



THE UNIVERSITY OF  
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Energy-based models

Constraints and generalised coordinates

Variational mechanics

# Analytical Mechanics

Newtonian mechanics uses momentum and force as fundamental quantities to describe system mechanics (statics and dynamics)

Leibnitz advocated for the use of two scalar fundamental quantities: Kinetic & Potential energy. Euler and Lagrange put this to a solid foundation, which was further elaborated by Hamilton and Jacobi.

The techniques that follow from this line of thought are known as **Analytical Mechanics** – a name that follows from Lagrange's seminal work *Mechanique Analytique* (1788).

The word analytical derived from the fact that all concepts are purely based on calculus or mathematical analysis.

The development of analytical mechanics was based on the highest philosophical speculation of great thinkers, which had been unparalleled in modern times—until Einstein's work.

# Configuration & Generalised Coordinates

The **configuration** of a mechanical system is specified by the location of all its particles:

$$x_i, y_i, z_i, \quad i = 1, 2, \dots, N$$

If the  $N$  particles move freely, we need  $3N$  coordinates to specify the configuration. In some cases, however, **when there are constraints on the positions and velocities, it may be possible to specify the configuration with less than  $3N$  coordinates.**

A set of  $n \leq 3N$  coordinates that fully specify the configuration of a system of  $N$  particles is called a set of **generalised coordinates** and denoted  $q_i$  with  $i = 1, 2, \dots, n$ . Then,

$$x_i = x_i(q_1, q_2, \dots, q_n, t)$$

$$y_i = y_i(q_1, q_2, \dots, q_n, t)$$

$$z_i = z_i(q_1, q_2, \dots, q_n, t)$$

# Degrees of Freedom

The number of generalised coordinates of a mechanical system is a unique characteristic of the system, and it is equal to the number of degrees of freedom (DOF) of the system.

The number of states required for a dynamic model of a mechanical system is, in general, twice the number of DOF.

The number of DOF can be reduced due to constraints:

$$g_k(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_N) = 0, \quad k = 1, 2, \dots, m,$$

Then we can, in general, choose a new set of  $n = N - m$ , generalised coordinates (independent), which describe the motion of the system in harmony with the constraints.

Example: Two particles rigidly attached (5DOF)

# Degrees of Freedom

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## Examples

- 1DOF: A piston, a single pendulum,
- 2DOF: A particle moving on a surface,
- 3DOF: A rigid body moving on a surface, a particle moving in space
- 4DOF: A robotic manipulator with 4 links with rotary joints,
- 5DOF: Two particles rigidly linked moving in space,
- 6DOF: A free rigid-body moving in space.

# Configuration Space

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Generalised coordinates are called **configuration variables**.

We can consider the vector of configuration variables (also known as **generalised positions**) and their derivatives (known as **generalised velocities**):

$$\mathbf{q} = [q_1, q_2, \dots, q_n]^\top \quad \dot{\mathbf{q}} = [\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n]^\top$$

This configuration vector belongs of a space called **configuration space**.

As the mechanical system moves in the 3-dimensional space, the vector  $\mathbf{q}$  traces a curve in the  $n$ -dimensional configuration space.

The use of generalised coordinates allows us to transfer the study of motion of a system to that of the motion of a single point in the configuration space:

*A useful link between mechanics and geometry.*

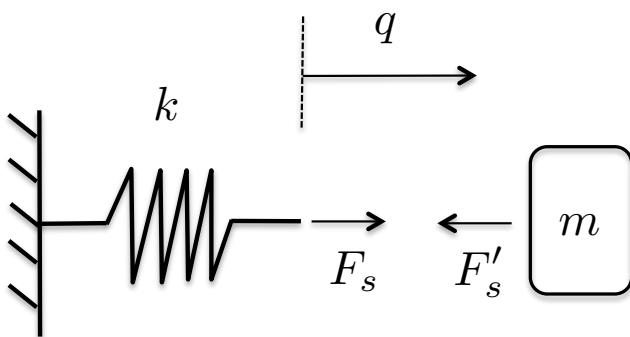
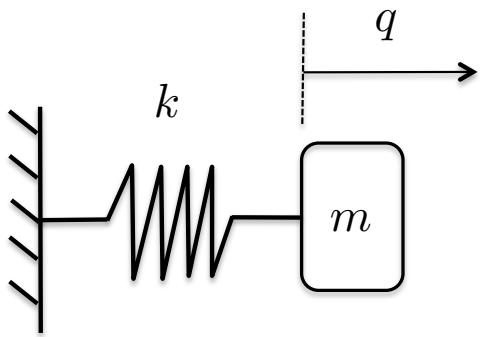


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# Energy-based Dynamic Models

Energy, co-energy, and the Legendre transform

# Potential energy



$$\text{Spring CCR: } e = F_s = k q, \quad f = \dot{q}$$

Note that I am considering the force experienced by the spring, not the mass.

