GRAPH THEORY

Chapter 10

INTRODUCTION

- Graph theory plays an important role in several areas of computer science such as:

  - Switching Theory and Logical Design
  - Artificial Intelligence
  - Formal Languages
  - Computer Graphics
  - Operating Systems
  - Compiler Writing
  - Information Organization and Retrieval.

GRAPH

- A graph is a non-empty set of points called vertices and

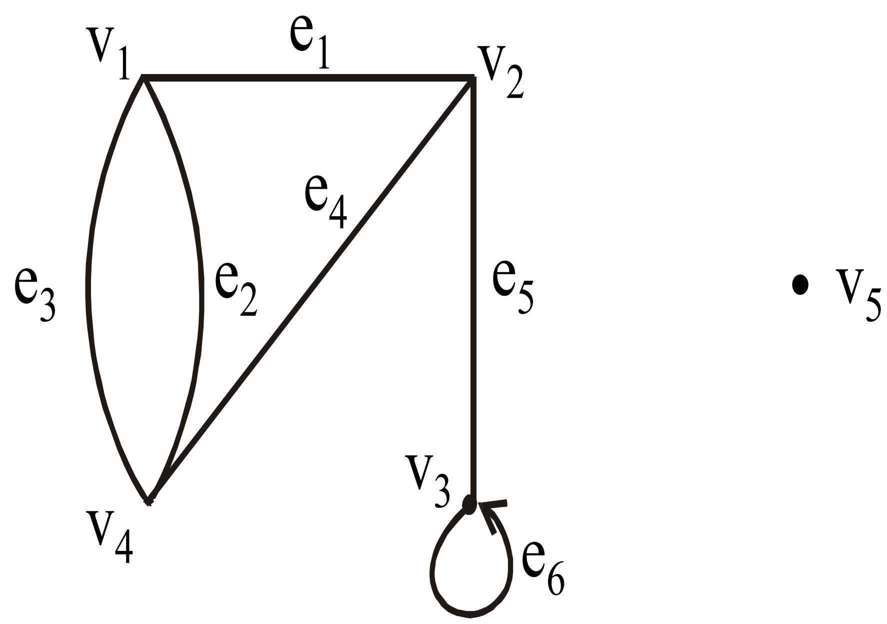
A set of line segments joining pairs of vertices called edges.

- Formally, a graph $G = (V, E)$ consists of two finite sets:

  - A set $V = V(G)$ of vertices (or points or nodes)
  - A set $E = E(G)$ of edges;

  where each edge corresponds to a pair of vertices.

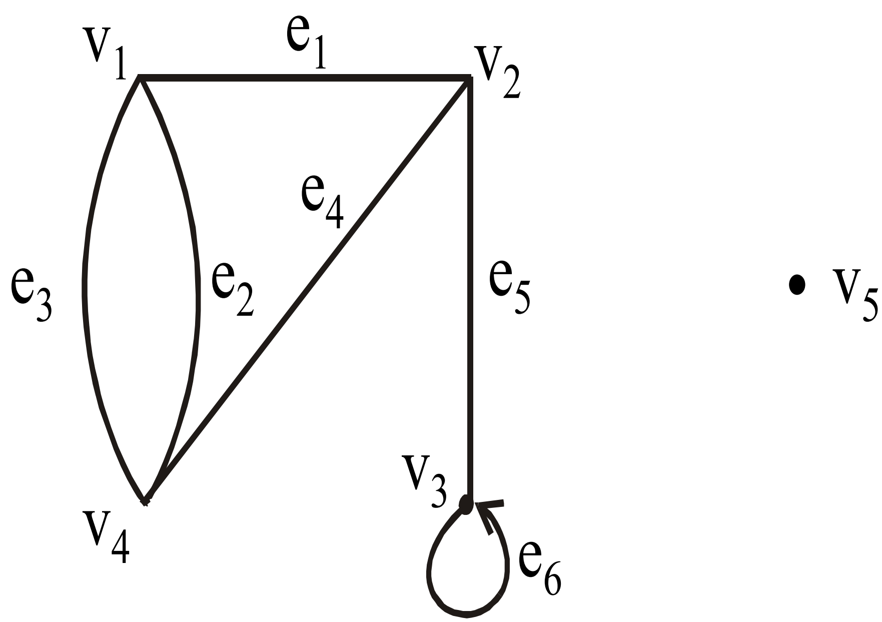
EXAMPLE



- We have five vertices labeled by v_1, v_2, v_3, v_4, v_5 .
- We have edges $e_1, e_2, e_3, e_4, e_5, e_6$.

- e_1 edge is for vertices v_1 and v_2 .
- e_2 and e_3 end points are v_1 and v_4.
- e_4 has end points v_2 and v_4 .
- e_5 has end points v_2 and v_3 .
- e_6 is a loop .

SOME TERMINOLOGY



- An edge connects either one or two vertices called its **endpoints** (edge e_1 connects vertices v_1 and v_2 described as $\{v_1, v_2\}$ i.e v_1 and v_2 are the endpoints of an edge e_1).

- An edge with just one endpoint is called a **loop**. Thus a loop is an edge that connects a vertex to itself (e.g., edge e_6).

