2.3 FUNCTIONS

FUNCTIONS

- A function ***f*** from a set **X** to a set **Y** is a ***relationship*** between elements of **X** and elements of **Y** such that ***each*** element of **X** is related to a ***unique*** element of **Y**, and is denoted $f: X \to Y$.

- The set **X** is called the **domain** of ***f*** and **Y** is called the **co-domain** of ***f***.

- **Functions** are sometimes also called **mappings** and **transformations**.

REMARK

- The unique element y of Y that is related to x by f is denoted $f(x)$ and is called

  the value of f at x,

  or

  the image of x under f

ARROW DIAGRAM OF A FUNCTION

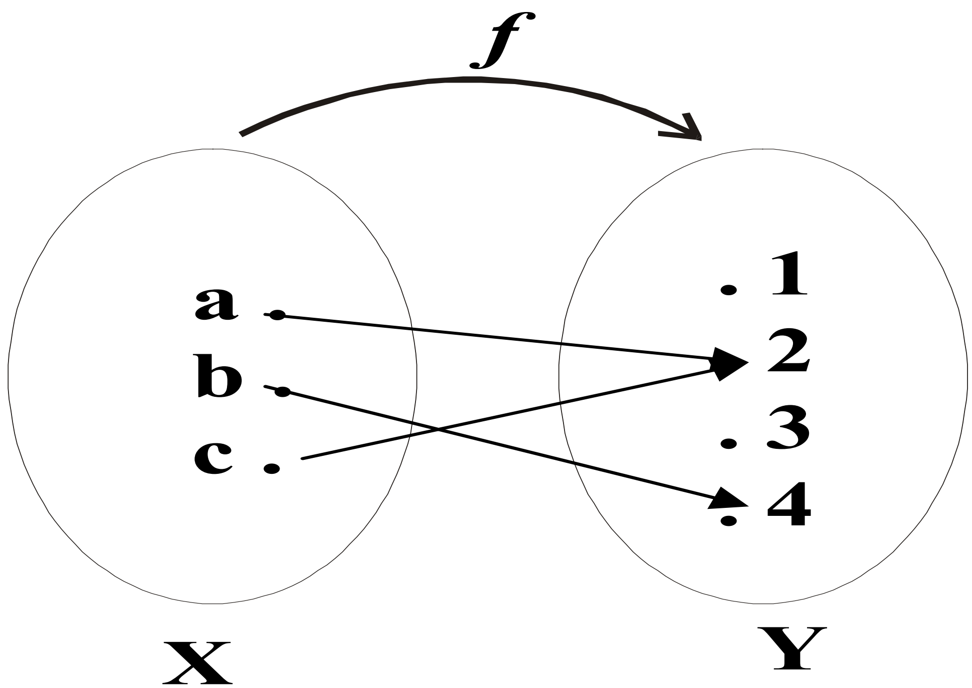
- The definition of a function implies that the **arrow diagram** for a function f has the following **two properties**:

1. Every element of X has an arrow coming out of it

2. No two elements of X has two arrows coming out of it that point to two different elements of Y.

EXAMPLE

- Let X = {a, b, c} and Y = {1, 2, 3, 4}.
- Define a function *f* from X to Y by the arrow diagram.

