Sequences and Series

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CONTENTS

SEQUENCE

Definition:

A set of numbers $a_1, a_2, \ldots a_n, \ldots$ such that to each positive inte- ger n, there corresponds a number a_n of the set , is called a sequence and it is denoted by $\{a_n\}$. In otherwords, a sequence of real numbers is a function s from the set of natural numbers N into the set of real numbers R.

Examples:

- If $an = \frac{1}{n}$, then the sequence is $1, \frac{1}{2}, \frac{1}{3} \ldots, \frac{1}{n} \ldots$
- If $an = (-1)n$, then the sequence is $-1, 1, -1, \ldots.$
- If $an = k$, then the sequence is $k, k, \ldots.$

Note:

1. A finite sequence has a finite number of terms.
2. A sequence which is not finite, is an infinite sequence.
3. Example 3 is a constant sequence.

Operations on Sequences:

If $\{s_n\}$ and $\{t_n\}$ are sequences then

- Sum sequence is $\{s_n + t_n\} = \{s_n\} + \{t_n\}$.
- Product sequence is $\{s_n t_n\} = \{s_n\}.\{t_n\}$.
- If $c \in R$, then $c\{s_n\} = \{cs_n\}$.
- $\{\frac{1}{s_n}\}$ is called the reciprocal of $\{s_n\}$.
- $\{\frac{s_n}{t_n}\}$ is defined as the quotient of sequence $\{s_n\}$ and $\{t_n\}$,

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there exist numbers m, M such that $m < a_n < M$ for all $n \in N$. Otherwise it is said to be unbounded.

Example:

1. $\{\frac{1}{n}\}$ which is bounded by 1.
2. $\{2^n\}$ which is unbounded.

Monotonic Sequence:

A sequence $\{s_n\}$ is said to be

(i) Monotonically increasing if $s_{n+1} \geq s_n$ for every n, $s_1 \leq s_2 \leq s_3 \ldots. \leq s_n \leq s_{n+1} \leq \cdots$.

(ii) Monotonically decreasing if $s_{n+1} \leq s_n$ for every n, $s_1 \geq s_2 \geq s_3 \ldots. \geq s_n \geq s_{n+1} \geq \cdots$.

(iii) Monotonic if it is either monotonically increasing or monoton- ically decreasing.

Example:

- 1,4,7,10,.... is a monotonic sequence.
- $1, \frac{1}{2}, \frac{1}{3} \ldots$ is monotonic sequence.
- 1,−1,1,−1,.... is not a monotonic sequence.

Limit of a sequence:

Let $\{s_n\}$ be a sequence. l is said to be limit of the sequence $\{s_n\}$, if to each $\varepsilon > 0$ there exists $m \in z^+$ such that $|s_n - l| < \varepsilon, \forall\, n \geq m$. That is $lim_{n \to \infty} s_n = l$.

Convergent Sequence:

A sequence $\{s_n\}$ is said to be convergent if it has a finite limit. That is $lim_{n \to \infty} s_n = l$.

Divergent Sequence:

If $lim_{n \to \infty} s_n = \infty$, $\{s_n\}$ is divergent.

Oscillatory Sequence:

If $lim_{n \to \infty}$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely) then $\{s_n\}$ is oscillatory sequence.

Examples:

- $\{\frac{1}{n^2}\}$ is a convergent sequence.
- $\{n\}$ is a divergent sequence.
- $\{(-1)^n\}$ oscillates finitely.
- $\{(-1)^n n^2\}$ oscillates infinitely.

Problems:

Which of the following sequence are convergent?

(i) $\{\frac{n}{n^2+1}\}$ (ii) $\{(-1)^{n+1}\}$ (iii) $\{\frac{n}{n+1}\}$ (iv) $\left\{1+\frac{(-1)^n}{n}\right\}$ (v) $\{(\frac{1}{2})^n\}$

Solution: (i)

$$lim_{n\to\infty} s_n = lim_{n\to\infty} \frac{n}{n^2+1}$$
$$= lim_{n\to\infty} \frac{n}{n(n+\frac{1}{n})}$$
$$= lim_{n\to\infty} \frac{n}{(n+\frac{1}{n})}$$
$$= 0$$

Hence the sequence is convergent.

