21MAB101T-CALCULUS AND LINEAR ALGEBRA

Unit II - Fuctions of Several Variables

by

Dr. B. Bira
Department of Mathematics
SRM Institute of Science and Technology
Kattankulathur - 603203.

INTRODUCTION

There are many practical situations in which a quantity of interest depends on the values of two or more variables.

For example (i) the volume of a cylinder is $V = \pi r^2 h$, where r is the radius of the base circle and h is the height of the cylinder. So, V is a function of two variables.

(ii) The volume of a rectangular parallelopiped is V = lbh, where l, b, h are the length, breadth and height. Here V is a function of three variables.

Similarly we can have functions of more than two or three variables. But, for simplicity, we shall deal with functions of two variables and the arguments and results can be extended for more than two variables.

Definitions

Function of two variables: The function of the form $z = f(x,\ y)$, where x and y are independent variables and z is dependent variable is called function of two variables.

Function of several variables: The function of the form $z = f(x_1,\ x_2,\ \cdots\ x_n)$, where $x_1,\ x_2, \cdots x_n$ are independent variables and z is dependent variable is called function of several variables.

Continuity of a function of several variables: A function $f(x,\ y)$ is said to be continuous at (a, b), if $lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Partial derivatives: The derivative of a functions of several variables w.r.t one variable keeping other variables as constant is called partial derivative i.e. if $z = f(x,\ y)$ is a function of x and y then the partial derivative of z w.r.t x is denoted by $\frac{\partial f}{\partial x}$ and the partial derivative of z w.r.t y is denoted by $\frac{\partial f}{\partial y}$. Similarly the highest order partial derivatives are given by $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and so on.

Examples

Example-1: Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.

Solution: Given $z = x^3 + y^3 - 3axy$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = 3x^2 - 3ay, \quad \frac{\partial z}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial y^2} = 6y$$

$$\frac{\partial^2 z}{\partial y \partial x} = -3a, \quad \frac{\partial^2 z}{\partial x \partial y} = -3a.$$

Example-2: Find the first and second partial derivatives of $z = x^3y^3$.

Solution: $\frac{\partial z}{\partial x} = 3x^2y^3$, $\frac{\partial z}{\partial y} = 3x^3y^2$ and $\frac{\partial^2 z}{\partial x^2} = 6xy^3$, $\frac{\partial^2 z}{\partial y^2} = 6yx^3$.

TOTAL DERIVATIVE

Total differential: If $z = f(x, y)$ is a function of two variables x and y, then total differential of z denoted by dz and defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Similarly, if $z = f(x_1,\ x_2,\ \cdots\ x_n)$, then the total differential of z is given by

$$dz = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \cdots + \frac{\partial z}{\partial x_n}dx_n.$$

Total differentiation: Let $z = f(x, y)$ is a function of two variables x and y where x and y are functions of another variable t, then total differentiation of z defined as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Similarly, if $z = f(x_1,\ x_2,\ \cdots\ x_n)$, then the total differentiation of z will be

$$\frac{dz}{dt} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt}dx_n.$$

EXAMPLES

Example: If $u = \sin(\frac{x}{y})$, $x = e^t$ and $y = t^2$, find $\frac{du}{dt}$.

Solution: Given $u = \sin(\frac{x}{y})$, $x = e^t$ and $y = t^2$. We know

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \cos\left(\frac{x}{y}\right)\frac{1}{y}e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)2t$$

Putting $x = e^t$, $y = t^2$ and simplifying we get:

$$\frac{du}{dt} = e^t\cos\left(\frac{e^t}{t^2}\right)\left[\frac{1}{t^2} - \frac{2}{t^3}\right] = e^t\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t-2}{t^3}\right)$$

Example: If $f(x,y) = x^2 + y^2$ where $x = r\cos\theta$ and $y = r\sin\theta$, then find (i) $\frac{df}{dr}$, (ii) $\frac{df}{d\theta}$ and (iii) df.

Solution: We know

$$\frac{df}{dr} = \frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr} = 2x\cos\theta + 2y\sin\theta.$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$ we get $\frac{df}{dr} = 2r(\cos^2\theta + \sin^2\theta) = 2r$.

$$\frac{df}{d\theta} = \frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} = 2x(-r\sin\theta) + 2y(r\cos\theta).$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{df}{dr} = -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0.$$

In order to calculate df, we consider $f(r,\theta) = x^2 + y^2 = r^2$. Hence

$$\frac{df}{dr} = 2r \ \Rightarrow df = 2rdr.$$

Example: A metal box without a top has inside dimensions $6ft$, $4ft$ and $2ft$. If the metal is $0.1ft$ thick, find the approximate values by using the differential.