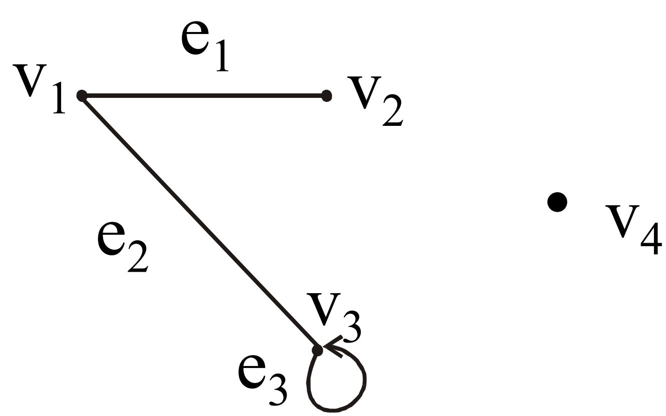
- Two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be adjacent to itself.

- An edge is said to be **incident** on each of it endpoints(i.e. e_1 is incident on v_1 and v_2).

- A vertex on which no edges are incident is called **isolated** (e.g., v_5)
- Two distinct edges with the same set of end points are said to be **parallel** (i.e. e_2 & e_3).

EXAMPLE

- Define the following graph formally by specifying its vertex set, its edge set, and a table giving the edge endpoint function.



SOLUTION

Vertex Set = $\{v_1, v_2, v_3, v_4\}$

Edge Set = $\{e_1, e_2, e_3\}$

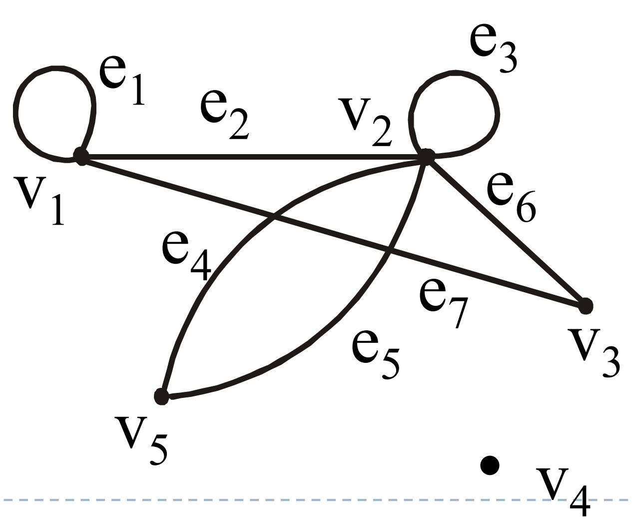
Edge - endpoint function is:

Edge	Endpoint
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

EXAMPLE

- For the graph shown below:

  - Find all edges that are incident on v_1;
  - Find all vertices that are adjacent to v_3;
  - Find all loops;
  - Find all parallel edges;
  - Find all isolated vertices;



SOLUTION

- Find all edges that are incident on v_1?
  - v_1 is **incident** with edges e_1, e_2 and e_7
- Find all vertices that are adjacent to v_3?
  - vertices **adjacent** to v_3 are v_1 and v_2
- Find all loops?
  - **Loops** are e_1 and e_3

- Find all parallel edges?
  - Only edges **e_4** and **e_5** are **parallel**

- Find all isolated vertices?
  - The only **isolated** vertex is **v_4** in this Graph.

EXERCISE

- Draw picture of Graph H having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge endpoint function:

Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

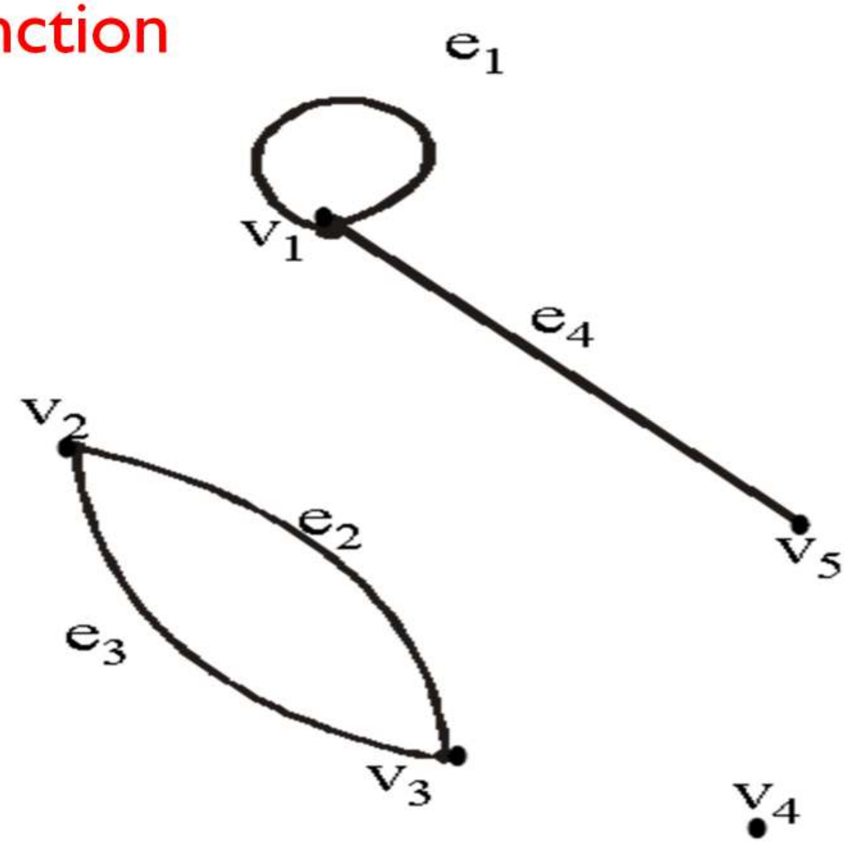
EXERCISE

- Given V(H) = $\{v_1, v_2, v_3, v_4, v_5\}$ and

  E(H) = $\{e_1, e_2, e_3, e_4\}$

  with edge endpoint function

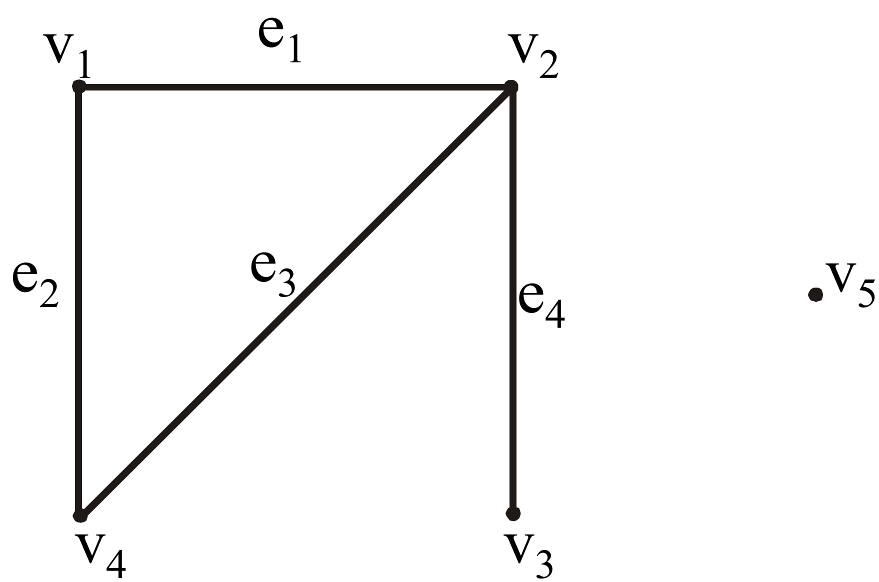
Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$



SIMPLE GRAPH

- A simple graph is a graph that does not have any loop or parallel edges.

- **Example:**



- It is a simple graph H

  $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ & $E(H) = \{e_1, e_2, e_3, e_4\}$

EXERCISE

- Draw all simple graphs with the four vertices {u, v, w, x} and two edges, one of which is {u, v}.

- **Solution:**

  We are given four vertices {u, v, w, x} and one edge is {u, v}.

  Since we are interested in simple graph so we cannot take {u, v} again.

- There are $C(4,2) = 6$ ways of choosing two vertices from 4 vertices.

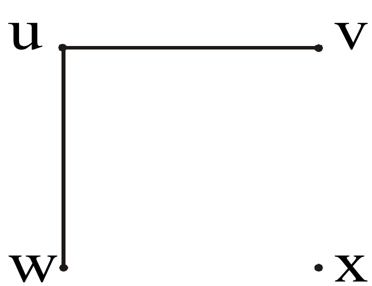
- These edges may be listed as:

  {u, v}, {u, w}, {u, x}, {v, w}, {v, x}, {w, x}

- One edge of the graph is specified to be {u, v}, so any of the remaining five from this list may be chosen to be the second edge.

- This required graphs are:

1.



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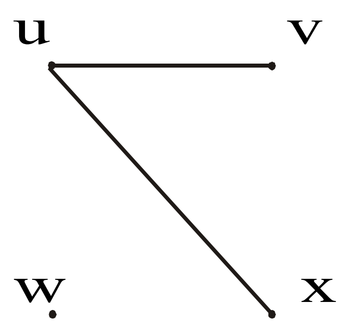

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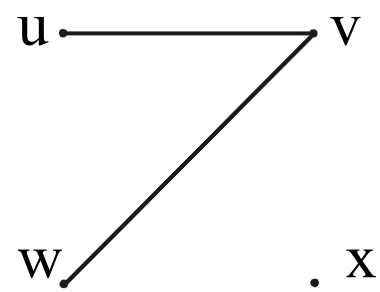
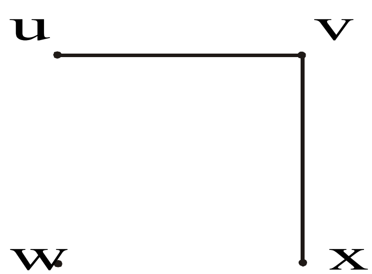
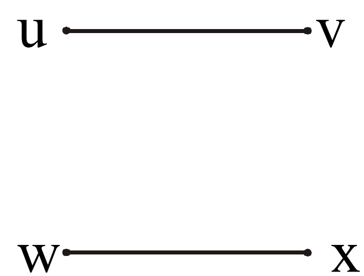


4.



5.



DEGREE OF A VERTEX

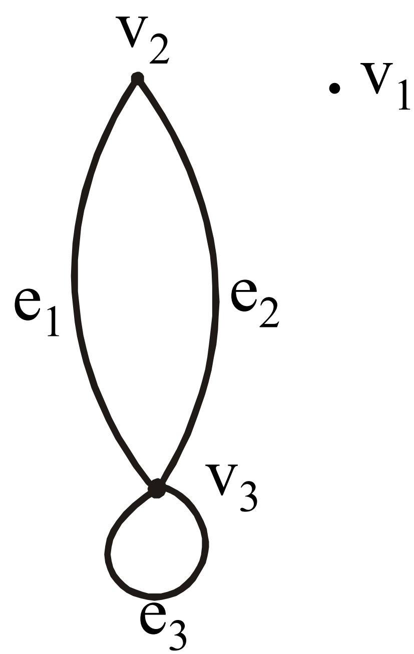
- Let G be a graph and ***v*** a vertex of G. The degree of ***v***, denoted **deg(*v*)**, equals the number of edges that are incident on ***v***, with an edge that is a loop counted twice.

- The **total degree** of G is the sum of the degrees of all the vertices of G.

