SUMMATION

Chapter 2 – Section 2.4

INTRODUCTION

SERIES:

The sum of the terms of a sequence forms a series. If

$$a_1, a_2, a_3, \ldots$$

represent a sequence of numbers, then the corresponding series is:

$$a_1 + a_2 + a_3 + \ldots$$

$$= \sum_{k=1}^{\infty} a_k$$

SUMMATION NOTATION

- The capital Greek letter sigma ∑ is used to write a sum in a short hand notation.
- Hence $\sum_{k=1}^{\infty} a_k$ represents the sum given in expanded form by

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

Here **k** is called the index of the summation;

Lower limit of the summation is **1.**

Upper limit of the summation is ∞ .

- More generally if m and n are integers and m ≤ n, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

- where k varies from 1 to n represents the sum given in expanded form by a_k

- Other Notations to represent the same Series are:

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

- reads as the sum from j = m to j = n of a_j
- Here, the variable j is called the index of summation,
- The choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

COMPUTING SUMMATIONS

- Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where

$$a_j = 1/j \qquad \text{for} \quad j = 1, 2, 3, \ldots .$$

SOLUTION

Lower limit of the summation is 1.

Upper limit of the summation is 100.

We write the Sum as:

$$\sum_{j=1}^{100} \frac{1}{j}.$$

COMPUTING SUMMATIONS

- Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.

  Compute each of the summations:

  1. $$\sum_{i=0}^{4} a_i$$
  2. $$\sum_{j=0}^{2} a_{2j}$$
  3. $$\sum_{k=1}^{1} a_k$$

Example 1

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$

Compute $\sum_{i=0}^{4} a_i$

SOLUTION

We will take $i = 0, 1, 2, 3, 4$

$$\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4$$

$$= 2 + 3 + (-2) + 1 + 0$$

$$\sum_{i=0}^{4} a_i = 4$$

Example 2

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$

Compute $\sum_{j=0}^{2} a_{2j}$

SOLUTION

$$\sum_{j=0}^{2} a_{2j} = a_0 + a_2 + a_4 \qquad \text{(Take } j = 0, 1, 2\text{)}$$

$$= 2 + (-2) + 0$$

$$\sum_{j=0}^{2} a_{2j} = 0$$

Example 3

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$

Compute $\sum_{k=1}^{1} a_k$

SOLUTION

$$\sum_{k=1}^{1} a_k = a_1 \qquad \text{(As } k = 1\text{)}$$

$$\sum_{k=1}^{1} a_k = 3$$

EXERCISE

- Compute the summations.

1. $$\sum_{i=1}^{3}(2i-1) = [2(1)-1]+[2(2)-1]+[2(3)-1]$$
$$= 1+3+5$$
$$= 9$$

2. $$\sum_{k=-1}^{1}(k^3+2) = [(-1)^3+2]+[(0)^3+2]+[(1)^3+2]$$
$$= [-1+2]+[0+2]+[1+2]$$
$$= 1+2+3$$
$$= 6$$

NOTE

- Note that in both the **examples** we have **constants**.
- In First example we have **constant '-1'** that constant appear in all three terms.
- Similarly **'+2'** in the second example.
- The **constant term** in series keep on **adding**.

SUMMATION NOTATION TO EXPANDED FORM

- Write the summation $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ to expanded form:
- **SOLUTION:**

Lower Limit = 0, Upper Limit = n

Total number of terms will be n + 1.

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

EXPANDED FORM TO SUMMATION NOTATION

- Write the following using summation notation:

$$\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}$$

- **SOLUTION**

Terms	Numerators
For first term	1
For second term	2
For third term	3
……	……
……	……
For last term	n + 1

Terms	Denominator
For first term	n
For second term	n + 1
For third term	n + 2
……	……
……	……
For last term	2n

1, 2, 3,…, n+1 are Numerators

- The numerators forms an arithmetic sequence

  a = first term = 1

  & d = common difference = 1

n, n+1, n+2, …, 2n are Denomenators.

