INVERSE & COMPOSITIONS OF FUNCTIONS

EQUALITY OF FUNCTIONS

- Suppose **f** and **g** are functions from **X** to **Y**. Then **f equals g,** written **f = g**, if, and only if,

$$f(x) = g(x) \quad \text{for all} \quad x \in X$$

- i.e.

image of x under f = image of x under g

- **Note:**

For functions to be equal, their ***domain*** and ***co-domain*** must be the ***same***. If domain and co-domain are not equal then their functions equality is not possible.

EXAMPLE

- Define **f: R →R** and **g: R→R** by formulas:

$$f(x) = |x| \quad \text{for all } x \in R$$
$$g(x) = \sqrt{x^2} \quad \text{for all } x \in R$$

Since the absolute value of a real number equals to square root of its square.

i.e., $|x| = \sqrt{x^2}$ for all $x \in R$

Therefore **f(x) = g(x)** for all $x \in R$

EXERCISE

- Define functions f and g from R to R by formulas:

$$f(x) = 2x \qquad \text{and} \qquad g(x) = \frac{2x^3 + 2x}{x^2 + 1}$$

for all $x \in R$.

Show that

f = g

SOLUTION

$$g(x) = \frac{2x^3 + 2x}{x^2 + 1}$$

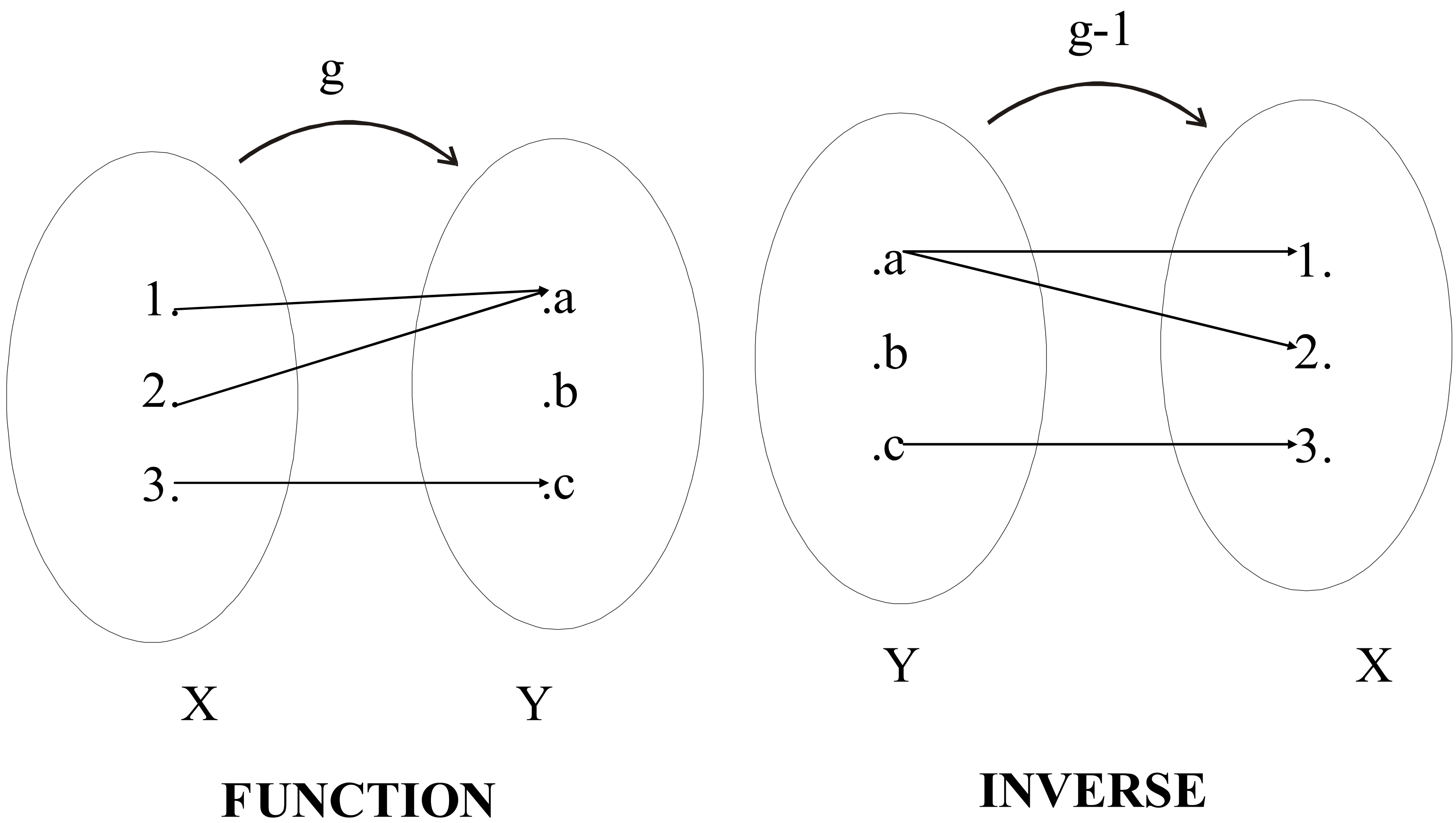
$$= \frac{2x(x^2+1)}{(x^2+1)}$$

$$= 2x \qquad [\because x^2 + 1 \neq 0]$$

$$= f(x) \qquad \text{for all } x \in R$$

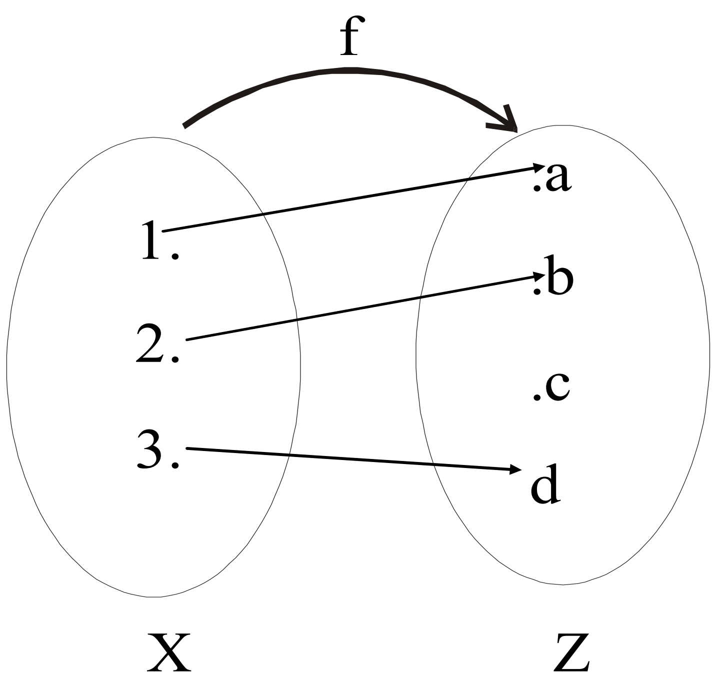
Accordingly **f = g**

INVERSE OF A FUNCTION

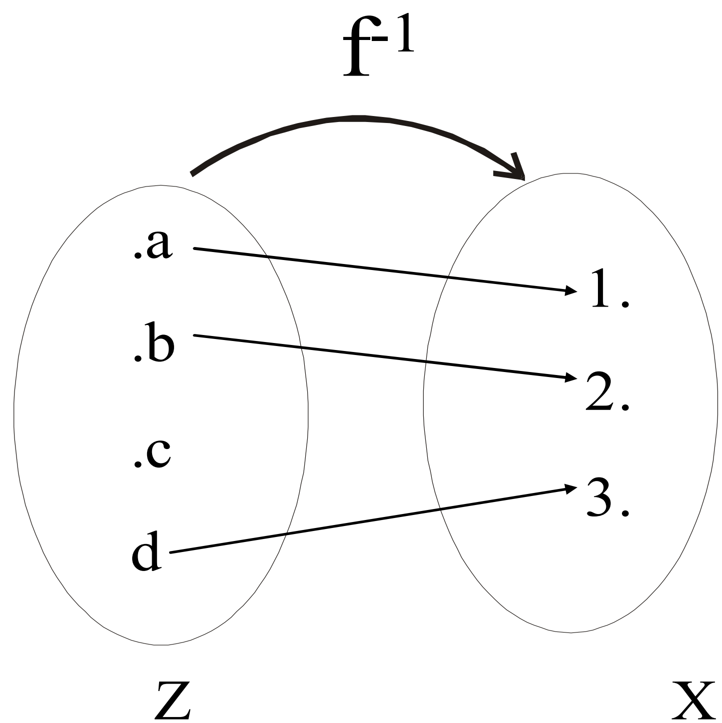


Inverse of a **function** may not be a **function**.

