Linear Programming Formulation and Solution

Example 1 A candy manufacturer has 130 pounds of chocolate-covered cherries and 170 pounds of chocolate-covered mints in stock. He decides to sell them in the form of two different mixtures. One mixture will contain half cherries and half mints by weight and will sell for \$2.00 per pound. The other mixture will contain one-third cherries and two-thirds mints by weight and will sell for \$1.25 per pound. How many pounds of each mixture should the candy manufacturer prepare in order to maximize his sales revenue?

Solution For simplicity, let us call A the mixture of half cherries and half mints, and B the mixture which is one-third cherries and two-thirds mints. Let *x* be the number of pounds of A to be prepared and *y* the number of pounds of B to be prepared. The revenue function can then be written as

$$z = 2x + 1.25y$$

Since each pound of A contains one-half pound of cherries and each pound of B contains one-third pound of cherries, the total number of pounds of cherries used in both mixtures is

$$\frac{1}{2}x + \frac{1}{3}y$$

Similarly, the total number of pounds of mints used in both mixtures is:

$$\frac{1}{2}x + \frac{2}{3}y$$

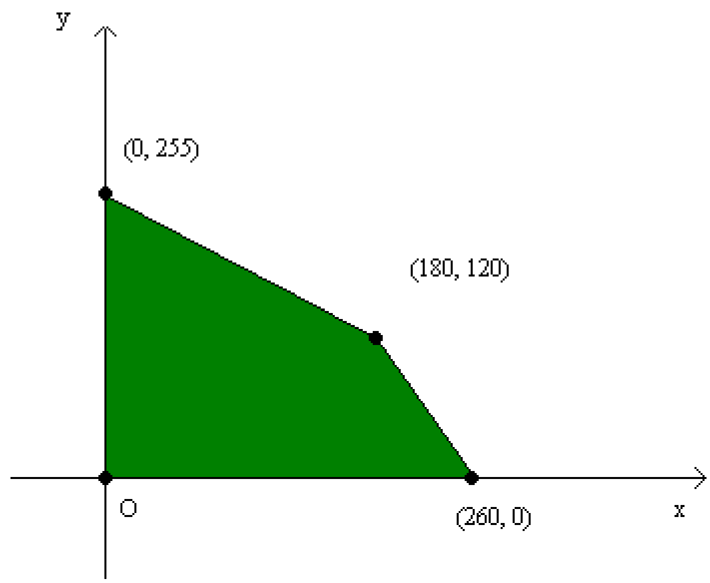
Now, since the manufacturer can use at most 130 pounds of cherries and 170 pounds of mints, we have the constraints:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &\leq 130 \\ \frac{1}{2}x + \frac{2}{3}y &\leq 170 \end{aligned}$$

Also, we must have $x \geq 0, \quad y \geq 0.$ Therefore, the above problem can be formulated as follows: find x and y that maximize $z = 2x + 1.25y$ subject to the constraints:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &\leq 130 \\ \frac{1}{2}x + \frac{2}{3}y &\leq 170 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

To solve the problem, we will use the technique of linear programming described above. We start by drawing the feasible region of the problem and locate our extreme points:



Since the feasible region is bounded in this case, we are sure that the optimal solution is attained at one of the extreme points shown on the diagram above. So we evaluate the objective function at each of these points:

Extreme point	Value of z=2x+1.25y
(0, 0)	0
(0, 255)	318.75
(180, 120)	510.00
(260, 0)	520.00

The table shows that the largest value for z is 520.00 and the corresponding optimal solution is (260, 0). Thus the candy manufacturer attains maximum sales of $520 when he produces 260 pounds of mixture A and none of mixture B.

Example 2 The Osgood County refuse department runs two recycling centers. Center 1 costs $40 to run for an eight hour day. In a typical day 140 pounds of glass and 60 pounds of aluminum are deposited at Center 1. Center 2 costs $50 for an eight-hour day, with 100 pounds of glass and 180 pounds of aluminum deposited per day. The county has a commitment to deliver at least 1540 pounds of glass and 1440 pounds of aluminum per week to encourage a recycler to open up a plant in town. How many days per week should the county open each center to minimize its cost and still meet the recycler's needs?

