RELATIONS

CHAPTER 9

ORDERED PAIR

- An ordered pair (a, b) consists of two elements "a" and "b" in which "a" is the first element and "b" is the second element.

- The ordered pairs (a, b) and (c, d) are equal if, and only if, a = c and b = d.

- Note that (a, b) and (b, a) are not equal unless a = b.

EXERCISE

- Find x and y given $(2x, x + y) = (6, 2)$

- **SOLUTION:**

  Two ordered pairs are equal if and only if the corresponding components are equal. Hence, we obtain the equations:

  $2x = 6$ ....(1)

  and $x + y = 2$ ....(2)

  Solving equation (1) we get $x = 3$ and when substituted in (2) we get $y = -1$.

ORDERED n-TUPLE

- The ordered n-tuple, $(a_1, a_2, \ldots, a_n)$ consists of elements a_1, a_2, ..a_n together with the ordering: first a_1, second a_2, and so forth up to a_n.

- In particular, an ordered 2-tuple is called an ordered pair, and an ordered 3-tuple is called an ordered triple.

- Two ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if each corresponding pair of their elements is equal, i.e., $a_i = b_j$, for all $i = 1, 2, \ldots, n$.

CARTESIAN PRODUCT OF TWO SETS

- Let A and B be sets. The Cartesian product of A and B, denoted $A \times B$ (read "A cross B") is the set of all ordered pairs (a, b), where a is in A and b is in B.

- Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

- **NOTE**

If set A has ***m*** elements and set B has ***n*** elements then $A \times B$ has $m \times n$ elements.

EXAMPLE

- Let $A = \{1, 2\}$, $B = \{a, b, c\}$ then

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2, b), (2, c)\}$$
$$B \times A = \{(a,1), (a,2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$
$$A \times A = \{(1, 1), (1,2), (2, 1), (2, 2)\}$$
$$B \times B = \{(a,a), (a,b), (a,c), (b,a), (b, b), (b, c), (c,a), (c,b),(c,c)\}$$

REMARK

- $A \times B \neq B \times A$ for non-empty and unequal sets A and B.
- $A \times \phi = \phi \times A = \phi$
- $|A \times B| = |A| \times |B|$

CARTESIAN PRODUCT OF MORE THAN TWO SETS

- The Cartesian product of sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$

  where

  $$a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n.$$

- Symbolically:

  $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ for } i = 1, 2, \ldots, n\}$

BINARY RELATION

- Let A and B be sets. A (binary) relation R from A to B is a subset of $A \times B$.

- When $(a, b) \in R$, we say a is related to b by R, written **a R b**.

  Otherwise if $(a, b) \notin R$, we write $\mathbf{a} \not\mathbf{R}\ \mathbf{b}$.

EXERCISE

- Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$

  Then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

  Let

  $R1 = \{(1,1), (1, 3), (2, 2)\}$

  $R2 = \{(1, 2), (2, 1), (2, 2), (2, 3)\}$

  $R3 = \{(1, 1)\}$

  $R4 = A \times B$

  $R5 = \varnothing$

  All being subsets of $A \times B$ are relations from A to B.

DOMAIN OF RELATION

- The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted $Dom(R)$.

- Symbolically:

$$Dom\ (R) = \{a \in A \mid (a, b) \in R\}$$

RANGE OF RELATION

- The range of a relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

- Symbolically:

$$Ran(R) = \{b \in B \mid (a,b) \in R\}$$

EXERCISE

- Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

  Define a binary relation R from A to B as follows:

  $$R = \{(a, b) \in A \times B \mid a < b\}$$

  Then

  - Find the ordered pairs in R.
  - Find the Domain and Range of R.
  - Is 1R3, 2R2?

SOLUTION

- Given

$A = \{1, 2\}$ and $B = \{1, 2, 3\}$,

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

a.) Find the ordered pairs in R?

$R = \{(a, b) \in A \times B \mid a < b\}$

$R = \{(1,2), (1,3), (2,3)\}$

b.) Find the domain and range of R?

$Dom(R) = \{1,2\}$ and $Ran(R) = \{2, 3\}$

c.) Is 1R3, 2R2?

