Computation of Homology

MATHEMATICS SENIOR SEMINAR

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Mathematics Senior Seminar

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Abstract

Topology's objective is to understand and formalize shape, whatever that may be. Despite its initial, seemingly miraculous departure from geometry—replacing distance metrics with open and closed sets—topology maintains a profound connection to it. Through a series of morphisms, a topology on a set can be transformed into simplicial sets, then into a chain of connected abelian groups. These groups reveal invariant characteristics of the topology, known as Betti numbers, which are crucial in comprehending the shape of mathematical objects. These algebraic manipulations allow for the mathematical objects to retain structure throughout the process. Due in large part to the concepts of persistence which brings back the notion of distance metrics, there have been extensions to the domain of various data structures including networks, images, and visual landscapes.

The paper's objective is to elucidate the computation of homology from various perspectives, demonstrating its application in both traditional topological settings and extended data structures. This exploration underlines both the elegance embedded within the abstract nature of homology in addition to the significance of topological methods in comprehending complex shapes.

Introduction

To understand persistent homology, the notion of topology must be extended into combinatorial structures known as simplicial complexes. This allows for a continuous abstract representation of a space to be represented in discrete structure with concrete points and connections. It turns out there are different grains of abstraction from which this process can be understood from.

The study of Homology allows for further identification of topological objects, illustrating the shape of the space with invariant properties.

The utility and importance of persistent homology and the introduction of persistence lies in the extension of such machinery into data captured in the real world. Beautiful surfaces like the sphere, the torus, the klein bottle, and many others have idealized objective manifestations, however for data collected from the real world in real time naturally spawn bizarre structures with varying degrees of randomness.

Preliminaries

2.1 From Point-Set to Combinatorial Topology

The conceptual viewpoint of point-set topology allows for a forumulaic continuous nature. The notions are abstracted from the concepts of open and closed sets which holster definitions that cater towards analytical perspectives.

2.1.1 Topological Spaces

To begin, we assume conceptual understanding of logic and set theory. Topology is founded on the ideas of inclusion and exclusion; this can be illustrated by means of set theory or geometry.

Definition 1. Topology A *topology* on a set X is a collection T of subsets of X having the following properties:

1. $\emptyset$ and X are in T.
2. The union of the elements of any subcollection of T is in T.
3. The intersection of the elements of any finite subcollection of T is in T.

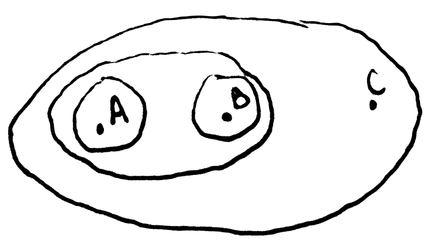

Figure 2.1: $X = \{A, B, C\}, T = \{\{A, B, C\}, \{A, B\}, \{A\}, \{B\}\}$.

Following then very cleanly from this are the ideas of openness and closedness.

Definition 2. Open-set Let X be a set equipped with a topology $\mathcal{T}$, then $S \in \mathcal{T}$ means that S is *open*. The complement of an open set is a *closed set*.

The example in Figure 2.1, illustrates an instance whereby the set-theoretic axioms of a topology hold. Of great importance in topology is the concept of continuous functions. Burgeoning from the analytical study of $\mathbb{R}$, an analytical perspective of ϵ, δ is met with mathematically equivalent representations using open/closed sets.

Definition 3. Continuity A function $f : X \to Y$ is *continuous* if for every open set A in Y, the pre-image $f^{-1}(A)$ is open in X.

The fundamental comparison between topological spaces X and Y is,

Definition 4. (Homeomorphism) A homeomorphism $f : X \to Y$ is a 1-1 onto function, such that both f and f^{-1} are continuous. We say that X is homeomorphic to Y, denoted $X \approx Y$, and that X and Y have the same topological type.

