2.3 TYPES OF FUNCTIONS

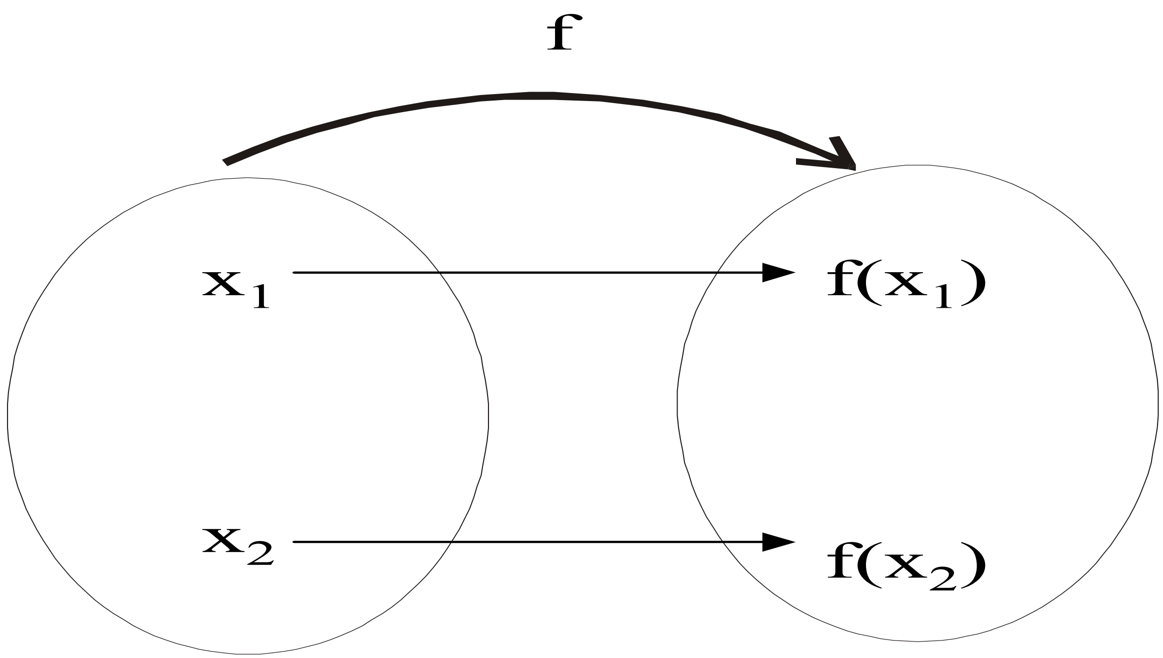
TYPES OF FUNCTIONS

- **Injective** or **ONE-TO-ONE Function**
- **Surjective** or **ON-TO Function**
- **Bijective Function**
- **Identity Function**
- **Constant Function**

ONE-TO-ONE FUNCTION / INJECTIVE FUNCTION

- Let **f**: $X \to Y$ be a **function**. **f** is **injective** or **one-to-one** if, and only if, $\forall\ x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

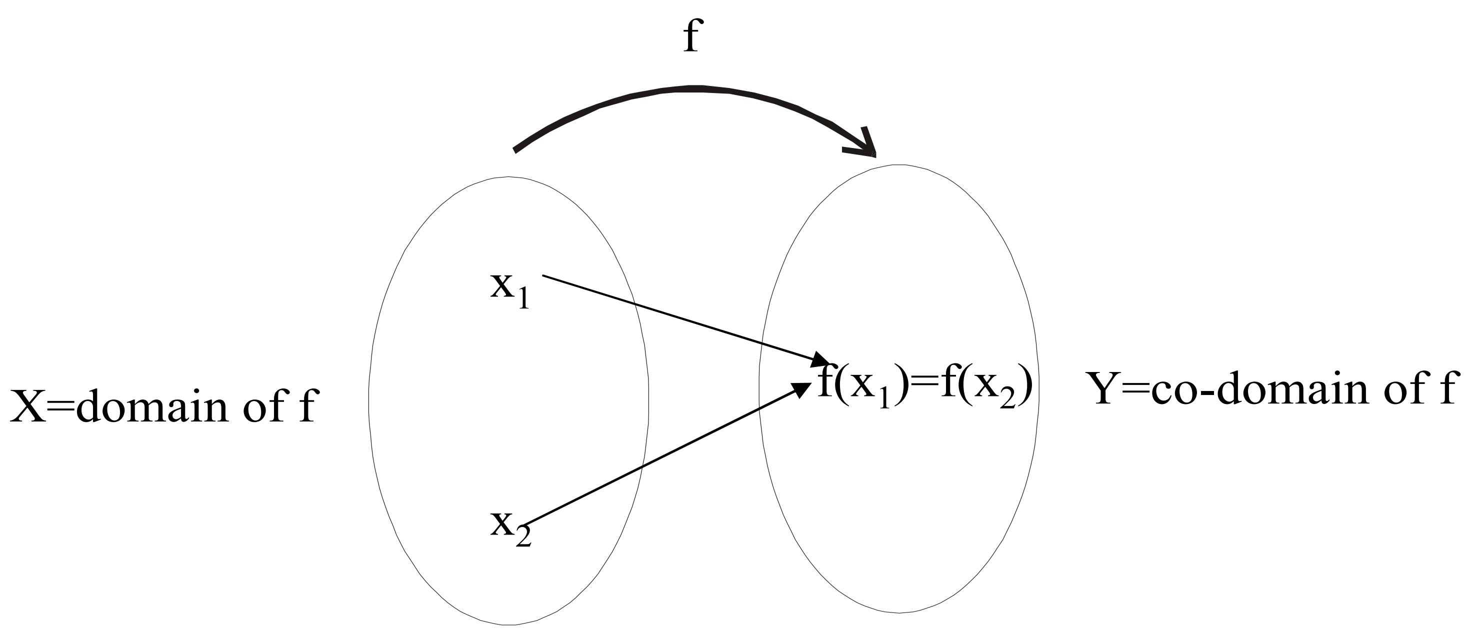
- That is, **f** is **one-to-one** if it maps **distinct points** of the **domain** into the **distinct points** of the **co-domain**.