Solution: If x, y, z are dimension of the metal box, then the volume is $V = xyz$. Then it's differential is given by

$$\begin{aligned} dv &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z} = yzdx + xzdy + xydz \\ &= 8 \times 0.2 + 12 \times 0.2 + 24 \times 0.1 = 6.4 sqft. (\because dx = dy = dz = 0.1) \end{aligned}$$

Example: Find $\frac{dz}{dt}$, when $z = xy^2 + x^2y$ at $x = at^2$ and $y = 2at$.

Ans: $2a^3t^3(8 + 5t)$.

Example: If $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \phi(u, v)$. Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$.

DEFINITION

Homogeneous Function: A function $f(x,y)$ is said to be homogeneous function of degree n if it can be expressed in the form

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right).$$

Example: Test for homogeneity $u = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$.

Solution: We can write it as $\sin u = \left(\frac{x^2+y^2}{x-y}\right) = \frac{x^2}{x}\left(\frac{1+y^2/x^2}{1-y/x}\right) = x\phi\left(\frac{y}{x}\right)$. Hence it is homogeneous.

Euler's Theorem: If u is homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

. Similarly,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Example: If $u = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Solution: We can write it as $\sin u = \left(\frac{x^2+y^2}{x-y}\right)$. Since it is a homogeneous function of degree 1, Hence

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u.$$

Chain Rule: $z = f(x,y)$ is a function of two variables x and y, where x and y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}.$$

DIFFERENTIATION OF IMPLICIT FUNCTIONS

Implicit Function: If x and y are connected by a relation $f(x, y) = c$, then it may not be possible to express y as a single valued function of x explicitly or viceversa. Such a functions are called as implicit functions.

The differentiation of such functions can be carried out by help of total differential. Let $f(x, y) = c$ be an implicit function, then the total differential is given by $df = 0$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad \Rightarrow \frac{dy}{dx} = -\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right), \quad \text{if} \quad f_y \neq 0.$$

This is the formula for the first differential coefficient of an implicit function. Similarly, formula for the second differential coefficient of an implicit function is given by $\frac{d^2y}{dx^2} = -\frac{q^2r-2pqs+p^2t}{q^3}$, where
$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}.$

EXAMPLES

Example: If $xe^{-y} - 2ye^{x} = 1$, then find $\frac{dy}{dx}$.

Solution:

$$f(x, y) = xe^{-y} - 2ye^{x} - 1 = 0 \quad \Rightarrow \frac{\partial f}{\partial x} = e^{-y} - 2ye^{x}, \ \frac{\partial f}{\partial y} = -xe^{-y} - 2e^{x}$$

$$\frac{dy}{dx} = \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

Example: If $u = xlogxy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$

Solution: We know $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$.

Given

$$f(x, y) = x^3 + y^3 + 3xy - 1 \quad \Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x}$$

$$\frac{du}{dx} = x\frac{1}{xy}y + logxy + x\frac{1}{xy}x\left(-\frac{x^2 + y}{y^2 + x}\right) = 1 + logxy - \frac{x(x^2 + y)}{y(y^2 + x)}$$

EXAMPLE:2

If $u = u(\frac{y-x}{xy}, \frac{z-x}{xz})$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

SOLUTION:

Let $v = \frac{y-x}{xy}, w = \frac{z-x}{xz}$

$\Rightarrow v = \frac{1}{x} - \frac{1}{y}, w = \frac{1}{x} - \frac{1}{z}$

So $u = u(v, w)$

Therefore $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}(\frac{-1}{x^2}) + \frac{\partial u}{\partial w}(\frac{-1}{x^2})$

$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$ ———— (1)

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}(\frac{-1}{y^2}) + \frac{\partial u}{\partial w}(0)$

$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}$ ———— (2)

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}(\frac{1}{z^2})$

$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w}$

———— (3)

Therefore $(1) + (2) + (3) \Rightarrow x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

EXAMPLE:3

EXAMPLE:3

If $u = f(x, y)$ where $x = rcos\theta$ and $y = rsin\theta$, prove that $(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$

SOLUTION:

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$

$= cos\theta \frac{\partial u}{\partial x} + sin\theta \frac{\partial u}{\partial y}$————-(1)

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

$= -rsin\theta \cdot \frac{\partial u}{\partial x} + rcos\theta \cdot \frac{\partial u}{\partial y}$

That is $\frac{1}{r}\frac{\partial u}{\partial \theta} = -sin\theta \frac{\partial u}{\partial x} + cos\theta \frac{\partial u}{\partial y}$————-(2)

Squaring both sides of (1) and (2) and adding, we get

$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$

EXAMPLE:4

Find $\frac{dy}{dx}$, when (i) $x^3+y^3=3ax^2y$ and (ii) $x^y+y^x=c$.

