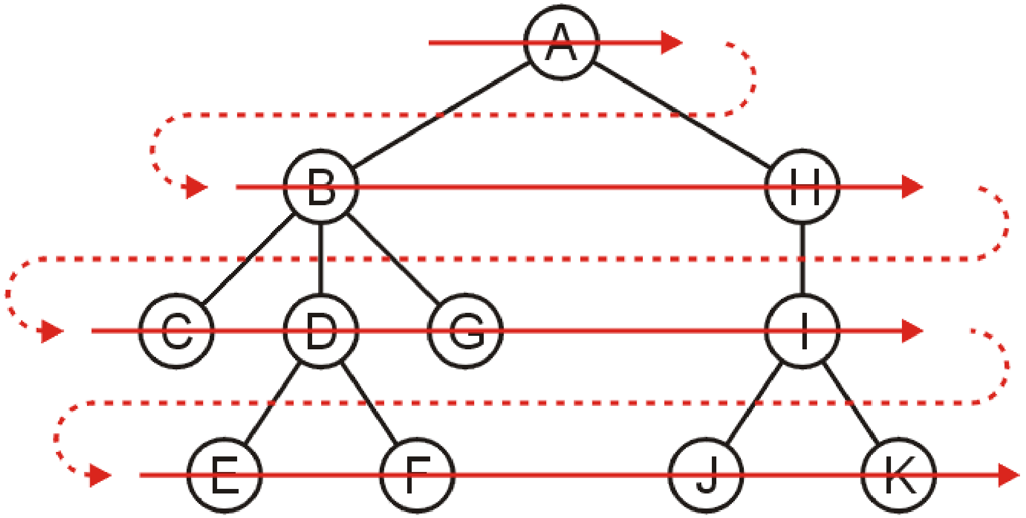
- All nodes at a given depth `d` are traversed before nodes at `d+1`
- Can be implemented using a queue



- Order: A B H C D G I E F J K

Breadth-First Traversal – Implementation

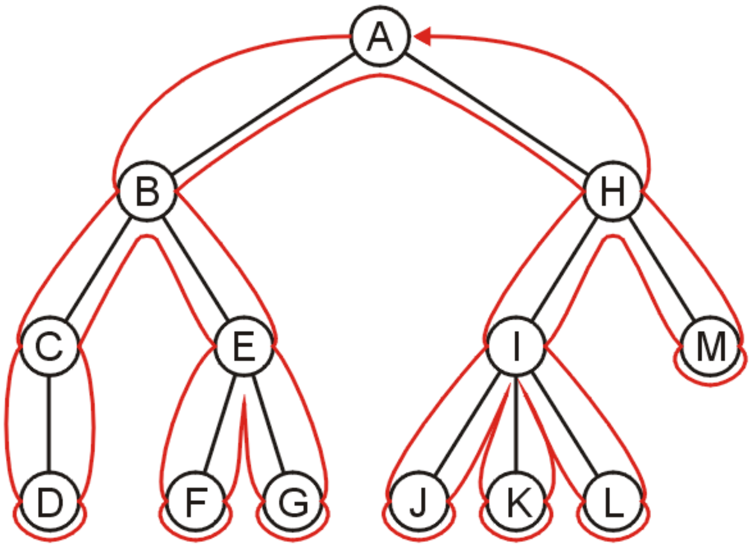
- Create a queue and push the root node onto the queue
- While the queue is not empty:
  - Enqueue all of its children of the front node onto the queue
  - Dequeue the front node



Depth-First Traversal (For Arbitrary Trees)

- Traverse as much as possible along the branch of each child before going to the next sibling
  - Nodes along one branch of the tree are traversed before backtracking

- Each node could be approached multiple times in such a scheme
  - The first time the node is approached (before any children)
  - The last time it is approached (after all children)