- The **degree** of a loop is counted twice.

EXAMPLE

- For the graph shown



- deg (v_1) = 0, since v_1 is isolated vertex.
- deg (v_2) = 2, since v_2 is incident on e_1 and e_2.
- deg (v_3) = 4, since v_3 is incident on e_1,e_2 and the loop e_3.
- Total degree of G = deg(v_1) + deg(v_2) + deg(v_3)

  = 0 + 2 + 4 = 6

NOTE

- The total degree of G, which is 6, equals twice the number of edges of G, which is 3.

- This is always the case the total degree of graph is always twice the number of edges in that graph. This is actually the theorem called Handshaking Theorem.

THE HANDSHAKING THEOREM

- If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

- Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a positive integer, then

- The total degree of

  $G = \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$

  $= 2 \,.\, (\text{the number of edges of } G)$

COROLLARY

- The total degree of G is an even number

EXERCISE

- Draw a graph with the specified properties or explain why no such graph exists.

(i) Graph with four vertices of degrees 1, 2, 3 and 3

(ii) Graph with four vertices of degrees 1, 2, 3 and 4

(iii) Simple graph with four vertices of degrees 1, 2, 3 and 4.

(i) Graph with four vertices of degrees 1, 2, 3 and 3

Total degree of graph = 1 + 2 + 3 + 3

= 9 an odd integer

Hence by Hand-Shaking Theorem, first graph is not possible.

(ii) Graph with four vertices of degrees 1, 2, 3 and 4

Total degree of graph = 1 + 2 + 3 + 4

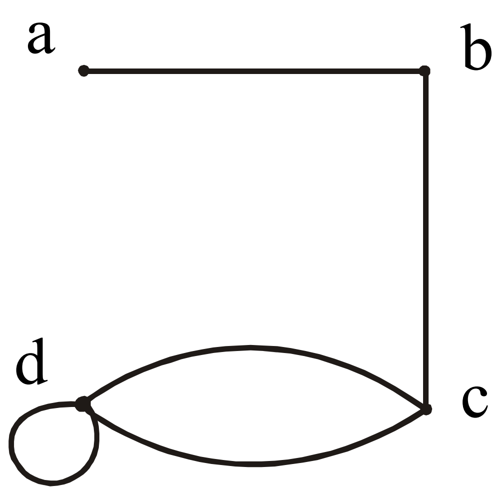
= 10 an even integer

- The vertices a, b, c, d have degrees 1, 2, 3, and 4 respectively (i.e. graph exists).

deg(a) = 1 deg(b) = 2

deg(c) = 3 deg(d) = 4

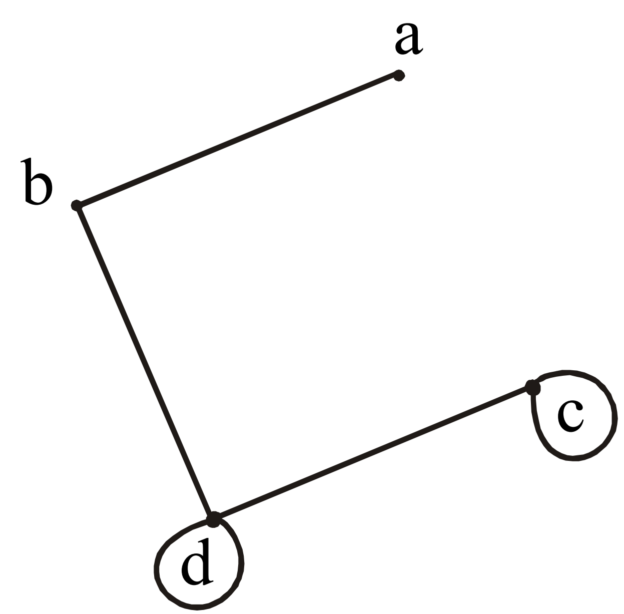
1.



deg(a) = 1 deg(b) = 2

deg(c) = 3 deg(d) = 4

2.



deg(a) = 1 deg(b) = 2

deg(c) = 3 deg(d) = 4

(iii) Simple graph with four vertices of degrees 1, 2, 3 and 4.

- Suppose there was a simple graph with four vertices of degrees 1, 2, 3, and 4. Then the vertex of degree 4 would have to be connected by edges to four distinct vertices other than itself because of the assumption that the graph is simple (and hence has no loop or parallel edges.) This contradicts the assumption that the graph has four vertices in total.
- Hence there is no simple graph with four vertices of degrees 1, 2, 3, and 4, so simple graph is not possible in this case.

EXERCISE

- Suppose a graph has vertices of degrees 1, 1, 4, 4 and 6. How many edges does the graph have?

