CALCULUS AND LINEAR ALGEBRA

MATHEMATICS-I

(21MAB101T)

DEPARTMENT OF MATHEMATICS

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Definition: The differential equations involving one independent variable and one or more than one dependent variables are called Ordinary differential Equations (ODEs).

Example: Let $y = y(x)$ be a function, where y is dependent variable and x is the independent variable, then

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-2x}$
2. $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 2y = \sin 3x$
3. $\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} + y = 5x$

are the ODEs.

Order: The order of a differential equation is the order of the highest derivative of the dependent variable present in the equation.

Degree: The degree of a differential equation is the degree of the highest derivative of the dependent variable present in the equation.

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-2x}$, (Here order=2, Degree=1)
2. $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 2y = \sin 3x$, (Here order=2, Degree=1)
3. $\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} + y = 5x$, (Here order=2, Degree=2)

Solution: The functional relation between the dependent variable and the independent variable satisfying differential equation is called the solution for that differential equation.

The ODE of the form

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1} y}{dx^{n-1}}+a_2\frac{d^{n-2} y}{dx^{n-2}}+\cdots\cdots+a_n y=f(x), \tag{1}$$

where $a_0,\ a_1,\ a_2,\cdots a_n$ are constants is called linear differential equation with constant coefficients.

Let us take

$$\frac{d}{dx}=D,\quad \frac{d^2}{dx^2}=D^2,\quad \frac{d^3}{dx^3}=D^3,\quad \cdots\cdots\frac{d^n}{dx^n}=D^n,$$

then the equation (1) can be written as

$$\begin{aligned}(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots\cdots+a_n)y&=f(x),\\ \Rightarrow F(D)y&=f(x).\end{aligned} \tag{2}$$

The complete solution of (2) can be obtain in two steps:

1. Finding the Complementary Function (C.F) from $F(D)y = 0$
2. Finding the Particular Integral (P.I) from $F(D)y = f(x)$

Then the complete solution can be written as $C.S = C.F + P.I.$

How to find C.F

Step-1: Write the auxiliary equation putting $D = m$ in $F(D) = 0$ i.e

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n = 0. \tag{3}$$

Step-2: Solve the auxiliary equation (4) and obtain n roots, let them be $m_1,\ m_2,\ \cdots\cdot m_n$, which may gives the following cases:

Case-I: Let all the roots $m_1,\ m_2,\ \cdots\cdot m_n$ are real and distinct i.e $m_1 \neq m_2 \neq m_3\ \cdots\cdot \neq m_n$, then write the C.F as

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots\cdot + c_n e^{m_n x}.$$

Case-II: Let all the roots are real where some roots are equal and some are distinct say $m_1 = m_2 = m_3$ and $m_4 \neq m_5\ \cdots\cdot \neq m_n$, then write the C.F as

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots\cdot + c_n e^{m_n x}.$$

Case-III: Let the roots are complex say $m = \alpha \pm i\beta$, then write the C.F as

$$C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Example: Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0.$

Solution: Taking $\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2$ in the given equation, we get

$$(D^2 - 7D + 12)y = 0 \quad \Rightarrow F(D)y = 0.$$

Now write the auxiliary equation by putting $D = m$ in

$$F(D) = 0 \quad \Rightarrow m^2 - 7m + 12 = 0 \quad \Rightarrow (m-3)(m-4) = 0 \quad \Rightarrow m = 3,\ 4.$$

Let $m_1 = 3$ and $m_2 = 4$, which are real and distinct i.e $m_1 \neq m_2$.

$\therefore$ The C.F is given by $\boxed{C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{3x} + c_2 e^{4x}.}$

Here there is no need to find P.I as $f(x) = 0$.

Example: Solve $(D^3+3D^2+2D+2)y=0$.

Solution: Now write the auxiliary equation by putting $D=m$ in

$$F(D)=0 \quad \Rightarrow m^3+3m^2+2m+2=0.$$

Solving the above auxiliary equation as:

$$(m+2)(m^2+m+2)=0 \quad \Rightarrow m=-2, \quad \frac{-1\pm i\sqrt{3}}{2}.$$

Let $m_1=-2$ and $m=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}=\alpha+i\beta$ with $\alpha=\frac{-1}{2}$, $\beta=\frac{\sqrt{3}}{2}$.

$$C.F=c_1e^{m_1x}+e^{\alpha x}\left(c_2\cos\beta x+c_3\sin\beta x\right).$$

$\therefore$ The C.F is given by

$$\boxed{C.F=c_1e^{-2x}+e^{-\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x+_3\sin\frac{\sqrt{3}}{2}x\right).}$$

Here there is no need to find P.I as $f(x)=0$.

