COUNTING

CHAPTER 6

COUNTING

- In combinatorics we generally solved counting problems.
- **Example:** How many books you have you can easily count them.
- Sometimes counting problems become complex and it is difficult to solve it.
- In counting problem the important step is we must realized that which thing we are counting, if you do that step then to solve the actual problem is not a big deal.

INTRODUCTION

- Combinatorics is the mathematics of counting and arranging objects.

- Counting of objects with certain properties (enumeration) is required to solve many different types of problem.

- **For example:** Counting is used to:
- Determine number of ordered or unordered arrangement of objects.

- Generate all the arrangements of a specified kind which is important in computer simulations.

- How many such passwords are there.
- Analyze the chance of winning games, lotteries etc.
- Determine the complexity of algorithms.

BASIC COUNTING PRINCIPLES

- The two basic counting principles are:

- Product Rule
- Sum Rule

- They can be used to solve many different counting problems.

EXAMPLE

- Your institute is offering 7 courses in computer science and 3 courses in mathematics. You are asked to choose only one course.
- How many choices you have?

- You can select either one computer science course or mathematics.

  7 cs courses = 7 choices

  3 math courses = 3 choices

  Total number of choices = 7 + 3 (basically by applying sum rule)

THE SUM RULE

- If one event can occur in n_1 ways.
- A second event can occur in n_2 (different) ways.

- Then the total number of ways in which exactly one of the events (i.e., first or second) can occur is

$$n_1 + n_2.$$

EXAMPLE

- A student can choose a computer project from one of the three lists. The three lists contain 23, 15 and 19 possible projects, respectively.

- How many possible projects are there to choose from?
- **SOLUTION:**

  23 choices are in first list,

  15 choices are in second list,

  19 choices are in third list. Hence, there are

  So, we have total number of choices

  $= 23 + 15 + 19 = 57$ projects to choose from.

GENERALIZED SUM RULE

- If one event can occur in n_1 ways,

  a second event can occur in n_2 ways,

  a third event can occur in n_3 ways,

  ……………………………………..

  then there are

$$n_1 + n_2 + n_3 + \ldots$$

  ways in which exactly one of the events can occur.

SUM RULE IN TERMS OF SETS

- If $A_1, A_2, \ldots, A_m$ are finite disjoint sets, then the number of elements in the union of these sets is the sum of the number of elements in them.

- If $n(A_i)$ denotes the number of elements in set A_i, then

- $n(A_1 \cup A_2 \cup \ldots \cup A_m) = n(A_1) + n(A_2) + \ldots + n(A_m)$

  where

$$A_i \cap A_j = \phi \quad \text{if} \quad i \# j$$

EXAMPLE

- Suppose

  There are 7 different optional courses in Computer Science and

  3 different optional courses in Mathematics.

  A student who wants to take one optional course of each subject, there are:

  $7 \times 3 = 21$ choices.

THE PRODUCT RULE

- If one event can occur in n_1 ways and if for each of these n_1 ways, a second event can occur in n_2 ways.

- Then the total number of ways in which both events occur is $n_1 \cdot n_2$

EXAMPLE

- The chairs of an auditorium are to be labeled with two characters, a letter followed by a digit.

- What is the largest number of chairs that can be labeled differently?

SOLUTION

- The procedure of labeling a chair consists of two events, namely,

  Assigning one of the 26 letters: A, B, C, …, Z and

  Assigning one of the 10 digits: 0, 1, 2, …, 9

  By product rule, there are $26 \times 10 = 260$ different ways that a chair can be labeled by both a letter and a digit.

GENERALIZED PRODUCT RULE

- If some event can occur in n_1 different ways, and if, following this event, a second event can occur in n_2 different ways, and following this second event, a third event can occur in n_3 different ways, …, then the number of ways all the events can occur in the order indicated is

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots$$

PRODUCT RULE IN TERMS OF SETS

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

- If $n(A_i)$ denotes the number of elements in set A_i, then

$$n(A_1 \times A_2 \times \ldots \times A_m) = n(A_1) \cdot n(A_2) \cdot \ldots \cdot n(A_m)$$

EXERCISE

- Find the number n of ways that an organization consisting of 15 members can elect a president, treasurer, and secretary. (assuming no person is elected to more than one position)

- **SOLUTION:**

The president can be elected in 15 different ways;

The treasurer can be elected in 14 different ways;

The secretary can be elected in 13 different ways.

