

# CALCULUS AND LINEAR ALGEBRA

MATHEMATICS-I

(21MAB101T)

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# Introduction

Matrices find many applications in scientific field and useful in many practical real life problem. For example:

- It is useful in the study of electrical circuits, quantum mechanics and optics
- Matrices play a role in calculation of battery power outputs, resistor conversion of electrical energy into another useful energy using Kirchhoff law of voltage and current
- Matrices can play a vital role in the projection of three dimensional images into two dimensional screens, creating the realistic decreasing motion
- It is useful in wave equation associated with transmitting power through transmission lines
- It can be used to crack or deformities in a solid

# Introduction

- In machine learning we often have to deal with structural data, which is generally represented by matrix
- Car designers analyze eigenvalues in order to damp out noise so that the occupant have a quite ride
- It is also used in structural analysis to calculate buckling margins of safty
- Matrices are used in the ranking of web pages in the Google search
- It can also be used in generalization of analytical motion like experimental and derivatives to their high dimensional
- The usages of matrices in computer side application are encryption of message codes with the help of encryptions in the transmission of sensitive and private data
- Matrices are also used in robotics and automation in terms of base elements for the robot movements which are programmed with the calculation of matrices

**Definition:** Let  $A$  be a square matrix. If there exists a scalar  $\lambda$  and non-zero column matrix  $X$  such that  $AX = \lambda X$ , then the scalar  $\lambda$  is called an eigenvalue/characteristic value/latent value of  $A$  and  $X$  is called the corresponding eigenvector of  $A$ .

**How to find:** We can obtain the eigenvalues and eigenvectors through the following steps:

**Step 1:** Write the characteristic equation as

$|A - \lambda I| = \lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \dots + (-1)^n S_n = 0, \quad n = 2, 3, 4, \dots$ ,  
where

$S_1$  = sum of the main diagonal elements of  $A$ .

$S_2$  = sum of the of minor of main diagonal elements of  $A$  . . . . .

$S_n$  = determinant of  $A$  i.e  $|A|$ .

**Step 2:** Find the eigenvalues by factorizing the characteristic equation as  $(\lambda_1 - a_1)(\lambda_2 - a_2) \cdots (\lambda_n - a_n) = 0$  or by synthetic division.

**Step 3:** Find the eigenvectors  $X$  for each value of  $\lambda$  from the linear system of equation  $(A - \lambda_i I)X = 0$ ,  $i = 1, 2, 3 \dots$

**Example:** Find the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

**Solution:**

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & -2 & 3 \\ 1 & 1 - \lambda & 1 \\ 1 & 3 & -1 - \lambda \end{vmatrix} = \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2 + 1 - 1 = 2,$$

$$S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -4 - 5 + 4 = -5$$

$$S_3 = |A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 6 \Rightarrow \lambda^3 - 2\lambda^2 - 5\lambda - 6 = 0$$

Which can be factorize as

$$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda = 1, -2, 3.$$

**Eigenvector for  $\lambda = 1$ :**

$$\begin{bmatrix} 2-1 & -2 & 3 \\ 1 & 1-1 & 1 \\ 1 & 3 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

i.e.  $\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$$x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 0 + x_3 = 0$$

$$x_1 + 3x_2 - 2x_3 = 0.$$

$$\Rightarrow \frac{x_1}{-3} = -\frac{x_2}{-3} = \frac{x_1}{3} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_1}{1} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Eigenvector for  $\lambda = -2$ :

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0.$$

Solving the above equation as  $x_3 = -(x_1 + 3x_2) \Rightarrow x_1 - 11x_2 = 0$ , then

we get  $X_2 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}.$

Eigenvector for  $\lambda = 3$ :

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + 3x_2 - 4x_3 = 0.$$

$$\Rightarrow \frac{x_1}{5} = -\frac{x_2}{-5} = \frac{x_3}{5} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**Example:** Find the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$



**Solution:**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 1 + 1 + 1 = 3, \quad S_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 + 1 + 1 = 2$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda-1)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 0, 1, 2.$$

**Eigenvector for  $\lambda = 0$ :**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0.$$

$\Rightarrow x_1 = 0$  and  $x_2 = -x_3$ . If we take  $x_3 = k \Rightarrow x_2 = -k$

$$\Rightarrow X_1 = \begin{bmatrix} 0 \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Eigenvector for  $\lambda = 1$ :