To facilitate understanding of homology, it would help to have examples illustrated from the perspective of 3-dimensional surfaces or 2-manifolds.

Homeomorphisms between between topological spaces allow for the morphing of one space into another. This can enable topological spaces to be studied from the lens of category theory; having objects and morphisms between objects. It turns out these point-set representations can be show to have equivalence to a more combinatorial representations.

2.1.2 Simplicial Complexes

Different objectives require differing frameworks or viewpoints on the same problem. Point-set topology enables the concepts of infinite sets to be well understood. It however isnt suited for the notion of homology or persistent homology that utilizes a finite amount of real-valued data points.

To reach a more computational methodology there must be geometric definitions that allow us to talk about combinations and there must be objects we want to combine. Said objects of interest are simplicial complexes, these manifestations are equivalent to the underlying space and allow for the ease of computation.

Definition 5. Abstract simplicial complex *An abstract simplicial complex* K consists of a set S of finite sets such that if $A \in S$, so is every subset of A. We say $A \in S$ is an (abstract) k-simplex of dimension k if $|A| = k + 1$.

To illustrate the connections to point-set topology consider the simplicial complex:
$\mathcal{K} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}\}$.

This definition shows that the inclusion and exclusion of separate entities within the complex are primordial. Allowing the capacity to understand intersection and union of subsets. Imparting the geometric notion into the above definition we attain,

Definition 6. Vertex scheme Let K be a simplicial complex with vertices V and let S be the collection of all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K. The collection S is called the *vertex scheme* of K.

Previously simplicial complex K was defined solely in terms of subsets of K. Defining the vertex set V, $V(\mathcal{K}) = \{a, b, c, d\}$. With this labelling along with the definition of a vertex scheme, the geometric understanding of d-dimensional simplices becomes understandable. Figure 2.2 shows the geometric interpretation or realization of the simplicial complex K. Figure 2.3 is provided to bridge the relationship between point-set topology and simplicial complexes. The linkages between layers allows for understanding of *face* and *co-face*.

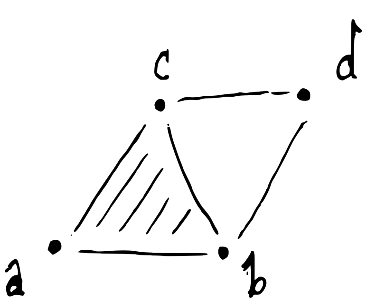

Figure 2.2: Geometric interpretation of K

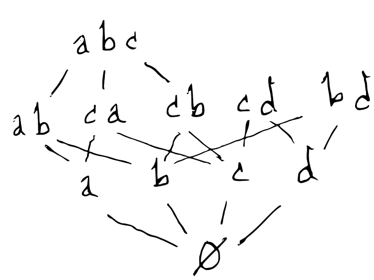

Figure 2.3: View of K through poset visualization.

Vertex schemes allow the specifics of spanning sets, combinations, and geometric realizations. Within the jump from abstract simplicial complex to vertex scheme, the concept of combination has varying properties of choice that construct wholy different spaces.

Definition 7. Combination Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{V}^d$. A *linear combination* is $x = \sum_{i=0}^{k} \lambda_i p_i$, for some $\lambda_i \in \mathbb{F}$. An *affine combination* is a linear combination with $\sum_{i=0}^{k} \lambda_i = 1$ with $\lambda_i \in \mathbb{R}$. A *convex combination* is an affine combination with $\lambda_i \geq 0$, for all i. The set of all convex combinations is the convex hull.

Definition 8. Isomorphism Let K_1, K_2 be abstract simplicial complexes with vertices V_1, V_2 and subset collections S_1, S_2, respectively. An *isomorphism* between K_1, K_2 is a bijection $\phi : V_1 \to V_2$, such that the sets in S_1 and S_2 are the same under the renaming of the vertices by ϕ and its inverse.

There now exists a full relationship between the abstract simplicial complex and the vertex schemes.