INVERSE OF INJECTIVE FUNCTION



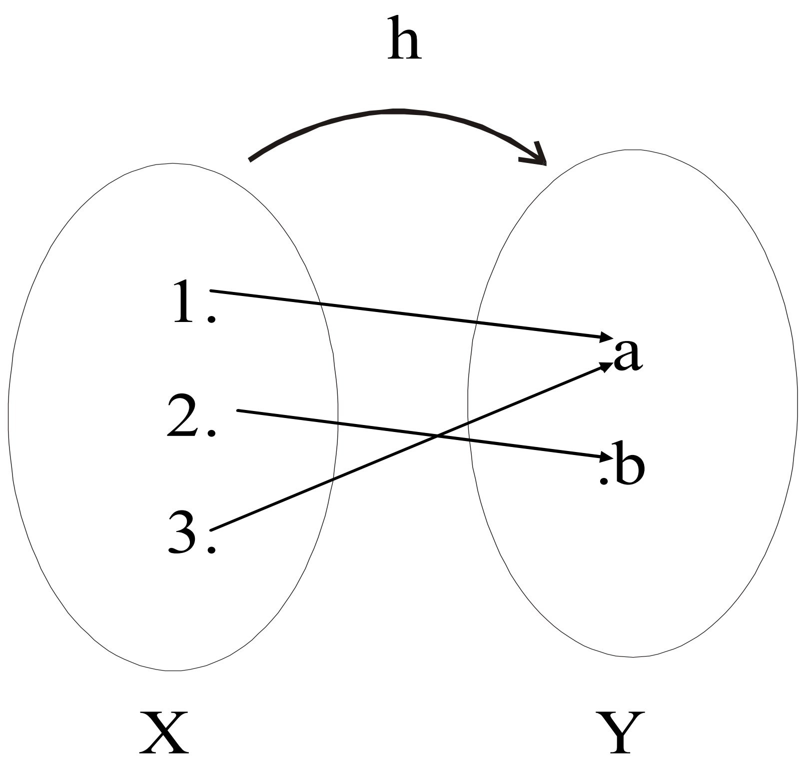
INJECTIVE FUNCTION



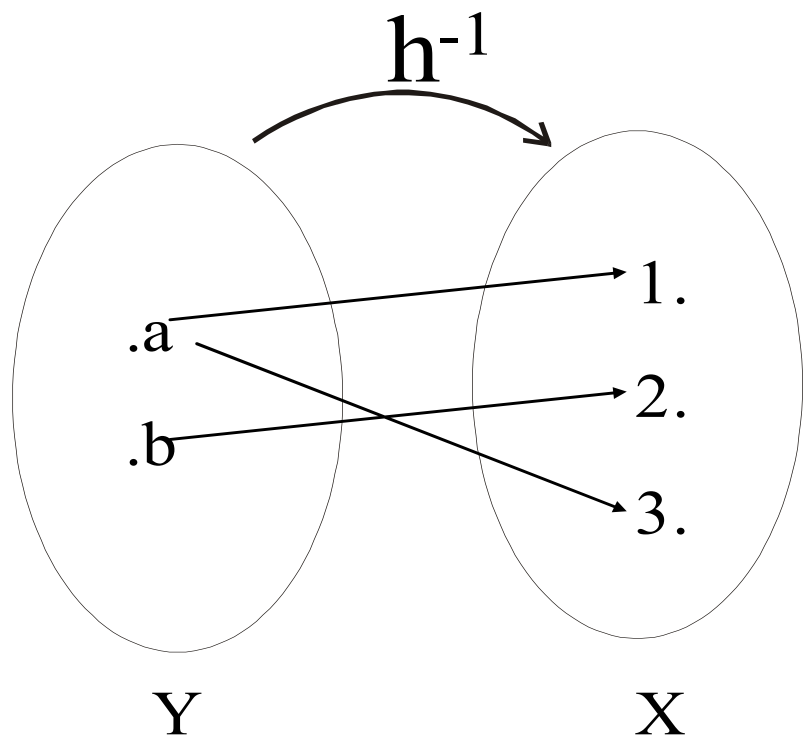
INVERSE

- **Inverse** of **Injective function** may not be a **function**.

INVERSE OF SURJECTIVE FUNCTION



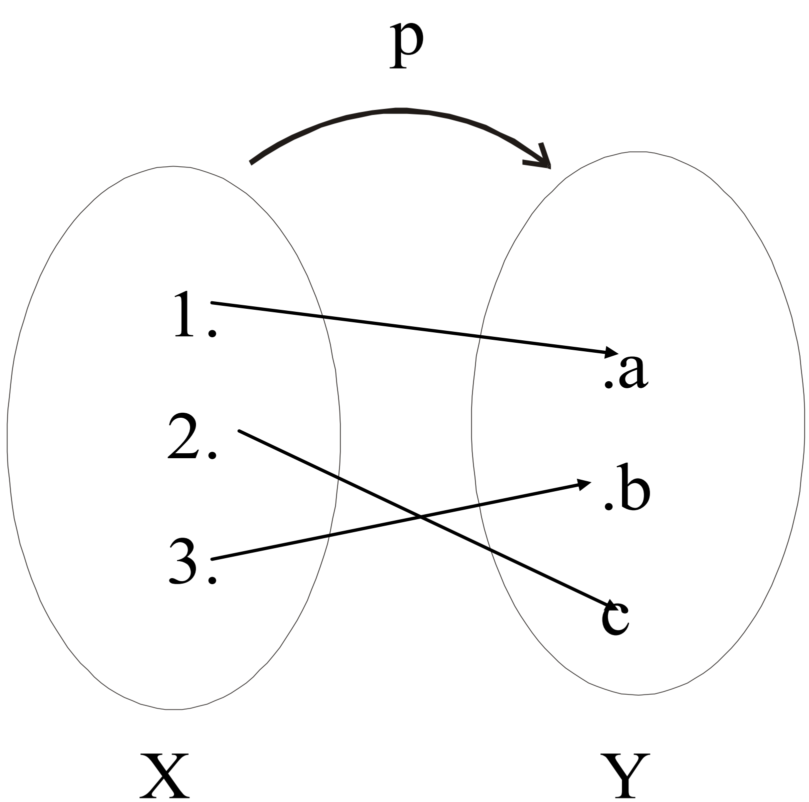
SURJECTIVE FUNCTION



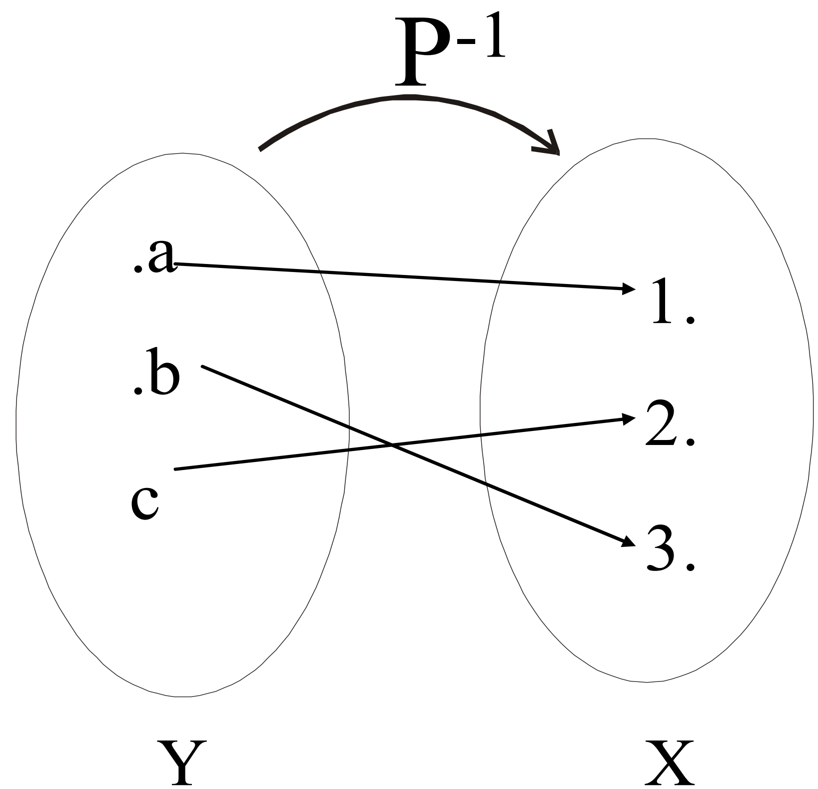
INVERSE

- **Inverse** of **Surjective function** may not be a **function**.