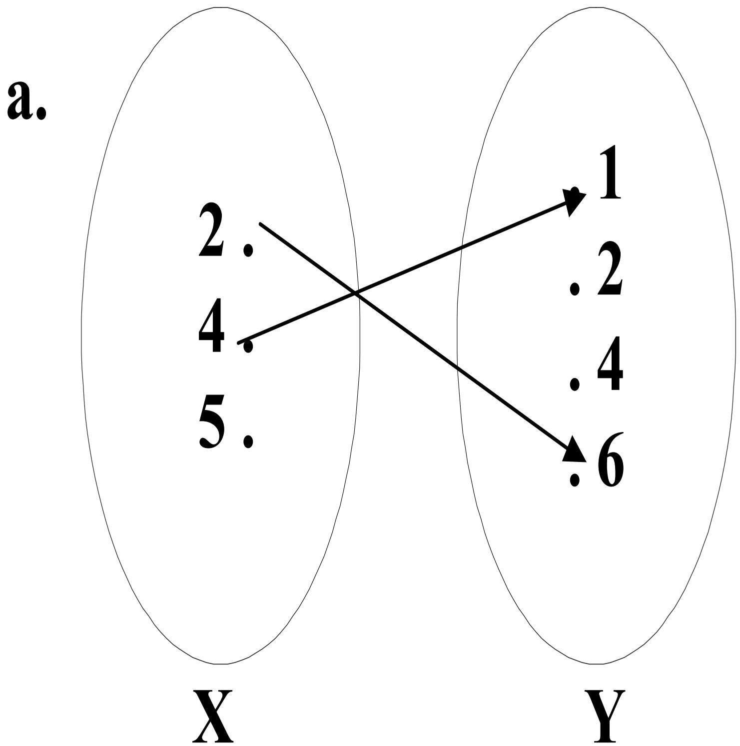


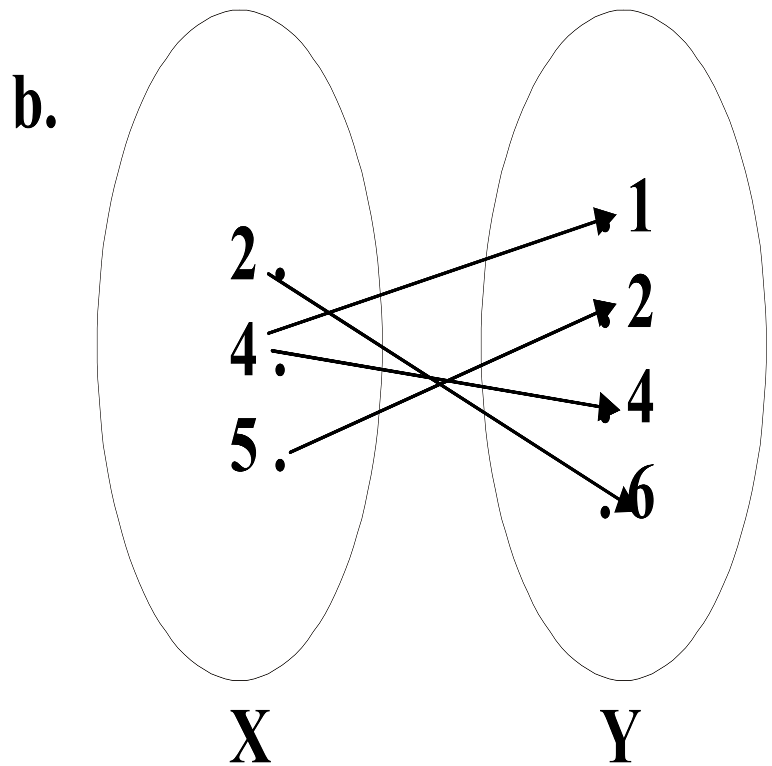
f(a) = 2, f(b) = 4,and f(c) = 2 OR

image of a = 2, image of b = 4, image of c = 2

NONFUNCTIONS

- X = {2, 4, 5} to Y = {1, 2, 4, 6}





NOT A FUNCTION

NOT A FUNCTION

RANGE OF A FUNCTION

- Let $f: X \to Y$. The range of f consists of those elements of Y that are images of elements of X.

- Symbolically:

