Generating Sets and Basis

In a vector space V, we are particularly interested in sets of vectors A that possess the property that any vector $\boldsymbol{v} \in V$ can be obtained by a linear combination of vectors in A. These vectors are special vectors, and in the following, we will characterize them.

Definition 1 (Generating Set/Span)**.** Consider a vector space V and set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq \mathcal{V}$. If every vector $\boldsymbol{v} \in V$ can be expressed as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, $\mathcal{A}$ is called a *generating set* or *span*, which spans the vector space V. In this case, we write $V = [\mathcal{A}]$ or $V = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$.

generating set
span

Generating sets are sets of vectors that span vector (sub)spaces, i.e., every vector can be represented as a linear combination of the vectors in the generating set. Now, we will be more specific and characterize the smallest generating set that spans a vector (sub)space.

Definition 2 (Basis)**.** Consider a real vector space V and $\mathcal{A} \subseteq \mathcal{V}$.

- A generating set $\mathcal{A}$ of V is called *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \subseteq \mathcal{A} \subseteq \mathcal{V}$ that spans V.

minimal

- Every linearly independent generating set of V is minimal and is called *basis* of V.

basis

Let V be a real vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Then, the following statements are equivalent:

- $\mathcal{B}$ is a basis of V
- $\mathcal{B}$ is a minimal generating set
- $\mathcal{B}$ is a maximal linearly independent subset of vectors in V.
- Every vector $\boldsymbol{x} \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i \tag{1}$$

  and $\lambda_i, \psi_i \in \mathbb{R}$, $\boldsymbol{b}_i \in \mathcal{B}$ it follows that $\lambda_i = \psi_i,\ i = 1, \ldots, k$.

Example:

- In $\mathbb{R}^3$, the *canonical/standard basis* is

canonical/standard basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{2}$$

- Different bases in $\mathbb{R}^3$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix}0.5\\0.8\\-0.4\end{bmatrix}, \begin{bmatrix}1.8\\0.3\\0.3\end{bmatrix}, \begin{bmatrix}-2.2\\-1.3\\3.5\end{bmatrix} \right\} \tag{3}$$

- The set

$$\mathcal{A} = \left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\\-4\end{bmatrix} \right\} \tag{4}$$

  is linearly independent, but not a generating set (and no basis): For instance, the vector $[1, 0, 0, 0]^\top$ cannot be obtained by a linear combination of elements in $\mathcal{A}$.

Remark. Every vector space V possesses a basis $\mathcal{B}$. The examples above show that there can be many bases of a vector space V, i.e., there is no unique basis. However, all bases possess the same number of elements, the *basis vectors*. basis vectors

We only consider finite-dimensional vector spaces V. In this case, the *dimension* of V is the number of basis vectors, and we write $\dim(V)$. If $U \subseteq V$ is a subspace of V then $\dim(U) \leq \dim(V)$ and $\dim(U) = \dim(V)$ if and only if $U = V$. Intuitively, the dimension of a vector space can be thought of as the number of independent directions in this vector space. dimension