SOLUTION:

Let $f(x,y)=x^3+y^3-3ax^2y$

$p=\frac{\partial f}{\partial x}=3x^2-6axy$

$q=\frac{\partial f}{\partial y}=3y^2-3ax^2$

Therefore $\frac{dy}{dx}=-\frac{p}{q}=-\frac{3(x^2-2axy)}{3(y^2-ax^2)}=\frac{x(2ay-x)}{y^2-ax^2}$

(ii) $f(x,y)=x^y+y^x=c$

$p=\frac{\partial f}{\partial x}=yx^{y-1}+y^x logy$

$q=\frac{\partial f}{\partial y}=x^y logx+xy^{x-1}$

Therefore $\frac{dy}{dx}=-\frac{p}{q}=-\frac{yx^{y-1}+y^x logy}{x^y logx+xy^{x-1}}$

Taylor's series expansion for function of two variables:
The Taylor's series expansion of a single variable function f(x) $f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots$ which is an infinite series of powers of h. This idea can be extended to expand $f(x+h, y+k)$ in an infinite series of powers of h and k.
STATEMENT:
If f(x, y) and all its partial derivatives are finite and continuous at all points (x, y), then
$f(x+h, y+k) = f(x,y) + \frac{1}{1!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})f(x,y) + \frac{1}{1!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f(x,y) + \cdots$ ——————-(1)
Taking x= a and y = b in (1), we get
$f(a+h, b+k) = f(a,b) + (\frac{h}{1!}\frac{\partial f(a,b)}{\partial x} + \frac{k}{1!}\frac{\partial f(a,b)}{\partial y}) + \frac{1}{2!}[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}] + \cdots$

put a+h= x, b+k= y so that h= x-a, k= y-b we get

$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$ ———-(2)

This is Taylor's expansion f(x, y) in powers of (x-a) and (y-b) or Taylor's series expansion of f(x, y) at the point (a, b).

put a= 0, b=0 in (2) we get

$f(x,y) = f(0,0) + \frac{1}{1!}[xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + \cdots$

This is called Maclaurin's series for f(x, y) in powers of x and y.

Example 1

EXAMPLE:1

Expand $e^x log(1+y)$ in powers of x and y upto terms of third degree.

SOLUTION:

Let $f(x,y) = e^x log(1+y)$ Therefore $f(0,0) = 0$

$f_x(x,y) = e^x log(1+y)$, Therefore $f_x(0,0) = 0$

$f_y(x,y) = e^x \frac{1}{1+y}$ Therefore $f_y(0,0) = 1$

$f_{xx}(x,y) = e^x log(1+y)$ Therefore $f_{xx}(0,0) = 0$

$f_{yy}(x,y) = -e^x \frac{1}{(1+y)^2}$ Therefore $f_{yy}(0,0) = -1$

$f_{xy}(x,y) = e^x \frac{1}{1+y}$ Therefore $f_{xy}(0,0) = 1$

$f_{xxx}(x,y) = e^x log(1+y)$ Therefore $f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \frac{1}{1+y}$ Therefore $f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x \frac{1}{(1+y)^2}$ Therefore $f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = 2e^x \frac{1}{(1+y)^3}$ Therefore $f_{yyy}(0,0) = 2$

Example 1 Cont...

Now Maclaurin's expansion of $f(x, y)$

$f(x, y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 yf_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$

$e^x log(1+y) = 0 + x(0) + y(1) + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(-1)] + \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)] + \cdots$

$e^x log(1+y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + \cdots$

$e^x log(1+y) = y + xy - \frac{y^2}{2} + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + \cdots$

EXAMPLE:2

Expand $x^2y+3y-2$ in powers of (x-1) and y+2 using Taylor's theorem.

SOLUTION:

Taylor's expansion of $f(x,y)$ in powers of (x-a) and (y-b) is given by

$f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]+\frac{1}{2!}[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)]+\frac{1}{3!}[(x-a)^3f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2f_{xyy}(a,b)+(y-b)^3f_{yyy}(a,b)]+\cdots$ —— —-(1)

Here a= 1, b= -2

$f(x,y)=x^2y+3y-2\Rightarrow f(1,-2)=-2-6-2=-10$

$f_x(x,y)=2xy\Rightarrow f_x(1,-2)=-4$

$f_y(x,y)=x^2+3\Rightarrow f_y(1,-2)=4$

$f_{yy}(x,y)=0\Rightarrow f_{yy}(1,-2)=0$

$f_{xx}(x,y)=2y\Rightarrow f_{xx}(1,-2)=-4$

$f_{xy}(x,y)=2x\Rightarrow f_{xy}(1,-2)=2$

$f_{xxx}(x,y)=0\Rightarrow f_{xxx}(1,-2)=0$

Example 2 Cont...

