CC141

Discrete Structures

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SDT-403

Section 1.4

PREDICATES and QUANTIFIERS

PROPOSITION

- Proposition: a declarative statement that is either true or false

  Example:
  - The sky is red
  - My name is Iqra Javed

- Boolean variable: A variable (usually p, q, r, etc.) that represents a proposition

PROPOSITION

$X+3 > 4$ is a PROPOSITION?

NO

Open Statement

because value of X is unknown

PROPOSITIONS ?

Statements involving variables.

Example:

1. $x > 3$
2. $x = y + 3$
3. $x + y = z$
4. computer x is under attack by an intruder

Need for additional rules

Propositional Functions & Predicates

Consider

$P(x) = x - 3 > 5$.

Let us call this propositional function P(x), where P is the **predicate** and x is the **variable**.

What is the truth value of P(2) ? false

What is the truth value of P(8) ? false

What is the truth value of P(9) ? true

Thus, P(x) will create a proposition when given a value

Truth value depends on value of variable

Predicates

Hence

A declarative sentence which contains one or more variables,and is not a proposition,

but becomes a proposition when the variables in it are replaced by certain allowable choices

Predicate

- *x* is greater than 3

$$\underline{x} \qquad \underline{\text{is greater than 3}}$$

variable   predicate

- Denote by P(x)
- statement *P(x) known as*

Propositional function *P* at *x*.

Example

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

$P(4) : 4>3$, *True*
$P(2)$: $2>3$, *False*

PREDICATES

P(x):= word x contains the letter *n*

- P(panama) T/F?
- P(swiss) T/F?

Example

Let *A(x)* denote the statement "Computer *x* is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of ***A*(CS1), *A*(CS2),** and ***A*(MATH1)?**

PREDICATES

- A predicate may involve more than one variables
  - *n*-ary Predicate
- Value of a Propositional Function
  - (True/False)

Example (more than 1 variable)

Let $Q(x, y)$ denote the statement "$x = y + 3$."

What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Q(1, 2) : *1=2+3, False*
Q(3, 0): *3=0+3 True*

Example (more than 1 variable)

Let $A(c, n)$ denote the statement
"Computer c is connected to network n,"
where c represents a computer and
n represents network.
Suppose that computer MATH1 is connected to network CAMPUS2,
but not to network CAMPUS1.
What are the values of A(MATH1, CAMPUS1) and
A(MATH1, CAMPUS2)?

PREDICATES

- Let R(x, y, z) denote the statement
  - “x + y = z.”
- What are the truth values of the **propositions**
  - R(1, 2, 3)
  - R(0, 0, 1)

Propositional functions

- Let P(x) = “x is a multiple of 5”
  - For what values of x is P(x) true?
- Let P(x) = x+1 > x
  - For what values of x is P(x) true?
- Let P(x) = x + 3
  - For what values of x is P(x) true?

Universe of Discourse

Consider the example $x > 3$.
Does it make sense to assign to x the value “blue”?

Predicates become propositions once every variable is bound by assigning it a value from the **Universe of Discourse** “U” (aka **domain**)

Example

Let U= Z, the integers = { . . -2, -1, 0 , 1, 2, . . .}

- P(x): x > 0 is the predicate.

propositions where x is assigned a value:

- P(-3)
- P(0)
- P(3)

PREDICATES

- Every student of this class is a bright student
  - *Ahmad is a student of this class*

- Ahmad is a bright student
  - **T / F ?**

Need for additional rules

Quantifiers

Another way to make a predicate into a proposition is to quantify it.

That is, the predicate is true (or false) for all possible values in the universe of discourse or for some value(s) in the universe of discourse

- A quantifier is "an operator that limits the variables of a proposition"
- Area of logic that deal with predicate and quantifiers is called predicate calculus.