How to find P.I

Let $F(D)y = f(x)$ be a given differential equation, in this case if $f(x) = 0$ i.e $F(D)y = 0$, then only C.F is the complete solution if $f(x) \neq 0$ then we need to find P.I. For that we discuss the following types of cases:

Type 1: If $f(x) = e^{ax}$, then

$$\text{P.I} = \frac{1}{F(D)} f(x) = \frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{F(a)} \quad \text{provided} \quad F(a) \neq 0.$$

Note: If $F(a) = 0$, then differentiate the denominator by D and multiply the numerator by x i.e. $\frac{x}{F'(D)} e^{ax} = \frac{xe^{ax}}{F'(a)}$ provided $F'(a) \neq 0$.
Similarly, if $F'(a) = 0$ continue the process

$$\frac{x^2}{F''(D)} e^{ax} = \frac{x^2 e^{ax}}{F''(a)} \quad \text{provided} \quad F''(a) \neq 0.$$

Example: Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-2x}$.

Solution: We can write it as
$(D^2+3D+2)y = e^{-2x} \;\Rightarrow F(D) = D^2+3D+2$ and $f(x) = e^{-2x}$. Now write the auxiliary equation by putting $D = m$ in

$$F(D) = 0 \;\Rightarrow m^2+3m+2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m_1 = -1,\;\; m_2 = -2$$

$\therefore$ The C.F is given by $\boxed{C.F = c_1e^{-x} + c_2e^{-2x}.}$

$$\text{P.I} = \frac{1}{D^2+3D+2}e^{-2x} \text{ put } D = -2 \Rightarrow \frac{1}{4-6+2}e^{-2x} = \frac{e^{-2x}}{0} \text{ i.e } F(a) = 0$$

Then differentiate the denominator by D and multiply the numerator by x i.e.

$$\frac{x}{2D+3}e^{-2x} \;\text{ now put }\; D = -2 \Rightarrow \frac{xe^{-2x}}{-4+3} = -xe^{-2x}.$$

$\therefore$ The C.S is given by $\boxed{C.S = C.F + P.I = c_1 e^{-x} + c_2 e^{-2x} - xe^{-2x}.}$

Example: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 3e^{4x}$.

Solution: We can write it as $(D^2 + 6D + 9)y = 3e^{4x} \;\Rightarrow F(D)y = e^{4x}$.
Now the auxiliary equation is given by

$$m^2 + 6m + 9 = 0 \Rightarrow (m+3)(m+3) = 0 \Rightarrow m_1 = -3, \quad m_2 = -3$$

$\therefore$ The C.F is given by $\boxed{C.F = (c_1 + c_2 x)e^{-3x}.}$

$$\text{P.I} = \frac{1}{D^2 + 6D + 9}(3e^{4x}) \;\; \text{putting} \;\; D = 4 \Rightarrow \frac{3}{16 + 24 + 9}e^{4x} = \frac{3e^{4x}}{49}.$$

$\therefore$ The C.S is given by $\boxed{C.S = (c_1 + c_2 x)e^{-3x} + \frac{3e^{4x}}{49}.}$

Example: Solve $(D^2+9)y = e^{-2x}$.

Solution: We can write it as

$$F(D)y = e^{-2x} \Rightarrow m^2+9=0 \Rightarrow m = \pm 3i = \alpha \pm i\beta, \Rightarrow \alpha = 0,\ \beta = 3$$

$\therefore$ The C.F is given by

$$\boxed{C.F = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right) = \left(c_1\cos 3x + c_2 \sin 3x\right).}$$

$$\text{P.I} = \frac{1}{D^2+9}e^{-2x} \ \text{ putting } \ D=-2 \Rightarrow \frac{1}{4+9}e^{-2x} = \frac{e^{-2x}}{13}.$$

$\therefore$ The C.S is given by $\boxed{C.S = \left(c_1\cos 3x + c_2\sin 3x\right) + \frac{e^{-2x}}{13}.}$

Note: If $F(D)y = A = constant$, then while finding P.I take $f(x) = Ae^{0.x}$ i.e put $D=0$ in $F(D)$.