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

$$\Rightarrow x_2 = 0 \text{ and } x_3 = 0. \text{ Taking } x_1 = k \Rightarrow X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Eigenvector for  $\lambda = 2$ :

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - x_2 + x_3 = 0$$

$$0x_1 + x_2 - x_3 = 0$$

$$\Rightarrow x_1 = 0 \text{ and } x_2 = x_3. \text{ If } x_3 = k \Rightarrow x_2 = k \Rightarrow X_3 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

**Example:** Find the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

**Solution:** Here  $|A - \lambda I| = \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0 \Rightarrow \lambda = 1, 1, 7$ , some eigenvalues are repeated. Therefore we find the eigenvectors as:

**Eigenvector for  $\lambda = 7$ :**

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{12-6} = -\frac{x_2}{-6-6} = \frac{x_3}{6+12} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3} \Rightarrow X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

**Eigenvector for  $\lambda = 1$ :**

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we observe that all rows are linearly dependent

$$\Rightarrow x_1 + x_2 + x_3 = 0.$$

Now we will construct two linearly independent eigenvectors from the same equation assuming the followings:

$$\text{Assume } x_1 = 0 \Rightarrow x_3 = -x_2 \quad \text{hence} \quad X_2 = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Similarly assuming

$$x_2 = 0 \Rightarrow x_3 = -x_1 \quad \text{hence} \quad X_3 = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

**Symmetric Matrix:** A real matrix  $A$  is said to be symmetric if  $A = A^T$ , where  $T$  stands for transpose.

**Orthogonal Matrix:** Let  $X_1$  and  $X_2$  be two column matrices of same order. Then  $X_1$  and  $X_2$  are said to be orthogonal if  $X_1^T X_2 = 0$

**Example:** Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

**Solution:** Here we can see that  $A = A^T$ , which implies it is a symmetric matrix.

Now  $|A - \lambda I| = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow \lambda = 2, 2, 8$ , some eigenvalues are repeated. Therefore we find the eigenvectors as:

**Eigenvector for  $\lambda = 8$ :**

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{25-1} = -\frac{x_2}{10+2} = \frac{x_3}{2+10} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \Rightarrow X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

**Eigenvector for  $\lambda = 2$ :**

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Here one}$$

can observe that all rows are linearly dependent  $\Rightarrow -2x_1 + x_2 - x_3 = 0$ .

Assume  $x_1 = 0 \Rightarrow x_3 = x_2$  hence  $X_2 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

For the next eigenvalue  $\lambda = 2$ , we consider  $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

As the matrix  $A$  is symmetric, so the eigenvectors are orthogonal.

$$\therefore X_1^T X_3 = 0 \Rightarrow \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow 2a - b + c = 0. \text{ again}$$

$$X_2^T X_3 = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow b + c = 0.$$

Solving the above two equations we get  $a = b = -c \Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

**Example:** Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$



**Property 1:** Every square matrix and its transpose has same eigenvalues.

**Example:** If  $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} \Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = 6, -1.$

$$A^T = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix} \Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = 6, -1.$$

**Property 2:** If  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$  then  $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$  are the eigenvalues of  $A^{-1}$ .

**Proof:** Let  $\lambda$  be the eigenvalue of a matrix  $A \Rightarrow AX = \lambda X$ , where  $X$  is an eigenvector  $X \neq 0$ . If we multiply  $A^{-1}$  with  $AX = \lambda X$  as below:

$$A^{-1}AX = A^{-1}\lambda X \Rightarrow IX = \lambda A^{-1}X \Rightarrow \frac{1}{\lambda}X = A^{-1}X.$$

$\therefore \frac{1}{\lambda}$  is the eigenvalue of  $A^{-1}$ .

**Property 3:** If  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$ , then  $\lambda_1^2, \lambda_2^2, \lambda_3^2, \dots, \lambda_n^2$  are the eigenvalues of  $A^2$ .

**Proof:** Let  $\lambda$  be the eigenvalue of a matrix  $A$ .

$\therefore AX = \lambda X$ , where  $X$  is an eigenvector  $X \neq 0$ . If we multiply  $A$  with  $AX = \lambda X$  as below:

$$AAX = A\lambda X \Rightarrow A^2X = \lambda AX \Rightarrow A^2X = \lambda^2X.$$

$\therefore \lambda^2$  is the eigenvalue of  $A^2$ .