$f_{xxy}(x,y) = 2 \Rightarrow f_{xxy}(1,-2) = 2$
$f_{xyy}(x,y) = 0 \Rightarrow f_{xyy}(1,-2) = 0$
$f_{yyy}(x,y) = 0 \Rightarrow f_{yyy}(1,-2) = 0$
All partial derivatives of higher order vanish.
Therefore (1)$\Rightarrow x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \frac{1}{3!}[(x-1)^2(0) + 3(x-1)^2(y+2)2 + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$
$\Rightarrow x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$

Example 3

EXAMPLE:3

Expand $f(x,y) = tan^{-1}(\frac{y}{x})$ in powers of (x-1) and (y-1) upto third degree terms. Hence compute $f(1.1, 0.9)$ approximately.

SOLUTION:

$f(x,y) = tan^{-1}(\frac{y}{x}), a = 1, b = 1$ and $f(1,1) = tan^{-1}(1) = \frac{\Pi}{4}$

$f_x(x,y) = \frac{1}{1+\frac{y^2}{x^2}}(-\frac{y}{x^2}) = \frac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = -\frac{1}{2}$

$f_y(x,y) = \frac{1}{1+\frac{y^2}{x^2}}(\frac{1}{x}) = \frac{x}{x^2+y^2} \Rightarrow f_y(1,1) = \frac{1}{2}$

$f_{xx}(x,y) = \frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \frac{2}{4} = \frac{1}{2}$

$f_{yy}(x,y) = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = -\frac{2}{4} = \frac{-1}{2}$

$f_{xy}(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1) = 0$

Example 3 Cont...

$f_{xxx}(x,y) = \frac{2y(x^2+y^2)^2 - 4xy(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$

$\Rightarrow f_{xxx}(1,1) = \frac{-4}{2^3} = \frac{-1}{2}$

$f_{xyy}(x,y) = \frac{-2y^3+6x^2y}{(x^2+y^2)^3} \Rightarrow f_{xyy}(1,1) = \frac{4}{2^3} = \frac{1}{2}$

$f_{xxy}(x,y) = \frac{2x(x^2+y^2)^2 - 4xy(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{2x(x^2+y^2) - 8xy^2}{(x^2+y^2)^3} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3} \Rightarrow f_{xxy}(1,1) =$
$\frac{-4}{8} = \frac{-1}{2}$

$f_{yyy} = \frac{-2x(x^2+y^2)^2 + 4xy(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{-2x(x^2+y^2) + 8xy^2}{(x^2+y^2)^3} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}$

$\Rightarrow f_{yyy}(1,1) = \frac{1}{2}$

Taylor's expansion of $f(x,y)$ in powers of (x-1) and (y-1) is given by
$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)] + \cdots$ ———(1)

$(1) \Rightarrow tan^{-1}(\frac{y}{x}) = \frac{\Pi}{4} + [(x-1)(\frac{-1}{2}) + (y-1)(\frac{1}{2})] + \frac{1}{2!}[(x-1)^2(\frac{1}{2}) + 2(x-1)(y-1)(0) + (y-1)^2(\frac{-1}{2})] + \frac{1}{3!}[(x-1)^3(\frac{-1}{2}) + 3(x-1)^2(y-1)(\frac{-1}{2}) + 3(x-1)(y-1)^2(\frac{1}{2})] + (y-1)^3(\frac{1}{2})$

MAXIMA AND MINIMA FOR FUNCTIONS OF TWO VARIABLES

You have learned maxima and minima of a function f(x) of a single variable in x. We shall extend these ideas to a function f(x, y) of two variables in x and y.

A function f(x, y) is said to have a relative maximum(or simply maximum) at x=a and y=b, if $f(a+h, b+k) < f(a, b)$ for all small values of h and k.

A function f(x, y) is said to have a relative minimum(or simply minimum) at x=a and y=b, if $f(a+h, b+k) > f(a, b)$ for all small values of h and k.

A maximum or minimum value of a function is called its extreme value.

The necessary condition for f(x, y) to have a maximum or a minimum at (a, b) are that $f_x(a, b) = 0, f_y(a, b) = 0$.

NOTE:

f(a, b) is said to be a stationary value of f(x, y) if $f_x(a, b) = 0, f_y(a, b) = 0$ that is the function is stationary at (a, b).

Procedure to find Extremum

Working rule to find maxima and minima of f(x, y):
Step 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial Y}$.
Step 2:Solve the eqeuations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial Y} = 0$ simultaneously to get the solutions $(a, b), (a_1, b_1), (a_2, b_2), \cdots$, which are stationary points of f(x, y)
Step 3: Evaluate $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at each stationary point and $\Delta = rt - s^2$.
Step 4: At the stationary point (a, b) (i) If $rt - s^2 > 0$ and r(or t) ¡ 0, then f(a, b) is a maximum value of f(x, y).
(ii) If $rt - s^2 > 0$ and r(or t) ¿ 0, then f(a, b) is a minimum value of f(x, y).
(iii) If $rt - s^2 < 0$, then f(x, y) has neither a maximum nor a minimum value at (a, b).In this case, the point (a, b) called a saddle point of the function f(x, y).
(iv) If $rt - s^2 = 0$, the case is doubtful and further investigations are required to decide the nature of the extreme values of the function f(x, y).