Theorem 3.2 Every abstract complex S is isomorphic to the vertex scheme of some simplicial complex K. Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes.

(a) $|K_1|$ (b) $|K_2|$

Figure 2.4: Geometric realizations

In Figure 2.4, the abstract simplicial complexes K_1 and K_2 have respective vertex schemes. Given there exists a bijection that maintains the set denominations between the abstract simplicial complexes, they are isomorphic.

The connection between topological spaces and simplicial complexes resides in the notion of triangulation. A way of carving up the soft and squishy surfaces into a vertex scheme that maintains the original topological nature.

Definition 9. Triangulation A *triangulation* of a topological space X is a homeomorphism $h : |K| \to X$, where K is a simplicial complex.

Here $|K|$ represents the geometric realization of abstract simplicial complex K. Triangluation connects back to the point-set notions of manifolds and surfaces, and presents equivalence towards the combinatorial representations.

Categorical Perspective

3.1 Simplical Homology to Singular Homology

Homology was originally presented from the standpoint of simplicial homology using the lens of simplicial complexes. This standpoint was initially fruitful, however through time, flaws in its expressability led to unintelligible complications. The introduction of simplicial sets allowed for non-finite but countable framing on the problem that allowed for a machinery of connectivity that wasn't present in the simplicial complexes. Building off the concept of simplicial complex, to understand homology and how it is achieved it very much helps to extend this idea to a different framing known as semisimplicial sets. This allows for a way to think about simplicial complex K as a sequence of sets $X_0, X_1, X_2, \ldots, X_n$, where K = $X_0 \cup X_1 \cup X_2 \cup ... \cup X_n$. The index of each sequenced set denotes the degree of simplices, thus X_0 holds the vertices of the simplicial complex.

Definition 10. Semisimplicial Set A semisimplicial set X is a sequence of sets $X_0, X_1, X_2, \ldots$, and functions $d_0, d_1 : X_1 \to X_0$, $d_0, d_1, d_2 : X_2 \to X_1$, and so on (in general, we have $n+1$ functions from $X_n \to X_{n-1}$ for every $n \geq 1$), such that the simplicial identities are satisfied:

$$d_i d_j = d_{j-1} d_i$$

whenever $i < j$ and those equations make sense.

Now with this notation, we can compute the homology of a semi-simplicial set. We are interested in the free abelian groups generated by all X_i. A free abelian group is a set S equipped with a binary operation + that is both a group and is communitative under the operation. If we denote the n-th abelian group to be the set of n-simplices from K or X_n, then $S_n(K)$. With this idea, we can then describe the boundary operator. It has a strong relationship to the functions defined on the semi-simplicial set.

Homology deals with the change between immediate dimensional simplices (0-dim ¡-¿ 1-dim, 1-dim ¡-¿ 2-dim and so on.) To formalize this we use the boundary operation:

Definition 11. Boundary Operators For all $n \geq 1$, the boundary operators

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

are defined by sending $\sigma \in X_n$ to

$$\sum_{k=0}^{n} (-1)^k d_k \sigma.$$

We also define $\partial_0 : S_0(X) \to 0$ to be the zero homomorphism.

With the notion of mapping between levels of simplices, we can understand homology. It is the quotient space between two underlying spaces or sets. Given set X_i, provided $i > 0$, with the boundary operation, the domain $S_i(X)$ can be split into two subspaces. The first space is the elements mapped to the 0 element, known as the kernel of ∂. The second space is the elements mapped to unique representations within S_{i-1}. Denoting $Z_i(X)$ as $ker(\partial_i)$ and $B_i(X)$ as $im(\partial_{i+1})$.

These subspaces or sub-abelian groups have important intuitive notions. $Z_i(X)$ is conceptually understood as the set of i-dimensional cycles because in 7, the boundary of a cycle equates to 0. $B_i(X)$ is understood as i-dimensional boundaries as they represent the boundary of a higher dimensional simplex. Homology in essence takes the quotient of these subspaces:

$$H_i(X) = ker(\partial)/im(\partial) = Z_i(X)/B_i(X)$$

Homology looks at the parts within dimensions that are holes. To understand the overall process of homology's computation from a topological space, I will use the concepts from category theory to illustrate the transition.