- Similarly, denominators forms an arithmetic sequence

  a = first term = n

  d = common difference = 1

1, 2, 3,…, n+1 are Numerators

- The numerators forms an arithmetic sequence

$$a = \text{first term} = 1$$

& $$d = \text{common difference} = 1$$

- So, For Numerator the k^{th} term will be:

$$a_k = a + (k - 1)d$$
$$= 1 + (k - 1)(1)$$
$$= 1 + k - 1$$
$$a_k = k$$

n, n+1, n+2, …, 2n are Denomenators.

- Similarly, denominators forms an arithmetic sequence

a = first term = n

d = common difference = 1

So, For Denominator the k^{th} term will be:

$$a_k = a + (k - 1)d$$

$$= n + (k - 1)(1)$$

$$a_k = k + n - 1$$

- Hence the kth term of the series is

$$\frac{k}{(n-1)+k}$$

- And the expression for the series is given by

$$\therefore \quad \frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}=\sum_{k=1}^{n+1}\frac{k}{(n-1)+k}$$

$$=\sum_{k=0}^{n}\frac{k+1}{n+k}$$

REMARK

Consider $\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$

And $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$

Hence $$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2$$

The index of a summation can be replaced by any other symbol. The index of a summation is therefore called a dummy variable.

EXERCISE

- Simplify the variables in summation as simplified as possible.
- Consider $\displaystyle\sum_{k=1}^{n+1} \frac{k}{(n-1)+k}$

If we put k = j + 1 then the denominator simplifies.

Substituting k = j + 1 so that j = k – 1

When k = 1, j = k - 1 = 1 - 1 = 0

When k = n + 1, j = k - 1 = (n + 1) - 1 = n

When k varies from 1 to n + 1 then j varies from 0 to n.

- We put j instead of k and summation becomes:

$$\sum_{k=1}^{n+1} \frac{k}{(n-1)+k} = \sum_{j=0}^{n} \frac{j+1}{(n-1)+(j+1)}$$

$$= \sum_{j=0}^{n} \frac{j+1}{n+j} = \sum_{k=0}^{n} \frac{k+1}{n+k} \quad \text{(changing variable)}$$

EXERCISE

- Transform by making the change of variable j = i - 1, in the summation:

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

SOLUTION

$$Set \quad j = i - 1 \qquad so\ that \quad i = j + 1$$

$$when \quad i = 1$$

$$j = i - 1 = 1 - 1 = 0$$

$$when \quad i = n - 1$$

$$j = i - 1 = (n - 1) - 1 = n - 2$$

$$\therefore \sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \sum_{j=0}^{n-2} \frac{j+1}{(n-(j+1))^2}$$

$$= \sum_{j=0}^{n-2} \frac{j+1}{(n-j-1)^2}$$

PROPERTIES OF SUMMATIONS

1. $$\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k; \quad a_k, b_k \in R$$

2. $$\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k \quad c \in \mathrm{R}$$

3. $$\sum_{k=a-i}^{b-i}(k+i) = \sum_{k=a}^{b} k \quad i \in N$$

4. $$\sum_{k=a+i}^{b+i}(k-i) = \sum_{k=a}^{b} k \quad i \in N$$

5. $$\sum_{k=1}^{n} c = c + c + \cdots + c = nc$$

EXERCISE

- Express the following summation more simply:

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

- **SOLUTION:**

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$= \quad \sum_{k=1}^{n}3(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$= \sum_{k=1}^{n} 3(2k-3) + \sum_{k=1}^{n} (4-5k)$$

$$= \sum_{k=1}^{n} [3(2k-3) + (4-5k)]$$

$$= \sum_{k=1}^{n} (k-5)$$

$$= \sum_{k=1}^{n} k - \sum_{k=1}^{n} 5$$

$$= \sum_{k=1}^{n} k - 5n$$

ARITHMETIC SERIES

- The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S).
- For example:

Arithmetic sequence is $1 , 3 , 5 , 7 , \ \ldots$

Then

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

- In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as:

$$a + (a+d) + (a+2d) + \ldots$$

SUM OF n TERMS OF AN ARITHMETIC SERIES

- Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \qquad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\, d]$$

$$= a + (a+d) + (a + 2d) + \ldots + a_n$$

Where $a_n = a + (n - 1)d$

$$= a + (a+d) + (a + 2d) + \ldots + (a_n - d) + a_n \ \ldots(1)$$

- Rewriting the terms in the series in reverse order.