Thus, by product rule, there are

$$n = 15 \times 14 \times 13 = 2730$$

different ways in which the organization can elect the officers.

EXERCISE

- There are four bus lines between A and B; and three bus lines between B and C.

- Find the number of ways a person can travel:

(a) By bus from A to C by way of B;

(b) Round trip by bus from A to C by way of B;

(c) Round trip by bus from A to C by way of B, if the person does not want to use a bus line more than once.

SOLUTION

(a) There are 4 ways to go from A to B and 3 ways to go from B to C;

hence there are

4 x 3 = 12 ways to go from A to C by way of B.

(b) The person will travel from A to B to C to B to A for the round trip.

i.e $(A \rightarrow B \rightarrow C \rightarrow B \rightarrow A)$

The person can travel 4 ways from A to B and 3 way from B to C and back.

i.e., $$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{3} B \xrightarrow{4} A$$

Thus there are 4 x 3 x 3 x 4 = 144 ways to travel the round trip.

(c) The person can travel 4 ways from A to B and 3 ways from B to C, but only 2 ways from C to B and 3 ways from B to A,

Since bus line cannot be used more than once. Thus

i.e., $$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Hence there are 4 x 3 x 2 x 3 = 72 ways to travel the round trip without using a bus line more than once.

EXERCISE

- How many bit strings of length 8

(i) begin with a "1"?

(ii) begin and end with a "1"?

SOLUTION

(i) If the first bit (left most bit) is a 1, then it can be filled in only one way.

Each of the remaining seven positions in the bit string can be filled in 2 ways (i.e., either by 0 or 1).

Hence,

there are $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$ different bit strings of length 8 that begin with a 1.

(ii) If the first and last bit in an 8 bit string is a 1, then only the intermediate six bits can be filled in 2 ways, i.e. by a 0 or 1.

Hence there are $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 = 2^6 = 64$ different bit strings of length 8 that begin and end with a 1.

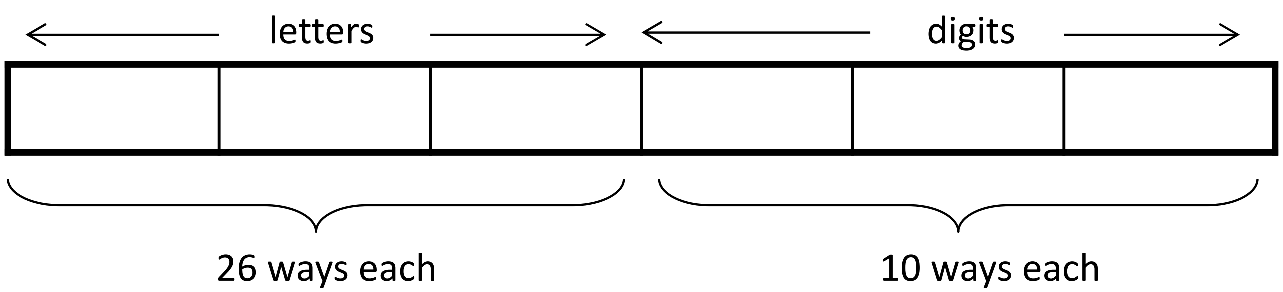
EXERCISE

- Suppose that an automobile license plate has three letters followed by three digits.

- How many different license plates are possible?

SOLUTION

Each of the three letters can be written in 26 different ways, and each of the three digits can be written in 10 different ways.