INVERSE OF BIJECTIVE FUNCTION



BIJECTIVE FUNCTION



INVERSE

Note: ***Inverse of a Bijective function will always be a function.***

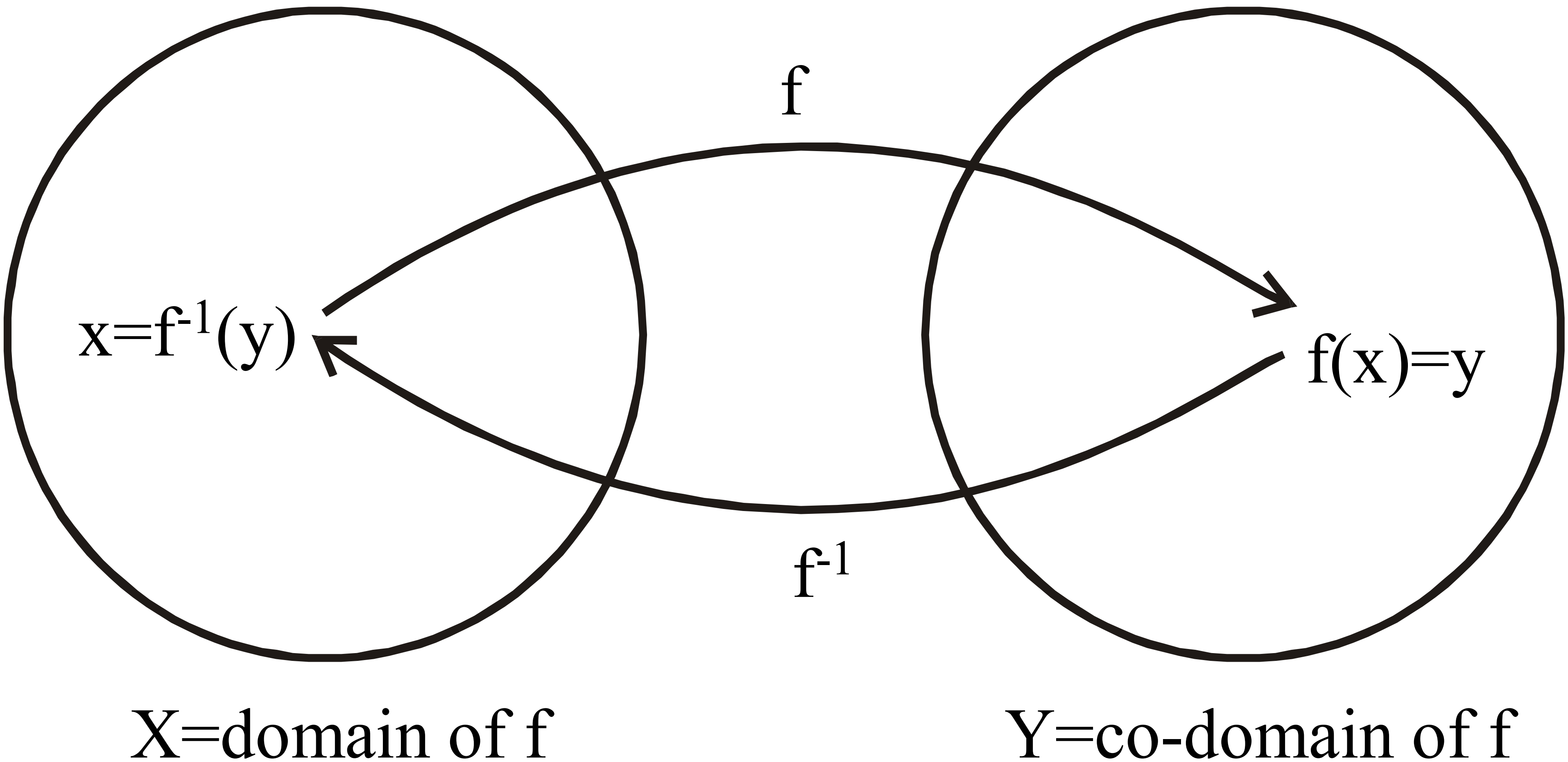
INVERSE FUNCTION

- Suppose $f: X \to Y$ is a Bijective function. Then the inverse function $f^{-1}: Y \to X$ is defined as:

$$f^{-1}(y) = x \Leftrightarrow y = f(x) \qquad \forall \quad y \in Y$$

- That is, f^{-1} sends **each element** of **Y** back to the element of **X** that it came from under **f**.