Range of f

$= \{y \in Y \mid y = f(x)$, for some $x \in X\}$

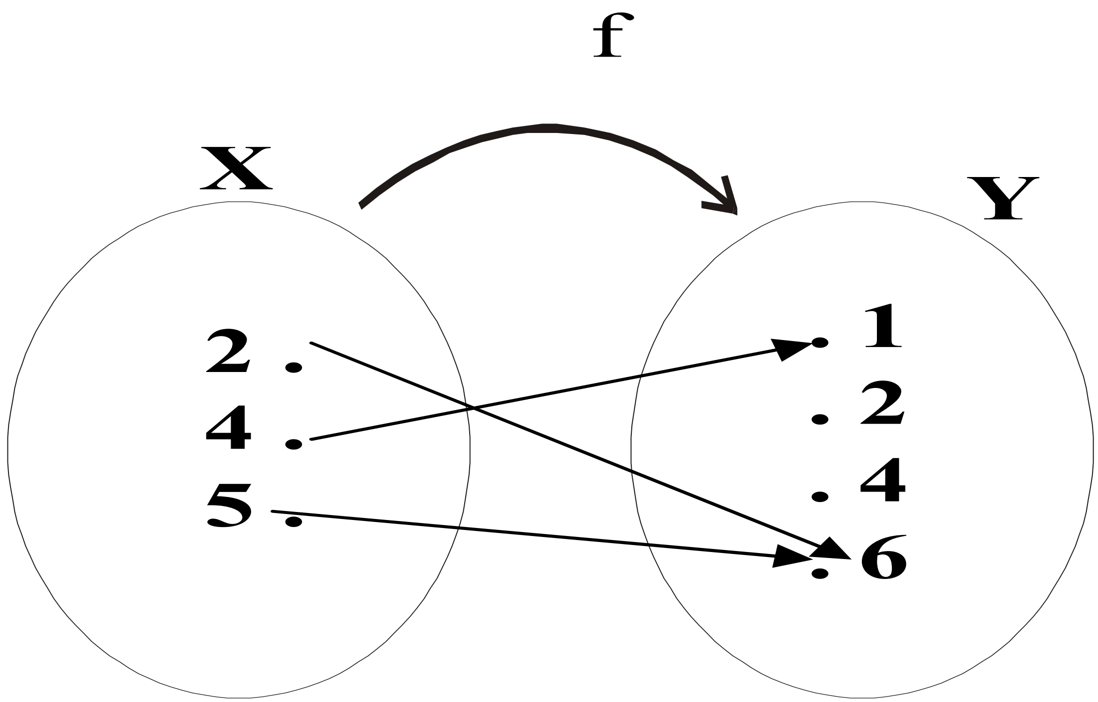
REMARKS

- The range of a function f is always a subset of the co-domain of f.
- The range of $f: X \rightarrow Y$ is also called the image of X under f.
- When $y = f(x)$, then x is called the pre-image of y.
- The set of all elements of X, that are related to some $y \in Y$ is called the inverse image of y.

EXERCISE

- Determine the **range** of the functions **f, g, h** from X = **{2, 4, 5}** to Y = **{1, 2, 4, 6}** defined as:

1.



- 2. g = { (2, 6), (4, 2), (5,1) }
- 3. h(2) = 4, h(4) = 4, h(5) = 1

SOLUTION

1. Range of f = {1, 6}

2. Range of g = {1, 2, 6}

3. Range of h = {1, 4}

GRAPH OF A FUNCTION

- Let **f** be a real-valued function of a real variable. i.e. **$f: R \rightarrow R$.** The graph of f is the set of all points (x, y) in the Cartesian coordinate plane with the property that x is in the **domain** of **f** and **$y = f(x)$.**