Potential energy:

$$\begin{aligned} \mathcal{V}(q(t)) &= \mathcal{V}(q(0)) + \int_0^t e f \, dt, \\ &= \mathcal{V}(q_0) + \int_{q_0}^q kq \, dq, \quad \Rightarrow \\ &= \mathcal{V}(q_0) + \frac{1}{2}kq^2 - \frac{1}{2}kq_0^2, \end{aligned}$$

$$\mathcal{V}(q) = \frac{1}{2}k q^2$$

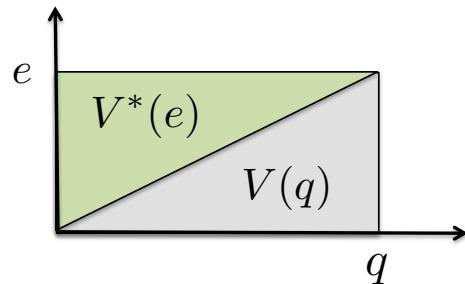
$$e = \frac{\partial \mathcal{V}}{\partial q} = k q$$

# Potential co-energy

The equation of the potential energy shows that the energy stored in the spring is the area under the curve of its constitutive relation  $e$  vs  $q$ :

$$e = k q$$

$$\mathcal{V}(q) = \frac{1}{2} k q^2$$

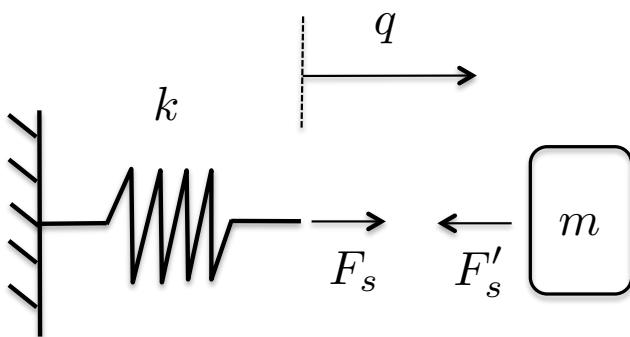
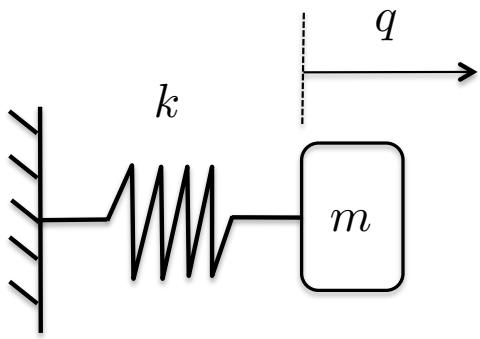


There is a complementary area which defines **the potential co-energy**:

$$\mathcal{V}^*(e) = eq - \mathcal{V}(q)$$

- The potential energy describes the capacity of the spring to do work,
- The potential co-energy can be handy for modelling,
- If the spring CCR is linear, both have the same numerical value.

# Kinetic energy



$$\text{Inertia CCR: } p = I f, \quad \dot{p} = e' = -e$$

Note that I am considering the force (effort) experienced by the spring as  $e$ , not that experienced by the mass.

Kinetic energy:

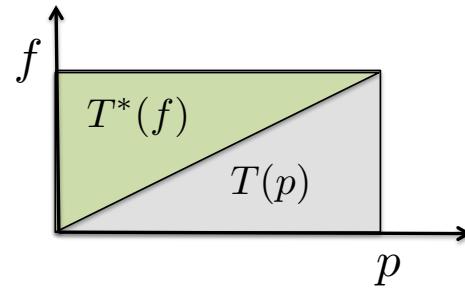
$$\begin{aligned}\mathcal{T}(p(t)) &= \mathcal{T}(p(0)) + \int_0^t f e' dt, \\ &= \mathcal{T}(p_0) + \int_{p_0}^p \frac{p}{I} dp\end{aligned}\Rightarrow$$

