SEQUENCES

Chapter 2 – Section 2.4

INTRODUCTION

- **SEQUENCE:**

  A sequence is just a list of elements usually written in a row.

- **EXAMPLES:**

1). 1, 2, 3, 4, 5, …
2). 4, 8, 12, 16, 20,…
3). 2, 4, 8, 16, 32, …
4). 1, 1/2, 1/3, 1/4, 1/5, …
5). 1, 4, 9, 16, 25, …
6). 1, -1, 1, -1, 1, -1, …

Note:
The symbol "…" is called ellipsis, and reads "so forth"

- **Sequences** are kind of **functions**.
- Second Sequence is

4, 8, 12, 16, 20, …

image of (1) = 4

image of (2) = 8

image of (3) = 12

image of (4) = 16

image of (5) = 20

FORMAL DEFINITION OF SEQUENCE

- A sequence is a function whose domain is the set of integers greater than or equal to a particular integer n_0

- Usually this set is the set of Natural numbers {1, 2, 3, …} or the set of whole numbers {0, 1, 2, 3, …}.

NOTATION

- We use the notation a_n to denote the **image** of the **integer n**, and call it a **term** of the **sequence**. Thus

$$a_1, a_2, a_3, a_4 \ldots, a_n, \ldots$$

represent the **terms** of a **sequence** defined on the set of **natural numbers N**.

- **Note:**

That a sequence is described by listing the terms of the sequence in order of increasing subscripts.

FINDING TERMS OF A SEQUENCE GIVEN BY AN EXPLICIT FORMULA

- An **explicit formula** or general formula for a **sequence** is a **rule** that shows how the values of a_k depends on k.

EXAMPLE

- Define a **sequence $a_1, a_2, a_3, \ldots$** by the explicit formula, find the **First Four terms** of the sequence:

$$a_k = \frac{k}{k+1} \qquad \text{for all integers } k \geq 1$$

- To find the **1st term** just replace **k** with **1** and so on…

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4}$$

$$and \quad fourth\ term\ is\ a_4 = \frac{4}{4+1} = \frac{4}{5}$$

EXAMPLE

- Write the first four terms of the sequence defined by the formula $b_j = 1 + 2^j$, for all integers $j \geq 0$

- **SOLUTION:**

$$b_0 = 1 + 2^0 = 1 + 1 = 2$$
$$b_1 = 1 + 2^1 = 1 + 2 = 3$$
$$b_2 = 1 + 2^2 = 1 + 4 = 5$$
$$b_3 = 1 + 2^3 = 1 + 8 = 9$$

- **REMARK:**

The formula $\mathbf{b_j = 1 + 2^j}$, for all integers $j \geq 0$ defines an infinite sequence having infinite number of values.

EXAMPLE

- Compute the first six terms of the sequence defined by the formula $C_n = 1 + (-1)^n$ for all integers $n \geq 0$

- **SOLUTION:**

$$C_0 = 1 + (-1)^0 = 1 + 1 = 2$$
$$C_1 = 1 + (-1)^1 = 1 + (-1) = 0$$
$$C_2 = 1 + (-1)^2 = 1 + 1 = 2$$
$$C_3 = 1 + (-1)^3 = 1 + (-1) = 0$$
$$C_4 = 1 + (-1)^4 = 1 + 1 = 2$$
$$C_5 = 1 + (-1)^5 = 1 + (-1) = 0$$

EXAMPLE

- Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \quad \text{for all integers } n \geq 1$$

- **SOLUTION:**

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$

$$And \quad fourth \quad term \ is C_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

- **REMARK:** A sequence whose terms alternate in sign is called an **alternating sequence**.

EXERCISE

- Find explicit formulas for sequences with the initial terms given:

1). **0, 1, -2, 3, -4, 5, …**

2). $1-\frac{1}{2},\frac{1}{2}-\frac{1}{3},\frac{1}{3}-\frac{1}{4},\frac{1}{4}-\frac{1}{5},\ldots$

3). **2, 6, 12, 20, 30, 42, 56, …**

4). **1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …**

1). **0, 1, -2, 3, -4, 5, …**

SOLUTION:

Its an alternating sequence. In alternating sequence always take the power of (-1)

$a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

when n is odd the term will become positive.