QUANTIFIERS

- ***Universal*** Quantifier
- ***Existential*** Quantifier

UNIVERSAL QUANTIFIER

- The universal quantification of a predicate P(x) is the proposition

"P(x) is true for all values of x in the universe of discourse (domain)"

- notation $\forall x\ P(x)$ which can be read "for all x"

UNIVERSAL QUANTIFIER

An element for which *P(x)* is false is called a **counterexample** of $\forall x\, P(x)$

- True
  - When No False example found
- False
  - When 1 False example found

Expressing words

We can state in following ways: –

$\forall x\, P(x) : x+1 > x$

- "for all values of x, P(x) is true"
- "for all values of x, x+1>x is true"

Expressing words:

"for all,"	"for every,"	"all of,"
"for each,"	"given any,"	"for arbitrary,"
"for any."		

Example 1

- Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

True

(no value of x found for which propositional function is false)

Example 2

- Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

False

Counter example

$Q(3)$: $3<2$, false.

If you found any false example then $\forall x\, Q(x)$ is false.

Example 3

Suppose that *P(x)* is "$x^2 > 0$." show statement $\forall x\, P(x)$ is false where the domain consists of all integers.

False

$x = 0$ is a counterexample.

Example 4

What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain is positive integers not exceeding 4?

Solution:

False

Counter example

- $P(4) : 4^2 < 10$, is false,
- Thus $\forall x\, P(x)$ is false.

Universal Quantifier

- What does the statement
  - $\forall x N(x)$

mean if $N(x)$ is
  - "Computer x is connected to the network"
  - and the *domain* consists of all computers on campus?

"Every computer on campus is connected to the network."

Example

- What is the truth value of ∀*x* *P*(*x*), where *P*(*x*) is the statement "$x^2 < 10$" and the domain is positive integers not exceeding 4?

Solution:

- *P*(1) ∧ *P*(2) ∧ *P*(3) ∧ *P*(4),
- as *P*(4), which is the statement
- "$4^2 < 10$," is false,
- Thus ∀*x* *P*(*x*) is false.

EXISTENTIAL QUANTIFIER

- The existential quantification of a predicate P(x) is the proposition "There exists an x in the universe of discourse such that P(x) is true."

- notation ∃x P(x) *existential quantifier* read as "there exists an x"

- True
  - When 1 true example found
- False
  - When No true example found

Expressing words

- We can state in following ways: -
  - $\exists x\, P(x)$
  - "there exists (a value of) x such that P(x) is true"
  - "for at least one value of x, x+1>x is true
  - For some *x P(x)* is true

Expressing words: "there exists," "for some," "for at least one," or "there is"

Example 1

1. Let *P(x)* denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\ P(x)$, where the domain consists of all real numbers?

True

Example

P(4): 4>3, True

Example 2

Let $Q(x)$ denote the statement "$x = x+1$." What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

False
(no value of x found for which propositional function is True)

Example 3

What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

True

Example

$P(4)$: $4^2>10$, True

UNIQUENESS QUANTIFIER

- There exists a unique *x* such that *P(x)* is true."

i.e. "there is exactly one" or "there is one and only one."

- denoted by ∃! or ∃1.
- Example: Let P(x) denote x - 1 = 0 and U are the integers. Then ∃!x P(x) is
- true.
- Example: Let P(x) denote x > 0 and U are the integers. Then ∃!x P(x) is
- false

EMPTY DOMAIN

Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty.

Note that if the domain is **empty**, then **$\forall x P(x)$ is true** for any propositional function P(x) *because there are no elements x in the domain for which P(x) is false.*

EMPTY DOMAIN

Note that if the domain is **empty**, then **$\exists x Q(x)$ is false** whenever Q(x) is a propositional function because when the domain is empty, *there can be no element x in the domain for which Q(x) is true.*

QUANTIFIERS OVER FINITE DOMAINS

If the universe of discourse is finite, say $\{n1, n2, \ldots, nk\}$,

- Then the universal quantifier is simply the conjunction of all elements:
  $\forall x\, P(x) \Longleftrightarrow P(n1) \wedge P(n2) \wedge \cdots \wedge P(nk)$

- And the existential quantifier is simply the disjunction of all elements:
  $\exists x\, P(x) \Longleftrightarrow P(n1) \vee P(n2) \vee \cdot\cdot \vee P(nk)$

QUANTIFIER

TABLE 1 Quantifiers.

Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Quantifiers with Restricted Domains

- An abbreviated notation is used to restrict the domain of a quantifier.
- a condition which variable must satisfy is included after the quantifier.