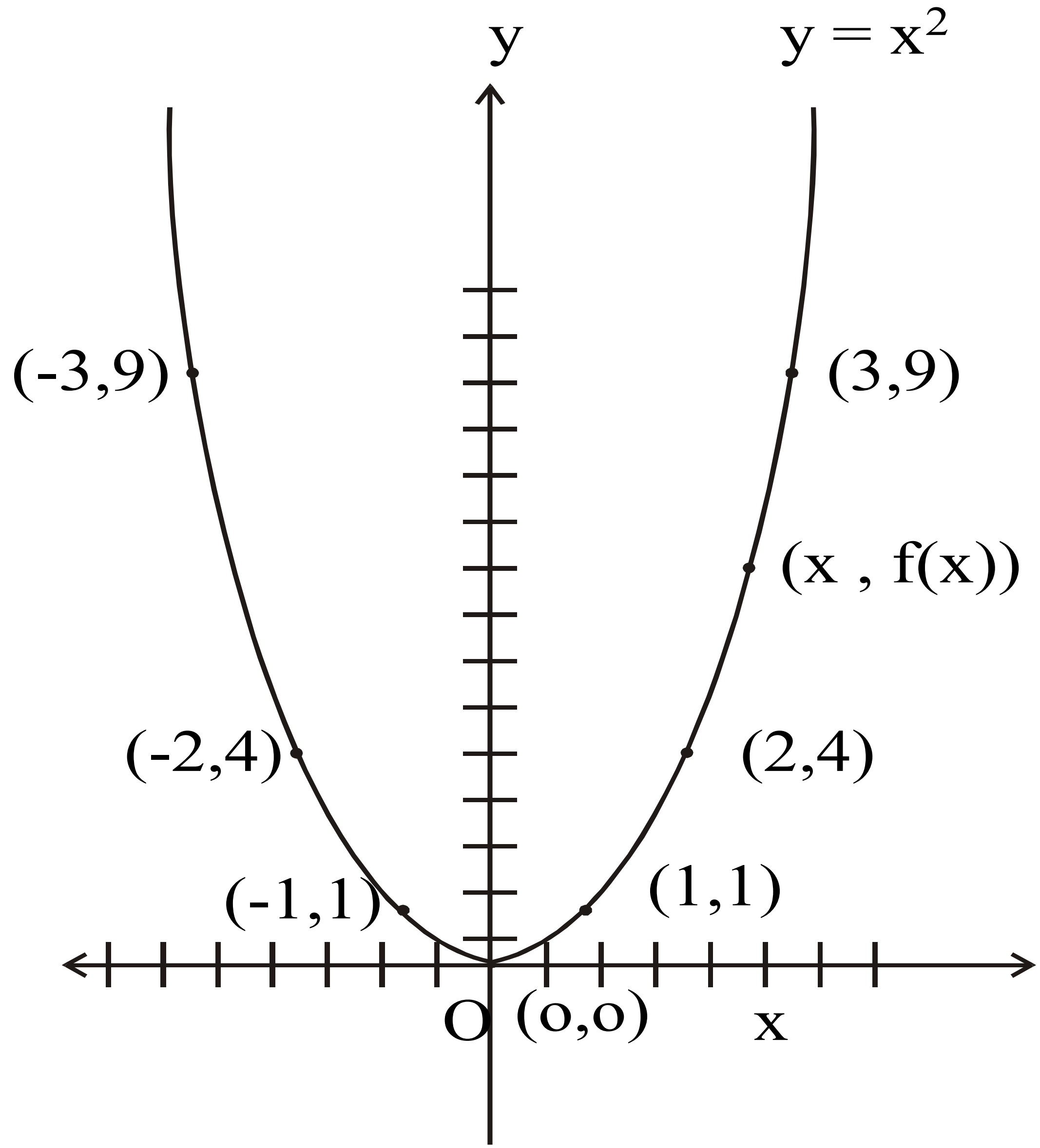
EXAMPLE

- We have the function

$$y = x^2$$

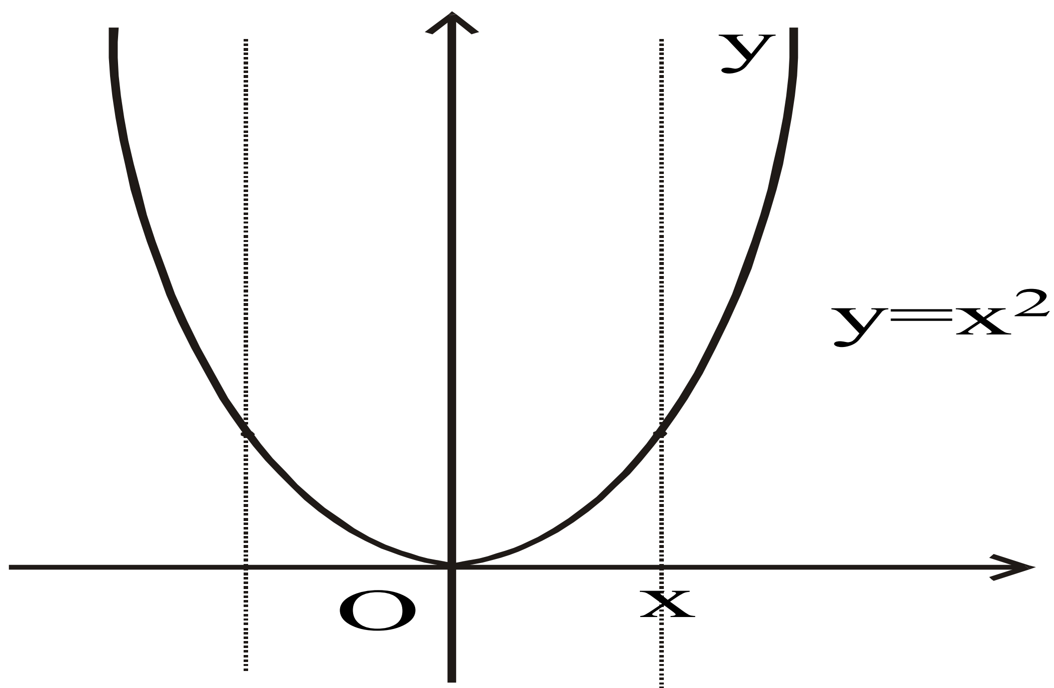
From set of real numbers to set of real numbers.

x	y=f(x)
-3	9
-2	4
-1	1
0	0
+1	1
+2	4
+3	9



VERTICAL LINE TEST FOR THE GRAPH OF A FUNCTION

- For a graph to be the graph of a function, any given vertical line in its domain intersects the graph in at most one point. **For $y = x^2$**



So, it's the **graph of a function**.