Hence, by the product rule, there is a total of

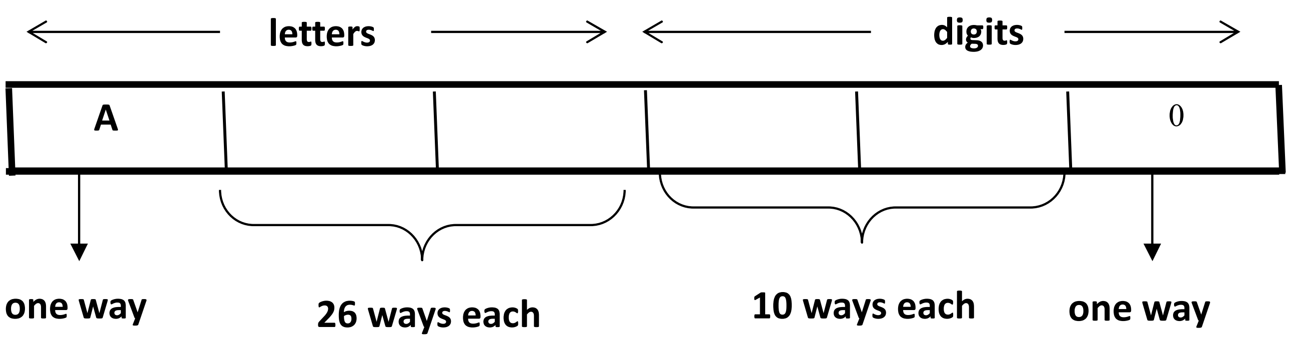
26 x 26 x 26 x 10 x 10 x 10 = 17,576,000

different license plates possible.

(b) How many license plates could begin with A and end on 0?

- **SOLUTION:**

The first and last place can be filled in one way only, while each of second and third place can be filled in 26 ways and each of fourth and fifth place can be filled in 10 ways.



Number of license plates that begin with A and end in 0 are 1 x 26 x 26 x 10 x 10 x 1 = 67600

EXERCISE

- A variable name in a programming language must be either a letter or a letter followed by a digit.
- How many different variable names are possible?

SOLUTION

- First consider variable names one character in length.
- Since such names consist of a single letter, there are 26 variable names of length 1.

- Next, consider variable names two characters in length.

  Since the first character is a letter, there are 26 ways to choose it. The second character is a digit, there are 10 ways to choose it.

  Hence, to construct variable name of two characters in length, there are 26 x10 = 260 ways.

- Finally, by sum rule, there are 26 + 260 = 286 possible variable names in the programming language.

EXERCISE

- A computer access code word consists of from one to three letters of English alphabets with repetitions allowed.

- How many different code words are possible.

SOLUTION

Number of code words of length 1 = 26^1

Number of code words of length 2 = 26^2

Number of code words of length 3 = 26^3

Hence, the total number of code words

$$= 26^1 + 26^2 + 26^3$$

$$= 18{,}278$$

NUMBER OF ITERATIONS OF A NESTED LOOP

- Determine how many times the inner loop will be iterated when the following algorithm is implemented and run

```
for     i: = 1 to 4
        for     j : = 1 to 3
        [number of statements]
        next j
next i
```

SOLUTION

- The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop.

- Hence, by product rules the total number of iterations of inner loop is $4 \cdot 3 = 12$

EXERCISE

- Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

```
for     i = 5 to 50
        for     j: = 10 to 20
                [number of statements]
        next j
next i
```

SOLUTION

- The outer loop is iterated 50 - 5 + 1 = 46 times and

- Inner loop iterate for 20 - 10 + 1 = 11 times.

- Hence by product rule, the total number of iterations of the inner loop is 46.11 = 506

EXERCISE

- Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

```
for     i = 1 to 4
        for     j: = 1 to i
                [number of statements]
        next j
next i
```

- The outer loop is iterated 4 times, but during each iteration of the outer loop, the inner loop iterates different number of times.

- For first iteration of outer loop, inner loop iterates 1 times.
- For second iteration of outer loop, inner loop iterates 2 times.
- For third iteration of outer loop, inner loop iterates 3 times.
- For fourth iteration of outer loop, inner loop iterates 4 times.

- Hence, total number of iterations of inner loop = 1 + 2 + 3 + 4 = 10

THE PIGEONHOLE PRINCIPLE

- Its very useful principle for many counting problems.

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, at least one of these 19 pigeonholes must have at least two pigeons in it.

- This illustrates a general principle called the **pigeonhole principle, which states that if there are more pigeons than pigeonholes**, then there must be at least one pigeonhole with at least two pigeons in it.