Example:

- $\forall x < 0\ (x^2 > 0)$
- $\exists z > 0\ (z^2 = 4)$

Quantifiers with Restricted Domains

1. $\forall x < 0\ (x^2 > 0)$

Means "For every x in the domain with x < 0, $x^2 > 0$"

Proposition is same as: $\forall x \quad (x < 0 \rightarrow x^2 > 0)$.

2. $\exists z > 0\ (z^2 = 4)$

Means "There is some z in the domain with $z > 0$, $z^2 = 4$

Proposition is same as: $\exists z\ (z > 0 \land z^2 = 4)$.

Translating from English into Logical Expressions

- Express statement "Every student in this class has studied calculus" using predicates and quantifier.

- Step 1: identify the appropriate quantifiers

- Every student in this class, that student has studied calculus

Translating from English into Logical Expressions

- Express statement "Every student in this class has studied calculus" using predicates and quantifier.

- Step 2: introduce a variable *x*

- For every student *x* in this class, *x* has studied calculus

Translating from English into Logical Expressions

- Express statement "Every student in this class has studied calculus" using predicates and quantifier.

- Step 3: introduce *C(x)*

- C(x)= *x* has studied calculus.

Translating from English into Logical Expressions

- Express statement "Every student in this class has studied calculus" using predicates and quantifier.

- Translate:

$$\forall x\ C(x)$$

Example

Express:" **Some student in this class has visited Mexico**"

Step 1: "There is a student in this class with the property that the student has visited Mexico"

Step 2: "There is a student x in this class having the property that x has visited Mexico."

Step 3: $M(x)$= "x has visited Mexico."

Step 4: $\exists x\, M(x)$

Precedence of Quantifiers

Higher than ***all*** *logical operators* from propositional calculus

Precedence of Quantifiers

The quantifiers ∀ and ∃ have **higher precedence** *than **all** logical operators* from propositional calculus.

Precedence of Quantifiers

For example,

$\forall x P(x) \vee Q(x)$ is the *disjunction* of $\forall x P(x)$ and $Q(x)$.

In other words, **it means** $(\forall x P(x)) \vee Q(x)$

- *rather than* $\forall x (P(x) \vee Q(x))$.

Logical Equivalences

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

Logical Equivalences

We use the notation

S ≡ T

to indicate that two statements **S** and **T** involving predicates and quantifiers are logically equivalent.

Logical Equivalences

- We can distribute a ***universal*** **quantifier** over a **conjunction**

$$\forall x\,(P(x) \land Q(x)) \equiv \forall x\,P(x) \land \forall x\,Q(x)$$

- We can also distribute an ***existential*** **quantifier** over a **disjunction**

$$\exists x\,(P(x) \lor Q(x)) \equiv \exists x\,P(x) \lor \exists x\,Q(x)$$

Logical Equivalences

- We cannot distribute a ***universal*** **quantifier** over a **disjunction**

$$\forall x\,(P(x) \vee Q(x)) \mathrel{!\equiv} \forall x\, P(x) \vee \forall x\, Q(x)$$

- Similarly,

we cannot distribute an ***existential*** **quantifier** over a **conjunction**

$$\exists x\,(P(x) \wedge Q(x)) \mathrel{!\equiv} \exists x\, P(x) \wedge \exists x\, Q(x)$$

Negating Quantified Expressions

De Morgan's Laws

Negating Quantified Expressions

- De Morgan's Laws for Quantifiers

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negating Quantified Expressions

- "Every student in your class has taken a course in calculus"

- $\forall x\, P(x)$= "x has taken a course in calculus"
- *Domain*=students in your class

- Negation "It is not the case that every student in your class has taken a course in calculus.
- This is equivalent to "There is a student in your class who has not taken a course in calculus."

so $\exists x\, \neg P(x)$.

$$\boldsymbol{\neg \forall x\, P(x) \equiv \exists x\, \neg P(x).}$$

Negating Quantified Expressions

- "There is a student in this class who has taken a course in calculus."
- $\exists x\, Q(x)$="x has taken a course in calculus"

- Find negation

$$\neg\exists x\, Q(x) \equiv \forall x\, \neg Q(x).$$

Example

What is negation of the “All Americans eat cheeseburgers”

- Let $C(x)$ =“x eats cheeseburgers.”
- Statement= “All Americans eat cheeseburgers”= $\forall x\, C(x)$
- Domain=all Americans.