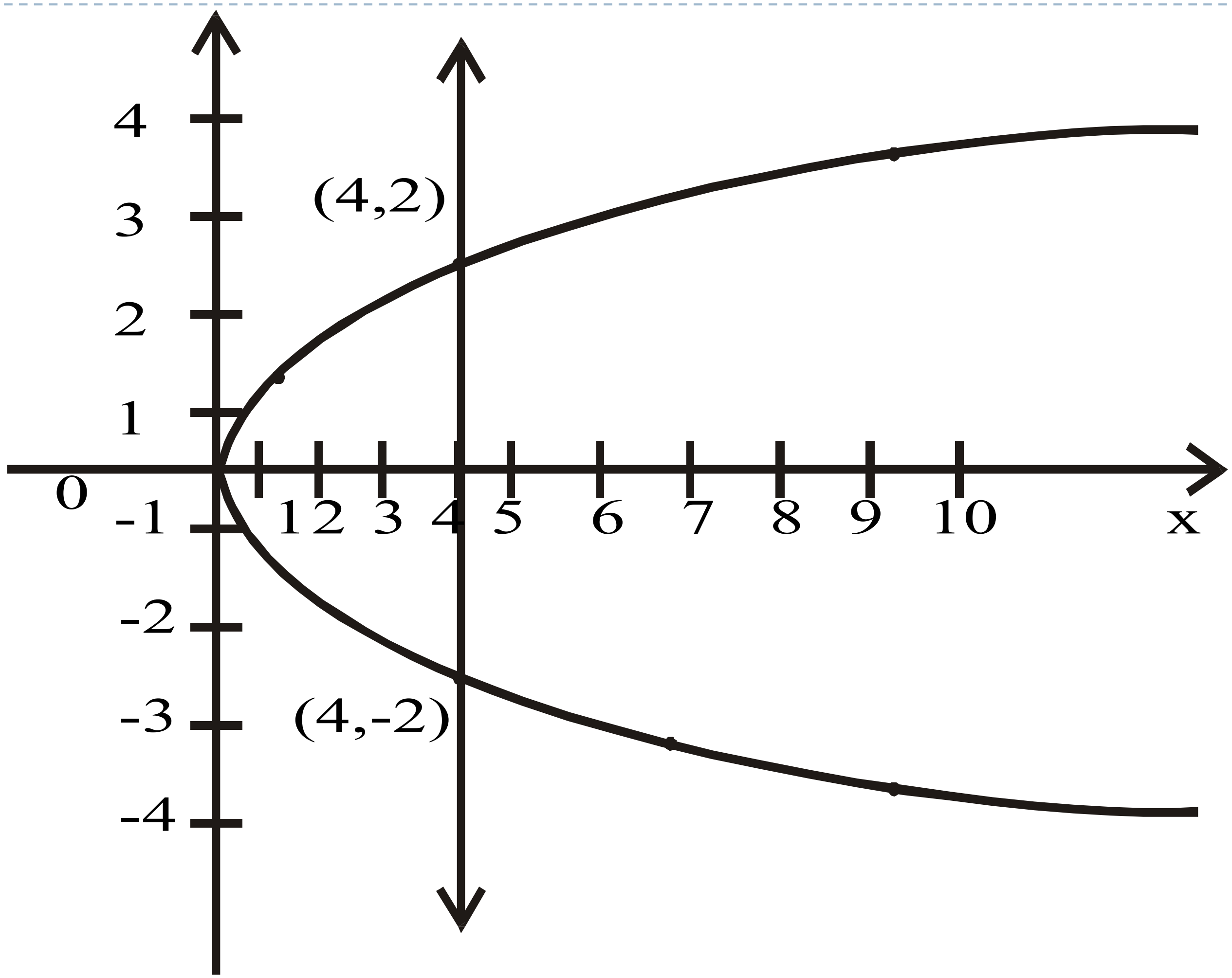
EXERCISE

Define a binary relation **P** from **R** to **R** as follows:

for all real numbers x and **y** $(x, y) \in P \Leftrightarrow x = y^2$

Is **P** a function?