Type 2: If $f(x) = \sin ax$ or $\cos ax$, then

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D)} f(x) = \frac{1}{F(D)} \sin ax \text{ or } \cos ax \\ &= \frac{1}{F(-a^2)} \sin ax \text{ or } \cos ax \quad \text{provided} \quad F(-a^2) \neq 0. \end{aligned}$$

i.e replace D^2 by $-a^2$ provided $F(-a^2) \neq 0$. If $F(-a^2) = 0$ then differentiate the denominator by D and multiply the numerator by x as:

$$\begin{aligned} \text{P.I} &= \frac{x}{F'(D)} \sin ax \text{ or } \cos ax \\ &= \frac{1}{F(-a^2)} \sin ax \text{ or } \cos ax \quad \text{provided} \quad F'(-a^2) \neq 0. \end{aligned}$$

If $F'(-a^2) = 0$, then continue the above process and

$$\frac{x^2}{F''(D)} \sin ax \text{ or } \cos ax = \frac{x^2 e^{ax}}{F''(-a^2)} \quad \text{provided} \quad F''(-a^2) \neq 0.$$

Example: Solve $(D^2+3D+2)y = \sin x$.

Solution: The auxiliary equation is

$$m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m_1=-1,\ m_2=-2.$$

$\therefore$ The C.F is given by $\boxed{C.F = c_1 e^{-x} + c_2 e^{-2x}.}$

$$\begin{aligned}
\text{P.I} &= \frac{1}{D^2+3D+2}\sin x \quad \text{putting} \quad D^2=-1^2 \Rightarrow \frac{1}{-1+3D+2}\sin x \\
&= \frac{1}{3D+1}\sin x = \frac{(3D-1)}{(9D^2-1)}\sin x = \frac{(3D-1)}{-10}\sin x \\
&= -\frac{(3D\sin x - \sin x)}{10} = -\frac{(3\cos x - \sin x)}{10}.
\end{aligned}$$

$\therefore$ The C.S is given by $\boxed{C.S = c_1 e^{-x} + c_2 e^{-2x} - \frac{(3\cos x - \sin x)}{10}.}$

Example: Solve $(D^2+6D+8)y = \cos^2 x$.

Solution: The auxiliary equation is

$$m^2+6m+8=0 \Rightarrow (m+2)(m+4)=0 \Rightarrow m_1=-2,\ m_2=-4.$$

$\therefore$ The C.F is given by $\boxed{C.F = c_1 e^{-2x} + c_2 e^{-4x}.}$

$$\begin{aligned}
\text{P.I} &= \frac{1}{D^2+6D+8}\cos^2 x = \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)\\
&= \frac{1}{2}.\frac{1}{D^2+6D+8}e^{0.x} + \frac{1}{2}.\frac{1}{D^2+6D+8}\cos 2x\\
&= \frac{1}{16} + \frac{1}{2}.\frac{1}{-4+6D+8}\cos 2x = \frac{1}{16} + \frac{1}{2}.\frac{1}{6D+4}\cos 2x\\
&= \frac{1}{16} + \frac{1}{2}.\frac{(6D-4)}{(36D^2-16)}\cos 2x = \frac{1}{16} + \frac{1}{2}.\frac{(6D\cos 2x - 4\cos 2x)}{(36(-2^2)-16)}
\end{aligned}$$

$$= \frac{1}{16} + \frac{1}{2}.\frac{(6D\cos 2x - 4\cos 2x)}{(-144-16)} = \frac{1}{16} - \frac{1}{320}.(-12\sin 2x - 4\cos 2x)$$
$$= \frac{1}{16} + \frac{1}{80}.(3\sin 2x + \cos 2x).$$

$\therefore$ The C.S is given by

$$\boxed{C.S = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{80}.(3\sin 2x + \cos 2x) + \frac{1}{16}.}$$

Example: Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution: The A.E is

$$m^2 - 4m + 3 = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m_1 = 1,\ m_2 = 3.$$

$\therefore$ The C.F is given by $\boxed{C.F = c_1 e^{x} + c_2 e^{3x}.}$

Note: We know $\sin A \cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$

$$
\begin{aligned}
\text{P.I} &= \frac{1}{D^2-4D+3}\sin 3x\cos 2x = \frac{1}{D^2-4D+3}\left(\frac{\sin 5x+\sin x}{2}\right)\\
&= \frac{1}{2}.\frac{1}{D^2-4D+3}(\sin 5x+\sin x)\\
&= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x+\frac{1}{D^2-4D+3}\sin x\right]\\
&= \frac{1}{2}\left[\frac{1}{-5^2-4D+3}\sin 5x+\frac{1}{-1^2-4D+3}\sin x\right]\\
&= \frac{1}{2}\left[-\frac{1}{4D+22}\sin 5x+\frac{1}{-4D+2}\sin x\right]\\
&= \frac{1}{4}\left[-\frac{1}{11+2D}\sin 5x+\frac{1}{-2D+1}\sin x\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\left[-\frac{(11-2D)}{(121-4D^2)}\sin 5x + \frac{(1+2D)}{(1-4D^2)}\sin x\right] \\
&= \frac{1}{4}\left[-\frac{(11\sin 5x - 2D\sin 5x)}{121-4(-5^2)} + \frac{(\sin x + 2D\sin x)}{1-4(-1^2)}\right] \\
&= \frac{1}{4}\left[-\frac{(11\sin 5x - 10\cos 5x)}{221} + \frac{(\sin x + 2\cos x)}{5}\right] \\
&= \left[-\frac{(11\sin 5x - 10\cos 5x)}{884} + \frac{(\sin x + 2\cos x)}{20}\right].
\end{aligned}
$$