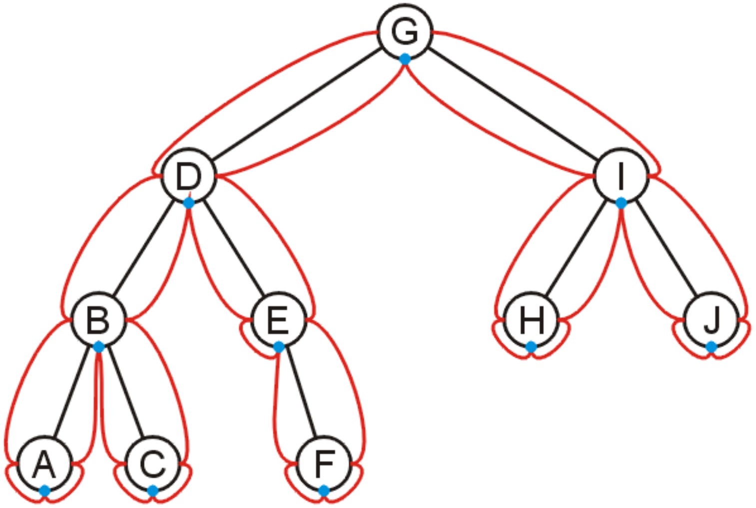
Depth-First Tree Traversal (Binary Trees)

- For each node in a binary tree, there are three choices
  - Visit the node first
  - Visit the node after left subtree
  - Visit the node after both the subtrees

- These choices lead to three commonly used traversals
  - Preorder traversal: visit Root (Left subtree) (Right subtree)
  - Inorder traversal: (Left subtree) visit Root (Right subtree)
  - Postorder traversal: (Left subtree) (Right subtree) visit Root

Inorder Traversal

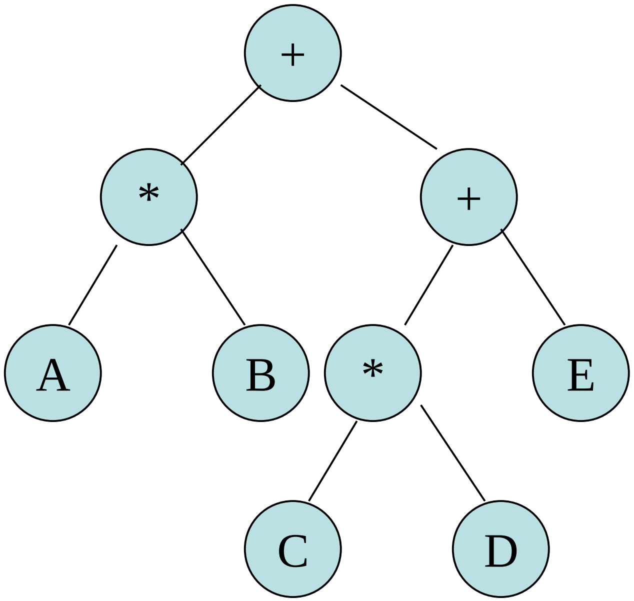
- Algorithm
  1. Traverse the left subtree in inorder
  2. Visit the root
  3. Traverse the right subtree in inorder



A, B, C, D, E, F, G, H, I, J

Inorder Traversal

- Algorithm
  1. Traverse the left subtree in inorder
  2. Visit the root
  3. Traverse the right subtree in inorder



- Example
  - Left + Right
  - [Left * Right] + [Left + Right]
  - (A * B) + [(Left * Right) + E)
  - (A * B) + [(C * D) + E]

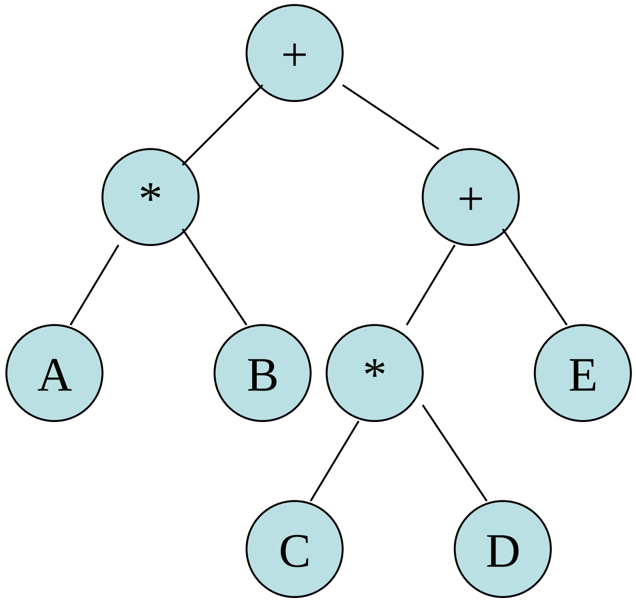
Inorder Traversal – Implementation

```
void inorder(Node *p) const
{
   if (p != NULL)
   {
      inorder(p->leftChild);
      cout << p->info << " ";
      inorder(p->rightChild);
   }
}

void main () {
   . . .
   inorder (root);
}
```

