CALCULUS AND LINEAR ALGEBRA

UNIT-IV

(Applications of Differential Calculus)

DEPARTMENT OF MATHEMATICS

SRM Institute of Science and Technology

Topics covered

- *Radius of Curvature -Cartesian coordinates.*
- *Radius of Curvature - Polar coordinates.*
- *Centre of curvature.*
- *Circle of curvature.*
- *Evolute of standard curves– Problems*
- *Envelope of standard curves.*
- *Beta Gamma Functions Definitions*
- *Beta Gamma Functions -simple problems.*

Introductio n

The rate of change of the direction of tangent with respect to arc lenngth as the point p moves along the curve is called curvature vector of the curve whose magnitude is called the curvature at p. Radius of curvature. The reciprocal of the curvature of a curve at any point P is called the radius of curvature at P and is denoted by rho.

- Radius of curvature for Cartesian Curve $y = f(x)$, is given by $\rho = \frac{[1+{y_1}^2]^{\frac{3}{2}}}{y_2}$, where y1 = dy/dx and y2 = d 2y/dx 2
- Radius of curvature for parametric equations $x = f(t)$, $y = g(t)$ is given by $\rho = \frac{(X'^2+Y'^2)^{\frac{3}{2}}}{X'Y''-Y'X''}$ where $Z' = dz/dt$ and Z' = d 2y/dx
- Radius of curvature for polar curve r = f (θ) is given by $\rho = \frac{(r^2+{r_1}^2)^{\frac{3}{2}}}{r^2-rr_2+2{r_1}^2}$

Problem-1

Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the Folium $x^3 + y^3 = 3axy$.

Soln: Differentiating with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \qquad (1)$$

$$\therefore \frac{dy}{dx} \text{at } (3a/2, 3a/2) = -1$$

Differentiating (1),

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x$$

$$\therefore \frac{d^2y}{dx^2} \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{32}{3a}$$

Hence ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1+(-1)^2]^{3/2}}{-32/3a}$

$= \frac{3a}{8\sqrt{2}}$ (in magnitude)

Problem-2

Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ is $4a\cos\theta/2$.

Soln: We have $dy/dx = a(1 + \cos\theta), dy/d\theta = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} + \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 + \cos\theta)} = \frac{2sin\theta/2\cos\theta/2}{2cos^2\theta/2} = \frac{tan\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{1}{2}sec^2\frac{\theta}{2}.\frac{1}{a(1+cos\theta)}$$

$$= \frac{1}{2}sec^2\frac{\theta}{2}.\frac{1}{2acos^2\theta/2} = \frac{1}{4a}sec^4\frac{\theta}{2}$$

$$\rho = \frac{\left[1 + (\frac{dy}{dx})^2\right]^{3/2}}{d^2y/dx^2} = \frac{4a(1 + tan^2\theta/2)^{3/2}}{sec^4\frac{\theta}{2}}$$

$$= 4a.\left(\frac{sec^2\theta}{2}\right)^{\frac{3}{2}}.\frac{cos^4\theta}{2} = 4a\,cos\theta/2$$

Problem-3

Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.

Soln: The parametric equation of the curve is

$$x = a\cos^3 t\,,\ y = a\sin^3 t$$

$$x'(= dx/dt) = -3a\cos^2 t\,\sin t\,,\ y' = 3a\sin^2 t\,\cos t.$$

$$x'' = -3a(\cos^3 t - 2\cos t\,\sin^2 t) = 3a\cos t\,(2\sin^2 t - \cos^2 t)$$

$$y'' = 3a(2\sin t\,\cos t - \sin^3 t)$$

$$= 3a\sin t\,(2\cos^2 t - \sin^2 t)\ x'^2 + y'^2$$

$$= 9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t) = 9a^2\sin^2 t\,\cos^2 t\ x'y'' - y'x''$$

$$= -9a^2\cos^2 t\sin^2 t\,(2\cos^2 t - \sin^2 t)$$

$$-9a^2\cos^2 t\,\sin^2 t\,(2\sin^2 t - \cos^2 t) = -9a\sin^2 t\cos^2 t$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27\,a^3\sin^3 t\cos^3 t}{-9a^2\sin^2 t\cos^2 t} = -3a\sin t\,\cos t.$$

Problem-3 Cont...

