1 Problem Statement

We will consider the problem of lid-driven cavity in 2D for incompressible fluid with a velocity slip on upper boundary. The domain of the problem is $[0, l_x] \times [0, l_y]$.

To model this problem, we use the reduced Navier-Stokes equations in 2D along with the condition for the incompressible fluid, which is derived from the continuity equation for fluids' flow.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

Where $\vec{u}(t, x, y)$ is the velocity vector, t is time, $p(t, x, y)$ is the pressure, ρ is the fluid density, and ν is the viscosity of the fluid.

$$\text{Let} \quad \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

Where u and v are the velocities in the x-direction and the y-directions, respectively.

Therefore,

$$\vec{u} \cdot \nabla \vec{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{bmatrix}, \qquad \nabla^2 \vec{u} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{bmatrix}$$

Substituting these in Eq. 1, we get the following equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

Looking Eq. 5, we notice that there is coupling between the velocity and the pressure, which makes it difficult to solve the system of differential equations, so we must find a way to couple the three equations. To so, differentiate the Eq. 3 with respect to x and Eq. 4 with respect to y. Then we add the equations together and include Eq. 5 as constraint. The procedure is the following.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2}+\nu\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x\partial y^2}\right) \tag{6}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial y\partial x}+\left(\frac{\partial v}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}+\nu\left(\frac{\partial^3 v}{\partial y\partial x^2}+\frac{\partial^3 v}{\partial y^3}\right) \tag{7}$$

We add Eq. 6 and Eq. 1 and we rearrange

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}\right)+v\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)\\ =-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\nu\left(\frac{\partial^3 u}{\partial x^2}+\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial y\partial x^2}+\frac{\partial^3 v}{\partial y^3}\right)\end{aligned} \tag{8}$$

The last step is to include Eq. 5 into Eq. 8. We will also include its first and second derivatives with respect to x and y, which are the following

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}&=0\\ \frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}&=0\\ \frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2\partial y}&=0\\ \frac{\partial^3 u}{\partial y^2\partial x}+\frac{\partial^3 v}{\partial y^3}&=0\end{aligned}$$

By substituting these equations along with Eq. 5 into Eq. 8, we get following

$$\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)^2 = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \tag{9}$$

Therefore, our final equations are

$$u_t + u\,u_x + v\,u_y = -\frac{1}{\rho}\,p_x + \nu\,(u_{xx} + u_{yy}) \tag{10}$$

$$v_t + u\,v_x + v\,v_y = -\frac{1}{\rho}\,p_y + \nu\,(v_{xx} + v_{yy}) \tag{11}$$

$$u_x^2 + 2u_y\,v_x + v_y^2 = -\frac{1}{\rho}\,(p_{xx} + p_{yy}) \tag{12}$$

Note: We have used u_x instead of $\frac{\partial u}{\partial x}$ for simplicity and similarly for other the derivatives.

The boundary conditions for the velocity are the following.

$$u(t,x,l_y) = 1, \quad u(t,x,0) = 0, \quad u(t,0,y) = 0, \quad u(t,l_x,y) = 0$$

$$v(t,x,l_y) = 0, \quad v(t,x,0) = 0, \quad v(t,0,y) = 0, \quad v(t,l_x,y) = 0$$

consider $l_x = l_y = 1$

The boundary condition for the pressure

$$\left.\frac{\partial p}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial p}{\partial x}\right|_{x=l_x} = 0, \quad \left.\frac{\partial p}{\partial y}\right|_{y=0} = 0, \quad p(t,x,l_y) = 0$$

For the initial condition we assume that the velocity and the pressure are zero everywhere except at the boundary (we apply the boundary conditions).