Check:

$a_0 = (-1)^{0+1} \cdot 0 = 0$, $a_1 = (-1)^{1+1} 1 = 1$,

$a_2 = (-1)^{2+1} \cdot 2 = -2$, $a_3 = (-1)^{3+1} \cdot 3 = 3$,

$a_4 = (-1)^{4+1} \cdot 4 = -4$, $a_5 = (-1)^{5+1} \cdot 5 = 5$, 2). $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \cdots$

SOLUTION:

Every term has two parts:

$b_1 = \frac{1}{1} - \frac{1}{2}$

$b_2 = \frac{1}{2} - \frac{1}{3}$

$b_3 = \frac{1}{3} - \frac{1}{4}$

General term is:

$$b_k = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all integers } n \geq 1$$

3). **2, 6, 12, 20, 30, 42, 56, …**

<u>SOLUTION:</u>

Note that we can write

$C_1 = 1.2 = 2$ $\qquad$ $C_2 = 2.3 = 6$

$C_3 = 3.4 = 12$ $\qquad$ $C_4 = 4.5 = 20$

In general n^{th} term is

$$C_n = n.(n+1) \text{ for all integers } n \geq 1$$

4). **1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …**

<u>SOLUTION:</u>

Every term has two parts numerator and denominator.

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers} \quad i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers} \quad j \geq 0$$

Both defined the same sequences.

ARITHMETIC SEQUENCE

- A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)

- The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by "d".

EXAMPLES

1). **5, 9, 13, 17, ...**

SOLUTION:

We need two things to define the sequence

First Term = 5

Common Difference = 4

2). **0, -5, -10, -15, …**

SOLUTION:

We need two things to define the sequence

First Term = 0

Common Difference = -5

3). **$x + a , x + 3a , x + 5a , \ldots$**

SOLUTION:

We need two things to define the sequence

First Term = $x + a$

Common Difference = $2a$

We need to add $2a$ every time to get the next term in the preceding term.

GENERAL TERM OF AN ARITHMETIC SEQUENCE

- Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is

$$a_1 = a$$
$$a_2 = a+d$$
$$a_3 = a+2d$$
$$a_4 = a+3d \ldots$$

$a_1 = a = a + (1-1)\,d$

$a_2 = a + d = a + (2-1)\,d$

$a_3 = a + 2d = a + (3-1)\,d$

By symmetry

$\mathbf{a_n = n^{th}\ term = a + (n-1)d}$ for all integers $n \geq 1$.

EXAMPLE

- Find the 20th term of the arithmetic sequence

  3, 9, 15, 21, …

- **SOLUTION:**

  Here a = first term = 3

  d = common difference = 9 - 3 = 6

  n = term number = 20

  $\mathbf{a_{20}}$ **= value of 20th term = ?**

- Since $\mathbf{a_n = a + (n - 1)\, d}$ $\quad n \geq 1$

  $\therefore \quad a_{20} = 3 + (20 - 1)\, 6$

  $= 3 + 114$

  $= 117$

EXAMPLE

- Which term of the arithmetic sequence

  4, 1, -2, …, is -77

- **SOLUTION:**

Here a = first term = 4

d = common difference = 1 - 4 = -3

a_n = value of nth term = -77

n = term number = ?

Since $\mathbf{a_n = a + (n - 1)\, d}$ $\quad n \geq 1$

$\Rightarrow \quad -77 = 4 + (n - 1)(-3)$

$\Rightarrow \quad -77 - 4 = (n - 1)(-3)$

OR

$$\frac{-81}{-3} = n - 1$$

OR

$$27 = n - 1$$

$$\mathbf{n = 28}$$

Hence **–77** is the 28th term of the given sequence.

EXERCISE

- Find the **36th** term of the arithmetic sequence whose **3rd** term is **7** and **8th** term is **17**.

- **SOLUTION:**

  Let **a** be the first term and **d** be the common difference of the arithmetic sequence.

  Then

$$\mathbf{a_n = a + (n - 1)d} \qquad n \geq 1$$

$$\Rightarrow \qquad a_3 = a + (3 - 1)\, d$$

and $$a_8 = a + (8 - 1)\, d$$

Given that $a_3 = 7$ and $a_8 = 17$.