FUNCTION NOT ONE-TO-ONE

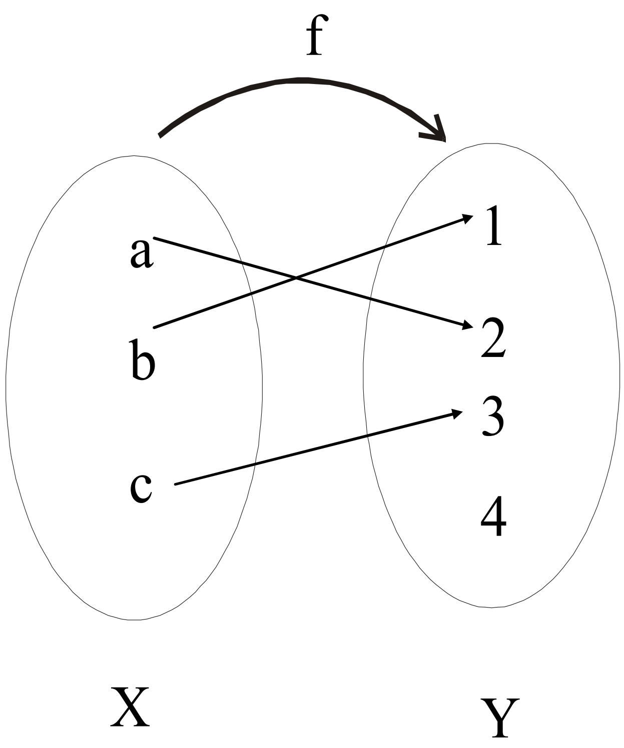
- A function $f: X \rightarrow Y$ is not one-to-one iff there exist elements x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

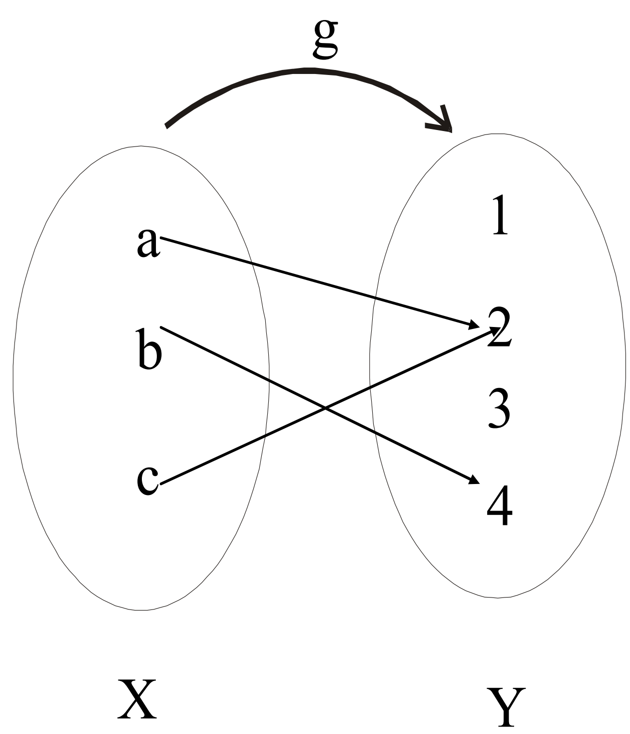
- That is, if distinct elements x_1 and x_2 can found in domain of f then they have the same function value.



EXAMPLE

- Which of the **arrow diagrams** define **one-to-one functions**?





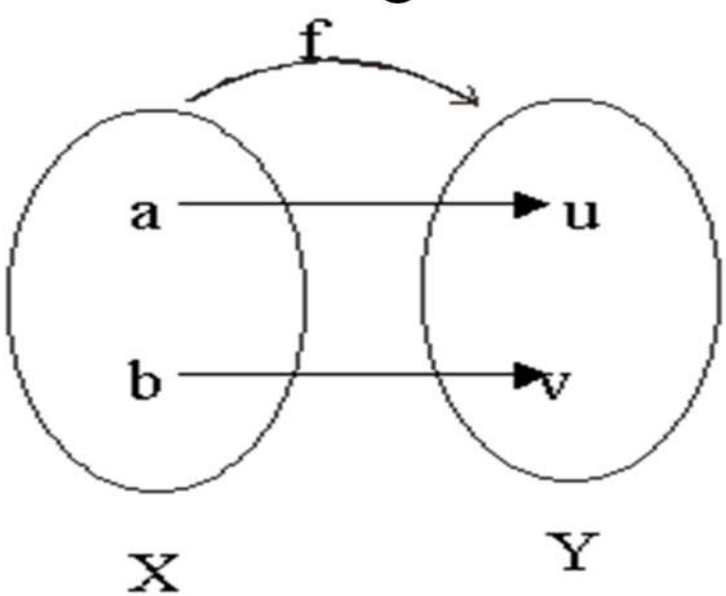
f is one-to-one function

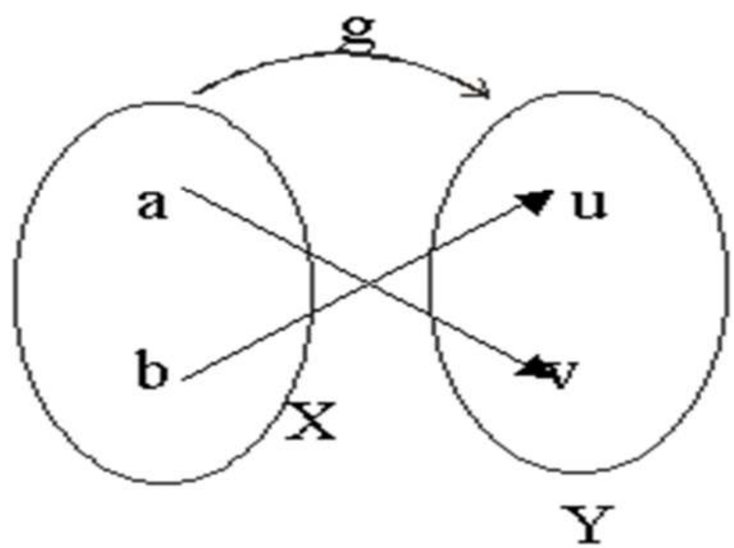
Image of (b) = 2

f is NOT one-to-one function

EXERCISE

- Find all **one-to-one** functions from X = { a, b } to Y = { u, v }
- **SOLUTION**:
- There are two one-to-one functions from X to Y defined by the arrow diagrams.





We have only **two one-to-one functions**.

EXERCISE

- How many **one-to-one** functions are there from a **set** with **three elements** to a **set** with **four elements**.
- **SOLUTION**

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$

x_1 may be mapped to any of the 4 elements of Y. Then x_2 may be mapped to any of the remaining 3 elements of Y & finally x_3 may be mapped to any of the remaining 2 elements of Y.