$\therefore$ The C.S is given by

$$\boxed{C.S = c_1 e^x + c_2 e^{3x} - \frac{(11\sin 5x - 10\cos 5x)}{884} + \frac{(\sin x + 2\cos x)}{20}.}$$

Remember:

$\sin A \sin B = \frac{\cos(A-B)-\cos(A+B)}{2}, \quad \cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$

Type 3: If $f(x) = x^n$, where n is positive integer, then

$$\text{P.I} = \frac{1}{F(D)}f(x) = \frac{1}{F(D)}x^n = \frac{1}{[1 \pm \phi(D)]}x^n = [1 \pm \phi(D)]^{-1}x^n.$$

i.e express $F(D) = 1 \pm \phi(D)$ and try to use one of the following formulae:

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots\cdots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 - \cdots\cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots$

Example: Solve $(D^2+5D+6)y = x^2$.

Solution: The A.E is

$$m^2+5m+6=0 \Rightarrow (m+2)(m+3)=0 \Rightarrow m_1=-2,\ m_2=-3.$$

$\therefore$ The C.F is given by $\boxed{C.F = c_1 e^{-2x} + c_2 e^{-3x}.}$

$$\begin{aligned}
\text{P.I} &= \frac{1}{F(D)} f(x) = \frac{1}{F(D)} x^2 = \frac{1}{(D^2+5D+6)} x^2 \\
&= \frac{1}{6\left[1+\frac{D^2+5D}{6}\right]} x^2 = \frac{1}{6}\left[1+\frac{D^2+5D}{6}\right]^{-1} x^2 \\
&= \frac{1}{6}\left[1-\left(\frac{D^2+5D}{6}\right)+\left(\frac{D^2+5D}{6}\right)^2+\cdots\right] x^2
\end{aligned}$$

$$= \frac{1}{6}\left[x^2 - \frac{5D}{6}(x^2) - \frac{D^2}{6}(x^2) + \frac{25D^2}{36}(x^2) + \frac{10D^3}{36}(x^2) + \frac{D^4}{6}(x^2) - \cdots\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{5}{6}(2x) - \frac{1}{6}(2) + \frac{25}{36}(2) + \frac{10}{36}(0) + \frac{1}{6}(0) - \cdots\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{5x}{3} - \frac{1}{3} + \frac{25}{18}\right] = \frac{1}{6}\left[x^2 - \frac{5x}{3} + \frac{19}{18}\right].$$

$\therefore$ The C.S is given by $\boxed{C.S = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{6}\left[x^2 - \frac{5x}{3} + \frac{19}{18}\right].}$

Example: Solve $(D^3 - D^2 - 6D)y = x^2 + 1$.

Solution: The A.E is

$m^3 - m^2 - 6m = 0 \Rightarrow m(m+2)(m-3) = 0 \Rightarrow m_1 = 0,\ m_2 = -2,\ m_3 = 3.$

$\therefore$ The C.F is given by $\boxed{C.F = c_1 + c_2 e^{-2x} + c_3 e^{3x}.}$

$$\begin{aligned}
\text{P.I} &= \frac{1}{F(D)} f(x) = \frac{1}{F(D)}(x^2+1) = \frac{1}{(D^3 - D^2 - 6D)}(x^2+1) \\
&= \frac{1}{-6D\left[1 - \frac{D^2-D}{6}\right]}(x^2+1) = -\frac{1}{6D}\left[1 - \frac{D^2-D}{6}\right]^{-1}(x^2+1) \\
&= -\frac{1}{6D}\left[1 - \left(\frac{D^2-D}{6}\right) + \left(\frac{D^2-D}{6}\right)^2 + \cdots\right](x^2+1) \\
&= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} - \cdots\right](x^2+1) \\
&= -\frac{1}{6D}\left[(x^2+1) - \frac{D}{6}(x^2+1) + \frac{7D^2}{6}(x^2+1)\right] \\
&= -\frac{1}{6D}\left[(x^2+1) - \frac{x}{3} + \frac{7}{3}\right] = -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]
\end{aligned}$$

$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right].$

$\therefore$ The C.S is given by

$$\boxed{C.S = c_1 + c_2e^{-2x} + c_3e^{3x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right].}$$

Type 4: If $f(x) = e^{ax}\phi(x)$ where $\phi(x) = x^n$ or $\sin ax$ or $\cos ax$, then

$$\text{P.I} = \frac{1}{F(D)}f(x) = \frac{1}{F(D)}e^{ax}\phi(x) = e^{ax}\frac{1}{(D+a)}\phi(x),$$

i.e replace D by $D+a$ solve $\frac{1}{(D+a)}\phi(x)$ using any one of previous methods.