- Negation = $\neg\forall x\, C(x) \equiv \exists x\, \neg C(x)$.
- Expressed as:
  - “Some American does not eat cheeseburgers”
  - It is not the case that All Americans eat cheeseburgers
  - “There is an American who does not eat cheeseburgers.”

Example

What is negation of "There is an honest politician"

- Let $H(x)$ ="x is honest."
- Statement "There is an honest politician"=$\exists x\, H(x)$
- Domain= all politicians.

- Negation=$\neg\exists x\, H(x) \equiv \forall x\, \neg H(x)$
- expressed as:
  - "Every politician is dishonest."
  - "Not all politicians are honest."

Negating Quantified Expressions

- What are the negations of the statements
  - $\forall x(x^2 > x)$
  - $\exists x(x^2 = 2)$

Example

- What are the negations of the statements $\forall x(x^2>x)$ and $\exists x(x^2 = 2)$?
- ***Solution***: $\forall x(x^2>x)$
- The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x2 > x)$,
- which is equivalent to $\exists x\neg(x^2 > x)$.
- This can be rewritten as $\exists x(x^2 \leq x)$.

Solution: $\exists x(x^2 = 2)$
The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$,
which is equivalent to $\forall x\neg(x^2 = 2)$.
This can be rewritten as $\forall x(x^2 \neq 2)$.

The truth values of these statements depend on the domain.

Negating Quantified Expressions

- Show this logical equivalence step by step:

$$\neg \forall x\, (P(x) \rightarrow Q(x)) \equiv \exists x\, (P(x) \wedge \neg Q(x))$$

Binding Variables

- If there is a quantifier used on a variable x, we say the variable is bound. Else it is free.

Ex: In $\exists x (x + y = 1)$, x is bound and y is free

- If all variables in a propositional function are bound or set equal to a particular value , the function becomes proposition
- – Ex: $\forall y \exists x (x + y = 1)$ is a proposition

Binding Variables (SCOPE)

The part of the logical expression where a quantifier is applied is called the scope of that quantifier

$\exists \mathbf{x}\ (P(x) \land Q(x)) \lor \forall x R(x)$

all variables are bound

- The scope of the first quantifier, $\exists \mathbf{x}$, is the expression $\mathbf{P(x) \land Q(x)}$
  - because $\exists x$ is applied only to $P(x) \land Q(x)$, and not to the rest of the statement.
- Similarly, the scope of the second quantifier, $\forall \mathbf{x}$, is the expression $\mathbf{R(x)}$

Binding Variables (SCOPE)

$$\exists \mathbf{x}(\mathbf{P}(\mathbf{x}) \wedge \mathbf{Q}(\mathbf{x})) \vee \forall \mathbf{x}\mathbf{R}(\mathbf{x})$$

we could have written above statement *using two different variables x and y*,

as

$$\exists \mathbf{x}(\mathbf{P}(\mathbf{x}) \wedge \mathbf{Q}(\mathbf{x})) \vee \forall \mathbf{y}\mathbf{R}(\mathbf{y}),$$

because the *scopes* of the two quantifiers do not overlap.

NESTED QUANTIFIERS

Section 1.5

Nested Quantifiers

- Nested Quantifiers
  - One Quantifier within scope of another Quantifier

- Examples:

proposition **$\forall y \exists x (x + y = 1)$** with two quantifiers,

where $\forall y$ is applied to $\exists x (x + y = 1)$, and $\exists x$ is applied to $x + y = 1$

Nested Quantifiers

- $\forall x \, \forall y \, (x + y = y + x)$
- $\forall x \, \exists y \, (x + y = 0)$
- $\forall x \forall y \forall z \, (x+(y + z) = (x + y) + z)$
- $\forall x \forall y \, ((x>0) \land (y < 0) \rightarrow (xy < 0))$

Order of Quantifiers

- Critical when *different* Quantifiers
- $P(x, y) := $ "$x + y = y + x$"
- $\forall x \forall y P(x, y)$ T/F?
- $\forall y \forall x P(x, y)$ T/F?