- **SOLUTION:**

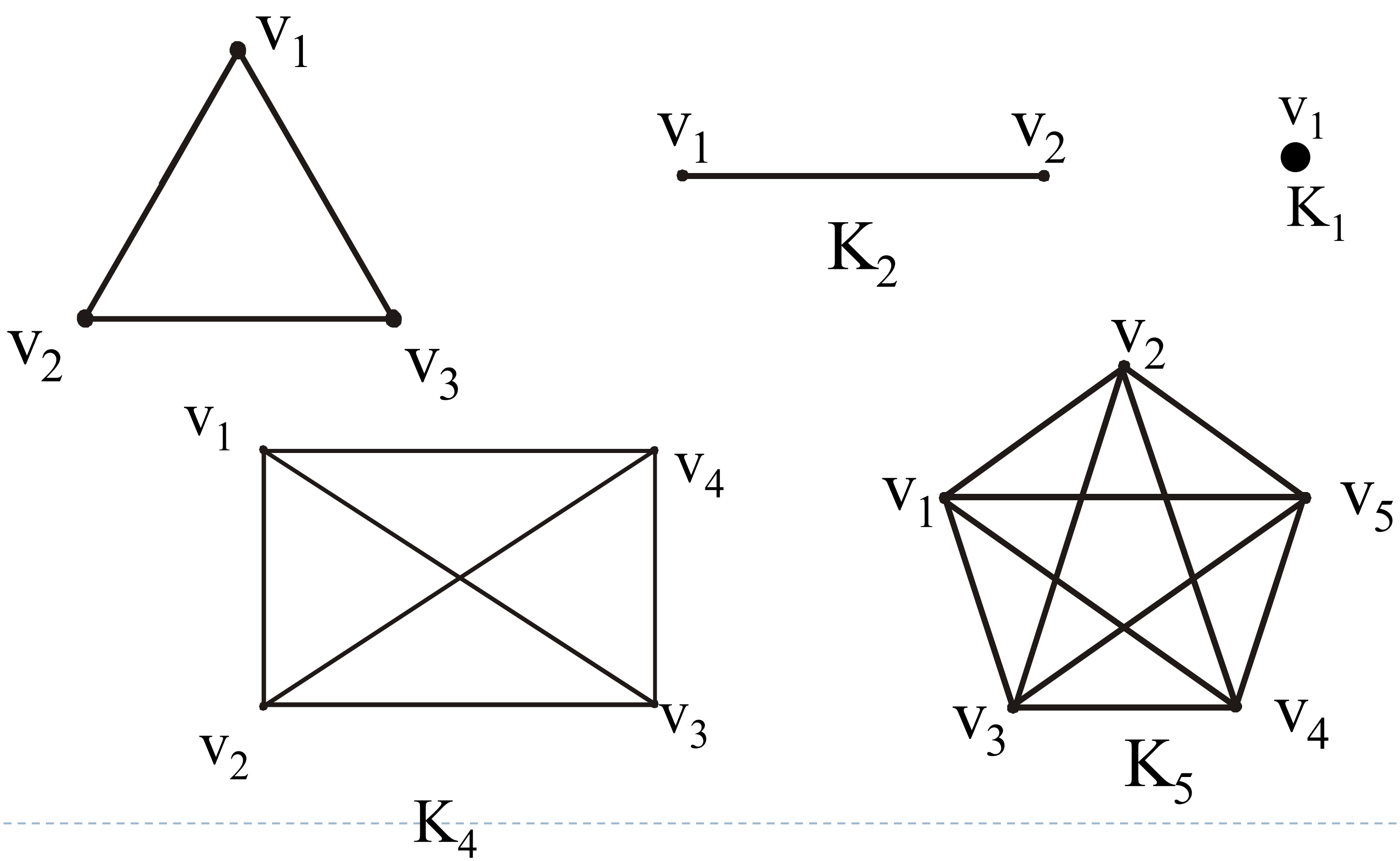
The total degree of graph = 1 + 1 + 4 + 4 + 6

= 16

$\Rightarrow$ Number of edges of graph = $\frac{16}{2} = 8$

COMPLETE GRAPH

- A complete graph on “n” vertices is a simple graph in which each vertex is connected to every other vertex and is denoted by K_n (K_n means that there are n vertices).

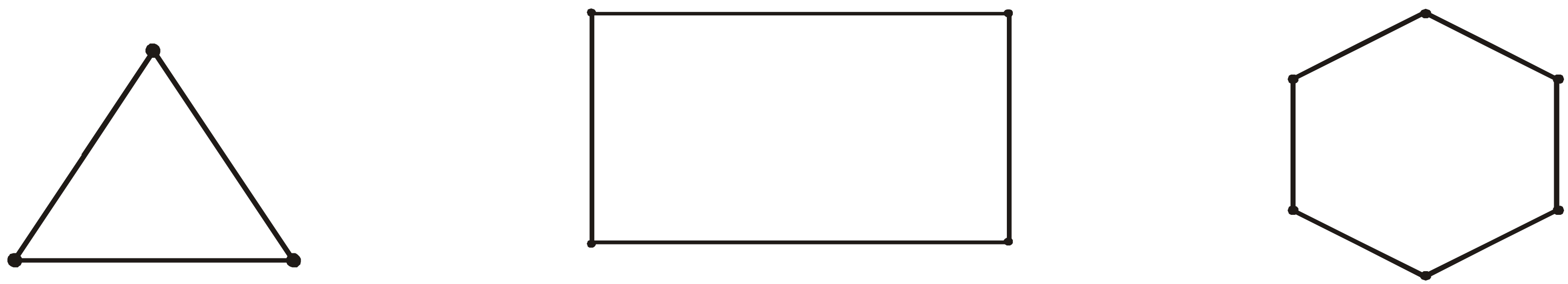


REGULAR GRAPH

- A graph G is regular graph of degree k or k-regular if every vertex of G has degree k.
- In other words, a graph is regular if every vertex has the same degree.
- Following are some regular graphs.