Since $\frac{dy}{dx} = \frac{y'}{x'} - tant,$

$\therefore$ Equation of the tangent at $(a\cos^3 t, a\sin^3 t)$ is

$$y - a\sin^3 t = -\tan t\,(x - a\cos^3 t)$$

$$x\tan t + y - a\sin t = 0$$

length of $\perp$ from (0,0) on (2) $= \frac{0+0-a\sin t}{\sqrt{(\tan^2 t+1)}} - a\sin t\cos t.$

Thus $\rho = 3p$.

Problem-4

Show that the radius of curvature at any point of the cardioid $r = a(1 - \cos\theta)$ varies as $\sqrt{r}$.

Soln: Differentiating w.r.t. θ, we get

$$r_1 = a\sin\theta, r_2 = a\cos\theta$$

$$\therefore (r^2 + r_1{}^2)^{3/2} = [a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^2[2(1 - \cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1{}^2 = a^2(1 - \cos\theta)^2 - a^2(1 - \cos\theta)\cos\theta + 2a^2\sin^2\theta$$

$$= 3a^2(1 - \cos\theta)$$

Thus $\rho = \frac{(r^2+r_1{}^2)^{3/2}}{r^2 - rr_2 + 2r_1{}^2} = \frac{a^3 2\sqrt{2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$

$$= \frac{2\sqrt{2}}{3}a(1 - \cos\theta)^{1/2} = \frac{2\sqrt{2a}}{3}\left(\frac{r}{a}\right)^{1/2} \alpha\sqrt{r}$$

CENTRE OF CURVATURE

Let Γ be a simple curve having tangent at each point. At any point P on this curve we can draw a circle having the same curvature at P as the curve Γ.

This circle is called the circle of curvature and its centre is called the centre of curvature and its radius is the radius of curvature of Γ at *P*. Centre of curvature at any point $P(x, y)$ on the curve $y = f(x)$ is given by $\bar{x} = x - \frac{y_1(1+{y_1}^2)}{y_2}$

$$\bar{y} = y - \frac{1 + {y_1}^2}{y_2}$$

Equation of the circle of curvature at *P* is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

EVOLUTE

The locus of centre of curvature of a given curve Γ is called the evolute of the curve. The given curve Γ is called an involute of the evolute. In fact, for the evolute there are many involutes

Procedure to Find the Evolute

Let $y = f(x)$ (1) be the equation of the given curve. If $(\bar{x}, \bar{y})$ is the centre of curvature at any point $P(x, y)$ on (1), then

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} \qquad (1)$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} \qquad (2)$$

Eliminating x, y using (1), (2) and (3), we get a relation in $\bar{x}, \bar{y}$.

Replacing $\bar{x}$ by x and $\bar{y}$ by y, we get the equation of locus of $(\bar{x}, \bar{y})$, which is the evolute of the given curve.

Problem :1

Find the centre of curvature at the point of P(2,4) on the parabola $y^2 = 4(x+2)$

Given the curve $y^2 = 4(x+2)$

tial

Differentiating (1) w. r. to x $2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$$\frac{d^2y}{dx^2} = -\frac{2}{y^2}\left(\frac{dy}{dx}\right) = \left(\frac{-2}{y^2}\right)\left(\frac{2}{y}\right) = -\frac{4}{y^3}$$

At (2,4), $y_1 = \frac{1}{2}$ *and* $y_2 = -\frac{1}{16}$

Let $C(\dot{x}, \bar{y})$*be the centre of curvature at the point* $P(2,4)$ *on the curve.* $y^2 = 4(x+2)$.

Problem :1 cont,...

Then

$$\bar{x} = x - \frac{y_1}{y_2}(1 + {y_1}^2) = 2 - \frac{1}{2}(-16)\left[1 + \frac{1}{4}\right] = 2 + 8\left(\frac{5}{4}\right) = 12$$

$$\bar{y} = y + \frac{(1 + {y_1}^2)}{y_2} = 4 + (-16)\left[1 + \frac{1}{4}\right] = 4 - \frac{5}{4}(16) = -16$$

The centre of curvature is $C(12, -16)$.