$$\mathcal{T}(p) = \frac{1}{2I} p^2$$

# Kinetic co-energy

The equation of the kinetic energy shows that the energy stored in the inertia is the area under the curve of its constitutive relation  $f$  vs  $p$ :

$$p = I f \quad \mathcal{T}(p) = \frac{1}{2I} p^2$$



There is a complementary area which defines **the kinetic co-energy**:

$$\mathcal{T}^*(f) = fp - \mathcal{T}(p)$$

- The kinetic energy describes the capacity of the inertia to do work,
- Unless we consider relativistic mechanics, the CCR of inertias are linear and therefore the numerical value of the kinetic and co-kinetic energy are the same.

# Legendre's Transformation

The energy and co-energy satisfy Legendre's Transformation:

$$\mathcal{V}^*(e) = eq - \mathcal{V}(q)$$

$$\mathcal{T}^*(f) = fp - \mathcal{T}(p)$$

## Legendre's Transformation

Scalar Function:  $\mathcal{F}(u_1, u_2, \dots, u_n)$

New variables:  $v_i \triangleq \frac{\partial \mathcal{F}}{\partial u_i}, \quad i = 1, 2, \dots, n$

New function:  $\mathcal{G}(v_1, v_2, \dots, v_n) = \sum_{i=1}^n u_i v_i - \mathcal{F}(u_1, u_2, \dots, u_n)$

Symmetry:  $\mathcal{F}(u_1, u_2, \dots, u_n) = \sum_{i=1}^n u_i v_i - \mathcal{G}(v_1, v_2, \dots, v_n)$

$$u_i \triangleq \frac{\partial \mathcal{G}}{\partial v_i}, \quad i = 1, 2, \dots, n$$

# Hamiltonian

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The **Lagrangian** is the difference between the co-kinetic energy and the potential energy

$$\mathcal{L}(\mathbf{f}, \mathbf{q}) = \mathcal{T}^*(\mathbf{f}, \mathbf{q}) - \mathcal{V}(\mathbf{q})$$

This is related to another function, the **Hamiltonian**, via the Legendre transform between momentum and flow variables:

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\top \mathbf{f} - \mathcal{L}(\mathbf{f}, \mathbf{q})$$

Substituting the Lagrangian yields

$$\begin{aligned}\mathcal{H}(\mathbf{p}, \mathbf{q}) &= \mathbf{p}^\top \mathbf{f} - [\mathcal{T}^*(\mathbf{f}, \mathbf{q}) - \mathcal{V}(\mathbf{q})] \\ &= \mathbf{p}^\top \mathbf{f} - \mathcal{T}^*(\mathbf{f}, \mathbf{q}) + \mathcal{V}(\mathbf{q}) \\ &= \mathcal{T}(\mathbf{p}, \mathbf{q}) + \mathcal{V}(\mathbf{q})\end{aligned}$$

*The Hamiltonian is the total energy!*

# Energy-based models

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Furthermore, by taking partial derivatives of

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T \mathbf{f} - \mathcal{L}(\mathbf{f}, \mathbf{q})$$

we discover that

$$\mathbf{f} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \quad \frac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{f}}$$

Extending this idea further, by relating the **co-Hamiltonian** to the Hamiltonian,

$$\mathcal{H}^*(\mathbf{f}, \mathbf{e}) = \mathbf{p}^T \mathbf{f} + \mathbf{e}^T \mathbf{q} - \mathcal{H}(\mathbf{p}, \mathbf{q})$$

additionally reveals that

$$\mathbf{e} = \frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

# Energy-based models

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Given the relations between the energy/power variables and gradients of the storage functions,

$$\mathbf{f} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \quad \mathbf{e} = \frac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{f}}$$

we can describe the dynamics of conservative lumped-parameter physical systems by using the following differential equations:

$$\begin{aligned}\dot{\mathbf{p}} &= -\mathbf{e}, \\ \dot{\mathbf{q}} &= \mathbf{f}\end{aligned}$$

# Euler-Lagrange equations revisited

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$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{f}}} \quad \mathbf{e} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

Taking the time derivative of momentum and substituting effort,

$$\dot{\mathbf{p}} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{f}}} \right) = -\mathbf{e} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

and introducing a generalised force yields the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{f}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}$$

# Hamilton's equations

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Using the effort and flow relations in terms of the Hamiltonian,

$$\mathbf{e} = \frac{\partial \mathcal{H}}{\partial \mathbf{q}} \quad \mathbf{f} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$$

leads to Hamilton's equations of motion:

$$\begin{aligned}\dot{\mathbf{p}} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}, \\ \dot{\mathbf{q}} &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}}\end{aligned}$$

A generalised force,  $\mathbf{Q}$ , may be added to the top equation to account for inputs and dissipative effects.