Order of Quantifiers

- Critical when *different* Quantifiers
- $Q(x, y)$:="$x + y = 0$"
- $\exists y \forall x Q(x, y)$ T/F?
- $\forall x \exists y Q(x, y)$ T/F?

More Levels of Nesting

- $Q(x, y, z)$:="$x + y = z$"

- $\forall x \forall y \exists z Q(x, y, z)$ T/F?
- $\exists z \forall x \forall y Q(x, y, z)$ T/F?

- Domain:=Real numbers

Nested Quantifiers

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Mathematical Statements to Nested Quantifiers

The sum of two positive integers is always positive

$$\forall x \forall y((x>0) \land (y>0) \rightarrow (x+y > 0))$$

Mathematical Statements to Nested Quantifiers

Every real number except zero has a multiplicative inverse

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$$

Nested Quantifiers to English

$$\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$$

- C(x):="x has a computer"
- F(x, y):="x and y are friends"
- Domain:=all students in UMT

Every student in UMT has a computer or has a friend who has a computer

English to Logical Expressions

- "If a person is female and is a parent, then this person is someone's mother"
  - F(x):="x is female"
  - P(x):="x is a parent"
  - M(x, y):="x is the mother of y."

$$\forall \mathbf{x}((\mathbf{F}\,(\mathbf{x}) \wedge \mathbf{P}(\mathbf{x})) \rightarrow \exists \mathbf{y}\mathbf{M}(\mathbf{x}, \mathbf{y}))$$

English to Logical Expressions

- "Everyone has exactly one best friend"
  - B(x, y):="y is the best friend of x"

English to Logical Expressions

- “There is a woman who has taken a flight on every airline in the world.”
  - P(w, f):=“w has taken f ”
  - Q(f, a):=“f is a flight on a”

Negating Nested Quantifiers

- Express the negation of the statement
  - $\forall x \exists y(xy = 1)$
- so that no negation precedes a quantifier.

- Hint: Successively apply De Morgan’s Laws

English to Logical Expressions

- "There is a woman who has taken a flight on every airline in the world."
    - $P(w, f)$:="w has taken f "
    - $Q(f, a)$:="f is a flight on a"

- $\exists w \forall a \exists f \, (P(w, f) \wedge Q(f, a))$

Negating Nested Quantifiers

- $\neg\forall x\exists y(xy = 1) \equiv \exists x\neg\exists y(xy = 1)$

$$\equiv \exists x\forall y\neg(xy = 1)$$

$$\equiv \exists x\forall y(xy \neq 1)$$

Negating Nested Quantifiers

- "There **does not** exist a woman who has taken a flight on every airline in the world."

Negating Nested Quantifiers

- RECALL
  - "There is a woman who has taken a flight on every airline in the world."
    - $P(w, f)$:="w has taken f "
    - $Q(f, a)$:="f is a flight on a"

- $\exists w \forall a \exists f\, (P(w, f) \land Q(f, a))$

Negating Nested Quantifiers

- "There **does not** exist a woman who has taken a flight on every airline in the world."
- $\neg \exists w \forall a \exists f \, (P(w, f) \land Q(f, a))$

Negating Nested Quantifiers

- $\neg\exists w \forall a \exists f\, (P(w, f) \wedge Q(f, a))$
- $\equiv \forall w \neg \forall a \exists f\, (P(w, f) \wedge Q(f, a))$
- $\equiv \forall w \exists a \neg \exists f\, (P(w, f) \wedge Q(f, a))$
- $\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a))$
- $\equiv \forall w \exists a \forall f\, (\neg P(w, f) \vee \neg Q(f, a))$

Section 1.5: Nested Quantifiers

- Questions for Practice
  - Exercise 1-20
  - All Odd Exercises form 21 to 47

Summary

The topics we covered so far:

- Predicate and Quantifiers
- Nested Quantifiers
- Negating Quantifiers
- Translation English to Logical statements