Problem :2

Find the equation to the circle of curvature of the curve $xy = c^2$ *at* (c,c)

$xy = c^2$ *Differentiating this twice w.r.to* x, $\frac{dy}{dx} = -\frac{c^2}{x^2}$

$\frac{d^2y}{dx^2} = \frac{c^2}{x^2}$

At the point (c,c) $y_1 = -1$ *and* $y_2 = \frac{2}{c}$

$\rho = \frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2} = \frac{c}{2}(1+1)^{\frac{3}{2}} = c\sqrt{2}e$

Let $C(\bar{x},\bar{y})$ *be the centre of curvature.*

Then $\bar{x} = x - \frac{y_1}{y_2}(1+y_1{}^2) = c + \frac{c}{2}(1+1) = 2c$

$$\bar{y} = y + \frac{1+y_1{}^2}{y_2} = c + \frac{c}{2}(1+1) = 2c$$

Equation of the circle of curvature is $(x-\bar{x})^2 = (y-\bar{y})^2 = \rho^2$

$$\Rightarrow (x-2c)^2 + (y-2c)^2 = 2c^2$$

Problem-1

Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is

$$27ay^2 = 4(x - 2a)^3$$

Soln: We have $2yy_1 = 4a$ i.e. $y_1 = 2a/y$ and

$$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the center of curvature, then

$$\begin{aligned} \bar{x} &= x - \frac{y_1(1 + y_1^2)}{y_2} = x - \frac{2a/y(1 + 4a^2/y^2)}{-4a^2/y^3} \\ &= x + \frac{y^2 + 4a^2}{2a} = x + \frac{4ax + 4a^2}{2a} = 3x + 2a \end{aligned}$$

and

Problem-1 Cont...

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^2}$$

$$= \quad y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}}$$

To find the evolute, we have to eliminate x from (4) and (5)

$$\therefore (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$$

Or $27a(\bar{y})^2 = 4(\bar{x}-2a)^3$

Thus the locus of $(\bar{x},\bar{y})$ i.e., evolute is $27ay^2 = 4(x-2a)^3$.

Problem-2

Show that the evolute of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ is another equal cycloid.

Soln: We have $y_1 = \frac{dy}{d\theta} + \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\frac{\theta}{2}$.

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}(\cot\theta/2) \cdot \frac{d\theta}{dx}$$
$$= -\operatorname{cosec}^2\theta/2 \cdot 1/2 \cdot \frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the center of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} =$$

$$a(\theta - \sin\theta) + \cot\theta/2(-4a\sin^4\theta/2)(1+\cot^2\theta/2)$$
$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2} \cdot 4a\sin^4\theta/2 \cdot \operatorname{cosec}^2\theta/2$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

Problem-2 Cont...

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = a(1 - \cos\theta) + (1 + \cot^2\theta/2)(-4a\sin^4\theta/2)$$

$$= a(1 - \cos\theta) - 4a\sin^4\theta/2 \cdot \operatorname{cosec}^2\theta/2$$

$$a(1 - \cos\theta) - 4\sin^2\theta/2$$

$$a(1 - \cos\theta) - 2a(1 - \cos\theta) = -a(1 - \cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ i.e., the evolute, is given by

$x = a(\theta + \sin\theta), y = -a(1 - \cos\theta)$ which is another equal cycloid.

ENVELOPE

Consider the system of straight lines $y = mx + \frac{1}{m}$ (1) where m is a parameter. For different values of m, we have different straight lines and so (1) represents a family of straight lines. Each member of this family touches the curve $y^2 = 4x$. So, these lines cover the curve $y^2 = 4x$. This curve is called the envelope of the family of lines. We shall now define envelope.

Definition:

Let $f(x, y, \alpha) = 0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

Problem-1

Find the envelope of the family of lines $y = mx + \sqrt{(1+m^2)}$, m being the parameter.

Soln: We have $(y - mx)^2 = 1 + m^2$

Differentiating (6) partially with respect to m,

$$2(y - mx)(-x) = 2m \text{ or } m = xy/(x^2 - 1)$$

Now eliminate m from (6) and (7).

Substituting the values of m in (6), we get

$\left(y - \frac{x^2 y}{x^2-1}\right)^2 = 1 + \left(\frac{xy}{x^2-1}\right)^2$ or $y^2 = (x^2 - 1)^2 + x^2y^2$

$x^2 + y^2 = 1$ which is the required equation of the envelope

Problem-2

Find the envelope of a system of concentric and coaxial ellipses of constant area.

Soln: Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are the parameters. The area of the ellipse $= \pi ab$ which is given to be constant, say $= \pi c^2$

$$ab = c^2 \text{ or } b = c^2/2$$

Substituting in (8),

$$\frac{x^2}{a^2} + \frac{y^2}{(c^2/a)^2} = 1 \text{ or } x^2a^{-2} + (y^2/c^4)a^2 = 0$$

which is given family of ellipses with a as the only parameter.