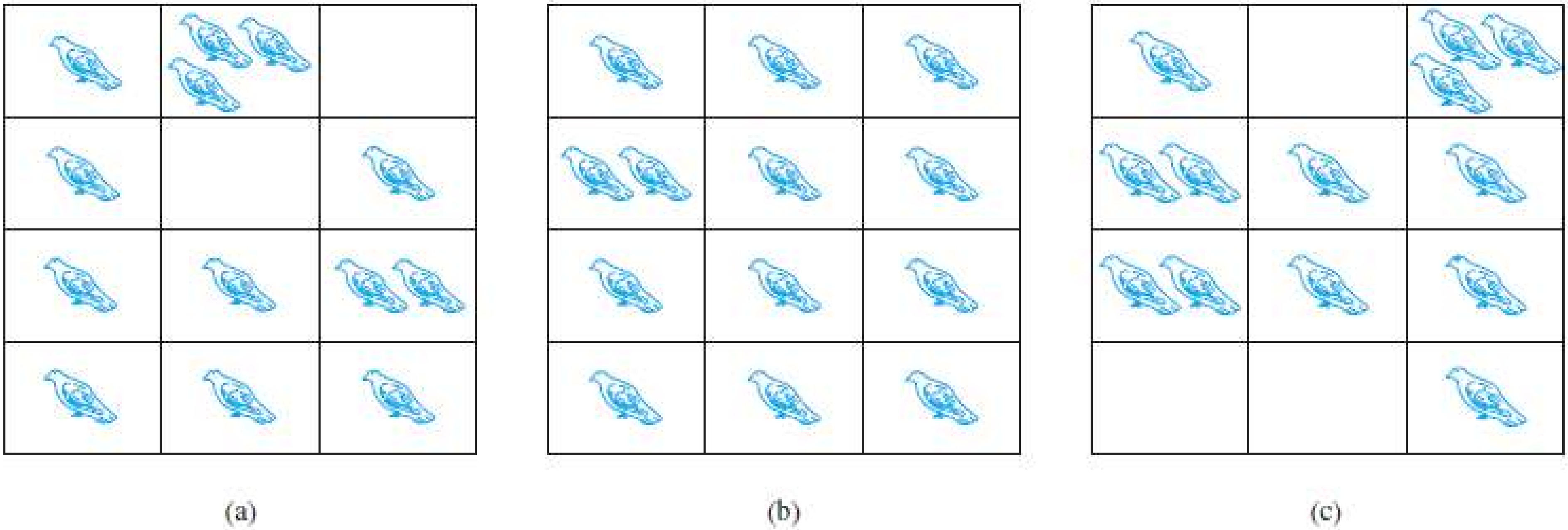
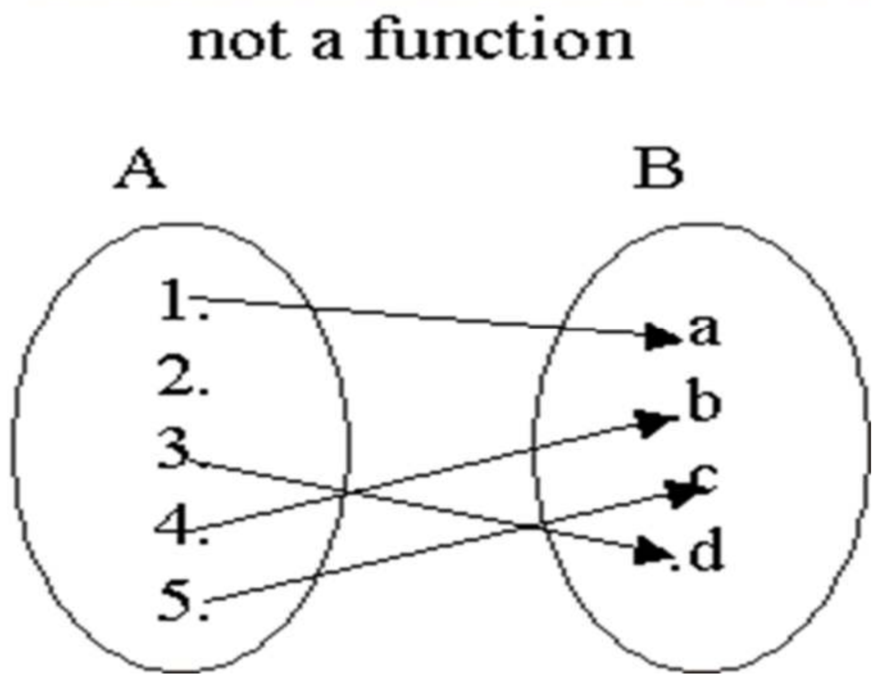
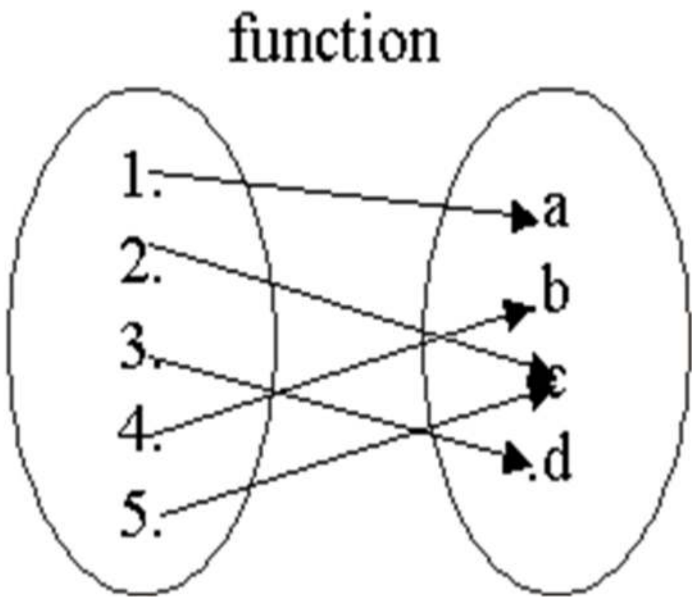