EXAMPLE:1

Examine the following function for extreme values

$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

SOLUTION:

$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y, \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$

$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4, s = \frac{\partial^2 f}{\partial x \partial y} = 4, t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

The stationary points are given by, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$\Rightarrow 4(x^3 - x + y) = 0, 4(y^3 + x - y) = 0$

$(x^3 - x + y) = 0 - - - - - - - - (1), (y^3 + x - y) = 0 —— (2)$

(1)+(2)$\Rightarrow$ $x^3 + y^3 = 0 \Rightarrow (x + y)(x^2 - xy + y^2) = 0 \Rightarrow x = -y$ or $x^2 - xy + y^2 = 0$

put x = -y in (2) we get $y^3 - y - y = 0 \Rightarrow y(y^2 - 2) = 0$

$\Rightarrow y = 0, y = \sqrt{2}, y = -\sqrt{2}$

corresponding values of x are 0, $-\sqrt{2}, \sqrt{2}$

Therefore the stationary points are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

Example 1 Cont..

$rt - s^2 = 4(3x^2 - 1)4(3y^2 - 1) - 16$
At $(0, 0)$, $rt - s^2 = 16 - 16 = 0$
Therefore the further investigation is required to find the nature of the extreme of f(x, y) at the origin.
At $(\sqrt{2}, -\sqrt{2}), rt - s^2 = 4(3x^2 - 1)4(3y^2 - 1) - 16 = 4(6-1)4(6-1) - 16 = 384 > 0$
$r = 4(3x^2 - 1) = 20 > 0$
Therefore the function attains minimum at $(\sqrt{2}, -\sqrt{2})$ and hence the minimum value is
$f(\sqrt{2}, -\sqrt{2}) = -8$
At $(-\sqrt{2}, \sqrt{2}), rt - s^2 = 384 > 0$
$r = 20 > 0$
Therefore the function attains minimum at $(-\sqrt{2}, \sqrt{2})$ and hence the minimum value is
$f(-\sqrt{2}, \sqrt{2}) = -8$

Example 2

EXAMPLE:2

In a plane triangle, find the maximum value of cos A cos B cos C

SOLUTION:

In triangle $A + B + C = \pi$

Using this condition we express the given function as a function of A and B. Thus

$cosAcosBcosC = cosAcosBcos[\pi - (A + B)] = -cosAcosBcos(A + B)$

Let f(A, B)= -cosA cos B cos (A+B)

$f_A = sinAcosBcos(A + B) + cosAcosBsin(A + B)$

= cos B [sin A cos (A+B)+ cos A sin (A+B)]

= cos B sin (2A+B)

$f_B = cosAsinBcos(A + B) + cosAcosBsin(A + B)$

= cos A[sin B cos (A+B)+ cos B sin (A+B)]

= cos A sin (A+2B)

Example 2 Cont..

$r = f_{AA} = 2cosBcos(2A + B)$
$s = f_{AB} = -sinAsin(2A + B) + cosBcos(2A + B)$
$= cos(2A + 2B)$
$t = f_{BB} = 2cosAcos(A + 2B)$
For maximum and minimum value $\frac{\partial f}{\partial A} = 0$ and $\frac{\partial f}{\partial B} = 0$
$\frac{\partial f}{\partial A} = 0 \Rightarrow cosBsin(2A + B) = 0$———(1)
$\Rightarrow \frac{1}{2}[sin(2A + 2B) + sin(2A)] = 0$
$\Rightarrow sin(2A + 2B) + sin(2A) = 0$———-(2)
$\frac{\partial f}{\partial B} = 0$
$\Rightarrow cosAsin(A + 2B) = 0$
$\Rightarrow \frac{1}{2}[sin(2A + 2B) + sin(2B)] = 0$
$\Rightarrow sin(2A + 2B) + sin(2B) = 0$——-(3)
(2)-(3)$\Rightarrow sin2A = sin2B \Rightarrow A = B$

Example 2 Cont..

substitute A= B in (1), we get cosA sin3A=0

$\Rightarrow cosA = 0, sin3A = 0$

$\Rightarrow A = \frac{\pi}{2}, \frac{3\pi}{2}, \cdots, 3A = 0, \pi, 2\pi, 3\pi, \cdots$

$\Rightarrow A = \frac{\pi}{2}, \frac{3\pi}{2}, \cdots, A = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \cdots$

Therefore $A = \frac{\pi}{3}$

The only stationary point is $A = \frac{\pi}{3}, \frac{\pi}{3}$

$rt - s^2 = \frac{3}{4} > 0, r = -1 < 0$

$r = -1 < 0$

Therefore f(x, y) has a maximum value at $(\frac{\pi}{3}, \frac{\pi}{3})$.

Therfore maximum value is $\frac{1}{8}$.