Definition 12. Category A Category consists of the following:

- A class ob(C) of objects in C.
- $\forall$ pair of objects $X, Y \in ob(C)$, a set of morphisms denoted $Hom_C(X, Y)$.
- $\forall X \in ob(C)$, an identity moprhism $1_X \in Hom_C(X, X)$
- $\forall (X, Y, Z) \in ob(C)$ a composition operation $Hom_C(X, Y) \times Hom_C(Y, Z) \to Hom_C(X, Z)$, written as $g\dot{f}$. Where the composition must satisfy identity and associativity.

The utility of expressing this process within this framing of Category Theory lies in the jumping between different mathematical structures. Beginning with some soft squishy topological space, the idea is to extract the dimensionality of certain abelian groups. To get from one to the other, we must have preservation of important information and we must have awareness of the structure of each category of objects.

Beginning, much like the simplicial complex methodology is a way to represent the topological spaces as more conceptually friendly beings, here we are considering simplicial sets. Topological spaces, specifically n-dimensional manifolds reside in Euclidean space; through this the relationship between geometric and topological objects spawns.

Definition 13. N-simplex For any $n \geq 0$, the standard n-simplex Δ_n is a subspace of $\mathbb{R}^{n+1}$, defined as the convex hull of the standard basis $\{e_0, e_1, \ldots, e_n\}$. In other words,

$$\Delta_n = \left\{ \sum_i t_i e_i : \sum t_i = 1, t_i \geq 0 \right\}$$

A very visualizable method, comparable to the notion of an abstract simplicial complex having a geometric realization. Here we couple this idea with both a topological space and a continuum:

Definition 14. Singular Simplices Let X be a topological space. Define $\mathrm{Sing}_n(X)$ to be the set of all continuous maps $\Delta_n \to X$.

Now instead of thinking about a finite set of simplices, we conceptualize a continuum of sets (or levels) whereby the sets contain at each n-level all continuous maps from a given n-simplex to the topological space X. Earlier within 10 there needed to be a function d_i that mapped n-simplices to the lower level. Here d can be defined with respect to the geometric underpinning of said topological space:

$$d_i : Sing_n(X) \to Sing_n(X)$$

This allows for the conveyance of topological space into a simplicial set. A way of translating from apples to oranges. The category Top of topological spaces has ob(Top) the class of all topological spaces, and if $X, Y \in ob(Top)$, then $Hom_{Top}(X, Y)$ (morphisms) are continuous functions from X to Y. The idea is to understand how to go from Topological spaces to Simplicial sets. This necessitates the following:

Definition 15. Functor A *functor* $F : \mathcal{C} \to \mathcal{D}$ of categories consists of

- An assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ from objects to objects, and
- for all $X, Y \in \mathrm{ob}(\mathcal{C})$, there is a function $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.

Furthermore, we must have $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{ob}(\mathcal{C})$, and for all composable pairs of morphisms $f, g \in \mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

Now $Sing_n$ can be understood as a functor that $Top \to Set$, however, it is apparent that within the concept of functor, one can think of the simplicial set as a sequence of functors itself. Then what does $Sing_n$ really do? It is a functor to a collection of functors; a collection of functors must be understood as a category. We need to further break down $Sing_n(X)$ constituent parts into categorical representations to give way to the functorial view of simplicial sets. The domain of such functors are presumed to be n-simplices, here we define it further:

Definition 16. Let Δ_{inj} denote the category with objects $\mathrm{ob}(\Delta_{\text{inj}}) = \{[0], [1], [2], \ldots\}$, and morphisms between objects given by

$$\mathrm{Hom}_{\Delta_{\text{inj}}}([a], [b]) = \{\text{injective functions } f : \{0, 1, \ldots, a\} \to \{0, 1, \ldots, b\} \text{ that preserve order}\}.$$

The category Set where ob(Set) is the class of all sets, and morphisms are functions between sets. The designated functor for simplicial sets that was imagined before can be represented as a category $\mathrm{Fun}(\Delta_{\text{inj}}, Set)$ with $\mathrm{ob}(\mathrm{Fun}(\Delta_{\text{inj}}, Set))$ as all functors between the designated categories and morphisms as natural transformations between the functors.

