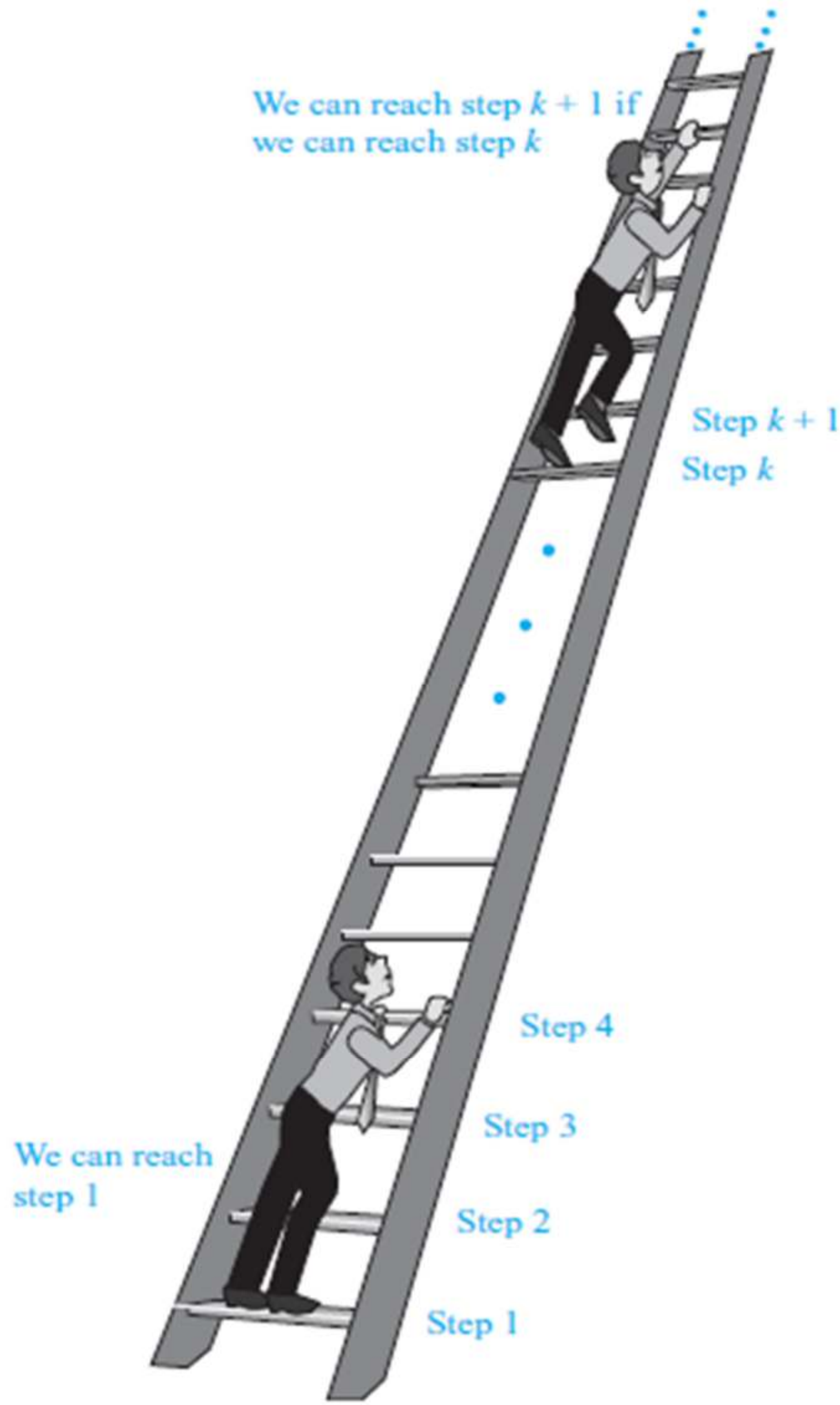
MATHEMATICAL INDUCTION

Chapter # 5

INTRODUCTION

- Suppose we have an infinite ladder, and we want to know whether we can reach every step on this ladder. We know two things:

  - We can reach first rung of the ladder.
  - If we can reach a particular rung of the ladder, then we can reach the next rung.