Solution: Let *x* be the number of days per week Center 1 is open and *y* be the number of days per week Center 2 is open. The linear programming problem associated with this question is as follows:

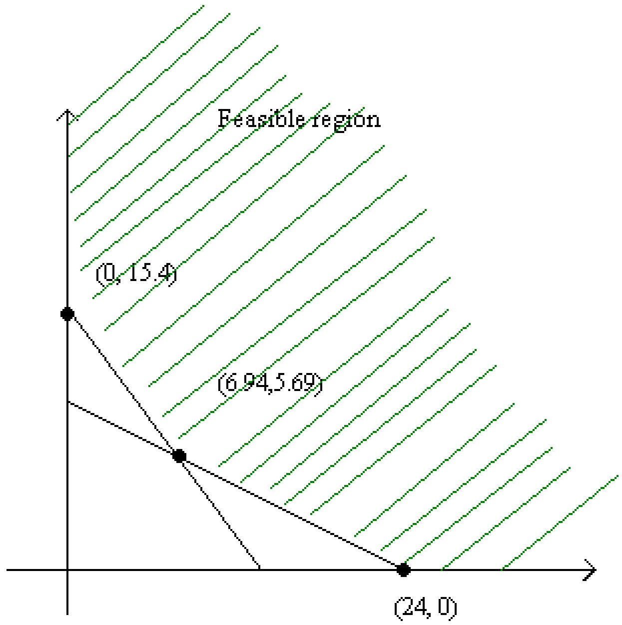
Minimize the function

$$z = 40x + 50y$$ (Cost of operating the two centers)

subject to the constraints:

$$\begin{array}{rcrcrl} 140x & + & 100y & \geq & 1540 & (glass) \\ 60x & + & 180y & \geq & 140 & Alm. \\ & & x & \geq & 0 & \\ & & y & \geq & 0 & \end{array}$$

The feasible region of the problem, together with the extreme points are shown in the following diagram



The region is unbounded, with three finite extreme points. Two are easy to obtain from the graph: (0, 15.4) and (24, 0). The third is found algebraically to be: (6.94, 5.69). Also, we can assume that there is other "infinite extreme points" since the region is unbounded. One can see that these infinite extreme points are not going to be "optimal" since we are minimizing and not maximizing. So we evaluate the objective function at each of the finite extreme points:

Extreme point	Value of z=40x+50y
(0, 15.4)	770
(24, 0)	960
(6.94, 5.69)	562.10

So, the minimal cost is $562.10 and it is attainted when Center 1 is open for about 7 days a week (eight-hours a day) and Center 2 is open for about 6 days a week (eight-hours a day).

<u>Example 3:</u> *Consider this scenario:* your school is planning to make toques and mitts to sell at the winter festival as a fundraiser. The school's sewing classes divide into two groups – one group can make toques, the other group knows how to make mitts. The sewing teachers are also willing to help out. Considering the number of people available and time constraints due to classes, only 150 toques and 120 pairs of mitts can be made each week. Enough material is delivered to the school every Monday morning to make a total of 200 items per week. Because the material is being donated by community members, each toque sold makes a profit of $2 and each pair of mitts sold makes a profit of $5.

In order to make the most money from the fundraiser, how many of each item should be made each week? It is important to understand that profit (the amount of money made from the fundraiser) is equal to the revenue (the total amount of money made) minus the costs: Proft = Revenue - Cost. Because the students are donating their time and the community is donating the material, the cost of making the toques and mitts is zero. So in this case, *profit* ≡ *revenue*.

If the quantity you want to optimize (here, profit) and the constraint conditions (more on them later) are linear, then the problem can be solved using a special organization called **linear programming**. Linear programming enables industries and companies to find optimal solutions to economic decisions. Generally, this means maximizing profits and minimizing costs. Linear programming is most commonly seen in operations research because it provides a "best" solution, while considering all the constraints of the situation. Constraints are limitations, and may suggest, for example, how much of a certain item can be made or in how much time.

