CC141

Discrete Structures

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Overview of Previous Lectures

- **Chapter 1 : The Foundations: Logic and Proofs**
- ***1.1 Propositional Logic***
  - Statement, Proposition, Truth value
  - Propositional symbol, Open proposition
  - Compound Propositions
  - Operators
  - Conditional statements
  - Truth tables of compound propositions
  - Precedence of logical operators
  - Logic and bit operation
- ***1.2 Application of Propositional Logic***
  - Translating English Statement
  - System Specifications
  - Boolean Searches
  - Logic Puzzles
  - Logic Circuits

Converse ($\mathbf{q} \rightarrow \mathbf{p}$)
Contrapositive ($\neg\mathbf{q} \rightarrow \neg\mathbf{p}$)
Inverse ($\neg\mathbf{p} \rightarrow \neg\mathbf{q}$)

Converse

- The proposition

$$q \rightarrow p$$

is called the converse of

$$p \rightarrow q$$

- Note that

$$\mathbf{q \rightarrow p \; !\equiv p \rightarrow q}$$

Converse

- “If it is raining, then the home team wins.”

The converse is

- “If the home team wins, then it is raining.”

$q \rightarrow p \not\equiv p \rightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Contrapositive

- The proposition

$$\neg q \rightarrow \neg p$$

is called the contrapositive of

$$p \rightarrow q$$

- Note that

$$\mathbf{\neg q \rightarrow \neg p \equiv p \rightarrow q}$$

Contrapositive

- “If it is raining, then the home team wins.”

The contrapositive is

- “If the home team does not win, then it is not raining.”

$\neg q \rightarrow \neg p \equiv p \rightarrow q$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	**T**	**T**	**F**	**F**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**

$$\neg q \rightarrow \neg p \equiv p \rightarrow q$$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Inverse

- The proposition

$$\neg p \rightarrow \neg q$$

is called the inverse of

$$p \rightarrow q$$

- Note that

$$\mathbf{\neg p \rightarrow \neg q \; !\equiv \; p \rightarrow q}$$

Inverse

- “If it is raining, then the home team wins.”

The inverse is

- “If it is not raining, then the home team does not win.”

$\neg p \rightarrow \neg q \; !\equiv \; p \rightarrow q$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

$\neg p \rightarrow \neg q \mathrel{!\equiv} p \rightarrow q$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Section 1.3: PROPOSITIONAL EQUIVALENCES

- Tautology, Contradiction & Contingency
- Logical Equivalence
- Propositional Satisfiability

TAUTOLOGY

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

Examples:

- $\mathbf{R \lor (\neg R)}$
- $\mathbf{\neg(P \land Q) \leftrightarrow (\neg P) \lor (\neg\, Q)}$

CONTRADICTION

A compound proposition that is always false is called a contradiction.

Examples:

- $\mathbf{R \wedge (\neg R)}$
- $\mathbf{\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))}$

CONTINGENCY

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

- $\mathbf{p \wedge q}$

Some Important Tautologies

$$p \wedge (p \rightarrow q) \rightarrow q$$

$$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$$

$$(p \vee q) \wedge \neg p \rightarrow q$$

Some Important Contingencies

$$\neg p \land (p \rightarrow q) \rightarrow \neg q$$

$$q \land (p \rightarrow q) \rightarrow p$$

LOGICAL EQUIVALENCE

- The compound propositions p and q are called logically equivalent if

  $$p \leftrightarrow q \text{ is a tautology.}$$

- The notation $\mathbf{p \equiv q}$ denotes that p and q are logically equivalent.

LOGICAL EQUIVALENCE

$$p \equiv q$$

- The symbol ≡ is not a logical connective
  - Hence, $p \equiv q$ is not a compound proposition, rather it is the statement that $p \leftrightarrow q$ is a tautology.
- The symbol ⇔ is sometimes used instead of ≡ to denote logical equivalence.