(i) 0-regular

(ii) 1-regular

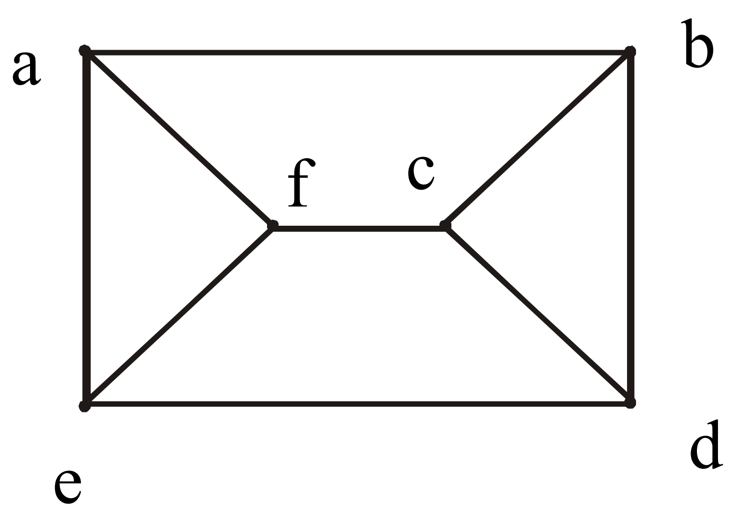


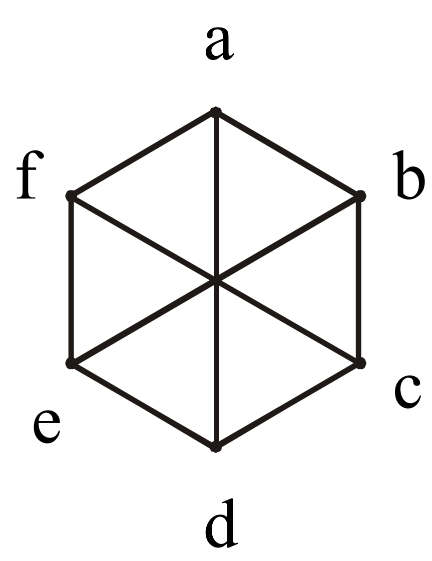
(iii) 2-regular

EXERCISE

- Draw two 3-regular graphs with six vertices.

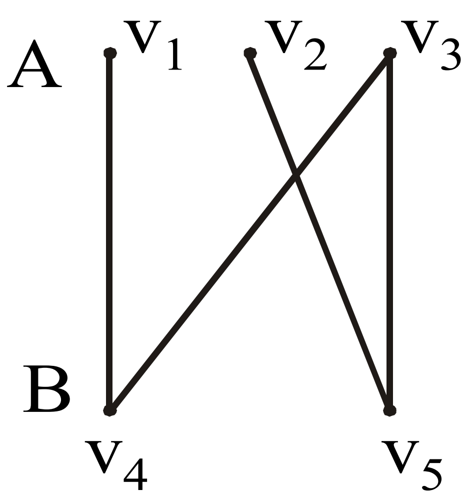
- **SOLUTION:**

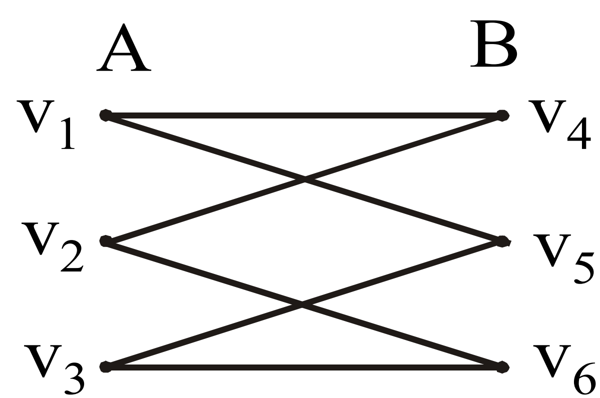




BIPARTITE GRAPH

- A bipartite graph G is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B such that the vertices in A may be connected to vertices in B, but no vertices in A are connected to vertices in A and no vertices in B are connected to vertices in B.





MATRIX REPRESENTATIONS OF GRAPHS

- To store graph in computer with pictorial representation is not possible rather you will store the graph with matrix representation.

- It is difficult to analyze a big complex graph with hundreds of vertices and thousands of edges, but in matrix form you can analyze big graph better.

MATRIX

- An $m \times n$ matrix A over a set S is a rectangular array of elements of S arranged into m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

j^{th} column of A

Briefly, it is written as: $\mathbf{A = [a_{ij}]_{m \times n}}$

EXAMPLE

$$A = \begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

- A is a matrix having 3 rows and 4 columns. We call it a 3×4 matrix, or matrix of size 3×4(or we say that a matrix having an order 3×4).
- Note:

a_{11} = 4 (11 means 1st row and 1st column),

a_{12} = -2 (12 means 1st row and 2nd column), a_{13} = 0,
a_{14} = 6 a_{21} = 2, a_{22} = -3, a_{23} = 1, a_{24} =9 etc.

SQUARE MATRIX

- A matrix for which the number of rows and columns are equal is called a square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal entries

- The main diagonal of A consists of all the entries

$$a_{11}, a_{22}, a_{33}, \ldots, a_{ii}, \ldots, a_{nn}$$

TRANSPOSE OF A MATRIX

- The transpose of a matrix A of size m × n, is the matrix denoted by A^t of size n × m, obtained by writing the rows of A, in order, as columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad then \quad A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

EXAMPLE

$$A = \begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

- Then

$$A^t = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -3 & 7 \\ 0 & 1 & 5 \\ 6 & 9 & -1 \end{bmatrix}$$

SYMMETRIC MATRIX

- A square matrix $A = [a_{ij}]$ of size $n \times n$ is called symmetric if, and only if, $A^t = A$ i.e., for all i, j = 1, 2, …, n, $a_{ij} = a_{ji}$

EXAMPLE

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 9 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

Then $A^t = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 7 & 9 \end{bmatrix}$, and $B^t = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

- Note that $B^t = B$, so that B is a symmetric matrix.