Since $(1,3) \in R$ so 1R3

But $(2, 2) \notin R$ so 2 is not related with 3.

EXERCISE

- Let $A = \{eggs, milk, corn\}$ and
  $B = \{cows, goats, hens\}$

- Define a relation R from A to B by $(a, b) \in R$ *iff* a is produced by b.

- Then $R = \{(eggs, hens), (milk, cows), (milk, goats)\}$

- Thus, with respect to this relation eggs R hens, milk R cows, and milk R goats.

EXERCISE

- Find all binary relations from $\{0,1\}$ to $\{1\}$

- **SOLUTION:**

Let $A = \{0,1\}$ & $B = \{1\}$

Then,

$$A \times B = \{(0,1), (1,1)\}$$

All binary relations from A to B are in fact all subsets of $A \times B$, which are:

$$R1 = \varnothing$$
$$R2 = \{(0,1)\}$$
$$R3 = \{(1,1)\}$$
$$R4 = \{(0,1), (1,1)\} = A \times B$$

REMARK

- If $|A| = m$ and $|B| = n$
- Then as we know that the number of elements in $A \times B$ are $m \times n$.
- Now as we know that the total number of and the total number of relations from A to B are $2^{m \times n}$.

EXERCISE

- Define a binary relation E on the set of the integers Z, as follows:

- for all $m, n \in Z$, $m \, E \, n \Leftrightarrow m - n$ is even.
  a) Is 0E0?
  b) Is 5E2?
  c) Is $(6,6) \in E$?
  d) Is $(-1,7) \in E$?
  e) Prove that for any even integer n, nE0.

SOLUTION

- $E = \{(m, n) \in Z \times Z \mid m - n \text{ is even}\}$

  (a) $(0, 0) \in Z \times Z$ and $0 - 0 = 0$ is even

  Therefore $0E0$.

  (b) $(5, 2) \in Z \times Z$ but $5 - 2 = 3$ is not even

  so $5 \not{E} 2$

  (c) $(6, 6) \in E$ since $6 - 6 = 0$ is an even integer.

  (d) $(-1, 7) \in E$ since $(-1) - 7 = -8$ is an even integer.

(e) For any even integer n, we have

$$n - 0 = n, \quad \text{an even integer}$$

so $(n, 0) \in E$

or

equivalently $nE0$

RELATION ON A SET

- A relation on the set A is a relation from A to A.

- In other words, a relation on a set A is a subset of $A \times A$.

EXERCISE

- Let

  $A = \{1, 2, 3, 4\}$

  Define a relation R on A as

  $(a, b) \in R$ *iff* a divides b {symbolically written as $a \mid b$}

  Then,

  $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

REMARK

- For any set A
  - $A \times A$ is known as the universal relation.
  - $\varnothing$ is known as the empty relation.

ARROW DIAGRAM OF A RELATION

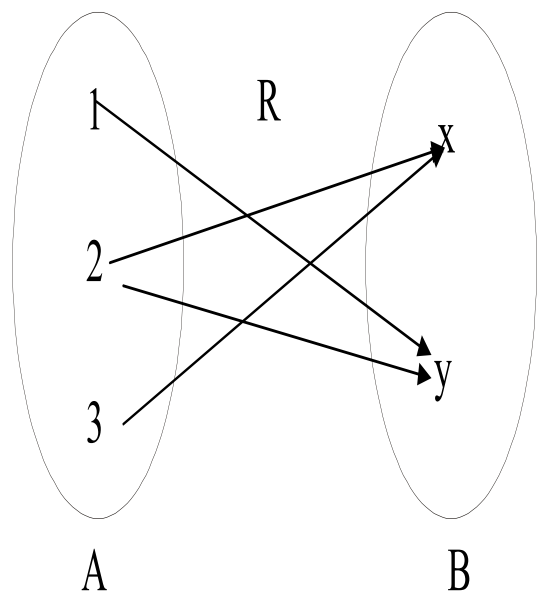
- Let

  $A = \{1, 2, 3\}$, $B = \{x, y\}$ and

  $R = \{(1, y), (2, x), (2, y), (3, x)\}$

  be a relation from A to B.