ARROW DIAGRAM

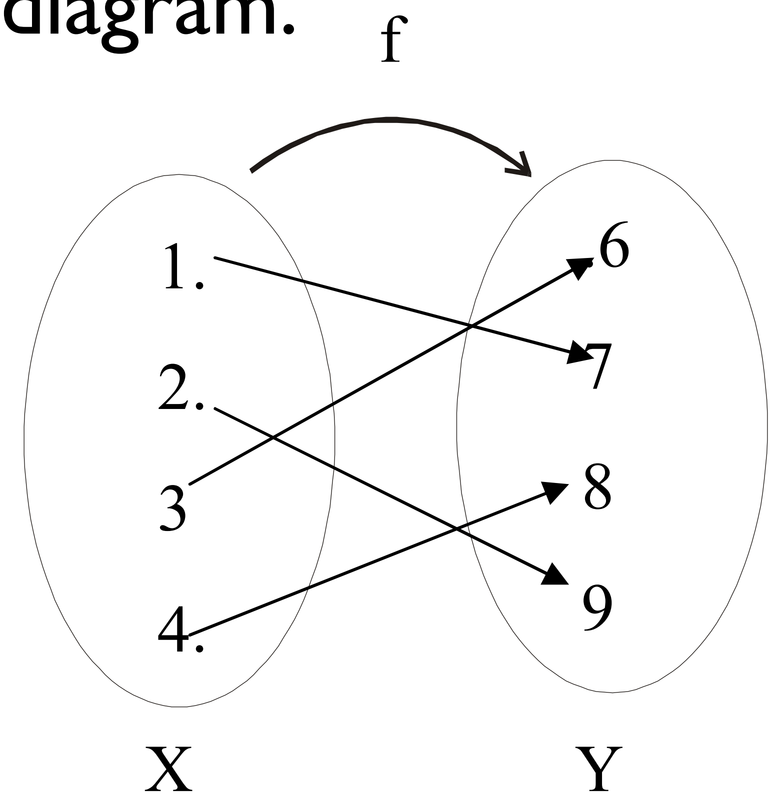


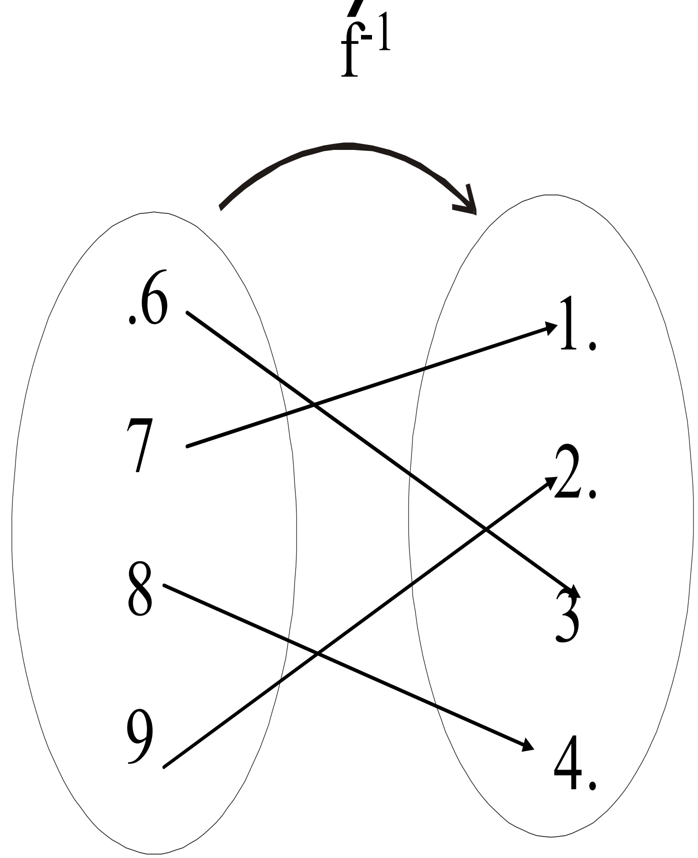
REMARK

- A function whose inverse function exists is called an **invertible function**.
- Only **Bijective functions** are **invertible functions**.

INVERSE FUNCTION FROM AN ARROW DIAGRAM

- Let the bijection $f:X\rightarrow Y$ be defined by the arrow diagram.





- The inverse function $f^{-1}:Y\rightarrow X$ is represented below by the arrow diagram.

INVERSE FUNCTION FROM A FORMULA

- Let **f: R→R** be defined by the formula

$$f(x) = 4x-1 \quad \forall\, x \in R$$

f^{-1} exists ???

We have to prove that f is bijective.

We have already proved that this function is bijective.

- By definition of f^{-1},

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Now solving $f(x) = y$ for x

$\Leftrightarrow$ $4x-1 = y$ (by definition of f)

$\Leftrightarrow$ $4x = y + 1$

$\Leftrightarrow$ $x = \dfrac{y+1}{4}$

- Hence, $f^{-1}(y) = \dfrac{y+1}{4}$ is the inverse of $f(x)=4x-1$ which defines $f^{-1} : R \to R$.

WORKING RULE TO FIND INVERSE FUNCTION

- First we write the function $f(x)$ and solve $f(x) = y$ for x (after putting the value of $f(x)$).

- Then we write $f^{-1}(y)$ = **Right hand side of our equation**.

EXAMPLE

- Let a function **f** be defined on a set of real numbers as

$$f(x) = \frac{x+1}{x-1}$$ for all real numbers $x \neq 1$.

1. Show that f is a bijective function on R - {1}.
2. Find the inverse function f^{-1}

SOLUTION

a) **f is injective**

Let $x_1, x_2 \in R - \{1\}$ and suppose

$f(x_1) = f(x_2)$ we have to show that

$x_1 = x_2$

$$\Rightarrow \frac{x_1 + 1}{x_1 - 1} = \frac{x_2 + 1}{x_2 - 1} \quad \text{(by definition of f)}$$

$\Rightarrow (x_1 + 1)(x_2 - 1) = (x_2 + 1)(x_1 - 1)$

$\Rightarrow x_1x_2 - x_1 + x_2 - 1 = x_1x_2 - x_2 + x_1 - 1$

$\Rightarrow -x_1 + x_2 = -x_2 + x_1$

$\Rightarrow x_2 + x_2 = x_1 + x_1$

$\Rightarrow 2x_2 = 2x_1$

$\Rightarrow x_2 = x_1$

Hence **f** is **injective**.

SOLUTION

b) **f is surjective**

Let $y \in R - \{1\}$. We look for an $x \in R - \{1\}$ such that

$f(x) = y$

$\Rightarrow x + 1 = y(x-1)$

$\Rightarrow x+1 = yx-y$

$\Rightarrow 1 + y = xy - x$

$\Rightarrow 1 + y = x(y-1)$

$\Rightarrow x = \dfrac{y+1}{y-1}$

Thus for each $y \in R - \{1\}$, there exists $x = \dfrac{y+1}{y-1} \in R - \{1\}$ such that $f(x) = f\left(\dfrac{y+1}{y-1}\right) = y$

Hence **f** is **surjective**.