*Every SS model you have studied in MCHA2000 is in this form.*

# Hamiltonian modelling for mechanical systems

$$\begin{aligned}\dot{\mathbf{p}} &= -\frac{\partial \mathcal{H}(\mathbf{p}, \mathbf{q})}{\partial \mathbf{q}} + \mathbf{Q}, \\ \dot{\mathbf{q}} &= \frac{\partial \mathcal{H}(\mathbf{p}, \mathbf{q})}{\partial \mathbf{p}}\end{aligned}$$

Procedure for modelling:

1. Choose a set of generalised coordinates  $q_i$  and form the Lagrangian  $L(\dot{\mathbf{q}}, \mathbf{q})$ ,
2. Define the momenta  $p_i = \partial L / \partial \dot{q}_i$ .
3. Write the Hamiltonian Using the Legendre transform:  $H(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T \mathbf{q} - L(\dot{\mathbf{q}}, \mathbf{q})$ ,
4. Use Hamilton's equations (4.147) and (4.148) to obtain the equations of motion.

For mechanical systems, Hamiltonian and conjugate momentum take the general form:

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \mathbf{p}^T \mathbf{M}^{-1}(\mathbf{q}) \mathbf{p} + \mathcal{V}(\mathbf{q}) \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}$$

# Energy-based modelling summary

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Storage functions

$$\mathcal{L}(\mathbf{f}, \mathbf{q})$$

$$\mathcal{H}(\mathbf{p}, \mathbf{q})$$

$$\mathcal{H}^*(\mathbf{f}, \mathbf{e})$$

$$\mathcal{L}^*(\mathbf{p}, \mathbf{e})$$

Power and energy variables

$$\mathbf{e} = \frac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

$$\mathbf{f} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = -\frac{\partial \mathcal{L}^*}{\partial \mathbf{p}}$$

$$\mathbf{p} = \frac{\partial \mathcal{H}^*}{\partial \mathbf{f}} = \frac{\partial \mathcal{L}}{\partial \mathbf{f}}$$

$$\mathbf{q} = \frac{\partial \mathcal{H}^*}{\partial \mathbf{e}} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{e}}$$

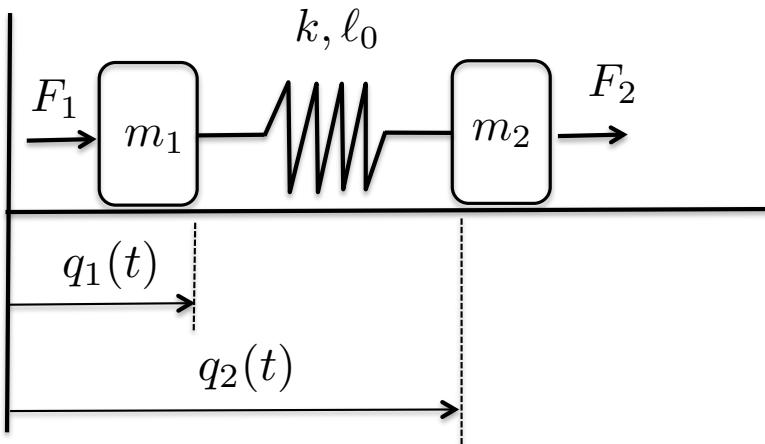
Dynamics

$$\dot{\mathbf{p}} = -\mathbf{e},$$

$$\dot{\mathbf{q}} = \mathbf{f}$$

# Example

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Lagrangian:

$$\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{1}{2}k(q_2 - q_1 - \ell_0)^2$$

$$\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} - \frac{1}{2}k(q_2 - q_1 - \ell_0)^2 \quad \mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

# Example

Lagrangian:

$$\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}k(q_2 - q_1 - \ell_0)^2$$

Conjugate momentum:  $\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{M}\dot{\mathbf{q}}$

$$\frac{\partial \mathbf{x}^\top \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^\top) \mathbf{x} = 2\mathbf{A} \mathbf{x}$$