Therefore,

$$7 = a + 2d \text{.......................}(1)$$

and $$17 = a + 7d \text{.......................}(2)$$

Subtracting (1) from (2), we get,

$$10 = 5d$$

$$\Rightarrow \quad d = 2$$

Substituting $d = 2$ in (1) we have

$$7 = a + 2(2)$$

which gives $a = 3$

Thus, $a_n = a + (n - 1)\, d$

$a_n = 3 + (n - 1)\, 2$ (using values of a and d)

Hence the value of 36th term is

$$a_{36} = 3 + (36 - 1)\, 2$$
$$= 3 + 70$$
$$= 73$$

GEOMETRIC SEQUENCE

- A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)

- The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by "r".

EXAMPLE

1). **1, 2, 4, 8, 16, ...**

- **SOLUTION:**

$a_1 = 1$, $a_2 = (1)(2) = 2$, $a_3 = (2)(2) = 4$

$a_4 = (4)(2) = 8$, $a_5 = (8)(2) = 16$

First Term = **1**

Common Ration = **2**

EXAMPLE

2). **3, -3/2, 3/4, -3/8, …**

- **SOLUTION:**

$a_1 = 3$, $a_2 = (1)(-1/2) = -3/2$, $a_3 = (-3/2)(-1/2) = 3/4$

$a_4 = (3/4)(-1/2) = -3/8$,

First Term = **3**

Common Ration = **-1/2**

EXAMPLE

3). **0.1, 0.01, 0.001, 0.0001, …**

- **SOLUTION:**

$a_1 = 0.1,\qquad a_2 = (0.1)(0.1) = 0.01$

$a_3 = (0.01)(0.1) = 0.001$

$a_4 = (0.001)(0.1) = 0.0001$

First Term = **0.1**

Common Ration = **0.1 = 1/10**

GENERAL TERM OF A GEOMETRIC SEQUENCE

- Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is

$$a\ ,\ ar\ ,\ ar^2,\ ar^3,\ \ldots$$

a_1 = first term $= a = ar^{1-1}$

a_2 = second term $= ar = ar^{2-1}$

a_3 = third term $= ar^2 = ar^{3-1}$

....

....

a_n = nth term $= ar^{n-1}$; for all integers $n \geq 1$

EXAMPLE

- Find the 8th term of the following geometric sequence

$$4, 12, 36, 108, \ldots$$

- **SOLUTION:**

Here a = first term = 4

r = common ratio = $\frac{12}{4} = 3$

n = term number = 8

a_8 = value of 8th term = ?

Since $\mathbf{a_n = ar^{n-1}}$ $\quad n \geq 1$

$\Rightarrow \quad a_8 = (4)(3)^{8-1}$

$= 4 .(2187)$

$= 8748$

EXAMPLE

- Which term of the geometric sequence is 1/8 if the first term is 4 and common ratio ½

- **SOLUTION:**

Given a = first term = 4

r = common ratio = ½

a_n = value of the nth term = 1/8

n = term number = ?

- Since $a_n = ar^{n-1}$ $n \geq 1$

$$\Rightarrow \quad \frac{1}{8} = 4\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \frac{1}{32} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad n-1=5 \qquad \Rightarrow n=6$$

- Since bases are same so powers must be equal

$$\Rightarrow \quad n - 1 = 5$$

$$\Rightarrow \quad n = 6$$

Hence **1/8** is the **6th** term of the given **G.P**.

EXERCISE

- Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.
- **SOLUTION:**

General Formula

$$a_n = ar^{n-1} \qquad n \geq 1$$

Now

$$a_2 = ar^{2-1}$$

$$\Rightarrow \quad 9 = ar \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also

$$a_4 = ar^{4-1}$$

$$\Rightarrow \quad 1 = ar^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

- **Dividing (2) by (1),** we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \qquad \frac{1}{9} = r^2$$

$$\Rightarrow \qquad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

- Substituting **r = 1/3** in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \qquad a = 9 \times 3 = 27$$

Hence the **geometric sequence** is

27, 9, 3, 1, 1/3, 1/9, …

SEQUENCES IN COMPUTER PROGRAMMING

- An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.