  The arrow diagram of R is:



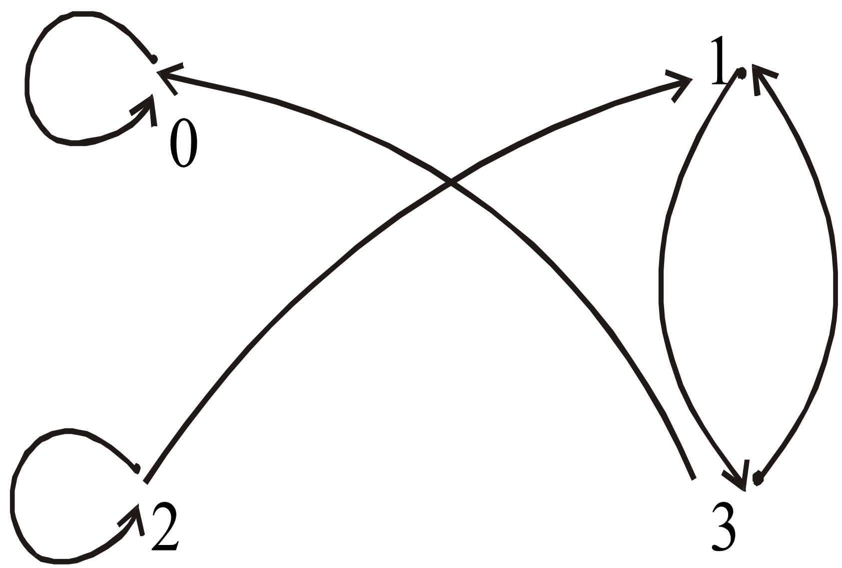
DIRECTED GRAPH OF A RELATION

- Let

  A = {0, 1, 2, 3} and

  R = {(0, 0), (1, 3), (2, 1), (2, 2), (3, 0), (3, 1)}

  be a binary relation on A.



DIRECTED GRAPH

MATRIX REPRESENTATION OF A RELATION

- Let

  A = $\{a_1, a_2, \ldots, a_n\}$ and B = $\{b_1, b_2, \ldots, b_m\}$.

  Let R be a relation from A to B.

  Define the $n \times m$ order matrix M by

$$m(i,j) = \begin{cases} 1 \text{ if } (a_i, b_i) \in R \\ 0 \text{ if } (a_i, b_i) \notin R \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$

EXAMPLE

- Let A = {1, 2, 3} and B = {x, y}

  Let R be a relation from A to B defined as

  R = {(1, y), (2, x), (2, y), (3, x)}

$$
M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cc} x & y \end{array} \\
\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2}
\end{array}
$$

EXAMPLE

- For the relation matrix.

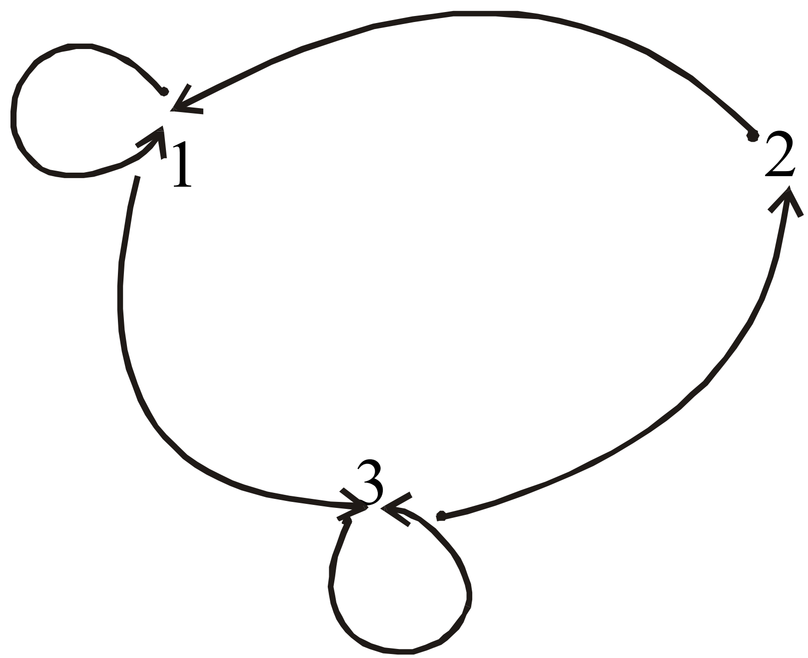
$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

- List the set of ordered pairs represented by M.
- Draw the directed graph of the relation.