Hence, total no. of one-to-one functions from X to Y are

$$4 \times 3 \times 2 = 24$$

EXERCISE

- How many one-to-one functions are there from a set with three elements to a set with two elements.

- **SOLUTION**

- Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

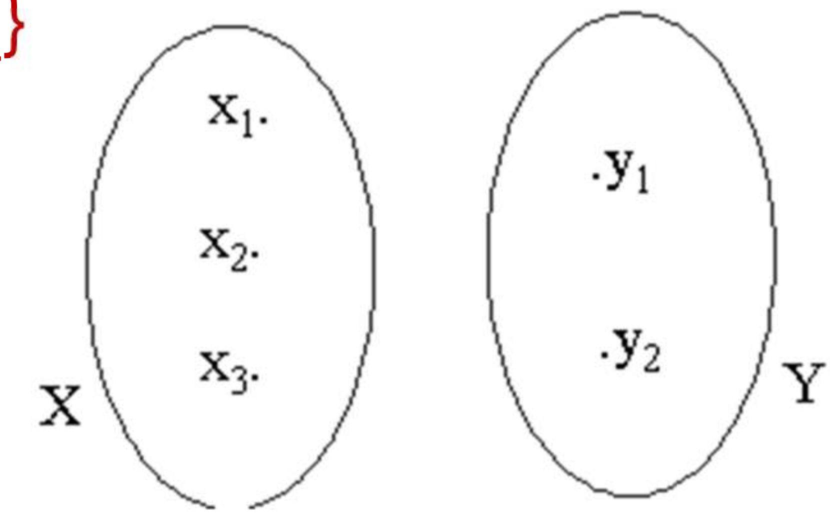
- Two elements in X could be mapped to the two elements in Y separately. But there is no new element in Y to which the third element in X could be mapped. Accordingly there is no one-to-one function from a set with three elements to a set with two elements.

EXERCISE

- How many one-to-one functions are there from a set with three elements to a set with two elements.

SOLUTION:

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

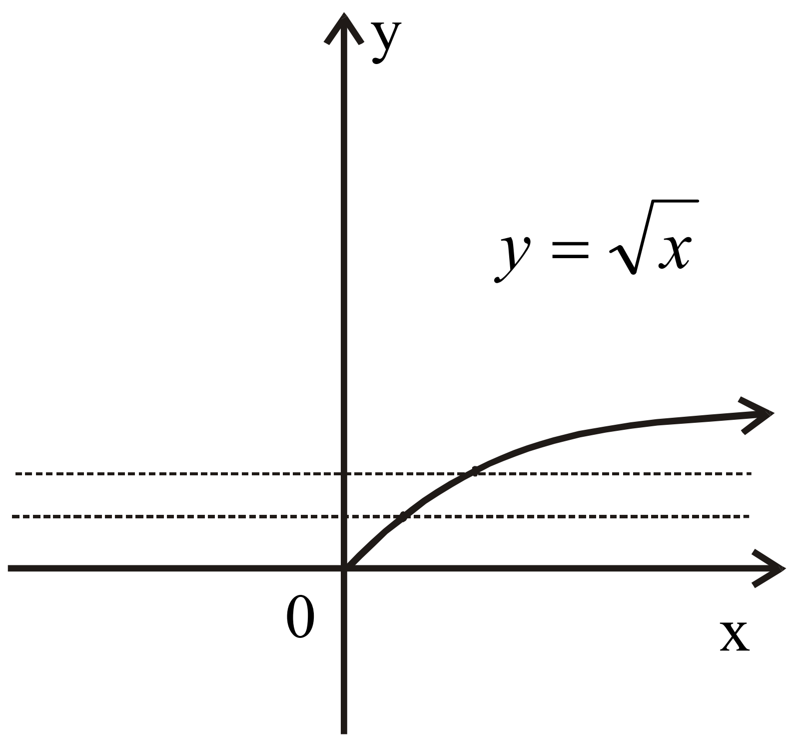


Two elements in X could be mapped to the two elements in Y separately. But there is no new element in Y to which the third element in X could be mapped. Accordingly there is no one-to-one function from a set with three elements to a set with two elements.

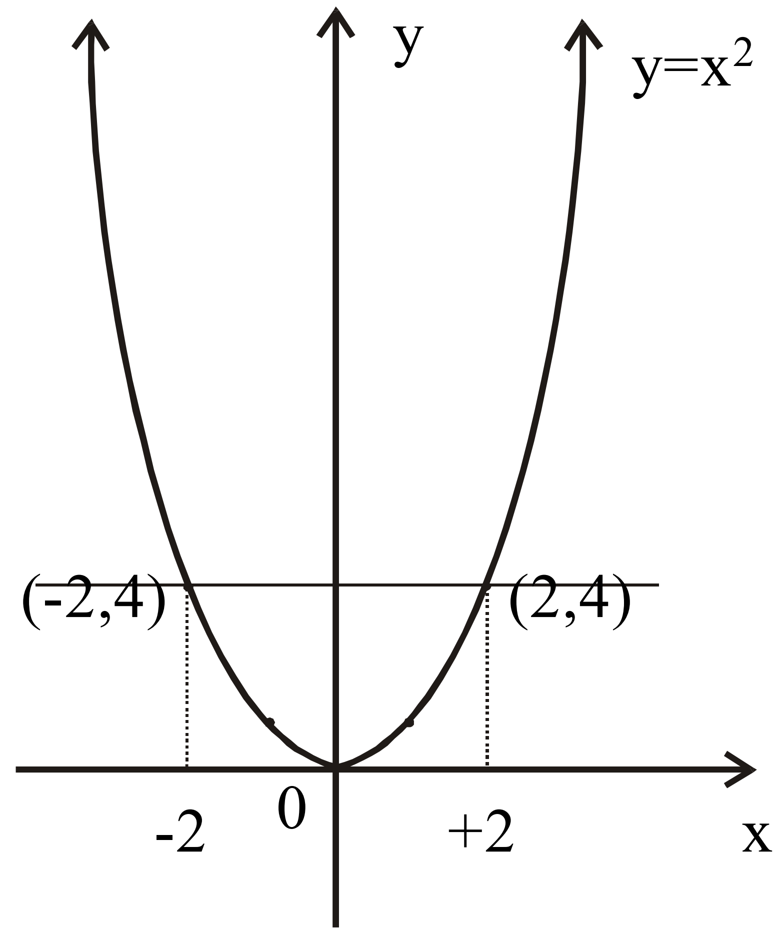
GRAPH OF ONE-TO-ONE FUNCTION

- A **graph** of a **function f** is **one-to-one** iff every **horizontal line** intersects the **graph** in **at most one point**.