FIGURE 1 **There Are More Pigeons Than Pigeonholes.**

REVISION OF FUNCTIONS



- Clearly the above relation is not a function because 2 does not have any image under this relation. Note that if want to made it relation we have to must map the 2 into some element of B which is also the image of some element of A. Now

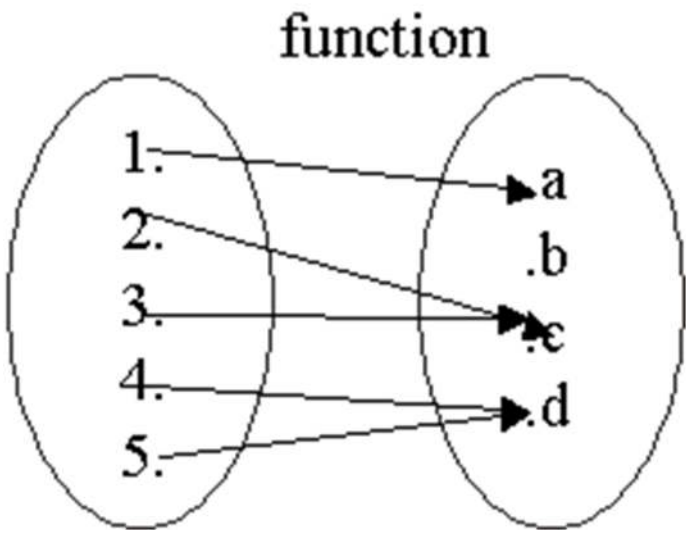


- The above relation is a function because it satisfy the conditions of functions(as each element of 1st set have the images in 2nd set).

PIGEONHOLE PRINCIPLE

- We define a function from A to B. If A has more elements then B. B must have element which is image of more than one element of A.

- The following is a function.



- The above relation is a function because it satisfy the conditions of functions(as each element of 1st set have the images in 2nd set). Therefore the above is also a function.

PIGEONHOLE PRINCIPLE

- A function from a set of $k + 1$ or more elements to a set of k elements must have at least two elements in the domain that have the same image in the co-domain.

- If $k + 1$ or more pigeons fly into k pigeonholes then at least one pigeonhole must contain two or more pigeons.

EXAMPLES

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

2. In any set of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet.

EXERCISE

- What is the minimum number of students in a class to be sure that two of them are born in the same month?

- **SOLUTION:**

There are 12 (= n) months in a year.

The pigeonhole principle shows that among any 13 (= n + 1) or more students there must be at least two students who are born in the same month.

EXERCISE

- Given any set of seven integers, must there be two that have the same remainder when divided by 6?

- **SOLUTION:**

  The set of possible remainders that can be obtained when an integer is divided by six is {0, 1, 2, 3, 4, 5}.