2. inverse function of f

The given function f is defined by the rule

$$f(x) = \frac{x+1}{x-1} = y \qquad \text{(say)}$$

$\Rightarrow \quad x + 1 = y\,(x-1)$

$\Rightarrow \quad x + 1 = yx-y$

$\Rightarrow \quad y + 1 = yx-x$

$\Rightarrow \quad y + 1 = x(y-1)$

$$\Rightarrow \quad x = \frac{y+1}{y-1}$$

Hence **$f^{-1}(y) =$** $\dfrac{y+1}{y-1}; \qquad y \neq 1$

EXERCISE

- Let **f: R→R** be defined by

$$f(x) = x^3 + 5$$

Show that **f** is **one-to-one** and **onto**.

Find a formula that defines the **inverse function f^{-1}**.

SOLUTION

- **f is one-to-one**

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$

$\Rightarrow \quad x_1^3 + 5 = x_2^3 + 5$ (by definition of f)

$\Rightarrow \quad x_1^3 = x_2^3$ (subtracting 5 on both sides)

$\Rightarrow \quad x_1 = x_2$

Hence f is one-to-one.

- **f is onto**

  Let $y \in R$. We search for an $x \in R$ such that $f(x) = y$.

  $\Rightarrow \quad x^3 + 5 = y$ (by definition of f)

  $\Rightarrow \quad x^3 = y - 5$

  $\Rightarrow \quad x = \sqrt[3]{y-5}$

- Thus for each $y \in R$, there exists $x = \sqrt[3]{y-5} \in R$

  such that

$$
\begin{aligned}
f(x) &= f\left(\sqrt[3]{y-5}\right) \\
&= \left(\sqrt[3]{y-5}\right)^3 + 5 \qquad \text{(by definition of f)} \\
&= (y-5)+5 = y
\end{aligned}
$$

Hence f is onto.

- **formula for f^{-1}**

f is defined by $y = f(x) = x^3 + 5$

$\Rightarrow \quad y-5 = x^3$

or $\quad x = \sqrt[3]{y-5}$

Hence $f^{-1}(y) = \sqrt[3]{y-5}$

which defines the inverse function.

COMPOSITION OF FUNCTIONS

- **Note:**
  a) Composition of function is always a function.

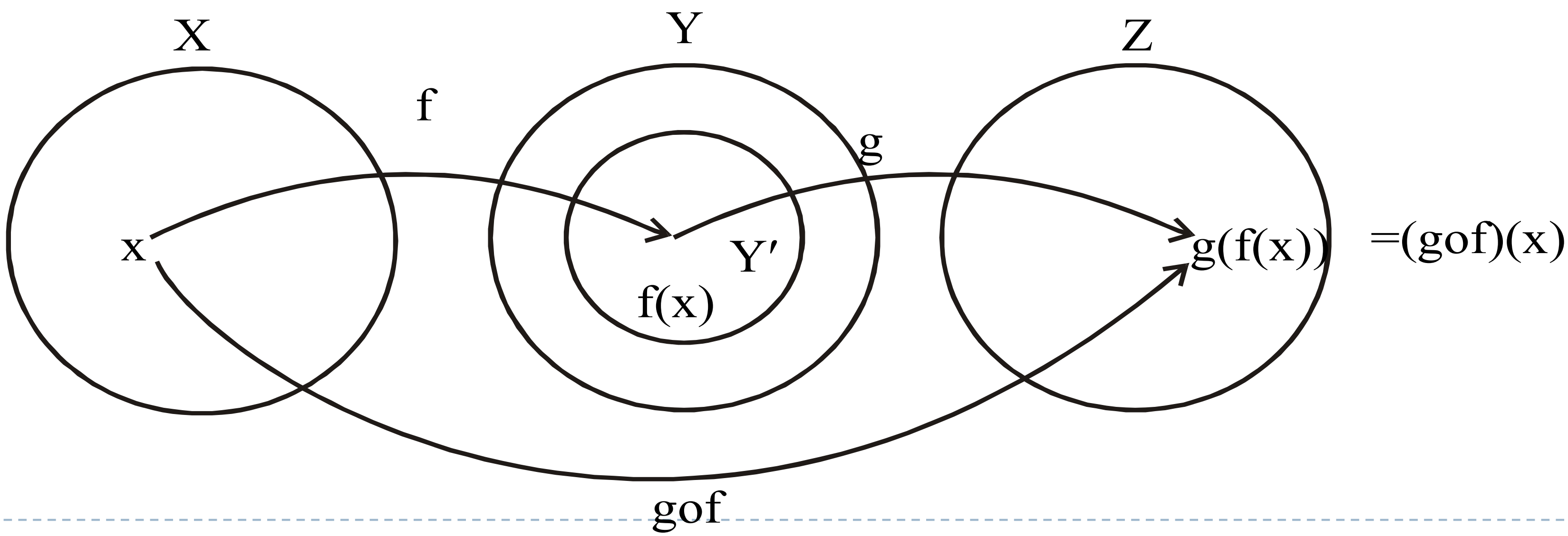
  a) The condition to apply composition of function is **1st function range** is **2nd function domain's subset**.

COMPOSITION OF FUNCTIONS

- Let **f: X →Y′** and **g:Y →Z** be functions with the property that the **range** of **f** is a **subset** of the **domain** of **g** i.e. **f(X) ⊆Y**.
- Define a new function **gof: X →Z** as follows:

$$(gof)(x) = g(f(x)) \qquad \text{for all } x \in X$$

The function gof is called the composition of f and g.