EXAMPLE



ONE-TO-ONE FUNCTION
from R^+ to R



NOT ONE-TO-ONE FUNCTION
From R to R^+

ALTERNATIVE DEFINITION OF ONE-TO-ONE FUNCTION

- Let **f**: $X \rightarrow Y$ be a **function**. **f** is **injective** or **one-to-one** if, and only if, $\forall\ x_1, x_2 \in X$, IF $x_1 \neq x_2$ THEN $f(x_1) \neq f(x_2)$.

- The contra-positive of definition is:

If $f(x_1) = f(x_2)$ then $x_1 = x_2$

If the contrapositive of the definition is also satisfied then the function is also one-to-one.

SURJECTIVE FUNCTION /ONTO FUNCTION

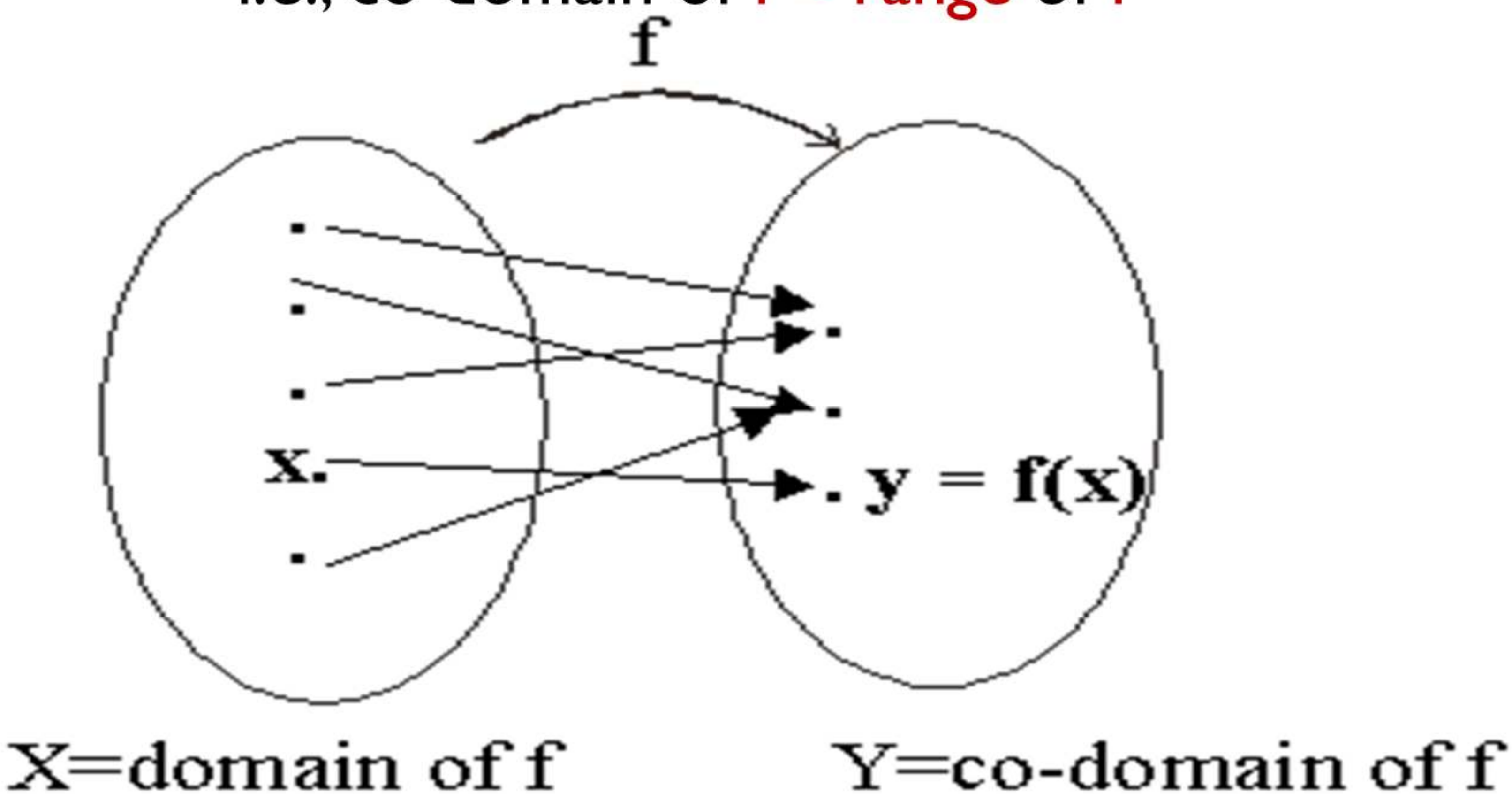
- Let $f: X \rightarrow Y$ be a function. f is surjective or onto if, and only if,

$$\forall\ y \in Y,\ \exists\ x \in X \text{ such that } f(x) = y.$$

That is, f is onto if every element of Y is the image of some element of X.

- That is, f is onto if every element of its co-domain is the image of some element(s) of its domain.