Example: Solve $(D^2 + D + 1)y = x^2e^{-x}$.

Solution: The A.E is

$$m^2 + m + 1 = 0 \Rightarrow m = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \alpha + i\beta.$$

$\therefore$ The C.F is given by $\boxed{C.F = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right).}$

Now

$$\begin{aligned}
\text{P.I} &= \frac{1}{F(D)}f(x) = \frac{1}{(D^2+D+1)}x^2e^{-x} \\
&= e^{-x}\frac{1}{[(D-1)^2+(D-1)+1]}x^2 \\
&= e^{-x}\frac{1}{[D^2-2D+1+D-1+1]}x^2 = e^{-x}\frac{1}{[D^2-D+1]}x^2 \\
&= e^{-x}\left[1+(D^2-D)\right]^{-1}x^2 \\
&= e^{-x}\left[1-(D^2-D)+(D^2-D)^2\right]x^2 \\
&= e^{-x}\left[1-D^2+D)+D^4-2D^3+D^2\right]x^2 = e^{-x}(x^2+2x).
\end{aligned}$$

$\therefore$ The C.S is given by

$$\boxed{C.S = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + (x^2+2x)e^{-x}.}$$

Example: Solve $(D^2+4D+4)y = e^{3x}\sin x$.

Solution: The A.E is

$$m^2+4m+4=0 \Rightarrow (m+2)(m+2)=0 \Rightarrow m_1=-2,\ m_2=-2.$$

$\therefore$ The C.F is given by $\boxed{C.F = (c_1+c_2x)e^{-2x}.}$

Now

$$\begin{aligned}
\text{P.I} &= \frac{1}{F(D)}f(x) = \frac{1}{(D+2)^2}e^{3x}\sin x \\
&= e^{3x}\frac{1}{(D+5)^2}\sin x = e^{3x}\frac{1}{(D^2+10D+25)}\sin x
\end{aligned}$$

$$
\begin{aligned}
&= e^{3x}\frac{1}{(-1^2+10D+25)}\sin x = e^{3x}\frac{1}{(10D+24)}\sin x \\
&= \frac{e^{3x}}{2}\frac{(12-5D)}{(12+5D)(12-5D)}\sin x = \frac{e^{3x}}{2}\frac{(12-5D)}{144-25D^2}\sin x \\
&= \frac{e^{3x}}{2}\frac{(12\sin x-5D\sin x)}{169} = \frac{e^{3x}(12\sin x-5D\cos x)}{338}.
\end{aligned}
$$

$\therefore$ The C.S is given by

$$\boxed{C.S = (c_1+c_2x)e^{-2x} + \frac{(12\sin x - 5D\cos x)}{338}e^{3x}.}$$

Example: Solve $(D^2+9)y = (x^2+1)e^{3x}\sin x$.

Ans:

$$\boxed{C.S = (c_1\cos 3x + c_2\sin 3x) + \frac{e^{3x}}{18}\left(x^2 - \frac{2x}{3} + \frac{10}{9}\right).}$$

How to find C.S

Step-1: Put
$x\frac{d}{dx} = D'$, $x^2\frac{d^2}{dx^2} = D'(D'-1)$, $x^3\frac{d^3}{dx^3} = D'(D'-1)(D'-2)\cdots$ with $D' = \frac{d}{dx}$ in the given equation and reduce it in to constant coefficients and write as $F(D')y = \phi(z)$, where z independent variable.