x	Y
9	-3
4	-2
1	-1
0	0
1	1
4	2
9	3

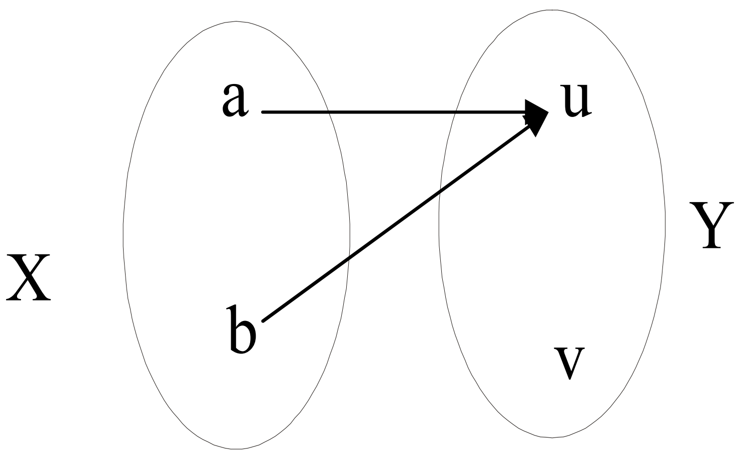


This is not a **graph of a function**.

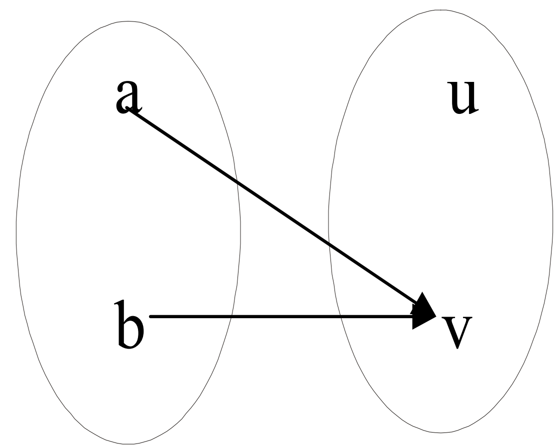
EXERCISE

- Find **all functions** from X = { a, b } to Y = { u, v }
- **SOLUTION**

1.

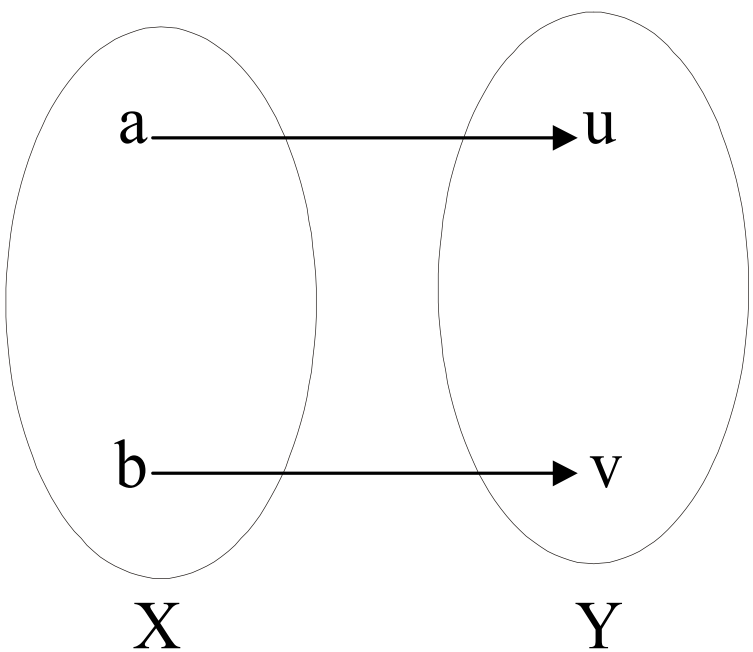


2.