SOLUTION

- The relation corresponding to the given Matrix is
  - R = {(1,1), (1,3), (2,1), (3,1), (3,2), (3,3)}

- And its Directed graph is given below



EXERCISE

- Let

  $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and

  define relations R and S from A to B as follows:

  for all $(x, y) \in A \times B$, $\quad x \, R \, y \Leftrightarrow x \mid y$

  for all $(x, y) \in A \times B$, $\quad x \, S \, y \Leftrightarrow y - 4 = x$

  State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$ and $R \cap S$.

SOLUTION

$A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$

$R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$

$S = \{(2, 6), (4, 8)\}$

$R \cup S = \{(2, 6), (2, 8), (2, 10), (4, 8)\} = R$

$R \cap S = \{(2, 6), (4, 8)\} = S$

PROPERTIES OF RELATION

- There are several properties that are used to classify relation on a set. We will introduce the most important of these here.

REFLEXIVE RELATION

- Let R be a relation on a set A. R is reflexive if, and only if, for all $a \in A$, $(a, a) \in R$. Or equivalently aRa.

  That is, each element of A is related to itself.

  REMARK

  R is not reflexive iff there is an element "a" in A such that $(a, a) \notin R$. That is, some element "a" of A is not related to itself.

EXAMPLE

- Let $A = \{1, 2, 3, 4\}$ and define relations R_1, R_2, R_3, R_4 on A as follows:

$$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$$
$$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$$
$$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$$
$$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$$

- Then,

  R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.

  R_2 is not reflexive, because $(4, 4) \notin R_2$.

  R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.

  R_4 is not reflexive, because $(1, 1) \notin R_4$, $(3, 3) \notin R_4$

DIRECTED GRAPH OF A REFLEXIVE RELATION

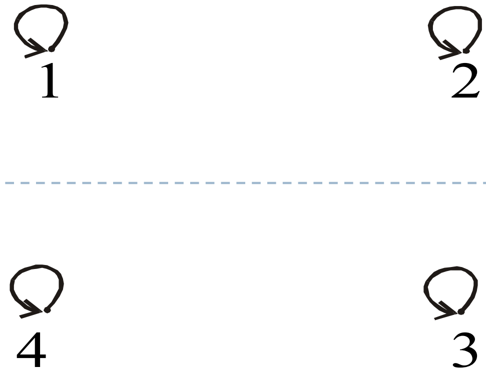
- The directed graph of every reflexive relation includes an arrow from every point to the point itself (i.e., a loop).

EXAMPLE

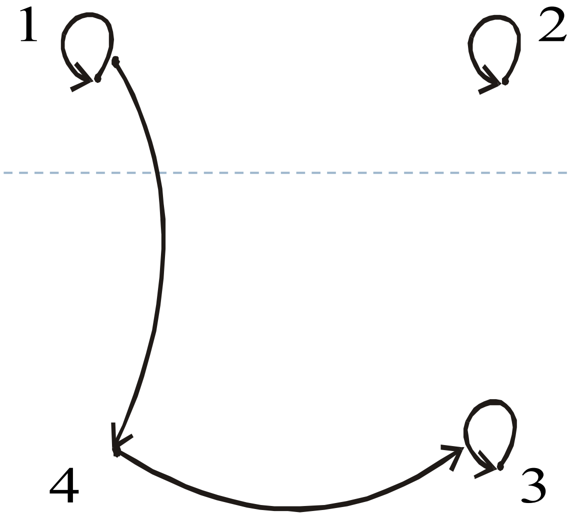
- Let $A = \{1, 2, 3, 4\}$ and define relations R_1, R_2, R_3 and R_4 on A by

$$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$$
$$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$$
$$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$$
$$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$$