(ii) $\{(-1)^{n+1}\} = 1, -1, 1, -1, \ldots.$ is an oscillating sequence.

(iv)

$$lim_{n\to\infty} s_n = lim_{n\to\infty} \frac{n}{n+1}$$
$$= lim_{n\to\infty} \frac{n}{n(1+\frac{1}{n})}$$
$$= lim_{n\to\infty} \frac{n}{(1+\frac{1}{n})}$$
$$= 1$$

Hence the sequence is convergent.

(iv) $lim_{n\to\infty} s_n = lim_{n\to\infty} 1 + \frac{(-1)^n}{n+1}$

$$= 1$$

Hence the sequence is convergent.

(v)The sequence $\{ (\frac{1}{2})^n \}$ is convergent.

Definition:

$u_1, u_2, \ldots, u_n \ldots$ is an infinite sequence. The expression $u_1 + u_2 + \cdots + u_n + \cdots$ is called the series. It is denoted by $\sum_{n=1}^{\infty} U_n$.

Note:

1.If the number of terms are finite in a series then the series is called a finite series.

2.If the number of terms are infinite in a series then the series is called an infinite series.

Definition:

The sum of a finite number of terms (the first n-terms) of a series is called the n^{th} partial sum of the series. $S_n = u_1 + u_2 + \ldots + u_n = \sum_{n=1}^{\infty} U_n$

1. If $lim_{n\to\infty} s_n = S$(finite), then the series $\sum_{n=1}^{\infty} U_n$ un converges.
2. If $lim_{n\to\infty} s_n = \pm\infty$, then the series $\sum_{n=1}^{\infty} U_n$ diverges.
3. If $lim_{n\to\infty} s_n$ is more than one limit (or) $\pm\infty$, then $\sum_{n=1}^{\infty} U_n$ is oscillatory (or) non converges

Problems:

1.Examine the nature of the series $1 + 3 + 5 + 7 + \cdots \infty$

Solution:

The n^{th} partial sum is $S_n = 1 + 3 + 5 + 7 + \ldots + n$. It is an arithmetic series with $a = 1, d = 2, s_n = \frac{n}{2}[2a + (n-1)d] = n^2$. It follows that $lim_{n\to\infty} S_n = \infty$. Hence the series is divergent.

2. Show that the series $1 + r + r^2 + \ldots \infty$ (i) Converges if $|r| < 1$ (ii) Diverges if $r \geq 1$ and (iii) Oscillatory if $r \leq -1$

Solution:

(i) If $|r| < 1$, the n^{th} partial sum is $S_n = 1 + r + r^2 + \cdots + r^{n-1}$ $= \frac{1(1-r^n)}{1-r}$. $lim_{n\to\infty} r^n = 0$, if $|r| < 1$. Thus the series is convergent.

(ii) If $r > 1$, $lim_{n \to \infty} r^n = \infty$. If $r = 1$, then $S_n = 1 + 1 + \cdot + 1 = n$. Hence $lim_{n \to \infty} r^n = \infty$. Hence the series is divergent if $r \geq 1$.

(iii) If $r < -1$, then $lim_{n \to \infty} r^n = \begin{cases} \infty, & \text{if n is odd} \\ -\infty, & \text{if n is even} \end{cases}$

If $r=-1$, then $S_n = 1-1+1-1+\cdots 1.\, S_n = \begin{cases} 1, & \text{if n is odd} \\ -1, & \text{if n is even} \end{cases}$

Hence the series $1+r+r^2+\cdots\infty$ is oscillatory if $r \leq -1$

3. Examine the converges of the following series.

(i) $5-4-1+5-4-1+\cdots\infty$ (ii) $1+\frac{5}{4}+\frac{6}{4}+\cdots\infty$ (iii) $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$

(iv) $1+\frac{1}{2}+\frac{1}{2^2}\cdots\infty$ (v) $1+\frac{4}{3}+\left(\frac{4}{3}\right)^2+\cdots\infty$

Solution:

(i) n^{th} partial sum is $S_n = 5$ or 1 or 0 . Hence the series is oscillatory.

