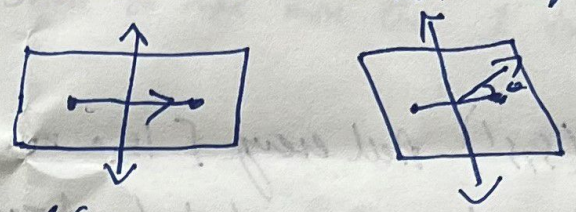
ARAP

·) estimating a linear transformation from surface points, ~~is not a good measure~~ ~~of~~ and then measuring its non orthogonality is not good measure of rigidity

↓

"measures the angle between two faces of the same cell value should be zero otherwise non-orthogonal"



1 face orthogonal

non orthogonal 'skew'

∴ direct optim of rigid transform

·) smooth means that local deformations are locally similar (fundamental forms preserved)

·) fundamental forms

$S \subset \mathbb{R}^3$ continuous surface

$x(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$, $(u,v) \in \mathbb{R}^2$ → x, y, z differentiable funct in u, v

partial derivatives x_u, x_v span tangent plane to S at x.

assuming $(x_u) \times (x_v) \neq 0$ $\quad n = \dfrac{(x_u \times x_v)}{\|x_u \times x_v\|}$ → normal

<u>first fundamental form of x</u>

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} x_u^T x_u & x_u^T x_v \\ x_u^T x_v & x_v^T x_v \end{bmatrix}$$

↓

inner product on tangent space of S

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$dA = \sqrt{EG - F^2}\,du\,dv$$

arc element → area element

1

1) Second fundamental form

$$\mathrm{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} x_{uu}^T n & x_{uv}^T n \\ x_{uv}^T n & x_{vv}^T n \end{bmatrix}$$

e.g. $x_{uu}^T n = -x_u^T n_u$

1) $E_s(S, S')$ stell energy [is a non linear function of surface positions]

1) results a min val but not zero for rigidity

·) optim problems

·) applications

1) k_s, k_b determine relative weight of tangential distortion (I against II's normal direction)

1) This paper formulates editing of discrete surfaces in 3D as a variational problem (end points of the curve which gives the extremum are fixed)

$\rightarrow \int (t, x, \dot{x}) dt \mid x(t_0) = x_0$, $x(t_1) = x_1$ initial, final points are given for curve $x(t)$

·) Discrete Surface Setup

S, triangular mesh, n, vertices, m, triangles

$N(i)$, set of vertices connected to vertex i, one ring

piecewise geometric embedding of S defined by vertex positions $p_i \in \mathbb{R}^3$

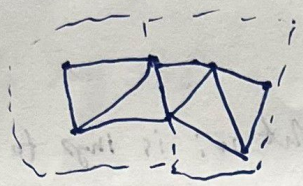
$S \mapsto S'$, same connectivity but different geometric embedding p_i'

$P = [p_1, \dots, p_v]$, $p_i := p(v_i) = \begin{pmatrix} x(v_i) \\ y(v_i) \\ z(v_i) \end{pmatrix} \in \mathbb{R}^3$

$v_i \in V$

1) Surface covered with small overlying cells

- cell overlap is necessary to avoid surface stretching or shearing
- All cells cover similar areas of surface or be properly weighted
- overlaps in a way that each part of surface is covered by same number of cells.



each cell to have more than 1 triangle

- Analyzing rigid transform between 2 cells

C_i corresponds to vertex i
C'_i] observe edges emanating from v_i in S and S'

rigid transform

if $C \to C'$ rigid then exists a rotation R_i such that

$$p_i' - p_j' = R_i(p_i - p_j) \quad , \forall j \in N(i)$$

even if not rigid we find the best approximation for R_i

$$E(C_i, C_i') = \sum_{j \in N(i)} w_{ij} \| (p_i' - p_j') - R_i (p_i - p_j) \|^2$$

weighted least squares

$e_{ij} := p_i - p_j$

$$\underset{R_i}{\operatorname{argmax}} \; Tr\left(R_i \sum_j w_{ij} e_{ij} e_{ij}'^T\right)$$

$$S_i = \sum_{j \in N(i)} w_{ij} e_{ij} e_{ij}'^T = P_i D_i P_i'^T$$

$\nearrow \begin{bmatrix} w_i & & \\ & w_{ij} & \\ & & \ddots \end{bmatrix}$

$\downarrow$

$3 \times \|N(i)\|$
e_{ij} as cols

$\therefore R_i$ maximizing $Tr(R_i S_i)$ when $R_i S_i$ symetric positive semi-definite

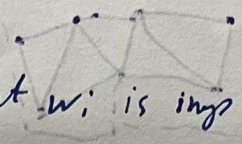
[if M is PSD then for any orthogonal R $Tr(M) \geq Tr(RM)$

$\therefore$ deriving R_i from SVD of $S_i = U_i \Sigma_i V_i^T$

$R_i = V_i U_i^T$

1) To measure local rigidity
eq (1) $E(S')$. in which only variables are the p_i'

1) $Tr(A)$ 'trace of A' sum of all diagonal entries
$\therefore tr(AB) = \sum_i^m \sum_j^n a_{ij} b_{ji}$