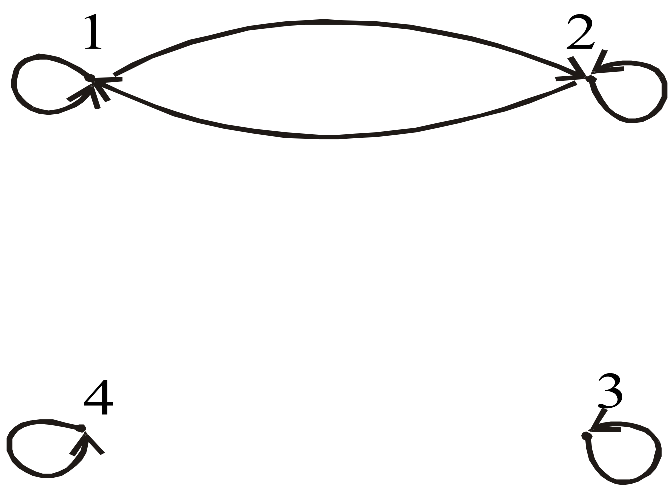
- Then their directed graphs are



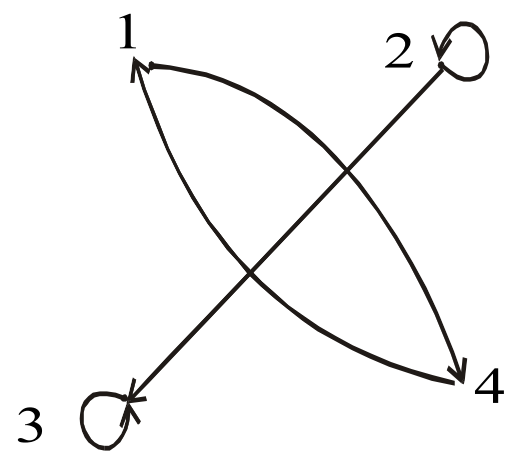
R_1 is reflexive because at every point of the set A we have a loop in the graph.



R_2 is not reflexive, as there is no loop at 4.



R_3 is reflexive



R_4 is not reflexive, as there are no loops at 1and 3.

MATRIX REPRESENTATION OF A REFLEXIVE RELATION

- Let $A = \{a_1, a_2, \ldots, a_n\}$. A Relation R on A is reflexive if and only if $(a_i, a_j) \in R \ \forall\ i=1,2, \ldots,n$.

- Accordingly, R is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing **R** are equal to **1**.

EXAMPLE

- The relation R = {(1,1), (1,3), (2,2), (3,2), (3,3)} on A = {1, 2, 3} represented by the following matrix M, is reflexive.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

SYMMETRIC RELATION

- Let R be a relation on a set A. R is symmetric if, and only if, for all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$.
- That is, if aRb then bRa.
- **REMARK**

  R is not symmetric iff there are elements *a* and *b* in A such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE

- Let $A = \{1, 2, 3, 4\}$ and define relations R_1, R_2, R_3, and R_4 on A as follows:

$$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$R_3 = \{(2, 2), (2, 3), (3, 4)\}$$
$$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$$

- Then R_1 is symmetric because for every order pair (a, b) in R_1

- We have (b, a) in R_1 for example we have (1, 3) in R_1 we also have (3,1) in R_1 similarly all other ordered pairs can be checked.

- R_2 is also symmetric we say it is vacuously true.
- R_3 is not symmetric, because $(2, 3) \in R_3$ but $(3, 2) \notin R_3$.
- R_4 is not symmetric because $(4, 3) \in R_4$ but $(3, 4) \notin R_4$.