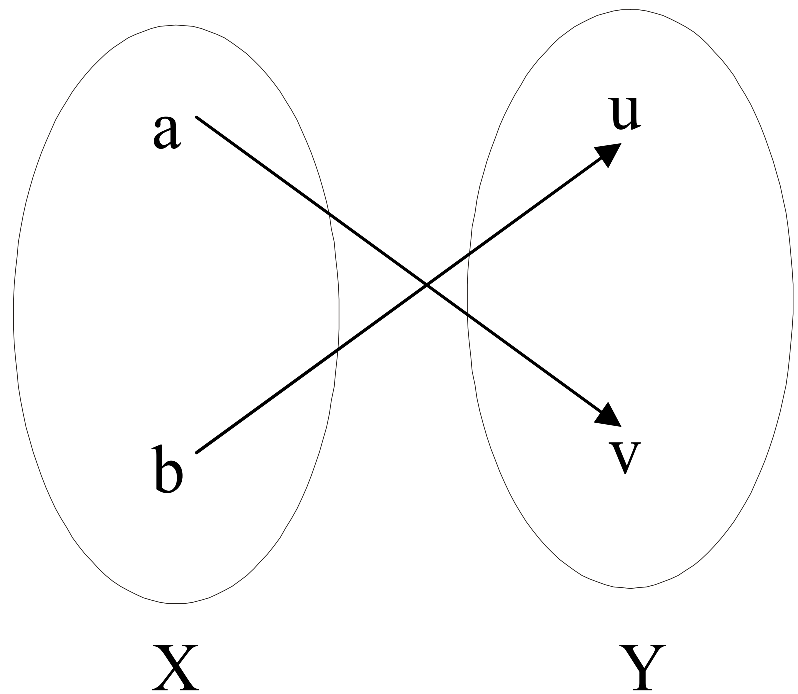


- Find **all functions** from X = { a, b } to Y = { u, v }

3.



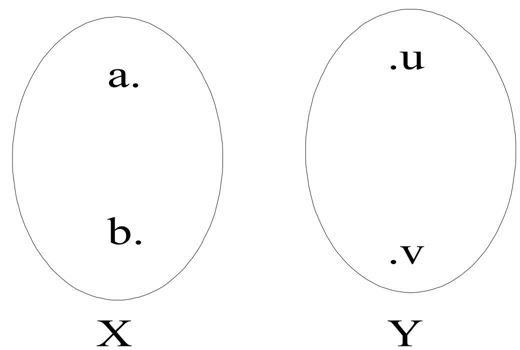
4.



EXERCISE

- Find four binary relations from X = { a, b }to Y = { u, v } that are not functions.
- **SOLUTION**

1.



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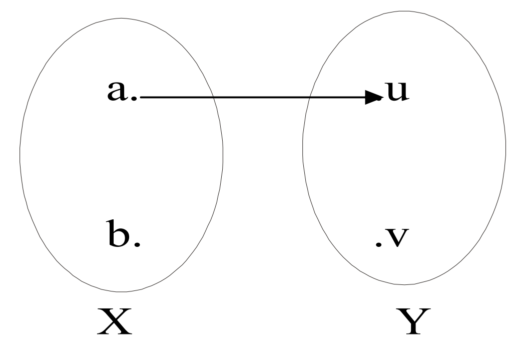

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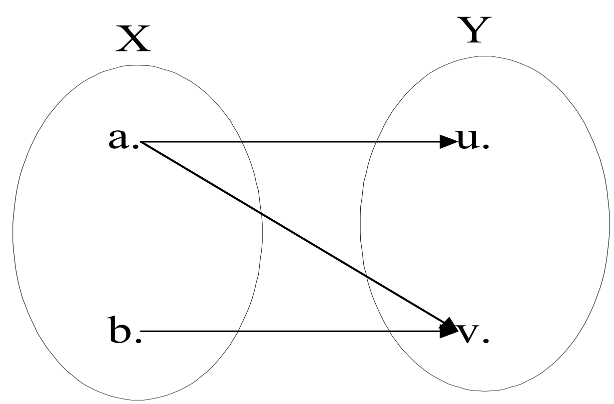

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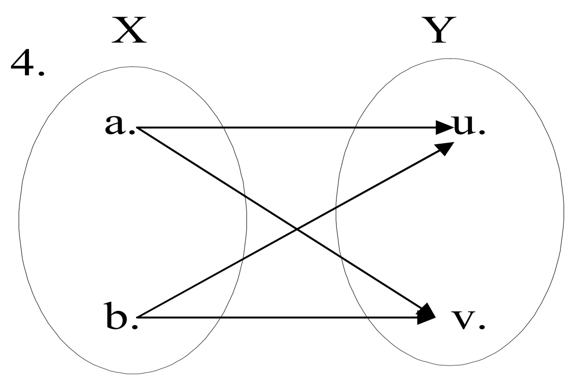

EXERCISE

- How many functions are there from a set with three elements to a set with four elements?