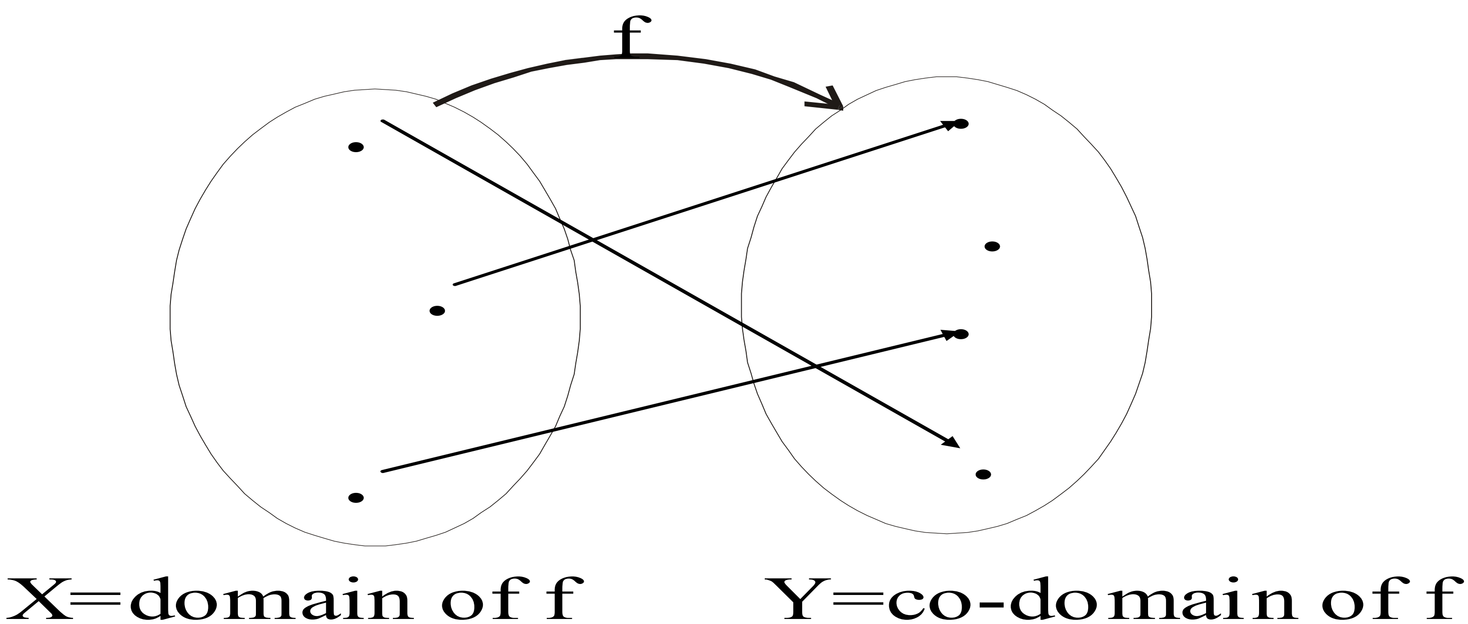
i.e., co-domain of f = range of f



FUNCTION NOT ONTO

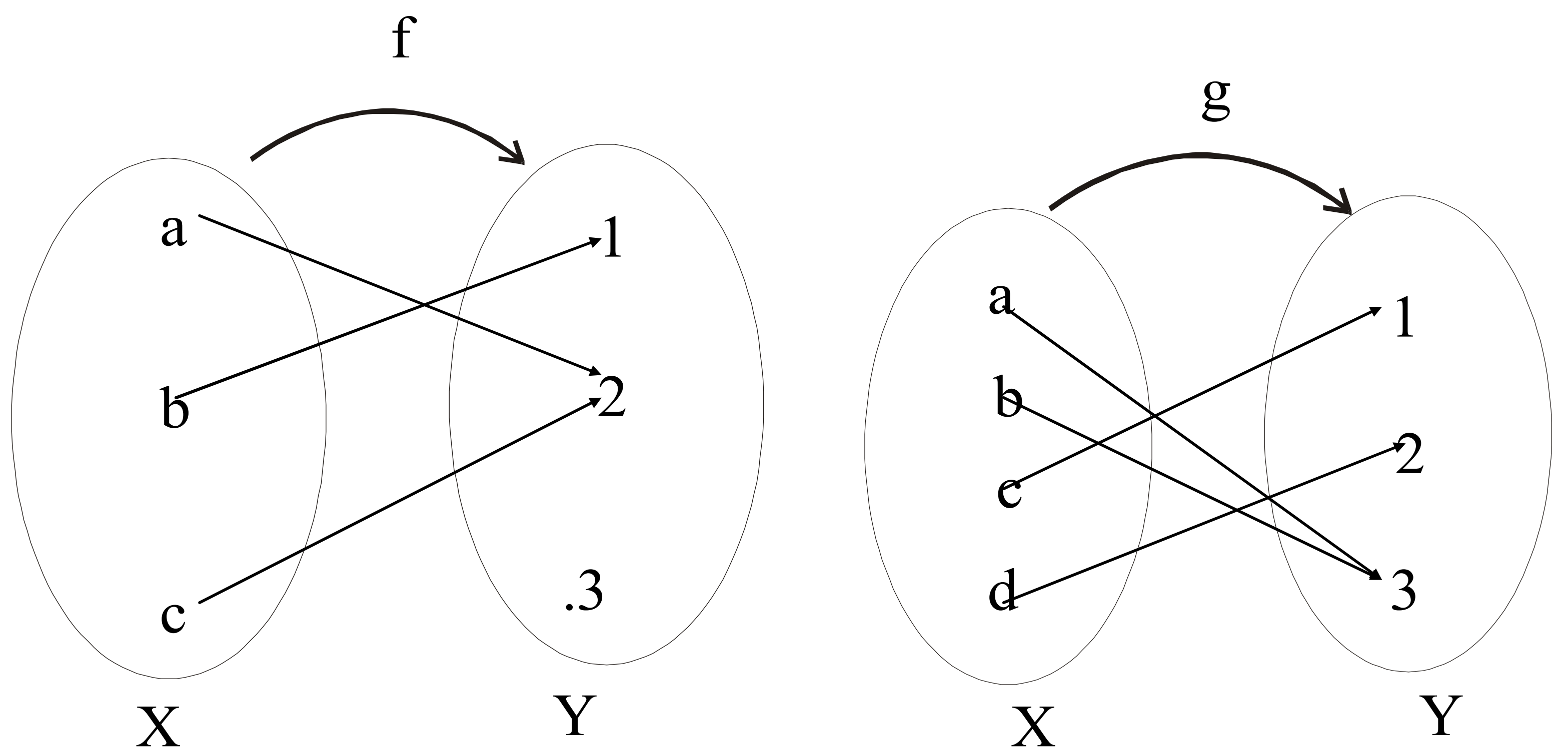
- A function $f:X \to Y$ is not onto iff there exists $y \in Y$ such that $\forall\, x \in X, f(x) \neq y$.

- That is, there is some element in Y that is not the image of any element in X.



EXAMPLE

- Which of the **arrow diagrams** define **onto functions**?



- **SOLUTION:**
- **f** is **not onto** because $3 \neq f(x)$ for any x in X. (3 an element which is not image of any element of set X)

- **g** is clearly **onto** because **each element** of **Y** equals **g(x)** for **some x** in **X**. (co-domain all elements are images of some elements of domain)

  as

$$1 = g(c)$$
$$2 = g(d)$$
$$3 = g(a) = g(b)$$

EXAMPLE

- Define **f**: R $\rightarrow$ R by the rule

$$f(x) = 4x - 1 \quad \text{for all } x \in R$$

- Is **f onto**?