Hamiltonian:  $\mathcal{H}(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\top \dot{\mathbf{q}} - \mathcal{L}(\dot{\mathbf{q}}, \mathbf{q})$

$$= \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p} - \mathcal{L}(\mathbf{M}^{-1} \mathbf{p}, \mathbf{q})$$

$$\begin{aligned} \mathcal{H}(\mathbf{p}, \mathbf{q}) &= \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p} - \frac{1}{2} \mathbf{p}^\top \mathbf{M}^{-\top} \mathbf{M} \mathbf{M}^{-1} \mathbf{p} + \frac{1}{2} k(q_2 - q_1 - \ell_0)^2 \\ &= \frac{1}{2} \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p} + \frac{1}{2} k(q_2 - q_1 - \ell_0)^2 \end{aligned}$$

Hamilton's Equations:

$$\dot{p}_1 = -k(q_2 - q_1 - \ell_0) + F_1,$$

$$\dot{p}_2 = k(q_2 - q_1 - \ell_0) + F_2,$$

$$\dot{q}_1 = m_1^{-1} p_1,$$

$$\dot{q}_2 = m_2^{-1} p_2$$



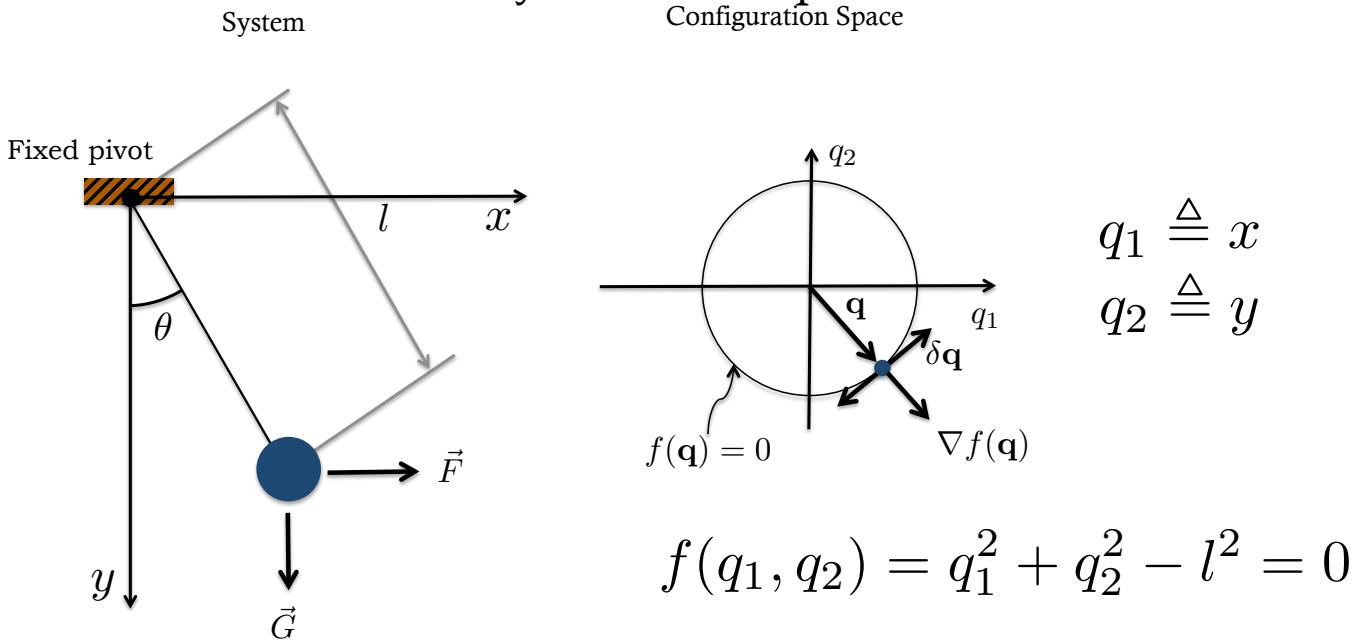
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# Constraints & Curvilinear Coordinates

# Curvilinear Spaces - Manifolds

When a system has constraints. The configuration variables live in a **manifold**—a curved space that resembles locally the Euclidean space (flat space).

For example, a circle is a manifold in  $\mathbb{R}^2$ , which locally resembles a line, and a sphere is a manifold in  $\mathbb{R}^3$ , which locally resembles a plane.