$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a$ ……….(2)

Adding (1) and (2) term by term, gives

$2S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n)$

(n terms)

$2S_n = n(a + a_n)$

$S_n = n(a + a_n)/2$

$\mathbf{S_n = n(a + l)/2}$…………………..(3) (l=last term)

If we write $a_n = a + (n - 1)d$

Therefore

$S_n = n/2\ [a + a + (n - 1)\ d]$

$\mathbf{S_n = n/2\ [2a + (n - 1)\ d]}$……….(4)

- We have two formulas for finding a sum.

$$S_n = n(a + l)/2 \qquad \text{..............(1)}$$

We use it when we are given first term *a* and the last term *l*.

$$S_n = n/2\,[2a + (n - 1)\,d] \qquad \text{...................(2)}$$

We will use it when first term and common difference is given.

EXERCISE

- Find the sum of first n natural numbers.
- **SOLUTION**

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right-hand side forms an arithmetic series with

$a = 1, \quad d = 2 - 1 = 1$ and $n = n$

$$
\begin{aligned}
\therefore \quad S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\
&= \frac{n}{2}\left[2(1) + (n-1)(1)\right] \\
&= \frac{n}{2}\left[2 + n - 1\right] \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

EXERCISE

- Find the sum of all two digit positive integers which are neither divisible by 5 nor by 2.
- **SOLUTION**

The series to be summed is:

$11 + 13 + 17 + 19 + 21 + 23 + 27 + 29 + \ldots + 91 + 93 + 97 + 99$

which is not an arithmetic series.

$11 + 21 + 31 + \ldots\ldots\ldots + 91$ is an arithmetic series.

with $a = 11$, $d = 10$

$13 + 23 + 33 + \ldots\ldots\ldots + 93$ is an arithmetic series.

with $a = 13$, $d = 10$

17 + 27 + 37 + ……… + 97 is an arithmetic series.

with $a = 17$, $d = 10$

19 + 29 + 39 + ……… + 99 is an arithmetic series.

with $a = 19$, $d = 10$

If we make group of four terms we get

(11 + 13 + 17 + 19) + (21 + 23 + 27 + 29) + (31 + 33 + 37 + 39) + … + (91 + 93 + 97 + 99)

= 60 + 100 + 140 + … + 380

which now forms an arithmetic series in which

$a = 60$; $d = 100 - 60 = 40$ and $l = a_n = 380$

- To find n, we use the formula

$$a_n = a + (n - 1)\,d$$

$$\Rightarrow \quad 380 = 60 + (n - 1)(40)$$

$$\Rightarrow \quad 380 - 60 = (n - 1)(40)$$

$$\Rightarrow \quad 320 = (n - 1)(40)$$

$$\frac{320}{40} = n - 1$$

$$8 = n - 1$$

$$\Rightarrow \quad n = 9$$

- Now

$a = 60 \qquad l = a_n = 380 \qquad n = 9$

$$S_n = \frac{n}{2}(a + l)$$

$$\therefore \quad S_9 = \frac{9}{2}(60 + 380) = 1980$$

GEOMETRIC SERIES

- The sum of the terms of a geometric sequence forms a geometric series (G.S.).

- For example

  Geometric Sequence

  1, 2, 4, 8, 16, …

  Geometric Series

  1 + 2 + 4 + 8 + 16 + …

- In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as:

$$a + ar + ar^2 + ar^3 + \dots$$

SUM OF n TERMS OF A GEOMETRIC SERIES

- Let **a** be the first term and **r** be the common ratio of a geometric series. Then its nth term is:

$$a_n = ar^{n-1}; \qquad n \geq 1$$

If S_n denotes the sum of first n terms of the G.S. then

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \ldots.(1)$$

Multiplying both sides by r we get.

$$r\,S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \ldots.(2)$$

- Subtracting (2) from (1) we get

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow \quad (1 - r)\, S_n = a\,(1 - r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

EXERCISE

- Find the sum of the geometric series

$$6-2+\frac{2}{3}-\frac{2}{9}+\cdots \quad + \quad \text{to } 10 \text{ terms}$$

- **SOLUTION**

$$a=6, \quad r=\frac{-2}{6}=-\frac{1}{3} \quad \text{and } n=10$$

$$\therefore \qquad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{6\left(1-\left(-\frac{1}{3}\right)^{10}\right)}{1-\left(-\frac{1}{3}\right)} = \frac{6\left(1+\frac{1}{3^{10}}\right)}{\left(\frac{4}{3}\right)}$$

$$= \frac{9\left(1+\frac{1}{3^{10}}\right)}{2}$$

RECURRENCE RELATION

- A recurrence relation for a sequence $a_0, a_1, a_2, \ldots,$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is a fixed integer and k is any integer greater than or equal to i.