(ii) The series is arithmetic series $a=1, d=\frac{1}{4}$, $S_n = \frac{n}{2}[2a+(n-1)d]$
$=\frac{n}{8}[7+n]$. Therefore $\lim_{n\to\infty} S_n = \infty$. Hence the series is divergent.

(iii) $u_n = \frac{1}{n(n+2)}$. By using partial fractions $\frac{1}{n(n+2)} = \frac{A}{n}+\frac{B}{n+2}$. Which implies
$1 = A(n+2)+Bn$. When $n=0, A=\frac{1}{2}$ and when $n=-2, B=\frac{-1}{2}$.

Therefore $u_n = \frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$. The n^{th} partial sum is $S_n = u_1 + u_2 + \cdots + u_n = \frac{1}{2} - \frac{1}{2(n+2)}$. $\lim_{n\to\infty} S_n = \frac{1}{2}$. Hence the given series is convergent.

(iv) The n^{th} partial sum is $S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1}$.
$S_n = \frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$. $\lim_{n\to\infty} S_n = 2$. Hence the given series is convergent.

(v) The n^{th} partial sum is $S_n = 1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{n-1}$.
$S_n = \frac{\left(\frac{4}{3}\right)^n - 1}{\frac{1}{3}} = 3\left(\left(\frac{4}{3}\right)^n - 1\right)$. $\lim_{n\to\infty} S_n = \infty$. Hence the given series is divergent.

SERIES OF POSITIVE TERMS

Properties of Series:

1.Convergence of a series remains unchanged by the replace- ment, inclusion or omission of a finite number of terms.

2.A series remains convergent, divergent or oscillatory when each term of it is multiplied by a fixed number other than zero.

3.A series of positive terms either converges or diverges to $+\infty$. That is omitting the negative terms the sum of first n terms tends to either a finite limit or $+\infty$

4. Every finite series is a convergent series.

Definition:

If all terms after few positive terms in an infinite series are posi- tive, such a series is a positive term series.

Example: $-10-6-1+5+12+20+\cdots$

Necessary Condition for Convergence:

If a positive term series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Note:

Converse of the above theorem is not true.

Example:

The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$ is divergent eventhough

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

Test for Divergence:

If $\lim_{n\to\infty} u_n \neq 0$, the series $\sum_{n=1}^{\infty} u_n$ must be divergent.

Comparison Test for Convergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ converges and $u_n \leq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also converges.

Comparison Test for divergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ diverges and $u_n \geq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also diverges.

Limit Comparision test:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ $=$ non zero finite value, then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges or diverges together.

Auxiliary Series:

(a) p-series:

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.

(b) Geometric series:

The geometric series $\sum_{n=1}^{\infty} r^{n-1}$ is convergent if $r > 1$ and divergent if $r \geq 1$.

PROBLEMS:

1. Examine the series $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\cdots$ for convergence.

Solution:

To find u_n :

1,2,3, ... is an arithmetic series where $a=1, d=1,$

$$t_n=a+(n-1)d=n$$

1,3,5, ... is an arithmetic series where $a=1, d=2,$

$$t_n=a+(n-1)d=2n-1$$

3,5,7, ... is an arithmetic series where $a=3, d=2,$

$$t_n=a+(n-1)d=2n+1$$

5,7,9, ... is an arithmetic series where $a=5, d=2,$

$$t_n=a+(n-1)d=2n+3$$

Therefore $u_n=\frac{n}{(2n-1)(2n+1)(2n+3)}=\frac{1}{n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}$.

Choose $v_n=\frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} = \frac{1}{8} \neq 0$. By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, implies that $\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)(2n+3)}$ is convergent.

2. Discuss the convergence or divergence of the series $\frac{2}{1p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots \infty$.

Solution:

$u_n = \frac{(n+1)}{n^p}$. Choose $v_n = \frac{1}{n^{p-1}}$. We have

$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is convergent if $p > 2$ and it is divergent if $p \leq 2$. Hence the given series $\sum_{n=1}^{\infty} \frac{(n+1)}{n^p}$ is convergent if $p > 2$ and it is divergent if $p \leq 2$.

3. Determine whether the following series is convergent or divergent $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$.