- **SOLUTION**

Let $X = \{x_1, x_2, x_3\}$ and

$Y = \{y_1, y_2, y_3, y_4\}$

- x_1 may be related to any of the four elements y_1, y_2, y_3, y_4 of Y.
- x_1 has four possibilities.
- x_2 has four possibilities.
- x_3 has four possibilities.

Total number of function $= 4 \times 4 \times 4 = 64$

EXERCISE

- Suppose **A** is a set with **m elements** and **B** is a set with **n elements**.
  - How many functions are there from A to B?

- **SOLUTION:**

  Number of functions from

  A to B = n.n.n. … .n (m times)

  = n^m

FUNCTIONS NOT WELL DEFINED

- Determine whether f is a function from Z to R if

$a. \quad f(n) = \pm n \qquad b. \quad f(n) = \dfrac{1}{n^2 - 4}$

$c. \quad f(n) = \sqrt{n} \qquad d. \quad f(n) = \sqrt{n^2 + 1}$

SOLUTION

a) **f** is **NOT well defined** since each **integer n** has two images **+n** and **–n**.

b) **f** is **NOT well defined** since **f(2)** and **f(-2)** are not defined.

c) **f is NOT defined** for **n < 0** since f then results in **imaginary** values (not real).

d) **f** is **well defined** because each integer has **unique** (one and only one) image in **R** under **f**.

EXERCISE

- Student **B** tries to define a function **h : Q → Q** by the rule.

$$h\left(\frac{m}{n}\right)=\frac{m^2}{n}$$

- for all integers **m** and **n** with **n ≠ 0**

- Students **C claims** that h is not well defined. Justify Student **C's claim**.

SOLUTION

- The function **h** is well defined if each **rational number** has a **unique** (one and only one) image.

$$\text{Consider} \quad \frac{1}{2} \in Q$$

$$h\left(\frac{1}{2}\right) = \frac{1^2}{2} = \frac{1}{2}$$

$$\text{Now} \quad \frac{1}{2} = \frac{2}{4} \quad \text{and}$$

$$h\left(\frac{2}{4}\right) = \frac{2^2}{4} = \frac{4}{4} = 1$$

Hence, an element of '**h**' has more than **two images** so **not** a function.

- Hence an element of **Q** has more than **one images** under h. Accordingly **h is not well defined**.

REMARK

- Mathematical Formulation of Function's Properties:

- A function $\mathbf{f: X \rightarrow Y}$ is well defined iff

$$\forall\, \mathbf{x_1, x_2} \in \mathbf{X},$$

$$\text{if } \mathbf{x_1 = x_2} \text{ then } \mathbf{f(x_1) = f(x_2)}$$

- Means an element can not have two images. Rephrase the properties of function.

EXERCISE

- Let **g: $R \rightarrow R+$** be defined by **$g(x) = x^2 + 1$**

1. Determine **g** is **well defined**?
2. Determine the **domain**, **co-domain** and **range** of **g**.

SOLUTION:

Domain of g = R (set of real numbers)

Co-domain of g = R^+ (set of positive real numbers)

Cont…

- **The range** of g consists of those elements of $\mathbf{R}^+$ that appear as image points.

Since $\quad x^2 \geq 0 \quad \forall\, x \in R$

$\quad x^2 + 1 \geq 1 \quad \forall\, x \in R$

i.e. $\quad g(x) = x^2 + 1 \geq 1 \quad \forall\, x \in R$

Hence the range of g is all real number greater than or equal to 1.

i.e $\quad \mathbf{[1, \propto)}$

SOLUTION: (we take two elements of R, and suppose these two elements are equal, by the definition of function we will prove that their images are also equal.)

Let $x_1, x_2 \in R$ and suppose $x_1 = x_2$

$\Rightarrow$ $x_1^2 = x_2^2$ (squaring both sides)

$\Rightarrow$ $x_1^2 + 1 = x_2^2 + 1$ (adding 1 on both sides)

$\Rightarrow$ $g(x_1) = g(x_2)$ (by definition of g)

Thus if $x_1 = x_2$ then $g(x_1) = g(x_2)$.

Accordingly $g:R \rightarrow R+$ is well defined.