LOGICAL EQUIVALENCE

- Definition: two propositional statements S1 and S2 are said to be (logically) equivalent, denoted S1 $\equiv$ S2 if
  - **They have the same truth table, or**
  - **S1 $\Leftrightarrow$ S2 is a tautology**

- Equivalence can be established by
  - **Constructing truth tables**
  - **Using equivalence laws**

LOGICAL EQUIVALENCE using TRUTH TABLES

Prove that **$\neg(P \wedge Q)$** and **$(\neg P) \vee (\neg Q)$** are logically equivalent

and **$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$** is a Tautology

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

A VERY IMPORTANT EQUIVALENCE

- $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent

TABLE 4 **Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.**

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

DE MORGAN'S LAWS

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

DE MORGAN'S LAWS

TABLE 3 **Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

DE MORGAN'S LAWS

Express the negations of

- “Miguel has a cellphone and he has a laptop computer”.
- “Heather will go to the concert or Steve will go to the concert.”

DE MORGAN'S LAWS

- "Miguel has a cellphone and he has a laptop computer"
- "Heather will go to the concert or Steve will go to the concert."

- "Miguel does not have a cellphone or he does not have a laptop computer."
- "Heather will not go to the concert and Steve will not go to the concert."

SOME IMPORTANT EQUIVALENCES

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

LOGICAL EQUIVALENCES of DISTRIBUTIVE LAW

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.
This is the distributive law of disjunction (OR) over conjunction (AND)

TABLE 5 **A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

LOGICAL EQUIVALENCES INVOLVING CONDITIONAL STATEMENTS

TABLE 7 **Logical Equivalences Involving Conditional Statements.**
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

LOGICAL EQUIVALENCES INVOLVING BICONDITIONAL STATEMENTS

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

CONSTRUCTING NEW LOGICAL EQUIVALENCES

Using already Established Equivalences

Exercises

Show that $\mathbf{(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)}$ by equivalence laws:

LHS = $\mathbf{(P \rightarrow Q) \wedge (P \rightarrow R)}$

= $\mathbf{(\neg P \vee Q) \wedge (\neg P \vee R)}$ Implication Law : $P \rightarrow Q = \neg P \vee Q$

= $\mathbf{\neg P \vee (Q \wedge R)}$ Distributive Law : $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

RHS = $\mathbf{P \rightarrow (Q \wedge R)}$

= $\mathbf{\neg P \vee (Q \wedge R)}$ Implication Law : $P \rightarrow Q = \neg P \vee Q$

Hence LHS = RHS

Show that **$\neg(p \rightarrow q)$** and **$p \wedge \neg q$** are logically equivalent using Equivalence Laws.

LHS = $\neg(p \rightarrow q)$

$\equiv \neg(\neg p \vee q)$ — Implication Law : $P \rightarrow Q = \neg P \vee Q$

$\equiv \neg(\neg p) \wedge \neg q$ — Second DeMorgan law: $\neg(P \vee Q) = \neg P \wedge \neg Q$

$\equiv p \wedge \neg q$ — Double Negation Law : $\neg(\neg P) = P$

= RHS

Show that ¬(p ∨ (¬p ∧ q)) and ¬p ∧¬q are logically equivalent.

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$$ Second DeMorgan law: ¬ (P ∨ Q)= ¬P ∧ ¬ Q

$$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$$ First DeMorgan law: ¬ (P ∧ Q)= ¬P ∨ ¬ Q

$$\equiv \neg p \wedge (p \vee \neg q)$$ Double Negation Law : ¬ (¬ P) = P

$$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$$ Distributive Law : P ∧ (Q ∨ R) = (P ∧ Q) ∨ (P ∧ R)

$$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$$ Negation Law : P ∧ ¬ P = F

$$\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$$ Commutative Law: P v Q = Q v P

$$\equiv \neg p \wedge \neg q$$ Identity Law : P v **F** = **P**

Summary

Chapter 1 : The Foundations: Logic and Proofs

1.3 Propositional Equivalence

- Tautology vs Contradiction
- Logical Equivalence
    - DeMorgan's Law,
    - Other Laws of Equivalence
- Constructing New Logical Equivalences
    - Using Truth Tables
    - Using Laws of Equivalences
- Propositional Satisfiability