- The initial conditions for such a recurrence relation specify the values of

$$a_0, a_1, a_2, \ldots, a_{i-1}.$$

EXAMPLE

- Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1, 2, 3, …, and suppose that $a_0 = 2$. What are a_1 , a_2 , a_3 ?

Solution:

$$a_n = a_{n-1} + 3$$

$$a_1 = a_0 + 3$$

$$a_1 = 2 + 3 = 5$$

$$a_2 = a_1 + 3$$

$$a_2 = 5 + 3$$

$$a_2 = 8$$

$$a_3 = a_2 + 3$$

$$a_3 = 8 + 3$$

$$a_3 = 11$$

EXAMPLE

- Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2, 3, 4, … and suppose that

  $a_0 = 3$ and $a_1 = 5$. What are a_2 , a_3 ?

Solution:

$$a_n = a_{n-1} - a_{n-2}$$

$$a_2 = a_1 - a_0$$

$$a_2 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1$$

$$a_3 = 2 - 5 = -3$$

THE FIBONACCI SEQUENCE

- The **Fibonacci sequence** is defined as follows.

  \- BASE

$$F_0 = 0, F_1 = 1$$

  \- Recursion

$$F_k = F_{k-1} + F_{k-2} \qquad \text{for all integers } k \geq 2$$

$$F_2 = F_1 + F_0 = 1 + 0 = 1$$
$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$
$$\ldots$$

EXAMPLE

- Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a **solution** of the **recurrence relation** that satisfies the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n = 2, 3, 4, … suppose that

Solution:

Suppose that $a_n = 3n$ for every nonnegative integer n.

$$a_n = 2a_{n-1} - a_{n-2}$$

So,

$$a_{n-1} = 3(n-1)$$

$$a_{n-2} = 3(n-2)$$

$$a_n = 2a_{n-1} - a_{n-2}$$

$$a_n = 2(3(n - 1)) - 3(n - 2)$$

$$a_n = 2(3n - 3) - 3n + 6$$

$$a_n = 6n - 6 - 3n + 6$$

$$a_n = 3n$$

So $a_n = 3n$ is solution of recurrence relation.

EXAMPLE

- Determine whether the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n, is a **solution** of the **recurrence relation** that satisfies the recurrence relation $\mathbf{a_n = 2a_{n-1} - a_{n-2}}$ for n = 2, 3, 4, … suppose that

Solution:

Suppose that $a_n = 2^n$ for every nonnegative integer n.

$$\mathbf{a_n = 2a_{n-1} - a_{n-2}}$$

So,

$\mathbf{a_{n-1} = 2^{n-1} =}$

$\mathbf{a_{n-2} = 2^{n-2}}$

$$a_n = 2a_{n-1} - a_{n-2}$$

$$a_n = 2.\ 2^{n-1} - 2^{n-2}$$

$$a_n = 2^{n-1+1} - 2^{n-2}$$

$$a_n = 2^n - 2^{n-2}$$

So $a_n = 2^n$ is **not a solution** of recurrence relation

EXAMPLE

- Determine whether the sequence $\{a_n\}$, where $a_n = 5$ for every nonnegative integer n, is a **solution** of the **recurrence relation** that satisfies the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n = 2, 3, 4, … suppose that

Solution:

Suppose that $a_n = 5$ for every nonnegative integer n.

$$a_n = 2a_{n-1} - a_{n-2}$$

So,

$a_{n-1} = 5$

$a_{n-2} = 5$

$$a_n = 2a_{n-1} - a_{n-2}$$

$$a_n = 2 . 5 - 5$$

$$a_n = 10 - 5$$

$$a_n = 5$$

So $a_n = 5$ is **a solution** of recurrence relation.