Definition 17. Natural-transformation Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\Theta : F \to G$ consists of maps $\Theta_X : F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$, such that for all maps $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\Theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Theta_Y"'] & G(Y) \end{tikzcd}$$

Now we have:

$$Sing_n(X) = Top \rightarrow Fun(\Delta_{\text{inj}}, Set)$$

With a functor to simplicial sets, there exists a functor from simplicial sets to chain complexes. There then exists a slew of functors that morph chain complexes of abelian groups into abelian groups. These abelian groups are then put into a quotient relation to compute the homology groups.

A semisimplical set morphs n-simplices into sets, and then to compute homology those sets need to be morphed into abelian groups. The category Ab where ob(Ab) is the class of Abelian groups, and $Hom_{Ab}(X, Y)$ are group homomorphisms between objects X and Y in ob(Ab).

Matrix Representation, Vector Spaces, and Algorithms

4.1 Computation of Homology for low dimensional surfaces

With the conceptual framework of combinatorial topology and the categorical morphisms into the homology groups. I will then express the conceptual analogs towards vector spaces and the resulting computation of such objects.

Shown below is a minimal triangulation of the Klein bottle. From left to right the progression is from 0-dim simplices to 2-dim simplices.

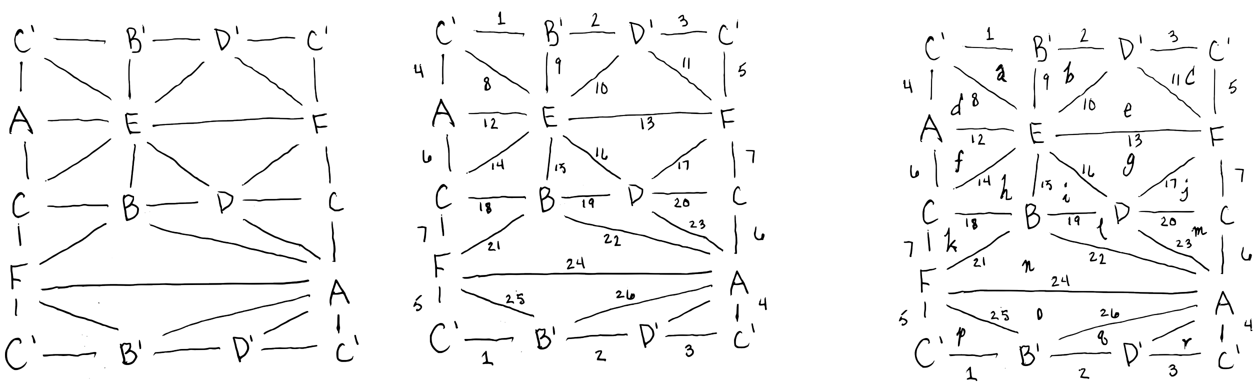

Figure 4.1: Klein Bottle

These higher dimension simplices can be represented in terms of their faces using the equipped boundary operator. Previously denoted as,

$\partial_p : C_p(K) \to C_{p-1}(K)$ (Boundary Operator)

Within the calculation of the Klein bottle, we have 18 2-dim simplices (Triangles) meaning $C_p(K)$ would be represented as a 18-dimensional vector, and 27 1-dim simplices (edges) equating to C_{p-1} being a 27-dimensional vector. Thus the boundary operation would be a 27 x 18 dimensional matrix.