MATRIX MULTIPLICATION

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{mi} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} \mathrm{b}_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \mathrm{b}_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & \\ \mathrm{b}_{\mathrm{p}1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & & & & \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

- **Note:**

  If the number of columns of A is not equal to the number of rows of B, then the product AB is not defined.

EXAMPLE

- Find the product AB and BA of the matrices

$$A=\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

- **SOLUTION:**

Size of A is 2×2 and of B is 2×3, the product AB is defined as a 2×3 matrix.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(3)(3) & (1)(0)+(3)(6) & (1)(-4)+(3)(6) \\ (2)(2)+(-1)(3) & (2)(0)+(-1)(-2) & (2)(-4)+(-2)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{bmatrix}$$

ADJACENCY MATRIX OF A GRAPH

- Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

  a_{ij} = the number of edges connecting v_i and v_j

  for all $i, j = 1, 2, \ldots, n$.

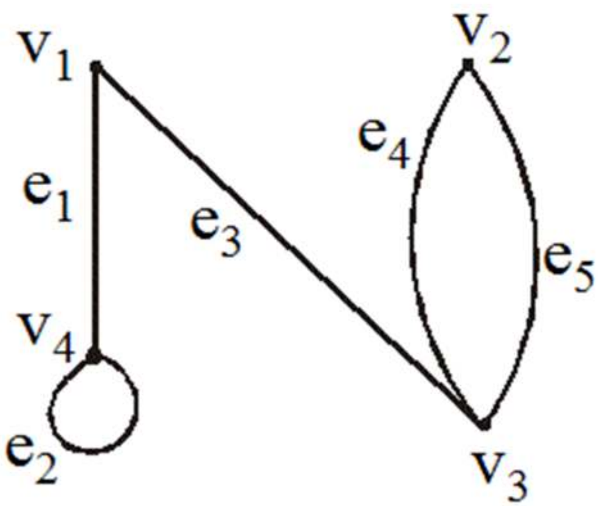
OR

- The adjancy matrix say $A = [a_{ij}]$ is also defined as

$$a_{ij} = \begin{cases} 1 \ \ if \ \{v_i, v_j\} \, is \, an \, edge \, of \ G \\ 0 \ \ otherwise \end{cases}$$

EXAMPLE

- A graph with it's adjacency matrix is shown.



$$A = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- Clearly graph has four vertices. It means that the corresponding square matrix will be order 4 x 4.

EXERCISE

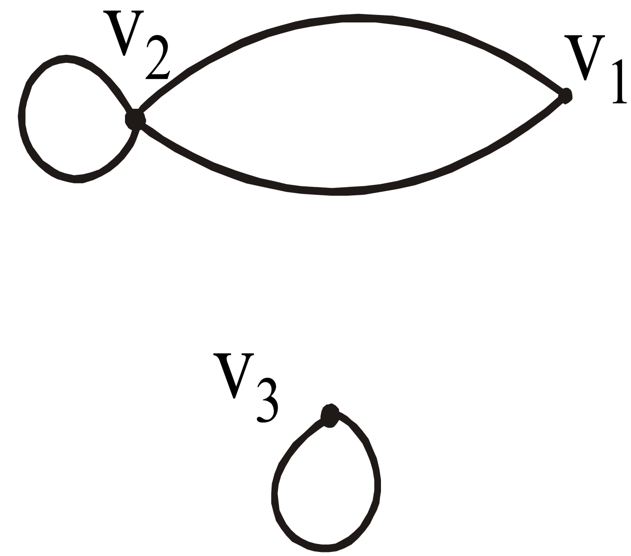
- Find a graph that have the following adjacency matrix.

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Its order is 3 x 3, it means its corresponding graph has three vertices.

- Let the three vertices of the graph be named v_1, v_2 and v_3

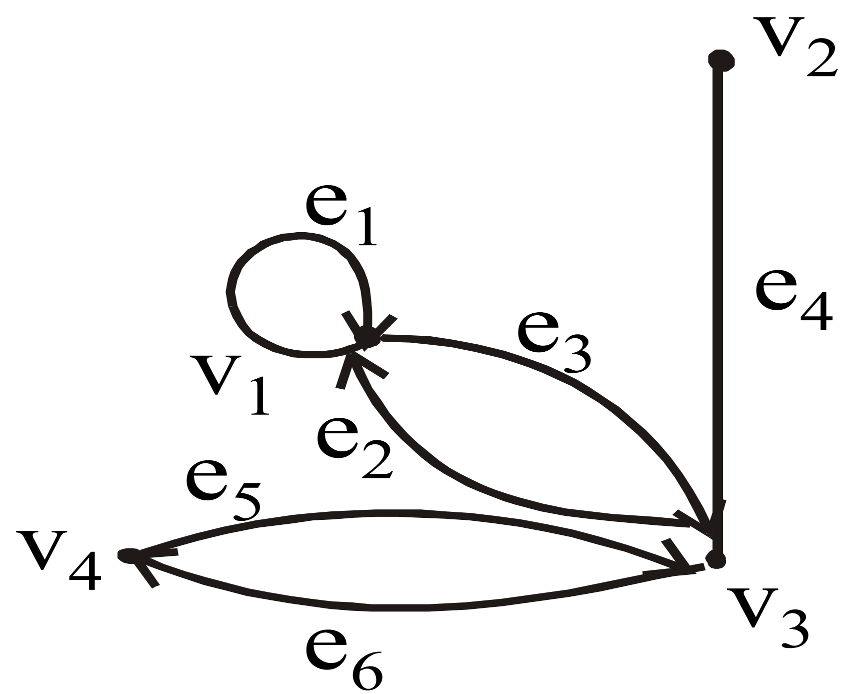
$$\begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 2 & 0 \\ v_2 & 2 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{array}$$



DIRECTED GRAPH

- A directed graph or digraph, consists of two finite sets: a set V(G) of vertices and a set D(G) of directed edges,
- where each edge is associated with an ordered pair of vertices called its end points.
- If edge e is associated with the pair (v, w) of vertices, then e is said to be the directed edge from v to w and is represented by drawing an arrow from v to w.