Step-2: Write the auxiliary equation putting $D' = m$ in $F(D') = 0$ i.e

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n = 0. \tag{4}$$

Step-3: Solve the auxiliary equation as before and depending upon the nature of roots write the C.F as below:

Case-I: Let all the roots $m_1,\ m_2,\ \cdots\cdot m_n$ are real and distinct i.e $m_1 \neq m_2 \neq m_3\ \cdots \cdot \neq m_n$, then write the C.F as

$$C.F = c_1x^{m_1} + c_2x^{m_2} + c_3x^{m_3} + \cdots\cdot + c_nx^{m_n}.$$

Case-II: Let all the roots are real where some roots are equal and some are distinct say $m_1 = m_2 = m_3$ and $m_4 \neq m_5 \cdots \neq m_n$, then write the C.F as

$$C.F = (c_1 + c_2 \log x + c_3(\log x)^2)x^{m_1} + c_4 x^{m_4} + \cdots + c_n x^{m_n}.$$

Case-III: Let the roots are complex say $m = \alpha \pm i\beta$, then write the C.F as

$$C.F = e^{\alpha x}(c_1 \cos \beta \log x + c_2 \sin \beta \log x).$$

Step-4: Find the P.I as in case of constant coefficient in terms of z and finally replace the z by $\log x$.

Step-5: Then write the complete solution as $C.S = C.F + P.I$.

Example: Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Solution: Given

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

Consider

$$x = e^z \Rightarrow \log x = z \quad \text{we} \quad \text{get} \quad x\frac{d}{dx} = D', \quad x^2\frac{d^2}{dx^2} = D'(D'-1).$$

Putting these in given equation we find

$$\left[D'(D'-1) + D'\right]y = 12z \Rightarrow (D'^2 - D' + D')y = 12z \Rightarrow D'^2 y = 12z.$$

The A.E is given by

$$m^2 = 0 \Rightarrow m = 0, 0 \Rightarrow C.F = (c_1 + c_2\log x)x^0 = c_1 + c_2\log x$$

To find P.I we have

$$F(D')y = D'^2 y = 12z = \phi(z).$$

So it can be obtain as:

$$\frac{1}{F(D')}\phi(z) = \frac{1}{D'^2}12z = 12\frac{1}{D'^2}z = 12\left(\frac{z^3}{6}\right) = 2z^3.$$

Replacing z by $\log x$, we get the $P.I = 2(\log x)^3$. Finally the C.S is

$$\boxed{C.S = c_1 + c_2 \log x + 2(\log x)^3.}$$

Example: Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4\sin(\log x)$

Solution: The given equation can be written as

$$(x^2D^2 + xD + 1)y = 4\sin(\log x) \text{ where } \frac{d}{dx} = D \text{ and } \frac{d^2}{dx^2} = D^2.$$

Considering

$$x = e^z \Rightarrow \log x = z \text{ we get } x\frac{d}{dx} = D', \quad x^2\frac{d^2}{dx^2} = D'(D'-1)$$

and putting these in given equation we find

$$\begin{aligned} & \left[D'(D'-1) + D' + 1\right] y = 4\sin z \\ \Rightarrow\ & (D'^2 - D' + D' + 1)y = 4\sin z \\ \Rightarrow\ & (D'^2 + 1)y = 4\sin z. \end{aligned}$$

The A.E is given by

$$
\begin{aligned}
& m^2 + 1 = 0 \Rightarrow m = \pm i \\
\Rightarrow \quad & C.F = (c_1 \cos z + c_2 \sin z) = c_1 \cos(\log x) + c_2 \sin(\log x).
\end{aligned}
$$

For the P.I we have $F(D')y = (D'^2 + 1)y = 4 \sin z = \phi(z)$, so

$$
\begin{aligned}
\frac{1}{F(D')}\phi(z) \quad &= \quad \frac{1}{(D'^2+1)} 4 \sin z = 4 \frac{1}{D'^2+1} \sin z = 4 \frac{1}{-1^2+1} \sin z \\
&\Rightarrow \quad 4 \frac{z}{2D'} \sin z = -2z \cos z.
\end{aligned}
$$

Taking $z = \log x$ we get $P.I = -2 \log x \cos(\log x)$. Finally the C.S is

$$\boxed{C.S = c_1 \cos(\log x) + c_2 \sin(\log x) - 2 \log x \cos(\log x).}$$

Example: Solve $(x^2D^2 + 4xD + 2)y = x \log x$

Solution: The given equation

$$(x^2D^2 + 4xD + 2)y = x \log x \quad \text{where} \quad \frac{d}{dx} = D \quad \text{and} \quad \frac{d^2}{dx^2} = D^2.$$

Considering $x = e^z \Rightarrow \log x = z$, we get

$$x\frac{d}{dx} = xD = D', \quad x^2\frac{d^2}{dx^2} = x^2D^2 = D'(D' - 1).$$

Putting these in given equation we find

$$\left[D'(D' - 1) + 4D' + 2\right] y = e^z z \Rightarrow (D'^2 + 3D' + 2)y = e^z z$$
$$\Rightarrow F(D')y = (D'^2 + 3D' + 2)y \quad \text{and} \quad \phi(z) = e^z z.$$