EXAMPLE:3

Examine the following function for extreme values

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.

SOLUTION:

Given $f(x, y) = 2x^2 - 2y^2 - x^4 + y^4$

$\frac{\partial f}{\partial x} = 4x - 4x^3, \frac{\partial f}{\partial y} = -4y + 4y^3$

$r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2, s = \frac{\partial^2 f}{\partial x \partial y} = 0, t = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

$\frac{\partial f}{\partial x} = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, x = \pm 1$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y(-1 + y^2) = 0 \Rightarrow y = 0, y = \pm 1$

The stationary points are (0, 0), (0, 1), (0, -1),(1, 0), (1, 1), (1, -1), (-1, 0), (-1, 1), (-1, -1)

Example 2 Cont..

points	r	t	s=0	$rt - s^2$	Nature
(0, 0)	4	-4	0	$-16 < 0$	neither maximum nor minimum
(0, 1)	$4 > 0$	8	0	$32 > 0$	minimum
(0, -1)	$4 > 0$	8	0	$32 > 0$	minimum
(1, 0)	$-8 < 0$	-4	0	$32 > 0$	maximum
(1, 1)	$-8 < 0$	8	0	$-64 < 0$	neither maximum nor minimum
(1, -1)	-8 < 0	8	0	$-64 < 0$	neither maximum nor minimum
(-1, 0)	$-8 < 0$	-4	0	$32 > 0$	maximum
(-1, 1)	-8	8	0	$-64 < 0$	neither maximum nor minimum
(-1, -1)	-8	8	0	$-64 < 0$	neither maximum nor minimum

Therefore the function f(x, y) has maximum at (1, 0), (-1, 0)

f(1, 0)= 1= f(-1, 0)

The function f(x, y) has minimum at (0, 1), (0, -1).

Therefore f(0, 1)= f(0, -1)= -1.

Lagranges multipliers method

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one , having least number of independent variables with the help of given relations. Then solve it by the previous method.

For example, Let u= f(x, y, z)———-(1) be the function whose extreme values are to be found subject to the restriction g (x, y, z) = 0 ——(2) between the variables x, y, z.

From (2), express z interms of x and y and substitute in (1), then u becomes function of x , y. Then easily we can solve this by the previous method

Lagranges multipliers method Cont.

WORKING RULE:

1.Construct the auxiliary function $F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$ where λ is an undetermined parameter independent of x, y, z. λ is called Lagranges multiplier.

2. Obtain the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

3.Solve the above equations together with $g(x, y, z) = 0.$

The values of x, y, z so obtained will give the stationary value of f(x, y, z).

EXAMPLE:1

A rectangular box, open at the top, is to have a volume of 32 cubic ft. Find dimensions of box which requires least amount of material for its construction.

SOLUTION:

Let x, y, z be the length, breadth and height of the box respectively. Given volume of the box is 32 cubic feet. xyz = 32 , x, y, z ¿ 0

We want to minimize the amount of material for its construction. That is surface area of the box is to be minimized subject to the condition that the volume of the box i.e. xyz=32.——-(1)

f(x, y, z)= Surface area = xy + 2xz + 2yz [since top is open]

g(x, y, z)= xyz-32

the auxiliary function is

$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$

$F(x, y, z) = xy + 2xz + 2yz + \lambda(xyz - 32)$

where λ is the Lagranges multiplier.

EXAMPLE:1 Cont...

$\frac{\partial F}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0$
$\Rightarrow y + 2z = -\lambda yz$
$-\lambda = \frac{1}{z} + \frac{2}{y}$———(2)
$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0$
$x + 2z = -2\lambda xz$
$-\lambda = \frac{1}{z} + \frac{2}{x}$———(3)
$\frac{\partial F}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda xy = 0$
$2x + 2y = -\lambda xy$
$-\lambda = \frac{2}{x} + \frac{2}{y}$———(4)
(2)-(3)$\Rightarrow \frac{1}{z} + \frac{2}{y} - \frac{1}{z} + \frac{2}{x} = -\lambda + \lambda$
$\frac{2}{y} - \frac{2}{x} = 0$
$\Rightarrow \frac{2}{y} = \frac{2}{x} \Rightarrow x = y$
(3)-(4)$\Rightarrow y = 2z$
Therefore x = y = 2z———(5)
Substitute (5) in (1) $\Rightarrow z^3 = 8 \Rightarrow z = 2$
Therefore by (5), x = 4 , y = 4, z = 2.