**Property 4:** If  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$ , then  $k\lambda_1, k\lambda_2, k\lambda_3, \dots, k\lambda_n$  are the eigenvalues of  $kA$ .

**Proof:** Let  $\lambda$  be the eigenvalue of a matrix  $A$ .

$$\therefore AX = \lambda X \Rightarrow kAX = k(\lambda X) = (k\lambda)X.$$

$\therefore k\lambda$  is the eigenvalue of  $kA$ .

**Property 5:** The eigenvalues of a real symmetric matrix are all real.

**Proof:** Let  $\lambda$  be the eigenvalue of a matrix  $A$ .

$$AX = \lambda X \quad (1)$$

Taking conjugate on both sides of (1) we get  $\bar{A}\bar{X} = \bar{\lambda}\bar{X}$ . As  $A$  is real

$\therefore A = \bar{A} \Rightarrow A\bar{X} = \bar{\lambda}\bar{X}$ . Taking transpose on both side one can get

$$(A\bar{X})^T = (\bar{\lambda}\bar{X})^T \Rightarrow \bar{X}^T A^T = \bar{\lambda}^T \bar{X}^T \Rightarrow \bar{X}^T A = \bar{\lambda} \bar{X}^T$$

( $\because A$  is symmetric  $A = A^T$  and  $\lambda$  is a scalar). Now post multiply by  $X$

$$\bar{X}^T AX = \bar{\lambda} \bar{X}^T X \Rightarrow \bar{X}^T \lambda X = \bar{\lambda} \bar{X}^T X \Rightarrow \lambda \bar{X}^T X = \bar{\lambda} \bar{X}^T X \Rightarrow \lambda = \bar{\lambda}.$$

**Property 6:** If  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$ , then trace of  $A$  = sum of eigenvalues =  $\lambda_1 + \lambda_2 + \lambda_3, \dots, + \lambda_n$  and product of eigenvalues of  $A$  =  $|A|$  i.e.  $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3, \dots, \lambda_n$ .

**Property 6:** Eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

**Proof:** Let  $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0.$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}.$$

**Example:** Find the sum and product of the eigenvalues of a matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}.$$

**Proof:** We know sum of eigenvalues of  $A$  = Sum of the leading diagonal elements of  $A$  = trace of  $A$  =  $-2 + 1 + 0 = -1$ .

Product of the

$$\text{eigenvalues} = |A| = -2(0 - 12) - 2(0 - 6) - 3(-4 + 1) = 45.$$

**Example:** Two of the eigenvalues of  $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$  are 3 and 6.

Find the eigenvalues of  $A^{-1}$ .

**Solution:** Let  $\lambda_1, \lambda_2, \lambda_3$  are eigenvalues of  $A$ .

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3 = 11$$

$$\text{As } \lambda_1 = 3, \lambda_2 = 6 \Rightarrow \lambda_3 = 2$$

$$\therefore \text{Eigenvalues of } A^{-1} \text{ are } \frac{1}{2}, \frac{1}{3}, \frac{1}{6}.$$

**Example:** If 2 and 3 are eigenvalues of  $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ . Find the eigenvalues of  $A^{-1}$  and  $A^3$ .

**Solution:** Let  $\lambda_1, \lambda_2, \lambda_3$  are eigenvalues of  $A$ .

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3 + 2 + \lambda_3 = 3 - 3 + 7 = 7 \Rightarrow \lambda_3 = 2$$

$$\therefore \text{Eigenvalues of } A^{-1} \text{ are } \frac{1}{2}, \frac{1}{2}, \frac{1}{3}.$$

$$\text{and eigenvalues of } A^3 \text{ are } 2^3, 2^3, 3^3.$$

**Example:** Find the constant  $a$  and  $b$  such that  $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$  matrix has 3 and -2 as eigenvalues.

**Solution:**  $a + b = 3 - 2 = 1$  and  $ab - 4 = 3 \times -2 = -6$

$$\therefore b = 1 - a \Rightarrow a(1 - a) - 4 = -6 \Rightarrow a(1 - a) = -2$$

$$\Rightarrow a = 2, -1 \Rightarrow b = -1, 2.$$

**Example:** Two eigenvalues of  $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$  are equal and they are double the third. Find the eigenvalues of  $A^2$ .