The A.E is

$$m^2+3m+2=0 \Rightarrow m=-1,-2 \Rightarrow C.F=(c_1x^{-1}+c_2x^{-2}).$$

P.I

$$\begin{aligned}
\frac{1}{F(D')}\phi(z) &= \frac{1}{(D'^2+3D'+2)}e^z z = e^z\frac{1}{[(D'+1)^2+3(D'+1)+2]}z \\
&\Rightarrow e^z\frac{1}{[D'^2+5D'+6]}z = e^z\frac{1}{6\left[1+\frac{D'^2+5D'}{6}\right]}z \\
&= \frac{e^z}{6}\left[1+\frac{D'^2+5D'}{6}\right]^{-1} z = \frac{e^z}{6}\left[1-\frac{D'^2+5D'}{6}+\cdots\right]z \\
&= \frac{e^z}{6}\left[1-\frac{D'^2}{6}-\frac{5D'}{6}+\cdots\right]z = \frac{e^z}{6}\left[z-\frac{5}{6}\right].
\end{aligned}$$

Taking $z=\log x$ we get $P.I=\frac{x}{6}\left[\log x-\frac{5}{6}\right]$. Finally the C.S is

$$\boxed{C.S=(c_1x^{-1}+c_2x^{-2})+\frac{x}{6}\left[\log x-\frac{5}{6}\right].}$$

Example: Solve $(x^2D^2 + 4xD + 2)y = x + \frac{1}{x}$

Solution: Taking
$x = e^z \Rightarrow \log x = z$ we get $xD = D'$, $x^2D^2 = D'(D'-1)$. Putting these in given equation we find

$$\left[D'(D'-1) + 4D' + 2\right] y = e^z + \frac{1}{e^z}$$
$$\Rightarrow \quad (D'^2 + 3D' + 2)y = e^z + e^{-z} \Rightarrow F(D')y = \phi(z).$$

From the previous problem we know $C.F = (c_1x^{-1} + c_2x^{-2})$. Now P.I is

$$\begin{aligned}
\frac{1}{F(D')}\phi(z) &= \frac{1}{(D'^2+3D'+2)}(e^z + e^{-z}) \\
&= \frac{1}{(D'^2+3D'+2)}e^z + \frac{1}{(D'^2+3D'+2)}e^{-z} \\
&= \frac{e^z}{6} + \frac{z}{(2D'+3)}e^{-z} = \frac{e^z}{6} + ze^{-z}.
\end{aligned}$$

Taking $z = \log x$ we get $P.I = \frac{x}{6} + \frac{\log x}{x}$. Finally the C.S is

$$\boxed{C.S = (c_1 x^{-1} + c_2 x^{-2}) + \frac{x}{6} + \frac{\log x}{x}.}$$

Example: Solve $(x^2D^2 - 7xD + 12)y = x^2$

Solution: This gives

$$\left[D'(D'-1) - 7D' + 12\right] y = e^{2z} \Rightarrow (D'^2 - 8D' + 12)y = e^{2z}.$$

The A.E is

$$m^2 - 8m + 12 = 0 \Rightarrow m = 2, 6 \Rightarrow C.F = (c_1 x^2 + c_2 x^6).$$

P.I is

$$\begin{aligned} \frac{1}{F(D')}\phi(z) &= \frac{1}{(D'^2 - 8D' + 12)} e^{2z} = \frac{z}{(2D' - 8)} e^{2z} \\ &= \frac{z}{-4} e^{2z} = -\frac{ze^{2z}}{4}. \end{aligned}$$

Taking $z = \log x$ we get $P.I = -\frac{x^2 \log x}{4}$. Finally the C.S is

$$\boxed{C.S = (c_1 x^2 + c_2 x^6) - \frac{x^2 \log x}{4}.}$$

This method is very useful for finding the particular integral of a second order linear differential equations whose complementary function is known. Let the equation be of the form

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x), \tag{5}$$

where $a_1,\ a_2$ are constants, $f(x)$ is a function of x. Let the complementary function of (5) be

$$C.F = c_1f_1 + c_2f_2, \tag{6}$$

where $c_1,\ c_2$ are constants, $f_1,\ f_2$ functions of x.

$$P.I = Pf_1 + Qf_2, \text{ where} \tag{7}$$

$$P = -\int \frac{f_2}{f_1f_2' - f_2f_1'}f(x)dx \text{ and } Q = \int \frac{f_1}{f_1f_2' - f_2f_1'}f(x)dx. \tag{8}$$

Using (8) in (7) we get the P.I.