EXAMPLE OF DIGRAPGH



ADJACENCY MATRIX OF A DIRECTED GRAPH

- Let G be a graph with ordered vertices

  $v_1, v_2, \ldots, v_n$.

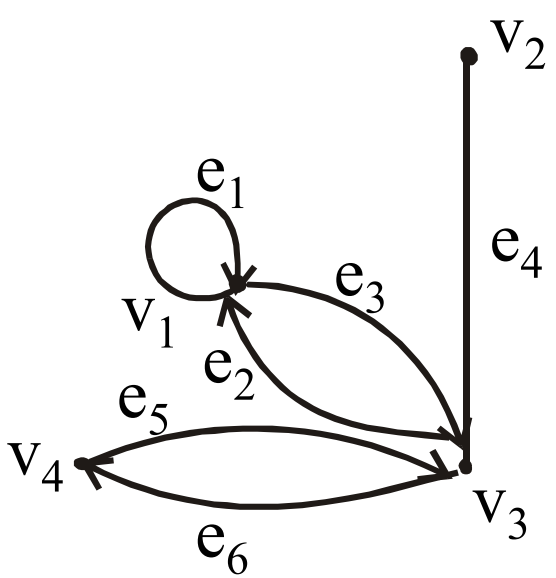
- The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

  a_{ij} = the number of arrows from v_i to v_j

  for all $i, j = 1, 2, \ldots, n$.

EXAMPLE

- A directed graph with its adjacency matrix is shown



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix

EXERCISE

- Find directed graph that has the adjacency matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

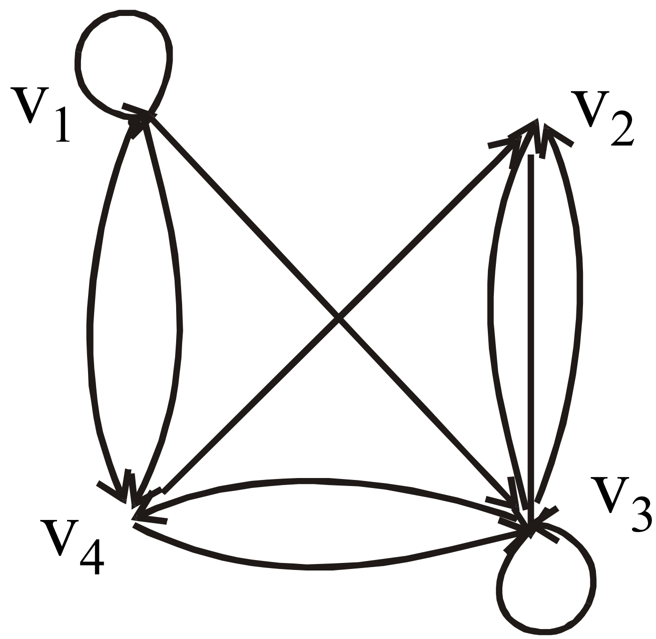
- The order of matrix is 4 x 4, it means it has 4 vertices.

SOLUTION

- The 4×4 adjacency matrix shows that the graph has 4 vertices say v_1, v_2, v_3 and v_4 labeled across the top and down the left side of the matrix.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- A corresponding directed graph is



- It means that a loop exists from v_1 and v_3 , two arrows go from v_1 to v_4 and two from v_3 and v_2 and one arrow go from v_1 to v_3 , v_2 to v_3 , v_3 to v_4, v_4 to v_2 and v_3.

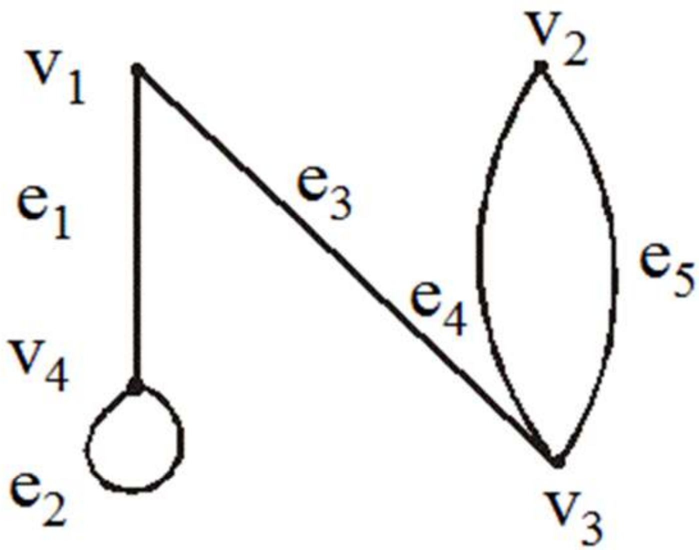
INCIDENCE MATRIX OF A SIMPLE GRAPH

- Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_n$.
- The incidence matrix of G is the matrix $M = [m_{ij}]$ of size $n \times m$ defined by:

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

EXERCISE

- A graph with its incidence matrix is shown.



$$
M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

REMARK

- In the incidence matrix:
  - Parallel edges are represented by columns with identical entries (in this matrix e_4 & e_5 are parallel edges).
  - Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop and other zeros (here e_2 is only a loop).