- Prove or give a **counter example**.

SOLUTION

- Let $y \in R$. We search for an $x \in R$ such that

$$f(x) = y$$

or $4x - 1 = y$ (by definition of f)

Solving it for x, we find

$$4x = y + 1$$

$$x = \frac{y+1}{4} \in R$$

Cont…

- Hence for every $y \in R$, there exists

$$x = \frac{y+1}{4} \in R \quad \text{such that}$$

$$f(x) = f\left(\frac{y+1}{4}\right)$$

$$= 4.\left(\frac{y+1}{4}\right) - 1 = (y+1) - 1 = y$$

Hence **f is onto**

EXAMPLE

- Define **h**: $Z \to Z$ by the rule

  h(n) = 4n - 1 for all $n \in Z$

  Is **h onto**?

  Prove or give a counter example.

SOLUTION

- Let $m \in Z$. We search for an $n \in Z$ such that

$$h(n) = m.$$

or $\quad 4n - 1 = m \quad$ (by definition of h)

Solving it for n, we find $n = \dfrac{m+1}{4}$

But $n = \dfrac{m+1}{4}$ is not always an integer for all $m \in Z$.

Cont…

- Let m = 3 then

$$\frac{m+1}{4} = \frac{4}{4} = 1 \text{ integer}$$

- Let m = 5 then

$$\frac{m+1}{4} = \frac{6}{4} \text{ not an integer}$$

- As a **counter example**, let m = 0 ∈ Z, then

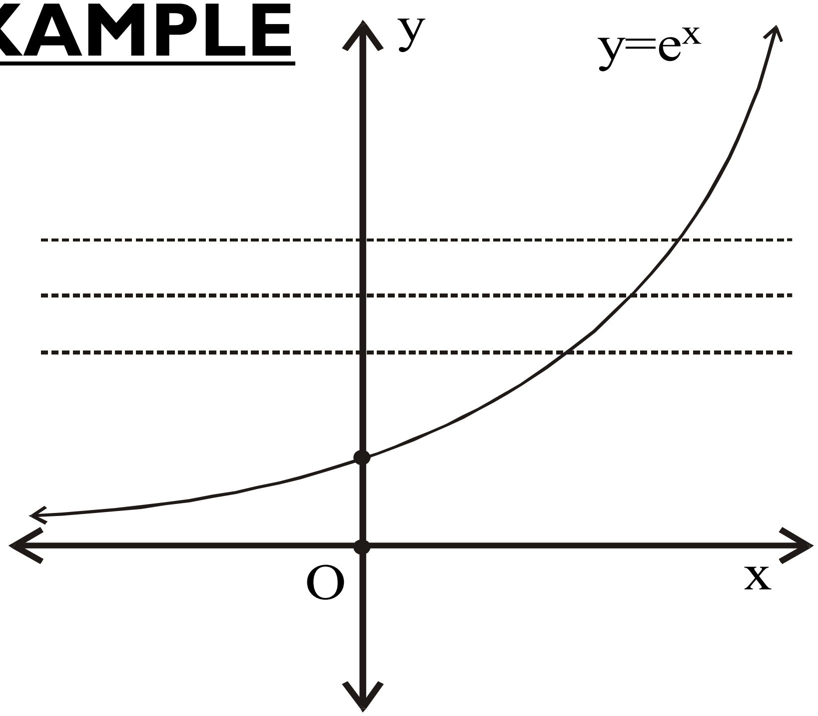
$$h(n) = 0$$
$$\Rightarrow \quad 4n-1 = 0$$
$$\Rightarrow \quad 4n = 1$$
$$\Rightarrow \quad n = \frac{1}{4} \notin Z$$

Hence there is no integer n for which h(n) = 0.
Accordingly, h is not onto.

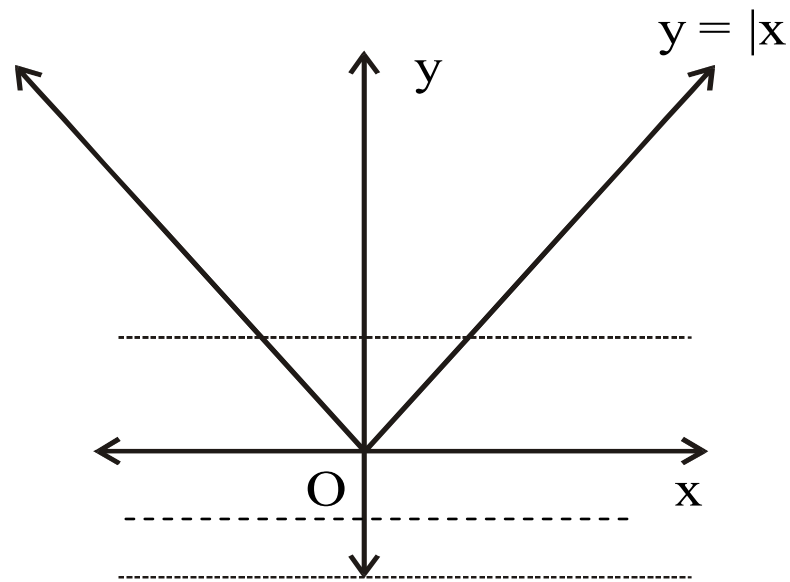
GRAPH OF ONTO FUNCTION

- A graph of a function **f** is **onto** iff every **horizontal line** intersects the graph in **at least one point** within the codomain.

EXAMPLE



ONTO FUNCTION
from R to R^+



NOT ONTO FUNCTION FROM
R to R

EXERCISE

- Let $X = \{1, 5, 9\}$ and $Y = \{3, 4, 7\}$.

  Define $g: X \to Y$ by specifying that

  $g(1) = 7, \quad g(5) = 3, \quad g(9) = 4$

  Is g one-to-one? Is g onto?