Consider two 0-1 vectors of dimension 6, $\mathbf{v}_1$ and $\mathbf{v}_2$. Adding these vectors with mod 2 coefficients gives us a vector of all zeros due to the cancellation at the second position:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now consider two different 0-1 vectors. Adding these vectors with mod 2 coefficients gives us a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathbf{v}_3 + \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using linear algebra, the corresponding matrices with mod-2 field coefficients are row-reduced into the following matrices.

Δ^1	$\partial(\Delta^1)$
1	$C' + B'$
2	$B' + D'$
3	$D' + C'$
4	$C' + A$
5	$C' + F$
6	$A + C$
7	$C + F$
8	$C' + E$
9	$B' + E$
10	$E + D'$
11	$D' + F$
12	$A + E$
13	$E + F$
14	$C + E$
15	$E + B$
16	$E + D$
17	$D + F$
18	$C + B$
19	$B + D$
20	$D + C$
21	$B + F$
22	$B + A$
23	$D + A$
24	$F + A$
25	$F + B'$
26	$A + B'$
27	$A + D'$

Figure 4.2: Higher Dimensional Simplices and Their Boundary Operations

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 4.3: $\partial(\Delta^1)$ matrix in RREF

Δ^2	$\partial(\Delta^2)$
a	$1+9+8$
b	$2+9+10$
c	$3+11+5$
d	$4+8+12$
e	$10+11+13$
f	$6+12+14$
g	$13+16+17$
h	$14+18+15$
i	$15+16+19$
j	$17+7+20$
k	$7+18+21$
l	$19+22+23$
m	$20+23+6$
n	$21+22+24$
o	$24+25+26$
p	$5+25+1$
q	$26+2+27$
r	$27+3+4$

$$\begin{pmatrix}
1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&1\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Figure 4.4: $\partial(\Delta^2)$ matrix in RREF

The n-dimensional simplicial complexes can have their boundary operation equivalents translated into 0-1 vectors to fill a matrix. This then allows for the matrix to be treated as a vector space with mod 2 coefficients.
After the matrices are row-reduced to echelon form, the coinciding homological information can be deduced.

Definition. For any $p \in \{0, 1, 2, \ldots\}$, the *pth homology* of a simplicial complex K is the quotient vector space

$$H_p(K) := \frac{\ker(\partial_p)}{\operatorname{im}(\partial_{p+1})}.$$

Its dimension

$$\beta_p(K) := \dim H_p(K) = \dim \ker(\partial_p) - \dim \operatorname{im}(\partial_{p+1})$$

is called the pth Betti number of K. Elements in the image of ∂_{p+1} are called p-boundaries, and elements in the kernel of ∂_p are called p-cycles.

An equivalent way to represent this from the matrices above:

- $\ker(\partial_p)$ are the linearly dependent vectors of the respective boundary operation, therefor $\dim \ker(\partial_p)$ is the number of zeroed out rows.
- $\operatorname{im}(\partial_{p+1}$ are the linearly independent vectors of the higher dimensions respective boundary operation, therefor $\dim \operatorname{im}(\partial_{p+1})$ is the number of non-zero rows.

Therefore, proceeding with the Klein Bottle:

$$\beta_2(K) := N_2 - R_3 = 1 - 0 = 1$$

$$\beta_1(K) := N_1 - R_2 = 19 - 17 = 2$$

$$\beta_0(K) := N_0 - R_1 = 9 - 8 = 1$$

Persistent Homology and Applications in Image Processing

Homology will be extended into the realm of $\mathbb{R}^N$ through the concepts of filtration and persistence.

5.1 Association of Simplicial Complexes to Point Clouds

In this section, we explore how point clouds, which are sets of points in a metric space, can be associated with simplicial complexes. This association is fundamental in topological data analysis, as it allows us to study the shape of the data by examining the topological properties of the corresponding simplicial complex.

Conclusion

unfinished.

Appendix

unfinished.

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