DIRECTED GRAPH OF A SYMMETRIC RELATION

- For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

EXAMPLE

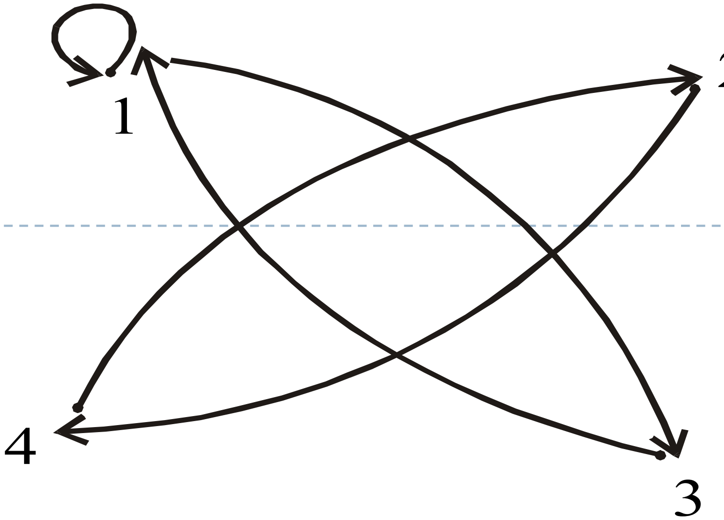
- Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A by the directed graphs:

R_1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4, 2)}

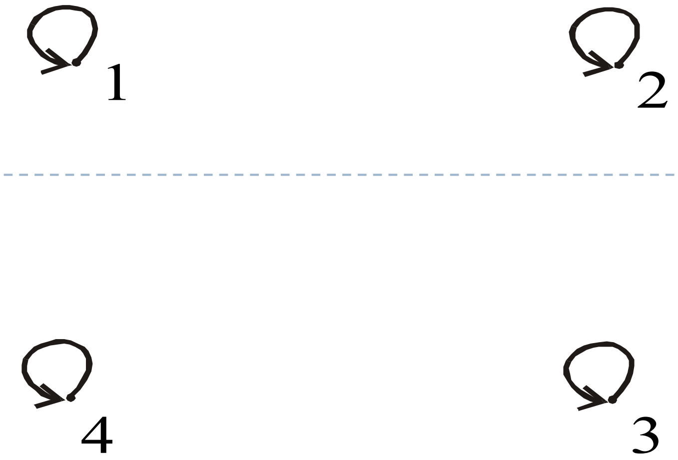
R_2 = {(1, 1), (2, 2), (3, 3), (4, 4)}

R_3 = {(2, 2), (2, 3), (3, 4)}

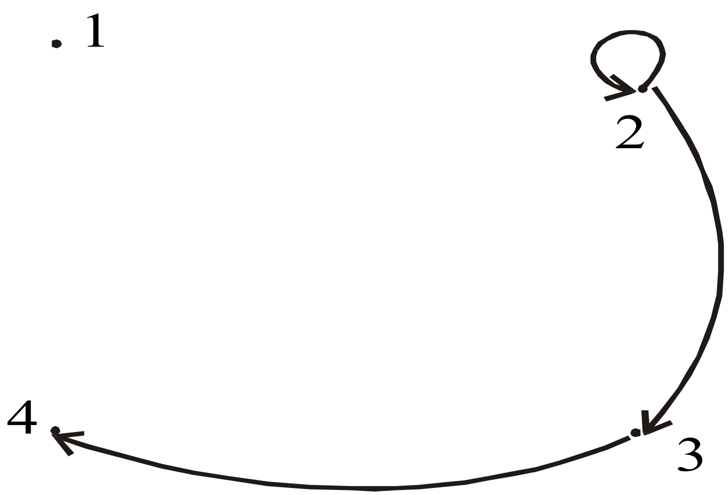
R_4= {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}



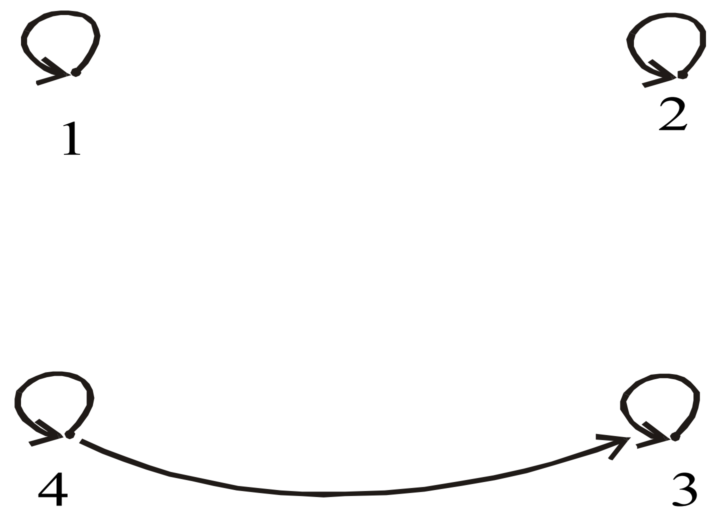
R_1 is symmetric



R_2 is symmetric



R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely



R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4

MATRIX REPRESENTATION OF A SYMMETRIC RELATION

- Let

  $A = \{a_1, a_2, \ldots, a_n\}$.