Problem-2 Cont...

Differentiating partially (10) with respect to a,

$$-2x^2a^{-3} + 2(y^2/c^4)a = 0 \text{ or } a^2 = c^2x/y$$

Eliminate a from (10) and (11)

Substituting the values of a^2 in (10), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope.

Problem-3

Find the evolute of the parabola $y^2 = 4ax$.

Soln: Any normal to the parabola is $y = mx - 2am - am^3$

Differentiating it with respect to m partially,

$0 = x - 2a - 3am^2$ or $m = [(x - 2a)/3a]^{1/2}$

Substituting this value of m in (12),

$$y = \left(\frac{x-2a}{3a}\right)^{1/2}\left[x - 2a - a\cdot\frac{x-2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2 = 4(x - 2a)^3$$

Which is the evolute of the parabola.

Beta, Gamma Functions

The beta function is defined as

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

where $m, n > 0$. Note that $\beta(p,q) = \beta(q,p)$.

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

where n > 0.

Note: (1) $\Gamma(1) = 1$, (2) $\Gamma(n+1) = n\Gamma(n) = n\,!$, (3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

The relation between Beta and Gamma functions is

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Problems Beta, Gamma Functions

Show that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy, (n > 0)$.

Soln: $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx (n > 0)$

$$= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} y \left(-\frac{1}{y} dy\right)$$

put $y = e^{-x}$

i.e., $x = \log(1/y)$

so that $dx = -(1/y)dy$

$= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$

Problems based on Beta, Gamma Functions

Show that $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}} dx$

Soln: $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}} \left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2} dy$$

put $x = \frac{1}{1+y}$ i.e., $y = \frac{1}{x} - 1$

so that $dx = \frac{-1}{(1+y)^2} dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+Y)^{p+q}} dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_1^0 \frac{1}{z^q - 1} \cdot \frac{1}{(1+1/z)^{p+q}} \left(-\frac{1}{z^2}\right) dz$$

$$= \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}} dz$$

Problems based on Beta, Gamma Functions

Express the following integral in terms of gamma function

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$$

Soln: $\int_0^1 \frac{dx}{\sqrt{(1-x)^4}}$ Put $x^2 = \sin\theta$, i.e., $x = \sin^{1/2}\theta$

so that $dx = 1/2\sin^{-1/2}\theta\cos\theta d\theta$

$$= \int_0^{\pi/2} \frac{1}{2}\frac{\sin^{-1/2}\theta\cos\theta d\theta}{\sqrt{(1-\sin^2\theta)}}$$

$$= 1/2\int_0^{\pi/2} \sin^{-1/2}\theta d\theta$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

Problems based on Beta, Gamma Functions

Express the following integral in terms of gamma function $\pi/2$

$$\int_0^{\pi} \sqrt{(\tan\theta) d\theta}$$

Soln: $\int_0^{\pi/2} \sqrt{(\tan\theta) d\theta} = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta d\theta$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right).$

Problems based on Beta, Gamma Functions

Evaluate $\int_0^\infty e^{-ax}x^{m-1}\sin bx dx$ in terms of Gamma functions.

Soln: We have $\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$

$$= \int_0^\infty e^{-ay}a^m y^{m-1}dy$$

$$\int_0^\infty e^{-ay}y^{m-1}dy = \Gamma(m)/a^m$$

Then $I = \int_0^\infty e^{-ax}x^{m-1}\sin bx dx$

$= \int_0^\infty e^{-ax}x^{m-1}$ (Imaginary part of e^{ibx})dx

$=$ I.P. of $\int_0^\infty e^{-(a-ib)x}x^{m-1}dx$

$=$ I.P. of $\{\Gamma(m)/(a-ib)^m$ by (1)

Problems based on Beta, Gamma Functions

$= \text{I.P. of } \{\Gamma(m)/[r^m(\cos\theta - i\sin\theta)^m]$ where $a = r\cos\theta, b = r\sin\theta$

$= \text{I.P. of } \Gamma(m)/[r^m(\cos m\theta - i\sin m\theta)]$ (Using Demoivre's theorem)

$= \text{I.P. of } \left\{\frac{\Gamma(m)\cdot(\cos m\theta + i\sin m\theta)}{r^m(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\right\}$

$= \frac{\Gamma(m)}{r^m}\sin m\theta$ Where $r = \sqrt{(a^2+b^2)}, \theta = \tan^{-1} b/a.$

Thank You