COMPOSITION OF FUNCTIONS DEFINED BY ARROW DIAGRAMS

- Let

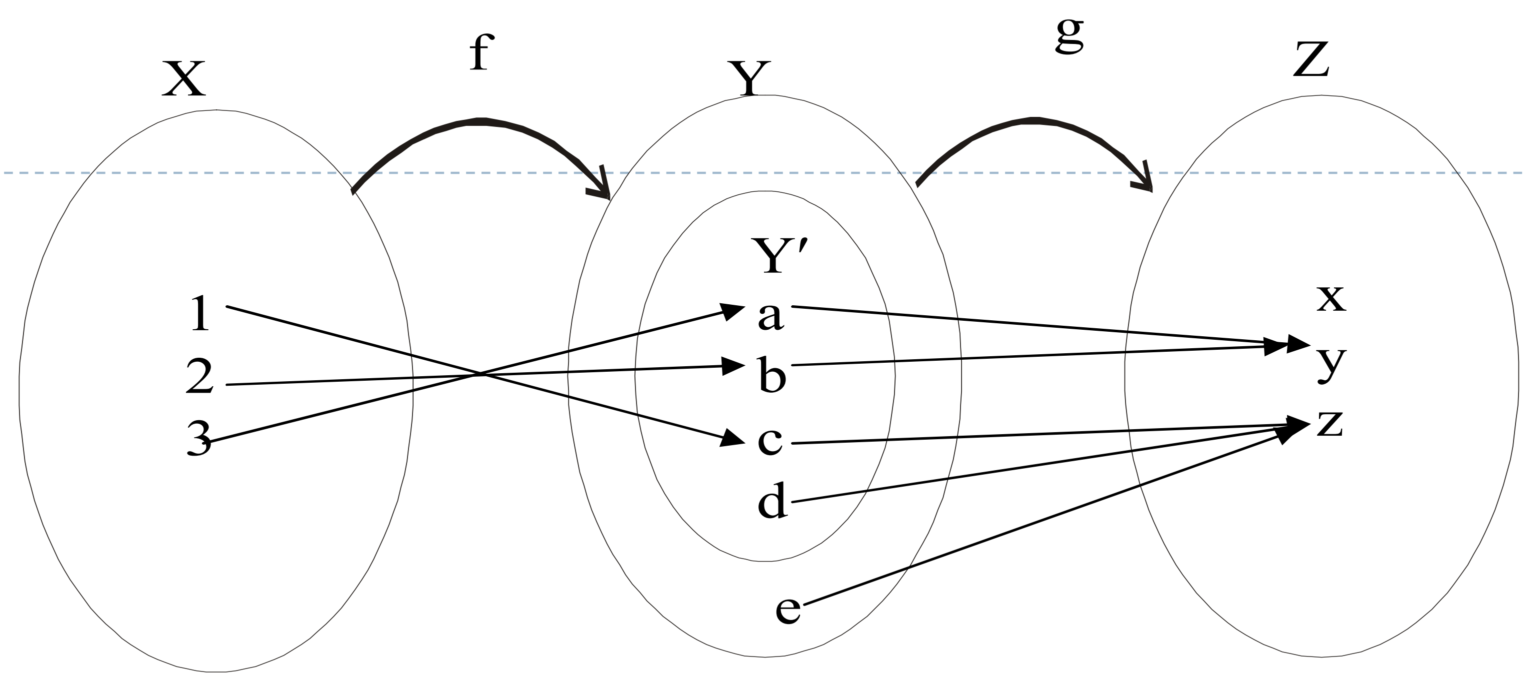
  $X = \{1, 2, 3\}$ and $Y' = \{a, b, c, d\}$

  $Y = \{a, b, c, d, e\}$ and $Z = \{x, y, z\}$.

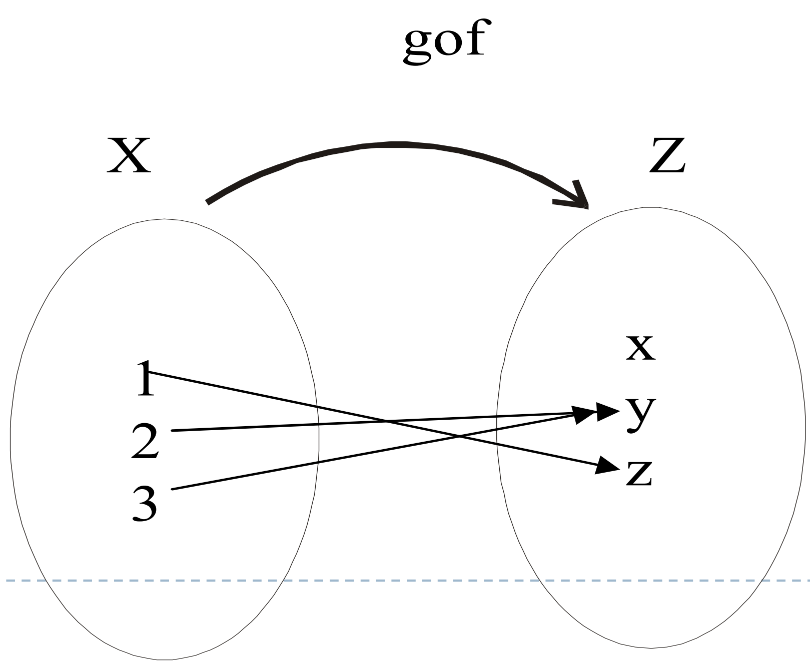
  Define functions

  $f: X \to Y'$ and $g: Y \to Z$

  by the arrow diagrams:



Then gof f: X →Z is represented by the arrow diagram.



EXERCISE

- Let $A = \{1, 2, 3, 4, 5\}$ and we define functions $f: A \to A$ and then $g: A \to A$:

$f(1)=3$, $f(2)=5$, $f(3)=3$, $f(4)=1$, $f(5)=2$

$g(1)=4$, $g(2)=1$, $g(3)=1$, $g(4)=2$, $g(5)=3$

Find the composition functions **fog** and **gof**.