Creating equations, or inequalities, and graphing them can help solve simple linear programming problems, like the one above. We can assign variables to represent the information in the above problem.

x = the number of toques made weekly
y = the number of pairs of mitts made weekly

Then, we can write linear inequalities based on the constraints from the problem.

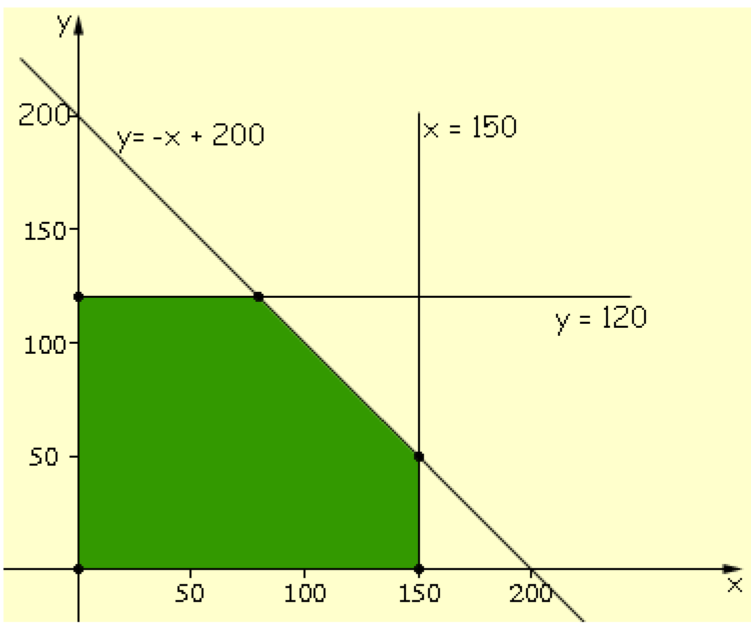
$x \leq 150$ and $y \leq 120$	The students can only make up to 150 toques and up to 120 pairs of mitts each week. This is one restriction.
$x + y \leq 200$	The total number of mitts and toques made each week cannot exceed 200. This is the material restriction.

We may also want to consider that $x \geq 0$ and $y \geq 0$. This means that we cannot make -3 toques.

Our final equation comes from the goal of the problem. We want to maximize the total profit from the toques and mitts. This can be represented by \$2x + \$5y = *P*, where *P* is the total profit, since there are no costs in production. If the school sells *x* toques, then they make \$2x from the sales of toques. If the school sells *y* mitts, then they make \$5y from the sales of mitts.

In some applications, the linear equations are very complex with numerous constraints and there are too many variables to work out manually, so they have special computers and software to perform the calculations efficiently. Sometimes, linear programming problems can be solved using matrices or by using an elimination or substitution method, which are common strategies for solving systems of linear equations.

Using the equations and inequations generated above, we can graph these, to find a *feasible region*. Our feasible region is the convex polygon that satisfies all of the constraints. In this situation, one of the vertices of this polygon will provide the optimal choice, so we must first consider all of the corner points of the polygon and find which pair of coordinates makes us the most money. From our toque and mitt example, we can produce the following graph:



We can see that our feasible region (the green area) has vertices of (0, 120), (150, 0), (150, 50), and (80, 120). By substituting these values for *x* and *y* in our revenue equation, we can find the optimal solution.

R = 2x + 5y
R = 2(80) + 5(120)
R = \$760

After considering all of the options, we can conclude that this is our maximum revenue. Therefore, the sewing students (and teachers) must make 80 toques and 120 pairs of mitts each week in order to make the most money. We can check that these solutions satisfy all of our restrictions:

$80 + 120 \leq 200$. This is true. We know that we will have enough material to make 80 toques and 120 pairs of mitts each week. We can also see that our values for x and y are less than 150 and 120, respectively. So, not only is our solution possible, but it is the best combination to optimize profits for the school. This is a fairly simple problem, but it is easy to see how this type of organization can be useful and very practical in the industrial world.