**Solution:** Let the third eigenvalue is  $\lambda$ . Therefore the three eigenvalues are  $\lambda, 2\lambda, 2\lambda$ .  $\Rightarrow \lambda + 2\lambda + 2\lambda = 4 + 3 - 2 \Rightarrow 5\lambda = 5 \Rightarrow \lambda = 1$

$\therefore$  The eigenvalues are 1, 2, 2 and eigenvalues of  $A^2$  are 1, 4, 4.

**Statement:** Every square matrix satisfies its own characteristic equation.

i.e If  $A$  is any  $n \times n$  matrix and

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - S_3\lambda^{n-3} \dots + (-1)^n S_n = 0$$

is the characteristic equation then

$$A^n - S_1A^{n-1} + S_2A^{n-2} - S_3A^{n-3} \dots + (-1)^n S_n = 0.$$

**Example:** Verify Cayley-Hamilton theorem and hence find  $A^{-1}$  for

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}.$$

**Solution:** The characteristic equation can be obtained from

$$\begin{vmatrix} 8 - \lambda & -8 & 2 \\ 4 & -3 - \lambda & -2 \\ 3 & -4 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$



Now we need to show that  $A^3 - 6A^2 + 11A - 6I = 0$ . For that we find the followings:

$$A^2 = A.A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -24 \\ 33 & -52 & 13 \end{bmatrix}$$

$$\begin{aligned} \text{Now } A^3 - 6A^2 + 11A - 6I &= \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -24 \\ 33 & -52 & 13 \end{bmatrix} - \begin{bmatrix} 156 & -192 & -12 \\ 84 & -90 & -24 \\ 66 & -93 & 18 \end{bmatrix} + \\ &\begin{bmatrix} 88 & -88 & -22 \\ 44 & -33 & -22 \\ 33 & 44 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0. \end{aligned}$$

**Finding  $A^{-1}$ :** Let us premultiply the equation  $A^3 - 6A^2 + 11A - 6I = 0$  by  $A^{-1}$ , then we get:  $A^2 - 6A + 11I - 6A^{-1} = 0 \Rightarrow 6A^{-1} = [A^2 - 6A + 11I]$ .

$$\Rightarrow 6A^{-1} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} - \begin{bmatrix} 48 & -48 & -12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{6} \begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}.$$

**Example:** Using Cayley-Hamilton theorem find the inverse of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}.$$

**Solution:** The characteristic equation can be obtain from

$$\begin{vmatrix} 2 - \lambda & 1 \\ 1 & -5 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 3\lambda - 11 = 0 \Rightarrow A^2 + 3A - 11I = 0.$$

$$\Rightarrow A + 3I = 11A^{-1} \quad \Rightarrow A^{-1} = \frac{1}{11}[A + 3I] = \frac{1}{11} \begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix}.$$

**Example:** Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ and use it to find } A^{-1} \text{ and } A^4.$$

**Solution:** The characteristic equation can be obtained from

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$\text{Now } A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5 \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} +$$

$$9 \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{Multiplying by } A^{-1} \text{ gives } A^{-1} = A^2 - 5A + 9I = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\text{Multiplying by } A \text{ gives } A^4 = 5A^3 - 9A^2 + A = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & -23 \end{bmatrix}.$$

**Diagonalizable Matrix:** A matrix  $A$  is diagonalizable if there exists an invertible matrix  $P$  such that  $P^{-1}AP = D$ . Where  $D$  is diagonal matrix.

**Remark 1:** A square matrix  $A$  of order  $n$  is diagonalizable if and only if it has  $n$  linearly independent eigenvectors.

**Remark 2:** A square matrix  $A$  of order  $n$  has always  $n$  linearly independent eigenvectors when it's eigenvalues are distinct.

**Remark 3:** If  $P^{-1}AP = D$ , then  $A^m = PD^mP^{-1}$

**Orthogonal Transformation:** Let  $A$  be a square symmetric matrix. Let  $N$  be the other square matrix whose columns are normalized eigenvectors of  $A$ . Then the transformation of the form  $N^TAN = D$  is called orthogonal reduction/orthogonal transformation and  $D$  is called diagonal matrix.

**Note:** This is possibly only for real matrix.

# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

**Example:** Diagonalize the matrix  $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$  by means of orthogonal transformations.