Preorder Traversal

- Algorithm
  1. Visit the node
  2. Traverse the left subtree
  3. Traverse the right subtree



- Example
  - + Left Right
  - + [* Left Right] [+ Left Right]
  - + (* AB) [+ * Left Right E]
  - +*AB + *C D E

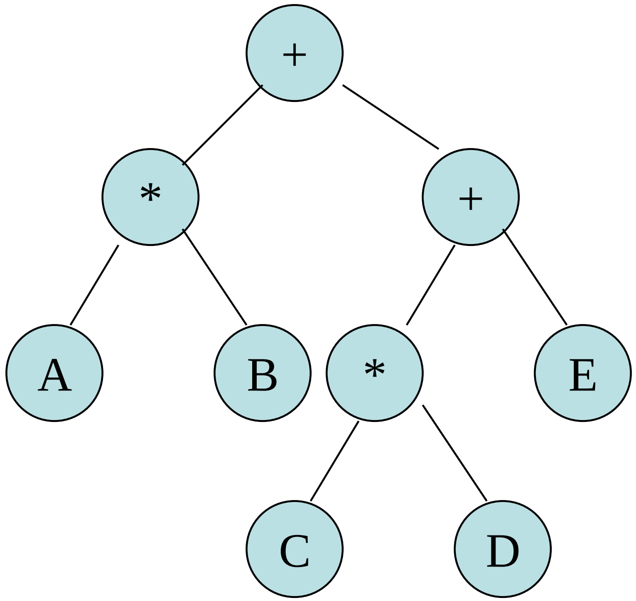
Preorder Traversal – Implementation

```
void preorder(Node *p) const
{
   if (p != NULL)
   {
      cout << p->info << " ";
      preorder(p->leftChild);
      preorder(p->rightChild);
   }
}

void main () {
   . . .
   preorder (root);
}
```

Postorder Traversal

- Algorithm
  1. Traverse the left subtree
  2. Traverse the right subtree
  3. Visit the node



- Example
  - Left Right +
  - [Left Right *] [Left Right+] +
  - (AB*) [Left Right * E +]+
  - (AB*) [C D * E +]+
  - AB* C D * E + +

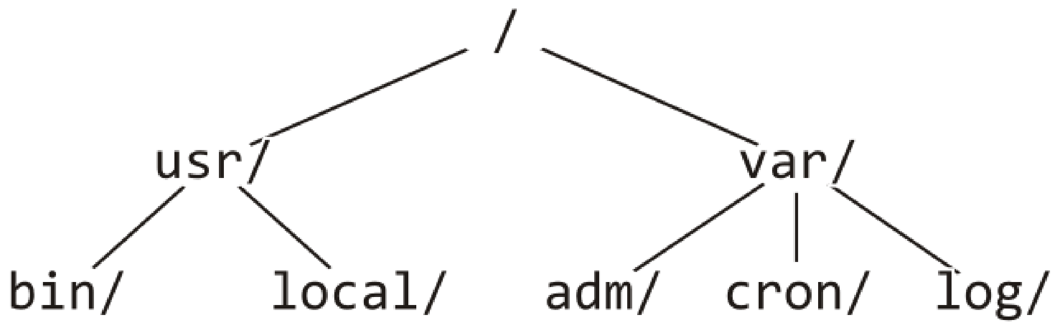
Postorder Traversal – Implementation

```
void postorder(Node *p) const
{
   if (p != NULL)
   {
      postorder(p->leftChild);
      postorder(p->rightChild);
      cout << p->info << " ";
   }
}

void main () {
   . . .
   postorder (root);
}
```

Example: Printing a Directory Hierarchy

- Consider the directory structure presented on the left
  - Which traversal should be used?




```
/
   usr/
      bin/
      local/
   var/
      adm/
      cron/
      log/
```

Any Question So Far?