SOLUTION

- We have the definition of the composition of functions and compute:

$f(1)=3,\quad f(2)=5,\quad f(3)=3,\quad f(4)=1,\quad f(5)=2$

$g(1)=4,\quad g(2)=1,\quad g(3)=1,\quad g(4)=2,\quad g(5)=3$

$$(fog)\ (1) = f(g(1)) = f(4) = 1$$
$$(fog)\ (2) = f(g(2)) = f(1) = 3$$
$$(fog)\ (3) = f(g(3)) = f(1) = 3$$
$$(fog)\ (4) = f(g(4)) = f(2) = 5$$
$$(fog)\ (5) = f(g(5)) = f(3) = 3$$

- Also

$$(gof)\ (1) = g(f(1)) = g(3) = 1$$
$$(gof)\ (2) = g(f(2)) = g(5) = 3$$
$$(gof)\ (3) = g(f(3)) = g(3) = 1$$
$$(gof)\ (4) = g(f(4)) = g(1) = 4$$
$$(gof)\ (5) = g(f(5)) = g(2) = 1$$

REMARK: The functions fog and gof are not equal.

COMPOSITION OF FUNCTIONS DEFINED BY FORMULAS

- Let $f: Z \to Z$ and $g: Z \to Z$ be defined by

$$f(n) = n+1 \quad \text{for } n \in Z$$

and

$$g(n) = n^2 \quad \text{for } n \in Z$$

a. Find the compositions gof and fog.

b. Is $gof = fog$?

SOLUTION

a. By definition of the composition of functions

$$(gof)\ (n) = g(f(n))$$

$$= g(n+1) \qquad \text{(Since } f(n) = n+1\text{)}$$

$$= (n+1)^2 \qquad \text{(Since } g(n) = n^2\text{)}$$

$(gof)\ (n) = (n+1)^2$ for all $n \in Z$ and

$(f \circ g)(n) = f(g(n))$

(By definition of composition of function)

$= f(n^2)$ (Since $g(n) = n^2$)

$= n^2+1$ (Since $f(n) = n+1$)

$(f \circ g)(n) = n^2+1$ for all $n \in Z$

b. Is $gof = fog$?

In this case,

For $n = 1$

$$(gof)(1) = g(f(1))$$

$= g(1 + 1)$ (Since $f(n) = n+1$)

$= (1 + 1)^2$ (Since $g(n) = n^2$)

$= 4$ where as

$$(fog)(1) = f(g(1)) = 1^2 + 1 = 2$$

Thus $fog \neq gof$

REMARK

- The **composition** of functions is **not** a **commutative** operation.

OPERATIONS ON FUNCTIONS

- SUM OF FUNCTIONS
- DIFFERENCE OF FUNCTIONS
- PRODUCT OF FUNCTIONS
- QUOTIENT OF FUNCTIONS

SUM OF FUNCTIONS

- Let **f** and **g** be real valued functions with the **same domain X.**

  That is $f: X \to R$ and $g: X \to R$.

  The sum of f and g denoted $f + g$ is a real valued function with the same domain X

  i.e. $f+g: X \to R$ defined by

$$(f+g)(x) = f(x) + g(x) \qquad \forall\, x \in X$$

EXAMPLE

- Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ defines functions f and g from R to R.

Then

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= (x^2 + 1) + (x + 2) \\ &= x^2 + x + 3 \qquad \forall x \in R\end{aligned}$$

which defines the sum functions $f+g: X \to R$

DIFFERENCE OF FUNCTIONS

- Let $f: X \to R$ and $g: X \to R$ be real valued functions. The difference of f and g denoted by $f-g$ which is a function from X to R defined by:

$$(f-g)(x) = f(x) - g(x) \quad \forall \quad x \in X$$

EXAMPLE

- Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ define functions f and g from R to R.

  Then

$$
\begin{aligned}
(f\text{-}g)(x) &= f(x) - g(x) \\
&= (x^2 + 1) - (x + 2) \\
&= x^2 - x - 1 \quad \forall \; x \in R
\end{aligned}
$$

which defines the difference function $f\text{-}g: X \rightarrow R$

PRODUCT OF FUNCTIONS

- Let $f: X \to R$ and $g: X \to R$ be real valued functions. The product of f and g denoted $f.g$ or simply fg is a function from X to R defined by:

$$(f \,.\, g)(x) = f(x) \,.\, g(x) \qquad \forall \qquad x \in X$$

EXAMPLE

- Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ define functions f and g from R to R.

  Then

$$
\begin{aligned}
(f \,.\, g)\,(x) &= f(x) \,.\, g(x) \\
&= (x^2 + 1) \,.\, (x + 2) \\
&= x^3 + 2x^2 + x + 2 \qquad \forall\, x \in R
\end{aligned}
$$

  which defines the product function $f \,.\, g : X \rightarrow R$

QUOTIENT OF FUNCTIONS

- Let f: X→R and g: X →R be real valued functions. The quotient of f by g denoted $\frac{f}{g}$ is a function from X to R defined by:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \ \ is\ \ not\, equal\ \ to\, 0$$

EXAMPLE

- Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ defines functions f and g from R to R.

Then

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \forall x \in X \,\&\, g(x) \neq 0$$

$$= \frac{x^2 + 1}{x + 2}$$

which defines the quotient function $\frac{f}{g} : X \rightarrow R$.

SCALAR MULTIPLICATION OF FUNCTIONS

- Let $f: X \to R$ be a real valued function and c is a non-zero number.
- Then the scalar multiplication of f is a function $c.f: R \to R$ defined by

$$(c.f)(x) = c.f(x) \qquad \forall x \in X$$

EXAMPLE

- Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ define functions f and g from R to R.

Then

$$\begin{aligned}(3f-2g)(x) &= (3f)(x) - (2g)(x)\\ &= 3.\,f(x) - 2.\,g(x)\\ &= 3(x^2 + 1) - 2(x + 2)\\ &= 3x^2 - 2x - 1 \quad \forall \quad x \in X\end{aligned}$$