Example: Solve $\frac{d^2y}{dx^2} + y = \sec x$ by the method of variation f parameters.

Solution: The given equation can be written as $(D^2 + 1)y = \sec x$.

The A.E is

$$m^2 + 1 = 0 \Rightarrow m = \pm i = \alpha + i\beta \Rightarrow \alpha = 0, \quad \beta = 1$$
$$\Rightarrow C.F = c_1 \cos x + c_2 \sin x = c_1 f_1 + c_2 f_2$$

where

$$f_1 = \cos x \quad \text{and} \quad f_2 = \sin x \Rightarrow f_1' = -\sin x, \quad f_2' = \cos x.$$

So $f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$, as we know $P.I = P\cos x + Q \sin x$ where P and Q are given by

$$\begin{aligned} P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} f(x)dx = -\int \frac{\sin x}{1} \sec x dx = -\int \frac{\sin x}{\cos x} dx \\ &= \log(\cos x). \end{aligned}$$

$$Q = \int \frac{f_1}{f_1 f_2' - f_2 f_1'} f(x) dx = \int \frac{\cos x}{1} \sec x dx = \int \frac{\cos x}{\cos x} dx = \int dx = x.$$

$\therefore$ The $P.I = \cos x \log(\cos x) + x \sin x$ and

$$\boxed{C.S = c_1 \cos x + c_2 \sin x + \cos x \log(\cos x) + x \sin x.}$$

Example: Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Solution: The given equation can be written as $(D^2 + 4)y = 4\tan 2x$.

The A.E is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i = \alpha + i\beta \Rightarrow \alpha = 0, \quad \beta = 2$$
$$\Rightarrow C.F = c_1 \cos 2x + c_2 \sin 2x = c_1 f_1 + c_2 f_2.$$

Now

$$f_1 = \cos 2x \quad \text{and} \quad f_2 = \sin 2x \Rightarrow f_1' = -2\sin 2x, \quad f_2' = 2\cos 2x.$$

So $f_1 f_2' - f_2 f_1' = 2\cos^2 2x + 2\sin^2 2x = 2$, as we know
$P.I = P\cos 2x + Q\sin 2x$ where P and Q are given by

$$\begin{aligned}
P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} f(x)dx = -\int \frac{\sin 2x}{2} 4\tan 2x dx = -2\int \frac{\sin^2 2x}{\cos 2x} dx \\
&= -2\int \frac{(1-\cos^2 2x)}{\cos 2x} dx = -2\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx \\
&= -2\int (\sec 2x - \cos 2x) dx = -2\int \sec 2x dx + 2\int \cos 2x dx \\
&= -2\left(\frac{1}{2}\right)\log(\sec 2x + \tan 2x) + 2\left(\frac{\sin 2x}{2}\right) \\
&= -\log(\sec 2x + \tan 2x) + \sin 2x.
\end{aligned}$$

$$\begin{aligned} Q &= \int \frac{f_1}{f_1 f_2' - f_2 f_1'} f(x)dx = \int \frac{\cos 2x}{2} 4\tan x dx = 2\int \sin 2x dx \\ &= -2\left(\frac{\cos 2x}{2}\right) = -\cos 2x. \end{aligned}$$

$$\begin{aligned} P.I &= \cos 2x[-\log(\sec 2x + \tan 2x) + \sin 2x] - \sin 2x \cos 2x \\ &= -\cos 2x \log(\sec 2x + \tan 2x) \end{aligned}$$

$$\boxed{C.S = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).}$$

Example: Solve $\frac{d^2y}{dx^2} + y = cosec\ x$.

Solution: The given equation can be written as $(D^2+1)y = cosec\ x$.

The A.E is

$$m^2 + 4 = 0 \Rightarrow m = \pm i = \alpha + i\beta \Rightarrow \alpha = 0, \quad \beta = 1$$

Now $f_1 = \cos x$ and $f_2 = \sin x \Rightarrow f_1' = -\sin x, \quad f_2' = \cos x$. So $f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$, as we know $P.I = P\cos x + Q\sin x$ where P and Q are given by

$$\begin{aligned} P &= -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} f(x)dx = -\int \frac{\sin x}{1} cosec\ xdx = -\int dx = -x \\ Q &= \int \frac{f_2}{f_1 f_2' - f_2 f_1'} f(x)dx = \int \frac{\cos x}{1} cosec\ xdx = \int \left(\frac{\cos x}{\sin x}\right) dx \\ &= \log(\sin x). \end{aligned}$$

$$P.I = -x\cos x + \sin x \log(\sin x)$$

$\therefore$ The complete solution is given by:

$$\boxed{C.S = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x).}$$