Solution:

$u_n=\frac{\sqrt{n+1}-1}{(n+1)^3-1}=\frac{\sqrt{1+\frac{1}{n}-\frac{1}{\sqrt{n}}}}{n^{\frac{5}{2}}\left(\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right)}$. Choose $v_n=\frac{1}{n^{\frac{5}{2}}}$. $\lim_{n\to\infty}\frac{u_n}{v_n}=$

$\lim_{n\to\infty}=\frac{\sqrt{1+\frac{1}{n}-\frac{1}{\sqrt{n}}}}{\left(\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right)}=1\neq 0$. Since $\sum_{n=1}^{\infty}v_n$ is convergent.

Hence by comparison test $\sum_{n=1}^{\infty}u_n$ is convergent.

4. Examine the nature of series $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ where a, b, p, q all positive.

Solution:

$$u_n = \frac{1}{(a+n)^p(b+n)^q}$$

$$= \frac{1}{n^{p+q}\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}$$

Choose $_n$ $$= \frac{1}{n^{p+q}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$ is convergent if $p+q>1$ and is divergent if $p+q \le 1$.

Which implies $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ is convergent if $p+q>1$ and is divergent if $p+q \le 1$.

5 . Determine whether the following series is convergent or divergent $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$.

Solution:

$$u_n = \left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\left(\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right)$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{n^2}\frac{1}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

Choose $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2}{2} = 1.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ is convergent.

6. Determine whether the following series is convergent or divergent

$\sum_{n=1}^{\infty}\sqrt{\frac{3^n-1}{2^n+1}}$.

Solution:

$$U_n = \sqrt{\frac{3^n-1}{2^n+1}}$$

$$= \left(\sqrt{\frac{3}{2}}\right)^n \sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}}$$

$$\text{let } V_n = \left(\sqrt{\frac{3}{2}}\right)^n$$

$$lim_{n\to\infty}\frac{U_n}{V_n} = lim_{n\to\infty}\sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}} = 1$$

Since $\sum_{n=1}^{\infty} V_n$ is divergent, $\sum_{n=1}^{\infty}\sqrt{\frac{3^n-1}{2^n+1}}$ is also divergent.

7. Examine the nature of the series $\sum_{n=1}^{\infty} sin\left(\frac{1}{n}\right)$

Solution:

$u_n = sin\left(\frac{1}{n}\right)$, choose $v_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence $\Sigma_{n=1}^{\infty} sin\left(\frac{1}{n}\right)$ is also divergent.

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Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and if $u(x) = f(x)$ be such that

- $f(x)$ is continuous in $1 < x < \infty$
- $f(x)$ decreases as x increases, then the series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

CAUCHY'S INTEGRAL TEST:

1. Apply Cauchy's integral test to discuss the nature of the harmonic series (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Solution:
Let $u_x = f(x) = \frac{1}{n^p}$. As x increases $f(x)$ decreases.

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{n^p} dx$$

$$= \int_1^\infty x^{-p} dx$$

$$= \left[\frac{x^{-p+1}}{-p+1}\right]$$

$$= \begin{cases} \frac{1}{p-1} & if\ p > 1 \\ \infty & if\ p \leq 1 \end{cases}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

CAUCHY'S INTEGRAL TEST:

2. Find the nature of the series Type equation here. $\sum_{n=2}^{\infty} \frac{1}{(logn)^p}$

Solution:

Let $u_x = f(x) = \frac{1}{x(logn)^p}$. As x increases $f(x)$ decreases.

$$\int_2^{\infty} \frac{1}{x(logn)^p} = \int_{log2}^{\infty} \frac{dt}{t^p}$$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]$$

$$= \begin{cases} \text{finite, if p=1} \\ \text{finite, if p>1} \\ \text{finite, if p≤1} \end{cases}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

CAUCHY'S INTEGRAL TEST:

3. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{nlogn}$

Solution:

let $u_x = f(x) = \frac{1}{xlogx}$. As x increase $f(x)$ decreases.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(logx)} dx$$

$$= \int_{log2}^{\infty} \frac{dt}{t}$$

$$= infinite$$

Therefore $\sum u_n$ is divergent.