EXAMPLE:2

EXAMPLE:2

Shoe that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

SOLUTION:

Let 2x, 2y, 2z be the length, breadth and height of the rectangular solid so that its volume $V = 8xyz = f(x, y, z)$

Let R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$——(1)

$g(x, y, z) = x^2 + y^2 + z^2 - R^2$

Let $F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + 2\lambda x = 0 \Rightarrow \lambda = \frac{-4yz}{x} - - - - - - - (2)$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8xz + 2\lambda y = 0 \Rightarrow \lambda = \frac{-4xz}{y} - - - - - - - (3)$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xy + 2\lambda z = 0 \Rightarrow \lambda = \frac{-4xy}{z} - - - - - - (4)$

From (2) and (3)

$\frac{yz}{x} = \frac{xz}{y}$

$\Rightarrow x^2 = y^2$————(5)

EXAMPLE:2 Cont...

From (3) and (4)
$\frac{xz}{y} = \frac{xy}{z}$
$y^2 = z^2$———(6)
From (5) and (6)
$x^2 = y^2 = z^2$——(7)
(1)$\Rightarrow 3x^2 = R^2 \Rightarrow x^2 = \frac{R^2}{3} \Rightarrow x = \frac{R}{\sqrt{3}}$
Therefore from (7) ,$y = z = \frac{R}{\sqrt{3}}$
Therefore the extreme point is $(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}})$ and the maximum volume $=$ $\frac{8R^3}{3\sqrt{3}}$
That is the rectangular solid is cube.

EXAMPLE:3

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

SOLUTION:

Let 2x, 2y, 2z be the dimensions of the rectangular parallelopiped.

We have to maximize the volume 8xyz subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$———-(1)

f(x, y, z)= 8xyz, $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$F(x,y,z)=8xyz+\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1)$

$\frac{\partial F}{\partial x}=0 \Rightarrow 8yz+\frac{2\lambda x}{a^2}=0 \Rightarrow \lambda=\frac{-4a^2yz}{x}$———-(2)

$\frac{\partial F}{\partial y}=0 \Rightarrow 8xz+\frac{2\lambda y}{b^2}=0 \Rightarrow \lambda=\frac{-4b^2xz}{y}$———-(3)

$\frac{\partial F}{\partial z}=0 \Rightarrow 8xy+\frac{2\lambda z}{c^2}=0 \Rightarrow \lambda=\frac{-4c^2yx}{z}$———-(4)

Equate (2) and (3)

$\frac{a^2yz}{x}=\frac{b^2xz}{y}$

$\Rightarrow \frac{y^2}{b^2}=\frac{x^2}{a^2}-----------(5)$

EXAMPLE:3 Cont...

Equate (3) and (4)
we get $\frac{z^2}{c^2} = \frac{y^2}{b^2}$———(6)
From (5) and (6), we get
$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$———-(7)
substitute (7) in (1), we get $\frac{3x^2}{a^2} = 1 \Rightarrow x = \pm\frac{a}{\sqrt{3}}$
Similarly $y = \pm\frac{b}{\sqrt{3}}$ and $z = \pm\frac{c}{\sqrt{3}}$
So the stationary points are given by $x = \pm\frac{a}{\sqrt{3}}, y = \pm\frac{b}{\sqrt{3}}, z = \pm\frac{c}{\sqrt{3}}$
Therefore there are 8 stationary points.
Since we want maximum value of V, choose the points with xyz positive.
This will occur at 4 of the points. They are
$(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}), (\frac{a}{\sqrt{3}}, \frac{-b}{\sqrt{3}}, \frac{-c}{\sqrt{3}}), (\frac{-a}{\sqrt{3}}, \frac{-b}{\sqrt{3}}, \frac{c}{\sqrt{3}}), (\frac{-a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{-c}{\sqrt{3}})$
Therefore maximum $V = \frac{8abc}{3\sqrt{3}}$

EXAMPLE: 4

Divide the number 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

SOLUTION:

Let 24 be divided into 3 parts x, y, z so that x + y + z = 24———(1) where $x, y, z > 0$

and the product is xy^2z^3

We have to maximize this product subject to (1) Let $f(x, y, z) = xy^2z^3$ and $g(x, y, z) = x + y + z - 24$

Form the auxiliary function $F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$

where λ is the Lagranges multiplier.

$F(x, y, z) = xy^2z^3 + \lambda(x + y + z - 24)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow y^2z^3 + \lambda = 0 \Rightarrow \lambda = -y^2z^3$——(2)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2xyz^3 + \lambda = 0 \Rightarrow \lambda = -2xyz^3$——(3)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 3xy^2z^2 + \lambda = 0 \Rightarrow \lambda = -3xy^2z^2$——(4)

EXAMPLE:4 Cont...

Equate (2) and (3) , $y^2z^3 = 2xyz^3 \Rightarrow y = 2x - - - - - - (5)$
Equate (3) and (4), $2xyz^3 = 3xy^2z^2 \Rightarrow z = 3x$——(6)
Substitute (5) and (6) in (1), we get
$x + 2x + 3x = 24 \Rightarrow 6x = 24 \Rightarrow x = 4$
by (5), y = 8
by (6), z= 12
Therefore the product is maximum if the parts are 4, 8, 12.