SOLUTION

X = {1, 5, 9} and Y = {3, 4, 7}.

g(1) = 7,   g(5) = 3,   g(9) = 4

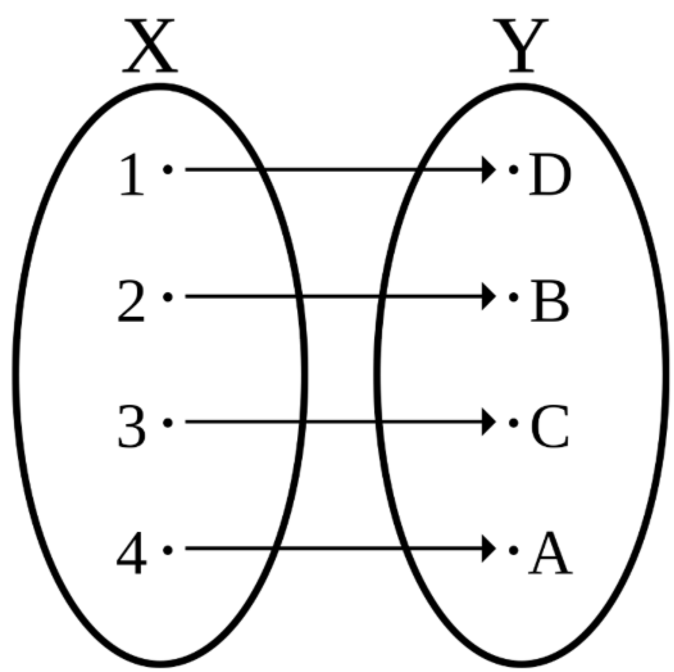
g is **one-to-one** because each of the **three elements** of **X** are **mapped** to a **different elements** of **Y** by **g**.

- **g** is **onto** as well, because **each** of the **three elements** of **co-domain Y** of **g** is the **image** of **some element** of the **domain** of **g**.

3 = g(5),   4 = g(9),   7 = g(1)

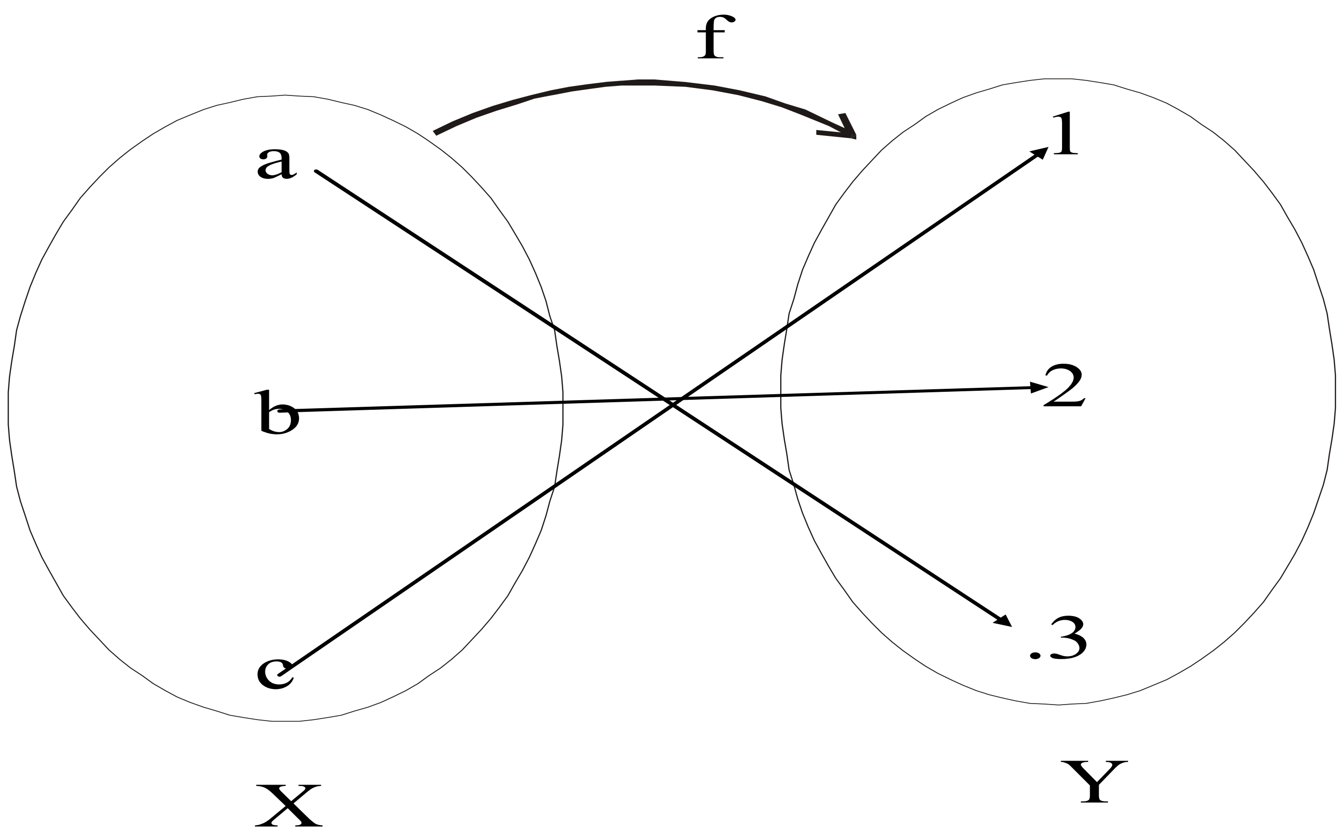
BIJECTIVE FUNCTION

- A function f: X→Y that is both one-to-one (injective) and onto (surjective) is called a bijective function or a one-to-one correspondence.



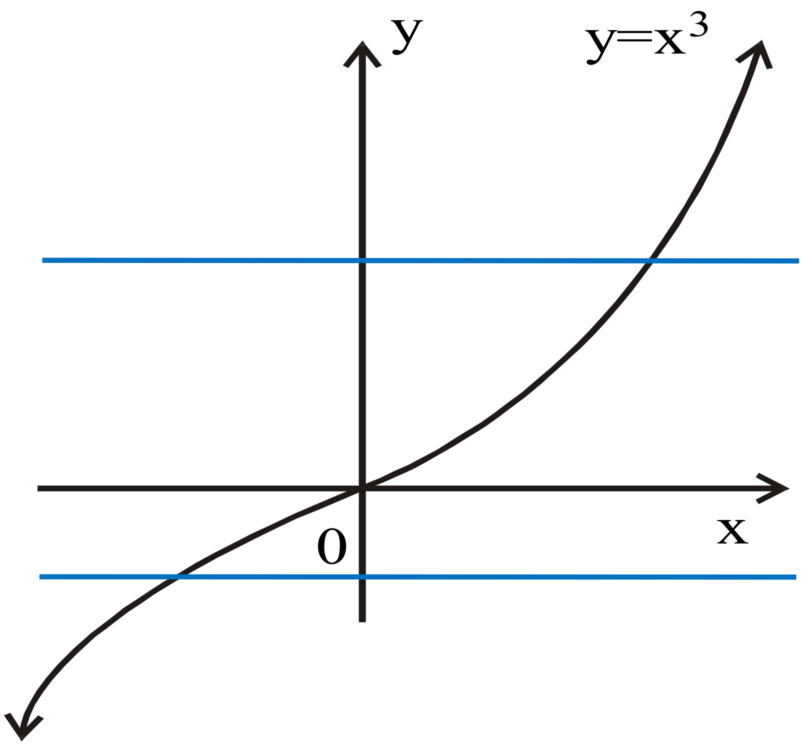
EXAMPLE

- The function f: X→Y defined by the arrow diagram is both one-to-one and onto; hence a bijective function.