CAUCHY'S INTEGRAL TEST:

4. Examine the convergence of the series $1+\frac{1}{4^{\frac{2}{3}}}+\frac{1}{9^{\frac{2}{3}}}+\frac{1}{16^{\frac{2}{3}}}+$

Solution:

Let $u_x = f(x) = \frac{1}{x^{\frac{4}{3}}}$. As x increases $f(x)$ decreases.

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^{\frac{4}{3}}}dx$$

$$= \int_1^\infty x^{\frac{-4}{3}}dx$$

$$= -3\left[x^{\frac{-1}{3}}\right]_1^\infty$$

$$= 3(\text{ finite })$$

Therefore $\sum u_n$ is convergent.

D'ALEMBERT'S RATIO TEST:

D'Alembert's Ratio Test

The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ is divergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$. If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the test fails.

1. Test the convergence of the series $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)} +$

Solution:

Let $u_n = \frac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)}{(1+\beta)(1+2\beta).....(1+(n)\beta)}$ then

$u_{n+1} = \frac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)(1+(n+1)\alpha)}{(1+\beta)(1+2\beta)....(1+(n)\beta)(1+(n+1)\beta)}$. It follows that

$\frac{u_{n+1}}{u_n} = \frac{(1+(n+1)\alpha)}{(1+(n+1)\beta)} \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{\alpha}{\beta}$. Thus the series converges if $\frac{\alpha}{\beta} < 1$ and diverges if $\frac{\alpha}{\beta} > 1$. If $\frac{\alpha}{\beta} = 1$, then $\alpha = \beta$.

Therefore the series $\sum_{n=1}^{\infty} u_n = 1 + 1 + 1 + \cdots$ is a divergent series.

D'ALEMBERT'S RATIO TEST:

2. Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots \ldots$

Solution:

Let $u_n = \frac{x^n n^n}{n!}$ then $u_{n+1} = \frac{x^{n+1}(n+1)^{(n+1)}}{(n+1)!}$. It follows that $\frac{u_{n+1}}{u_n} = x\left(\frac{n+1}{n}\right)^n \Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = xe$. Thus

the series converges if $ex < 1$ and diverges if $ex > 1$. If $ex = 1$, then the ratio test fails.

3. Find the nature of the series $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1\cdot 5.9.13} + \cdots$

Solution:

$u_n = \frac{2.5\cdot 8....(3n-1)}{1\cdot 5\cdot 9....(4n-3)}$ and $u_{n+1} = \frac{2\cdot 5\cdot 8....(3n-1)(3n+2)}{1\cdot 5\cdot 9....(4n-3)(4n+1)}$. Thus $\frac{u_{n+1}}{u_n} = \frac{3n+2}{4n+1}$. Hence $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{3+\frac{2}{n}}{4+\frac{1}{n}} = \frac{3}{4} < 1$. Hence

the given series is convergent.

Raabe's Test

The positive termed series $\sum_{n=1}^{\infty} u_n$ is convergent or deivergent according as $\lim_{n\to\infty} n\left(\frac{u_{n+1}}{u_n}\right) - 1 > 1$ or < 1. If D'Alemberts test fails then use Raabe's test.

1. Test the convergence of the series $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3\cdot5.7.8} + \cdots$ ....

Solution:

Let $u_n = \frac{2\cdot4\cdot6\cdot8....2n}{3\cdot5.7....(2n+1)}\frac{1}{2n+2}$ then

$u_{n+1} = \frac{2\cdot4\cdot6\cdot8...\cdot2n\cdot2(n+1)}{3\cdot5\cdot7....(2n+1)\cdot(2n+3)}\frac{1}{2n+4}$. It follows that

$\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{3}{2n}\right)\left(1+\frac{4}{2n}\right)} \Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$. Hence the ratio test fails. $n\left(\frac{u_{n+1}}{u_n}\right) - 1 = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right] \to \frac{3}{2}$ as $n \to \infty$.

Therefore by Raabe's test the given series is convergent.

RAABE'S TEST:

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3 \cdot 5 \ldots .(2n-1)}{2 \cdot 4.6 \ldots .(2n)} x^n$.