We can reach step $k + 1$ if
we can reach step k
Step $k + 1$
Step k
Step 4
Step 3
We can reach
step 1
Step 2
Step 1

- Many theorems state that P(n) is true for all positive integers ‘n’. Where P(n) is a propositional function.

- **Example:**

$$1 + 2 + 3 + \ldots.. + n = n(n + 1)/2$$

or

$$n \leq 2^n$$

- Mathematical induction is a technique for proving theorems of this kind.

- In other words, Mathematical Induction is used to prove propositions of the form $\forall n P(n)$.

  When

  Universe of discourse is the set of positive integers.

PRINCIPLE OF MATHEMATICAL INDUCTION

- Let $P(n)$ be a propositional function defined for all positive integers n. $P(n)$ is true for every positive integer n if:
- **Basis Step:**

  The proposition $P(1)$ is true.

- **Inductive Step:**

  If $P(k)$ is true then $P(k + 1)$ is true for all integers $k \geq 1$.

  i.e. $\forall k\, P(k) \rightarrow P(k + 1)$

EXAMPLE

- Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$ for all integers n ≥1

SOLUTION

Let $$P(n):1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

So the proposition is true for n = 1.

Inductive Step: Suppose P(k) is true for, some integers k ≥ 1.

(1) $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

To prove P(k + 1) is true. That is,

(2) $$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

- Consider L.H.S. of (2)

$$
\begin{aligned}
1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)} \\
&= (k+1)\left[\frac{k}{2}+1\right] \\
&= (k+1)\left[\frac{k+2}{2}\right] \\
&= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}
\end{aligned}
$$

- Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE

- Use mathematical induction to prove that

  $1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

- **SOLUTION:**

  Let $P(n)$ be the equation $1+3+5+\ldots+(2n-1) = n^2$

  Basis Step:

  $P(1)$ is true

  For $n = 1$, L.H.S of $P(1) = 1$ and

  R.H.S $= 1^2 = 1$

  Hence the equation is true for $n = 1$

Inductive Step:

Suppose P(k) is true for some integer $k \geq 1$. That is,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2 \text{ ....(1)}$$

To prove P(k+1) is true; i.e.,

$$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2 \quad \text{..........(2)}$$

- Consider L.H.S. of (2)

$$
\begin{aligned}
1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1) \\
&= 1+3+5+\cdots+(2k-1)+(2k+1) \\
&= k^2+(2k+1) \qquad\qquad \text{using (1)} \\
&= (k+1)^2 \\
&= \text{R.H.S. of (2)}
\end{aligned}
$$

- Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers n≥1.

EXERCISE

- Use mathematical induction to prove that

  $1+2+2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all integers $n \geq 0$

- **SOLUTION:**

  Let

  $P(n)$: $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

  Basis Step:

  P(0) is true.

  For $n = 0$

  L.H.S of $P(0) = 1$

  R.H.S of $P(0) = 2^{0+1} - 1 = 2 - 1 = 1$

  Hence P(0) is true.

Inductive Step:

Suppose P(k) is true for some integer $k \geq 0$; i.e.,

$1+2+2^2+\ldots+2^k = 2^{k+1} - 1$ ....(1)

To prove P(k+1) is true, i.e.,

$1+2+2^2+\ldots+2^{k+1} = 2^{k+1+1} - 1$ ....(2)

- Consider LHS of equation (2)

$$1+2+2^2+\ldots+2^{k+1} = (1+2+2^2+\ldots+2^k) + 2^{k+1}$$
$$= (2^{k+1} - 1) + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+1+1} - 1$$
$$= \text{R.H.S of (2)}$$

- Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers $n \geq 0$.

EXERCISE

- Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$ for all integers n ≥1.