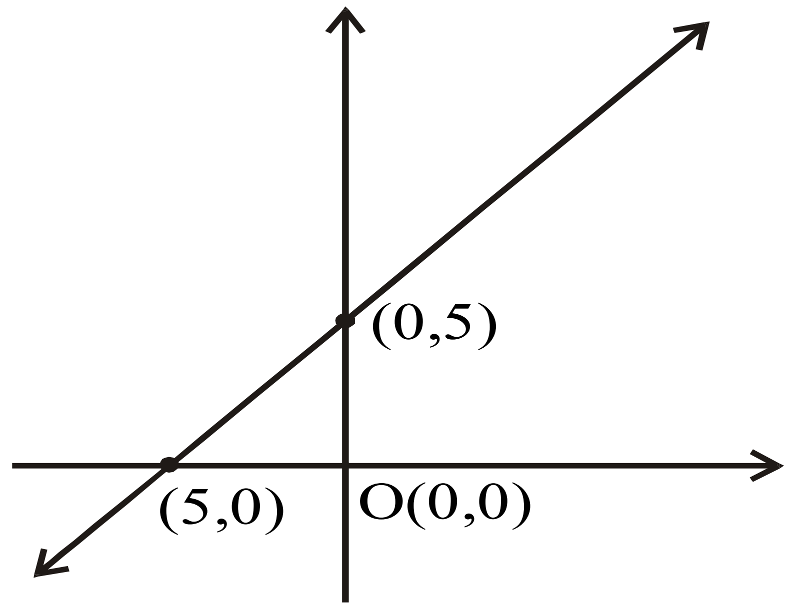


GRAPH OF BIJECTIVE FUNCTION

- A graph of a function f is bijective iff every horizontal line intersects the graph at exactly one point.



BIJECTIVE FUNCTION
from R to R



BIJECTIVE FUNCTION
from R to R

IDENTITY FUNCTION ON A SET

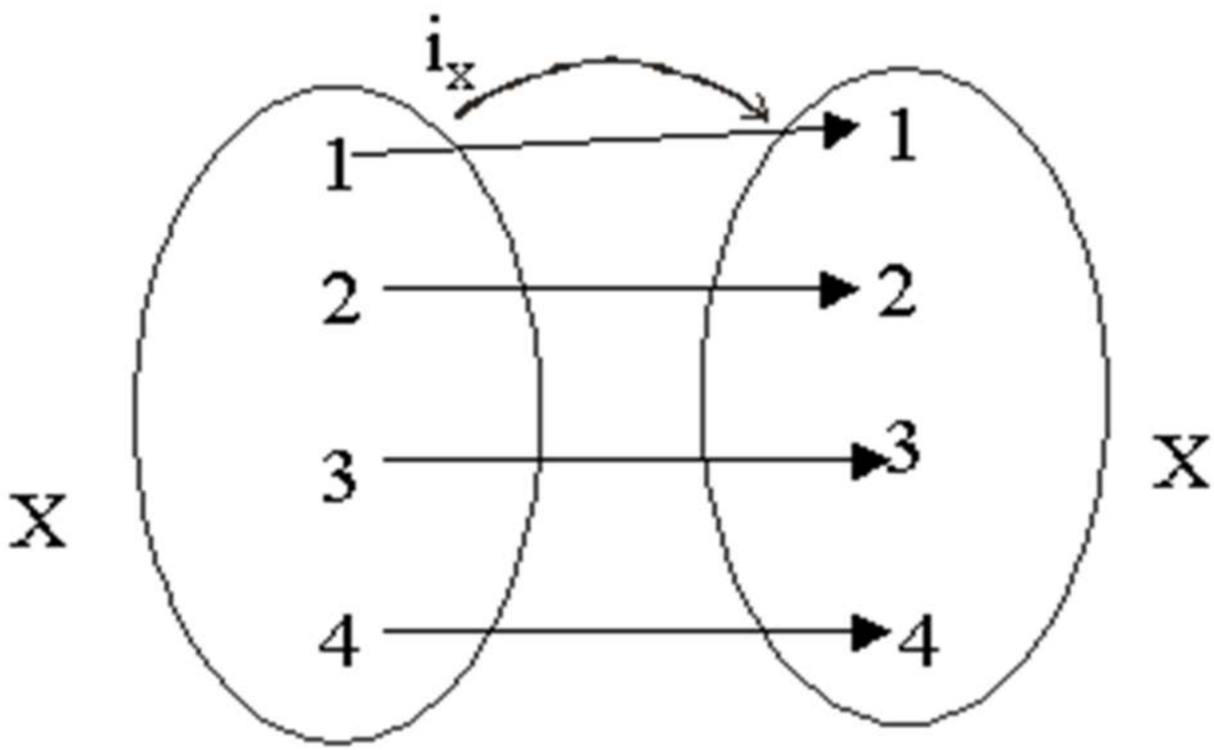
- Given a **set X**, define a **function i_x** from **X** to **X** by

  $$i_x(x) = x \text{ from all } x \in X.$$

- The **function i_x** is called the **identity function** on **X** because it sends **each element** of **X** mapped to **itself**.

EXAMPLE

- Let X = {1, 2, 3, 4}. The identity function $\mathbf{i_x}$ on X is represented by the arrow diagram.



CONSTANT FUNCTION ON A SET

- A function $f{:}X\rightarrow Y$ is constant function if it maps (sends) all elements of X to one element of Y i.e.

$$x \in X, f(x) = c \text{ for some } c \in Y.$$

- The constant function is one-to-one iff its domain is a singleton.
- A constant function is onto iff its co-domain is a singleton.