JACOBIANS

Jacobians have many important applications such as functional dependence, transformation of variable in multiple integrals, problems in partial differentiation and in the study of existence of implicit functions determined by a system of functional equations.

Definition:

(1)If u and v are continuous functions of two independent variables x and y, having first order partial derivatives, then the determinant

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian determinant or Jacobian of u and v with respect to x and y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(\frac{u}{x}, \frac{v}{y})$ or J.

Thus , $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

JACOBIANS Cont...

(2)If u, v, w are continuous functions of three independent variables x, y, z having first order partial derivatives then the Jacobian of u, v, w with respect to x, y, z is defined as $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Similarly, we can define Jacobians for functions of 4 or more variables.

Properties of Jacobians

Property 1: If u and v are functions of x and y, then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

Property 2: Jacobians of composite functions or chain rule If u and v are functions of x and y, where x and y are functions of r and θ, then

$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

functionally dependent:

The functions u, v, w are said to be functionally dependent if each can be expressed interms of others or equivalently f(u, v, w)=0.

Property 3:

If u, v, w are functionally dependent functions of three independent variables x, y, z then

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

Jacobian of implicit functions:

If $y_1, y_2, y_3, \cdots y_n$ are implicitly given as functions of $x_1, x_2, \cdots, x_n$ by the functional equations $f_i(x_1, x_2, \cdots x_n, y_1, y_2, y_3, \cdots y_n) = 0$ for $i = 1, 2, \cdots n$, then

$\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(x_1,x_2,\cdots x_n)} = (-1)^n \frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(y_1,y_2,\cdots y_n)} \times \frac{\partial(y_1,y_2,\cdots,y_n)}{\partial(x_1,x_2,\cdots,x_n)}$

EXAMPLE:1

(i)If $x = rcos\theta, y = rsin\theta$, then find the Jacobian of x and y with respect to r and θ.

SOLUTION:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = rcos^2\theta + rsin^2\theta = r$$

(ii) In cylindrical coordinates $x = \rho cos\phi, y = \rho sin\phi, z = z$ Show that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$

SOLUTION:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} cos\phi & -\rho sin\phi & 0 \\ sin\phi & \rho cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

EXAMPLE:1 Cont...

(iii) In spherical coordinates $x = rsin\theta cos\phi, y = rsin\theta sin\phi, z = rcos\theta$, Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 sin\theta$

SOLUTION:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} sin\theta cos\phi & rcos\theta cos\phi & -rsin\theta sin\phi \\ sin\theta sin\phi & rcos\theta sin\phi & rsin\theta cos\phi \\ cos\theta & -rsin\theta & 0 \end{vmatrix} = sin\theta cos\phi[0+$$

$$r^2 sin^2\theta cos\phi] - rcos\theta cos\phi[0 - rsin\theta cos\theta cos\phi] - rsin\theta sin\phi[-rsin^2\theta sin\phi - rcos^2\theta sir$$

$$= r^2 sin^3\theta[cos^2\phi + sin^2\phi] + r^2 cos^2\theta sin\theta[cos^2\phi + sin^2\phi]$$

$$= r^2 sin^3\theta + r^2 cos^2\theta sin\theta$$

$$= r^2 sin\theta$$

EXAMPLE:2

If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$, Show that the jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

SOLUTION:

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix} = \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$\frac{x_2^2x_3^2x_1^2}{x_1^2x2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

EXAMPLE:3

If $u = x + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$,evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1, 0).

SOLUTION:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & 2z^2 - x & 4z \end{vmatrix}$$

Therefore at the point (1, -1, 0), $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 20$

EXAMPLE:4

EXAMPLE:4

If $u = x^2 - y^2, v = 2xy$ and $x = rcos\theta, y = rsin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

SOLUTION:

We have $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

since $u = x^2 - y^2, v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

$x = rcos\theta, y = rsin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

Hence $\frac{\partial(u,v)}{\partial(x,y)} = 4(x^2 + y^2) \times r^3$.

EXAMPLE:5

If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

SOLUTION:

Let $f_1 = u - xyz, f_2 = v - x^2 + y^2 + z^2, f_3 = w - x + y + z$

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0 & 1 \end{vmatrix}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

EXAMPLE:6

EXAMPLE:6

If $u = \frac{x+y}{1-xy}$ and $v = tan^{-1}x + tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Are u and v are functionally (related)dependent? If so find the relationship.

SOLUTION:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} & \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \begin{vmatrix} \frac{(1+y^2}{(1-xy)^2} & \frac{(1+}{(1-} \\ \frac{1}{1+x^2} & \frac{1}{1+y} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence u and v are functionally related.

We have $v = tan^{-1}x + tan^{-1}y = tan^{-1}(\frac{x+y}{1-xy}) = tan^{-1}u$

That is $u = tanv$ which is the required relationship between u and v.