**Solution:** The characteristic equation of  $A$  is given by  $|A - \lambda I| = 0$ . i.e

$$\begin{vmatrix} 3 - \lambda & 1 & 1 \\ 1 & 3 - \lambda & -1 \\ 1 & -1 & 3 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$$

Solving this we get  $\lambda = 1, 4, 4$

**Eigenvector for  $\lambda = 1$ :**

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

Which gives  $\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{-3} \Rightarrow X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ .

Eigenvector for  $\lambda = 4$ :

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = 0, \quad x_1 = x_2 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

As the eigenvalue  $\lambda = 4$  is repeated so the other vector can be evaluated

as below. Let us consider  $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

Then find  $X_1^T X_3 = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow -a + b + c = 0.$

Similarly  $X_2^T X_3 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow a + b = 0 \Rightarrow a = -b.$

Solving the above two equations we get

$a = -b$  and  $c = -2b \Rightarrow X_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$  The normalized vectors are

$$\tilde{X}_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$



# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

$$\Rightarrow N = [\tilde{X}_1 \quad \tilde{X}_2 \quad \tilde{X}_3] = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}. \text{ Now}$$

$$D = N^T A N$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

**Example:** Reduce the matrix  $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$  to a diagonal form using orthogonal transformations.

**Solution:** The characteristic equation of  $A$  is given by  $|A - \lambda I| = 0$ . i.e

$$\begin{vmatrix} 8 - \lambda & -6 & 2 \\ -6 & 7 - \lambda & -4 \\ 2 & -4 & 3 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0.$$

Solving this we get  $\lambda = 0, 3, 15$

Eigenvectors are given as for

$$\lambda = 0 \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \text{ for } \lambda = 3 \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ for } \lambda = 15 \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

# Diagonalisation of Matrix by Orthogonal Transformation

## Unit-I

∴ The normalized vectors are

$$\tilde{X}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad \tilde{X}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

$$\text{Hence } N = [\tilde{X}_1 \quad \tilde{X}_2 \quad \tilde{X}_3] = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

$$D = N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

**Definition:** An homogeneous expression of second degree in any number of unknowns is called quadratic form.

**Example:**  $Q = x_1^2 - 2x_2^2 + 4x_3^2 - 3x_1x_2 + 4x_2x_3 + 6x_1x_3$  is Q.F in three unknowns  $x_1, x_2, x_3$ .

**Matrix representation:** Let  $Q = ax^2 + 2hxy + by^2$ , then

$$Q = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^T A X$$

where  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$  (A is symmetric matrix).

**Note :** The diagonal entries of the the symmetric matrix A are the square terms in Q.

**Note :** The non-diagonal entries of the the symmetric matrix A are the half of the product terms in Q.

**Example:** Write the quadratic form as product of matrices

$$Q = x_1^2 - 2x_2^2 + 3x_3^2 - 4x_1x_2 + 5x_2x_3 + 6x_1x_3.$$

**Solution:** The required form is

$$Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T A X.$$

**Example:** Write the quadratic form where  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 3 \end{bmatrix}$ .

**Solution:** The required form is

$$Q = x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1x_2 + 6x_1x_3 + 18x_2x_3.$$

**Canonical Form:** The transformed Q.F is called as canonical form.

**Index:** The number of positive terms in the canonical form is called the index of the form and it is denoted by  $p$ .

**Rank:** The number of non-zero eigenvalues is called the rank of the form and it is denoted by  $r$ .

**Signature:** The difference between positive terms  $p$  and negative terms  $(r - p)$  in the canonical form is called the signature of the form and it is denoted by  $p - (r - p) = 2p - r$ .

**Positive Definite:** A Q.F  $Q = X^T A X$  in  $n$  variables where  $|A| \neq 0$  is called positive definite  $Q = X^T A X > 0$  i.e  $r = p = n$ .

**Positive Semi-Definite:** A Q.F  $Q = X^T A X$  in  $n$  variables where  $|A| \neq 0$  is called positive semi-definite  $r = p < n$ .

**Negative Definite:** A Q.F  $Q = X^T A X$  in  $n$  variables where  $|A| \neq 0$  is called negative definite  $p = 0$ , and  $r = n$ .

**Negative Semi-Definite:** A Q.F  $Q = X^T A X$  in  $n$  variables where  $|A| \neq 0$  is called negative semi-definite  $p = 0$ , and  $r < n$ .