  This set has 6 elements.

  Thus by the pigeonhole principle if 7 = 6 + 1 integers are each divided by six, then at least two of them must have the same remainder.

EXERCISE

- How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?

- **SOLUTION:**

There are 20 integers from 1 through 100 that are divisible by 5.

Hence there are 80 integers from 1 through 100 that are not divisible by 5.

Thus by the pigeonhole principle 81 = 80 + 1 integers from 1 though 100 must be picked in order to be sure of getting one that is divisible by 5.

EXERCISE

- Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Suppose six integers are chosen from A. Must there be two integers whose sum is 11.

- **SOLUTION:**

  The set A can be partitioned into five subsets:

  {1, 10}, {2, 9}, {3, 8}, {4, 7}, and {5, 6} each consisting of two integers whose sum is 11.

  These 5 subsets can be considered as 5 pigeonholes.

  If you pick six elements. Then you have to pick one set from above sets.

  If 6 = (5 + 1) integers are selected from A, then by the pigeonhole principle at least two must be from one of the five subsets. But then the sum of these two integers is 11.

GENERALIZED PIGEONHOLE PRINCIPLE

- Imagine we have 10 pigeonholes, and we want to ensure that at least one pigeonhole have more than 2 pigeons in it.

- If you have 20 pigeons then you can have 2 pigeons in each pigeonholes.

- If you have 21 pigeons then you can have 2 pigeons in each pigeonholes and in 1 pigeonhole you have more than 2 pigeons in it.

GENERALIZED PIGEONHOLE PRINCIPLE

- A function from a set of $n \cdot k + 1$ or more elements to a set of n elements must have at least $k + 1$ elements in the domain that have the same image in the co-domain.

- If $n \cdot k + 1$ or more pigeons fly into n pigeonholes then at least one pigeonhole must contain $k + 1$ or more pigeons.

EXERCISE

- Suppose a laundry bag contains many red, white, and blue socks.

- Find the minimum number of socks that one needs to choose in order to get two pairs (four socks) of the same color.

SOLUTION

- Here there are $n = 3$ colors (pigeonholes)

  and $k + 1 = 4$ or $k = 3$.

  Thus among any $n \cdot k + 1 = 3 \cdot 3 + 1 = 10$ socks (pigeons),

  at least four have the same color.

FLOOR & CEILING FUNCTIONS

- Given any real number x, the **floor** of ***x***, denoted $\lfloor x \rfloor$, is the largest integer smaller than or equal to x.

- Example: $\lfloor 3.4 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$

- Given any real number x, the **ceiling** of ***x***, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x.

- Example: $\lceil 3.4 \rceil = 4$ and $\lceil 5 \rceil = 5$

EXAMPLE

- Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x.

a. 25/4 **b.** 0.999 **c.** –2.01

- **SOLUTION:**

a. $\lfloor 25/4 \rfloor = \lfloor 6 + 1/4 \rfloor = 6$

$\lceil 25/4 \rceil = \lceil 6 + 1/4 \rceil = 6 + 1 = 7$

b. $\lfloor 0.999 \rfloor = \lfloor 0 + 0.999 \rfloor = 0$

$\lceil 0.999 \rceil = \lceil 0 + 0.999 \rceil = 0 + 1 = 1$

c. $\lfloor -2.01 \rfloor = \lfloor -3 + 0.99 \rfloor = -3$

$\lceil -2.01 \rceil = \lceil -3 + 0.999 \rceil = -3 + 1 = -2$

PIGEONHOLE PRINCIPLE

- If N pigeons fly into k pigeonholes then at least one pigeonhole must contain $\lceil N/k \rceil$ or more pigeons.

EXAMPLE

- We have 100 people and we want to know how many of them are born in same month?

- **Solution:**

n = 12 months (pigeonholes)

The required no. is $\lceil 100/12 \rceil = \lceil 8 + 1/3 \rceil = 9$ who were born in the same month.

EXERCISE

- What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

SOLUTION

- We want to ensure that there are 6 students who have the same grade that is

$$\lceil N/5 \rceil = 6 \text{ (given)}$$

- The smallest such integer is

$$N = 5(6-1)+1 = 5 \cdot 5 + 1 = 26$$

- Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grades.