Solution:

$\frac{u_{n+1}}{u_n} = x\left(\frac{2n+1}{2n+2}\right) \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$. Thus the series converges if $0 < x < 1$ and diverges if $x > 1$. If $x = 1$, then the D'Alemberts ratio test fails. Apply Raabe's test.

$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1} \to \frac{1}{2}$ as $n \to \infty$. Therefore when $x = 1$ the given series is divergent.

Cauchy's root Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms, then the series is conver-gent or deivergent according as $\lim_{n\to\infty} u_n^{\frac{1}{n}} < 1$ or > 1.

1. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution:

Let $u_n = \frac{1}{(\log n)^n}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{(\log n)^n}^{\frac{1}{n}} = \frac{1}{\log n}$. It follows that $u_n \frac{1}{n} \to 0$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution:

Let $u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$. It follows that $u_n \frac{1}{n} \to \frac{1}{e} < 1$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

ALTERNATING SERIES-LEBNITZ'S TEST:

Alternating Series

A series in which the terms are alternatively positive or negative is called an alternating series.

Lebnitz's Rule

An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$ converges if $u_n - u_{n-1} < 0$ and $\lim_{n\to\infty} u_n = 0$. The alternating series is not convergent if one of the condition is satisfied. If $\lim_{n\to\infty} u_n \neq 0$, then the given series is oscillatory.

1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \ldots$

Solution:

$u_n = \frac{1}{n}, u_{n-1} = \frac{1}{n-1}$. Then $u_n - u_{n-1} = \frac{-1}{n(n-1)} < 0. \lim_{n\to\infty} u_n = 0.$

Therefore by Lebnitz's Rule the given series is convergent.

ALTERNATING SERIES-LEBNITZ'S TEST:

2. Examine the nature of the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}x^n}{n(n-1)}, 0 < x < 1$.
   Solution:

$u_n = \frac{x^n}{n(n-1)}, u_{n-1} = \frac{x^{n-1}}{(n-2)(n-1)}. u_n - u_{n-1}$ is less than zero.

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{n(n-1)} = 0$. Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

3. Examine the nature of the series $\sum_{n=1}^{\infty} (-1)^{n-1}\left[\sqrt{n^2+1} - n\right]$.
   Solution:

$u_n = \sqrt{n^2+1} - n, u_{n-1} = \sqrt{n^2-2n+2} - n + 1$. Also $u_n - u_{n-1} < 0$.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left[\sqrt{n^2+1} - n\right] = \lim_{n\to\infty}\left[\frac{\left(\sqrt{n^2+1}\right)^2 - n^2}{\sqrt{n^2+1}+n}\right] = 0.$

Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

Absolute Convergence: If the series of arbitrary terms $u_1 + u_2 + \cdots . + u_n + \cdots$ be such that the series $|u_1| + |u_2| + \cdots . + |u_n| + \cdots$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. Conditional Convergence: If the series $\sum_{n=1}^{\infty} |u_n|$ is divergent but $\sum_{n=1}^{\infty} u_n$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

1. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$ for (i) Absolute Convergence (ii) Conditional Convergence.

Solution:

(i) $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$.. Thus $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

Which harmonic p-series with $p = \frac{3}{2} > 1$. Hence the series $\sum_{n=1}^{\infty} |u_n|$ is convergent which implies $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. (ii) $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$ is an alternating series.

$u_n = \frac{1}{n\sqrt{n}}$ and $u_{n-1} = \frac{1}{n-1\sqrt{n-1}}$. Also $u_n - u_{n-1} < 0$ and $\lim_{n\to\infty} u_n = 0$. Hence $\sum_{n=1}^{\infty} u_n$ is also convergent. Hence the given series is not conditionally convergent.

Absolute Convergence:

2. Prove that the exponential series $1 + \frac{x}{1!} + \frac{x}{2!} + \cdots + \frac{x}{n!} + \cdots$ is absolutely convergent and hence convergent for all values of x.
Solution:

Let $u_n = \frac{x^{n-1}}{(n-1)!}, u_{n+1} = \frac{x^n}{n!}$. Thus $\frac{u_{n+1}}{u_n} = \frac{x}{n}$. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1, \forall x$. Hence the series is absolutely convergent and hence convergence for all real x.

Thank You