**Indefinite:** A Q.F  $Q = X^T A X$  in  $n$  variables where  $|A| \neq 0$  is called indefinite if none of the above things happened.

The following steps are followed to construct the canonical form:

**Step 1:** First write the Q.F as  $Q = X^T A X$ .

**Step 2:** Find the eigenvalues and corresponding eigenvectors of  $A$ .

**Step 3:** Normalize the eigenvectors as  $\bar{X}_1, \bar{X}_2, \bar{X}_3$  and write the normalized modal matrix  $P = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \end{bmatrix}$ .

**Step 4:** Find  $P^T A P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ .



**Step 5:** Assume the transformation  $X = PY$  where  $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ . Then write  $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ .

$$\begin{aligned} \text{If we take } X = PY &\Rightarrow Q = X^T A X = (PY)^T A (PY) = Y^T P^T A P Y \\ &= Y^T (P^T A P) Y = Y^T \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2. \end{aligned}$$

As  $P$  is orthogonal  $\Rightarrow X = PY$  is orthogonal.

**Example:** Reduce the Q.F  $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$  to a diagonal canonical form and hence find it's nature, rank, index and signature.

**Solution:** Given

$$Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T A X.$$

The characteristic equation of  $A$  is given by  $|A - \lambda I| = 0$ . i.e

$$\begin{vmatrix} 3 - \lambda & -1 & 1 \\ -1 & 5 - \lambda & -1 \\ 1 & -1 & 3 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

Solving this we get  $\lambda = 2, 3, 6$

Eigenvectors are given as for  $\lambda = 2 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$

for  $\lambda = 3 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$  for  $\lambda = 6 \Rightarrow X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

∴ The normalized vectors are

$$\bar{X}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

$$\text{Hence } P = [\bar{X}_1 \quad \bar{X}_2 \quad \tilde{X}_3] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}.$$

Now  $P$  is orthogonal. Let  $X = PY$  be orthogonal transformation then

$$Q = X^T A X = (PY)^T A (PY) = Y^T P^T A P Y = Y^T (P^T A P) Y$$

$$\text{As } D = P^T A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$\Rightarrow Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow Q = 2y_1^2 + 3y_2^2 + 6y_3^2.$$

Here the number of non-zero eigenvalues are 3, therefore the rank  $r = 3$ .

Again the number of positive eigenvalues are 3, therefore index  $p = 3$ .

$\therefore$  The signature is,  $2p - r = 6 - 3 = 3$ .

The quadratic form is positive definite, since all the eigenvalues are positive.

**Example:** Reduce the Q.F  $Q = x_1^2 + 2x_2x_3$  into a canonical form by means of an orthogonal transformation. Determine it's nature, rank, index and signature.

**Solution:** Given  $Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T A X.$

The characteristic equation of  $A$  is given by  $|A - \lambda I| = 0$ . i.e

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 0 - \lambda & 1 \\ 0 & 1 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Solving this we get  $\lambda = -1, 1, 1$ .

Eigenvectors are given as for  $\lambda = -1 \Rightarrow X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$

For  $\lambda = 1 \Rightarrow X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ . As the matrix is symmetric and the eigenvalues are repeated, so the third eigenvalue is orthogonal to other two

and can be determined consider  $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\Rightarrow X_1^T X_3 = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow -b + c = 0 \Rightarrow b = c.$$

$$\text{again } X_2^T X_3 = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow -a + b + c = 0.$$

Solving the above two equations we get

$$a = 2c, \text{ and } b = c \Rightarrow X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

∴ The normalized vectors are

$$\bar{X}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

$$\text{Hence } P = [\bar{X}_1 \quad \bar{X}_2 \quad \tilde{X}_3] = \begin{bmatrix} 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}.$$

Now  $P$  is orthogonal. Let  $X = PY$  be orthogonal transformation then

$$Q = X^T A X = (PY)^T A (PY) = Y^T P^T A P Y = Y^T (P^T A P) Y$$

$$\text{As } D = P^T A P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow Q = y_1^2 + y_2^2 - y_3^2.$$

Here the number of non-zero eigenvalues are 3, therefore the rank  $r = 3$ .

Again the number of positive eigenvalues are 2, therefore index  $p = 2$ .

$\therefore$  The signature is,  $2p - r = 4 - 3 = 1$ .

As one eigenvalue is negative and two eigenvalues are positive, the given quadratic form is indefinite.