- **PROOF:**
- Let P(n) denotes the given equation

Basis step:

P(1) is true

For n = 1

L.H.S of P(1) = 1^2 = 1

R.H.S of P(1) $= \frac{1(1+1)(2(1)+1)}{6}$

$$= \frac{(1)(2)(3)}{6} = \frac{6}{6} = 1$$

So L.H.S = R.H.S of P(1).Hence P(1) is true

Inductive Step:

Suppose P(k) is true for some integer k ≥1;

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots\ldots\ldots(1)$$

To prove P(k+1) is true; i.e.;

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \qquad \ldots\ldots\ldots(2)$$

Consider LHS of above equation (2)

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \end{aligned}$$

$$= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

EXERCISE

- Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ for all integers n≥1

- **PROOF:**
- Let P(n) be the given equation.

Basis Step:

P(1) is true

For n = 1

L.H.S of P(1) $=\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) $=\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

Inductive Step:

Suppose P(k) is true, for some integer k≥1.

That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1}$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+2)}$$

$$=\text{RHS of (2)}$$

- Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers $n \geq 1$.

EXERCISE

- Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i2^i = n \cdot 2^{n+2} + 2, \qquad \text{for all integers } n \geq 0$$

- **PROOF:**

Basis Step:

To prove the formula for n = 0,

we need to show that

$$\sum_{i=1}^{0+1} i.2^i = 0 \cdot 2^{0+2} + 2$$

Now, L.H.S $= \sum_{i=1}^{1} i \cdot 2^i = (1)2^1 = 2$

$\text{R.H.S} = 0 \cdot 2^2 + 2 = 0 + 2 = 2$

Hence the formula is true for n = 0

Inductive Step:

Suppose for some integer n=k ≥0

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 \qquad \text{.........(1)}$$

We must show that

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+1+2} + 2 \qquad \text{...............(2)}$$

- Consider LHS of (2)

$$
\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^{i} &= \sum_{i=1}^{k+1} i \cdot 2^{i} + (k+2)\cdot 2^{k+2} \\
&= (k \cdot 2^{k+2} + 2) + (k+2)\cdot 2^{k+2} \\
&= (k+k+2)2^{k+2} + 2 \\
&= (2k+2)\cdot 2^{k+2} + 2 \\
&= (k+1)2 \cdot 2^{k+2} + 2 \\
&= (k+1)\cdot 2^{k+1+2} + 2 \\
&= \text{RHS of equation (2)}
\end{aligned}
$$

- Hence the inductive step is proved as well. Accordingly by mathematical induction the given formula is true for all integers n≥0.

EXERCISE

- Use mathematical induction to prove that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$ for all integers n ≥2

- **PROOF:**

Basis Step:

For n = 2

L.H.S $=1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$

R.H.S $=\frac{2+1}{2(2)}=\frac{3}{4}$

Hence the given formula is true for n = 2

- **Inductive Step:**

  Suppose for some integer k ≥2

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

  We must show that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)} \qquad \ldots\ldots(2)$$

- Consider L.H.S of (2)

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left[\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)}\right)$$

$$=\frac{k^2+2k}{2k(k+1)}=\frac{k(k+2)}{2k(k+1)}$$

$$=\frac{k+1+1}{2(k+1)}= \quad \text{RHS of (2)}$$

- Hence by mathematical induction the given equation is true

EXERCISE

- Prove by mathematical induction

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$ for all integers n≥1

- **PROOF:**

Basis step:

For n = 1

L.H.S $= \sum_{i=1}^{n} i(i!) = (1)(1!) = 1$

R.H.S = (1+1)! - 1 = 2! - 1

= 2 -1 = 1

Hence $\sum_{i=1}^{1} i(i!) = (1+1)! - 1$

which proves the basis step.

Inductive Step:

Suppose for any integer k ≥1

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1 \qquad \text{...........(1)}$$

We need to prove that

$$\sum_{i=1}^{k+1} i(i!) = (k+1+1)! - 1 \qquad \text{...........(2)}$$

Consider LHS of (2)

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$$
$$= (k+1)! - 1 + (k+1)(k+1)!$$
$$= (k+1)! + (k+1)(k+1)! - 1$$
$$= [1 + (k+1)](k+1)! - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$
$$= \text{RHS of } (2)$$

- Hence the inductive step is also true.
- Accordingly, by mathematical induction, the given formula is true for all integers $n \geq 1$.