1) Choice of A per edge weight w_{ij} and cell weight w_i is imp to make deformation energy of mesh independent as possible

Modelling

1) Solve for p' of S' to minimise $E(S')$
for each step
1) optimal rigid transformations $\{R_i\}$ in p' [computed for each cell]
R_i that minimizes eq(3) by using eq(6)

2) To compute optimal vertex positions from given rotations, compute gradient of $E(S')$ to p_i' $\Rightarrow$
$Lp' = b$

1) Laplace Beltrami
Δ_S
1) Is called the divergence of the gradient $\Delta = \nabla^2 = \nabla \cdot \nabla$
sum of second partial derivatives in Euclidean space
$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$
f defined on a manifold surface (S)
$$\Delta_S f = \operatorname{div}_S \nabla_S f$$

The Algo:

1) geometric embedding is just assigning ~~ed~~ 3D val to vertex p_i

1) eq(3) minimizing using weighted least squares

1) for finding R_i using SVD

$\vec{v}R = u$ → this has solution and strong ⇒ $\vec{v}R = u\sigma$ → mag
↓ unit

in Big Matrix

$R \cdot V = U \cdot \Sigma$

unit Mat transpose = their inverse

$\therefore R = \underbrace{U\Sigma}_{\text{mag}} V^T$ → vec directions

SVD, $A = U\Sigma V^T$

$A^{-1} = V\Sigma^{-1}U^T$

In my understand Since Σ is "scaling" ∴ we keep it unitary and compute rotations multiplying by identity

- R_i computation Summary

$$\sum_i^n w_i \|(Rp_i + t) - q_i\|^2$$

translation and Rot that minimize

1) Compute weighted centroids of both point sets p_i, q_i

$\bar{p}, \bar{q}$

2) compute centered vec

$x_i := p_i - \bar{p}$, $y_i = q_i - \bar{q}$

3) 2x2 covariance Matrix

$S = XWY^T$ X, Y = 2xn matrix that have x_i, y_i as column, $W = \text{diag}(w_1, w_2 \ldots)$

4) SVD of $S = U\Sigma V^T$

$R = V\begin{bmatrix} 1 & \\ & \det(VU^T) \end{bmatrix} U^T$

5) optimal translation

$t = \bar{q} - R\bar{p}$

2.2 local rigidity

·) eq (7) is for our local rigidity

·) only variables are p' since R_i are ~~[illegible]~~ functions of p'

·) weights w_{ij} w_i chosen to make deformation independent to compensate for non-uniformly shaped cells.

3) Modelling framework

·) Solve for p' to minimize $E(S')$
∴ we don't know $\{R_i\}$

·) So for any rigid transformations which the user makes
1) find p' to min $E(S')$
2) find R_i that min $E(S')$ for the above p'
3) continue until local energy min reached

·) Since in eq (7) $R_i L_i$ is per cell
∴ we can compute optimal transform ~~wth~~ for each cell without regard for others
- thats why we minimise (3) using (6)

·) So to compute optimal vertex positions from given rotations we compute $E(S')$ gradient ~~not~~ wrt p'. this partial derivatives
∴ eq (8) comes in the form of laplace beltrami operator
$Lp' = b$ -> right hand side expression in (8)

·) Incorporating constraints like static and handle vertices
$p'_j = c_k$ $k \in f$ -> constrained vertices -> put into (9)

means erasing rows and columns from 'L' and updating right hand side with c_k

·) $\{R_i\}$ only influence right hand side of (8) whereas Lp' depends on initial mesh
∴ Lp' only factored once

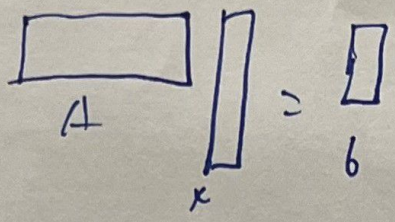
·) Since p' has 3 cols for 3 coordinate functions
∴ we perform 3 times back substitution to solve for each coordinate. We use sparse cholesky factorization with fill reducing reordering as solver

SVD

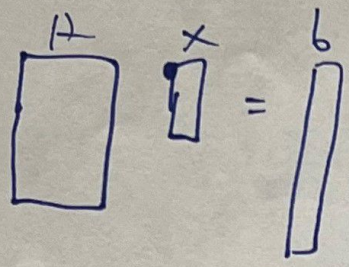
1) $A\tilde{x} = b$ (square)

$\downarrow$ known $\quad \downarrow$ known

SVD allows for generalization to nonsquare A



underdetermined $\qquad \min \|\tilde{x}\|_2, \; A\tilde{x} = b$



overdetermined $\qquad$ LLS $\quad \min \|A\tilde{x} - b\|_2$

$$Ax = b$$

$$U \Sigma V^T \tilde{x} = b$$

$$V \Sigma^{-1} U^T U \Sigma V^T \tilde{x} = \underbrace{V \Sigma^{-1} U^T}_{A^{-1}} b$$

$$\therefore \tilde{x} = A^{-1} b$$