  A relation R on A is symmetric if and only if for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

- Accordingly, R is symmetric if the elements in the i^{th} row are the same as the elements in the i^{th} column of the matrix M representing R.

- More precisely, M is a symmetric matrix. i.e. $M = M^t$

EXAMPLE

- The relation R = {(1, 3), (2, 2), (3,1), (3, 3)} on A = {1, 2, 3} represented by the following matrix M is symmetric.

$$M = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$

TRANSITIVE RELATION

- Let R be a relation on a set A. R is transitive if and only if for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$
  then $(a, c) \in R$.

- That is, if aRb and bRc then aRc.

- In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third.

- Note that the "first", "second" and "third" elements need not to be distinct.

REMARK

- R is not transitive iff there are elements a, b, c in A such that If $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$.

EXAMPLE

- Let $A = \{1, 2, 3, 4\}$ and define relations R_1, R_2 and R_3 on A as follows:

$$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$
$$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$$

- Then R_1 is transitive because $(1, 1), (1, 2)$ are in R then to be transitive relation $(1,2)$ must be there and it belongs to R Similarly for other order pairs.
- R_2 is not transitive since $(1, 2)$ and $(2, 3) \in R_2$ but $(1,3) \notin R_2$.
- R_3 is transitive.

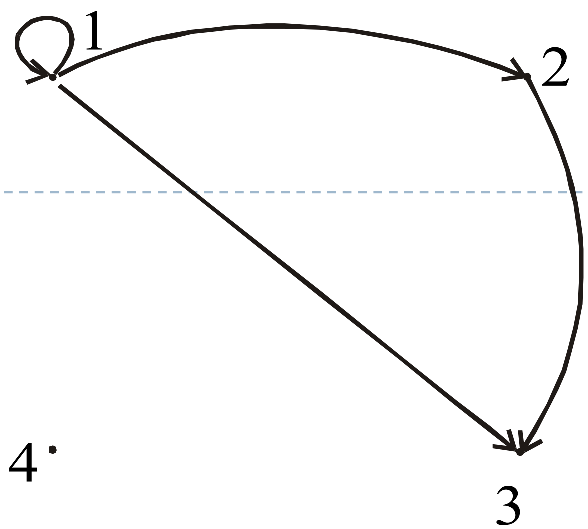
DIRECTED GRAPH OF A TRANSITIVE RELATION

- For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the third, there is an arrow going directly from the first to the third.

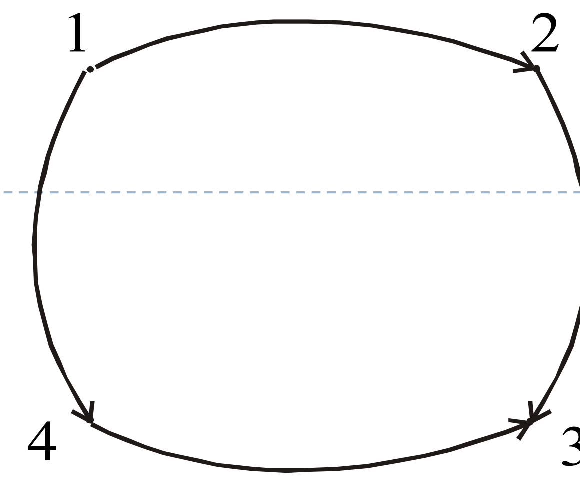
EXAMPLE

- Let $A = \{1, 2, 3, 4\}$ and define relations R_1, R_2 and R_3 on A by the directed graphs:

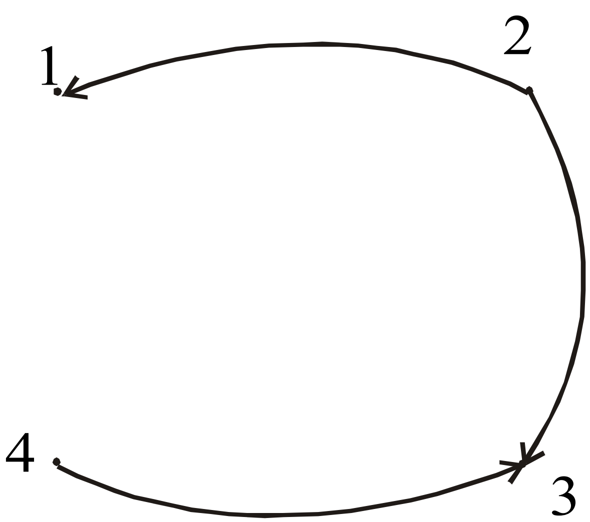
$$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$
$$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R_3 = \{(2, 1), (2, 4), (2, 3), (3, 4)\}$$



R_1 is transitive



R_2 is not transitive since there is an arrow from 1 to 2 and from 2 to 3 but no arrow from 1 to 3 directly



R_3 is transitive

EXERCISE

- Let $A = \{1, 2, 3, 4\}$ and define the null relation ϕ and universal relation $A \times A$ on A. Test these relations for reflexive, symmetric and transitive properties.

SOLUTION

- **Reflexive:**
  - $\varnothing$ is not reflexive since $(1,1), (2,2), (3,3), (4,4) \notin \varnothing$.
  - $A \times A$ is reflexive since $(a, a) \in A \times A$ for all $a \in A$.

- **Symmetric**
  - For the null relation $\varnothing$ on A to be symmetric, it must satisfy the implication:

    $$\text{if } (a, b) \in \varnothing \text{ then } (b, a) \in \varnothing.$$

    Since $(a, b) \in \varnothing$ is never true, the implication is vacuously true or true by default.

    Hence $\varnothing$ is symmetric.

  - The universal relation $A \times A$ is symmetric, for it contains all ordered pairs of elements of A. Thus,

    $$\text{if } (a, b) \in A \times A \text{ then } (b, a) \in A \times A \text{ for all } a, b \text{ in } A.$$

- **Transitive**
  - The null relation $\varnothing$ on A is transitive, because the implication.

    if $(a, b) \in \varnothing$ and $(b, c) \in \varnothing$ then $(a, c) \in \varnothing$ is true by default,

    Since the condition $(a, b) \in \varnothing$ is always false.

  - The universal relation $A \times A$ is transitive for it contains all ordered pairs of elements of A.

    Accordingly, if $(a, b) \in A \times A$ and $(b, c) \in A \times A$ then $(a, c) \in A \times A$ as well.

EXERCISE

- Let $A = \{0, 1, 2\}$ and

  $R = \{(0, 2), (1, 1), (2, 0)\}$ be a relation on A.

  - Is R reflexive? Symmetric? Transitive?
  - Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION

- R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$ and also $2 \in A$ but $(2, 2) \notin R$.

- R is clearly symmetric.

- R is not transitive, since $(0, 2)$ & $(2, 0) \in R$ but $(0, 0) \notin R$.

- For R to be reflexive, it must contain ordered pairs (0,0) and (2,2).

- For R to be transitive,

  we note $(0, 2)$ and $(2, 0) \in R$ but $(0, 0) \notin R$.

  Also $(2, 0)$ and $(0, 2) \in R$ but $(2, 2) \notin R$.

- Hence $(0, 0)$ and $(2, 2)$. Are needed in R to make it a transitive relation.

EQUIVALENCE RELATION

- Let A be a non-empty set and R a binary relation on A.

  R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

EXAMPLE

- Let $A = \{1, 2, 3, 4\}$ and
  $R = \{(1,1), (2, 2), (2, 4), (3, 3), (4, 2), (4, 4)\}$
  be a binary relation on A.

- Note that R is reflexive, symmetric and transitive, hence an equivalence relation.