# Constraints

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**Kinematic constraints** are relations among coordinates of the type

$$f_k(q_1, q_2, \dots, q_n, t) = 0, \quad k = 1, 2, \dots, m, \quad (1)$$

$$g_k(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) = 0, \quad k = 1, 2, \dots, m \quad (2)$$

- Constraints of the form (1) and constraints of the form (2) that can be integrated into the form (1) are called **holonomic**. A system that is subject to holonomic constraints is called a **holonomic system**.
- Constraints of the form (2) that cannot be integrated into the form (1) are called **nonholonomic**. A system that is subject to nonholonomic constraints is called a **nonholonomic system**.

# Pfaffian Constraints

Nonholonomic constraints are often affine in the generalised velocities:

$$g_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{i=1}^n G_{ki}(\mathbf{q}, t)\dot{q}_i + G_{kt}(\mathbf{q}, t) = 0, \quad k = 1, 2, \dots, m$$

which can also be written in differential form

$$\sum_{i=1}^n G_{ki}(\mathbf{q}, t) dq_i + G_{kt}(\mathbf{q}, t) dt = 0, \quad k = 1, 2, \dots, m$$

Holonomic constraints can be differentiated to

$$\sum_{i=1}^n \frac{\partial f_k(\mathbf{q}, t)}{\partial q_i} \dot{q}_i + \frac{\partial f_k(\mathbf{q}, t)}{\partial t} = 0, \quad k = 1, 2, \dots, m$$

Differential constraints of the form above are known as **Pfaffian Constraints**

# Conditions for Integrability

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The necessary and sufficient conditions for integrability of Pfaffian constraints of the form

$$g_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{i=1}^n G_{ki}(\mathbf{q}, t)\dot{q}_i + G_{kt}(\mathbf{q}, t) = 0, \quad k = 1, 2, \dots, m$$

or

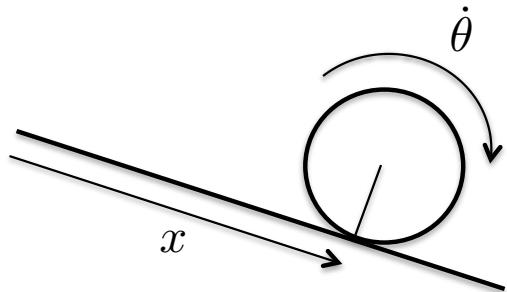
$$\sum_{i=1}^n G_{ki}(\mathbf{q}, t) dq_i + G_{kt}(\mathbf{q}, t) dt = 0, \quad k = 1, 2, \dots, m$$

are given by

$$\frac{d}{dt} \left( \frac{\partial g_k}{\partial \dot{q}_i} \right) - \frac{\partial g_k}{\partial q_i} \equiv 0, \quad i = 1, 2, \dots, n$$

# Examples

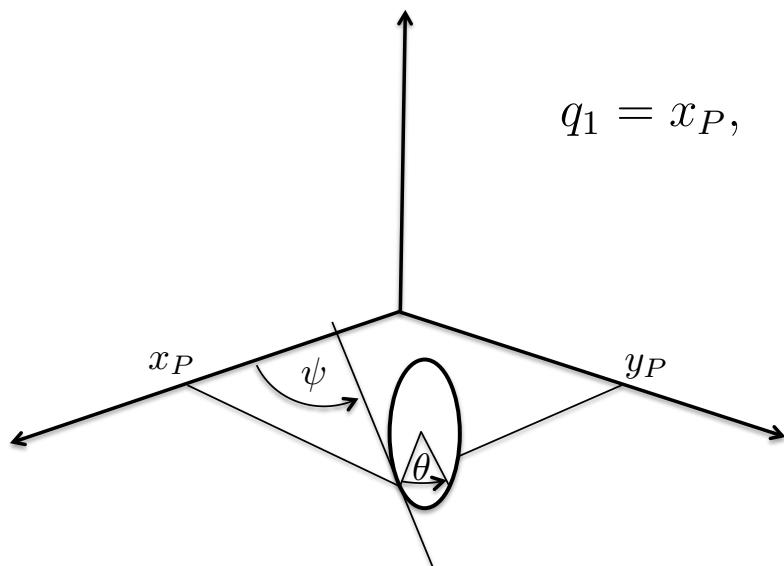
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$$\begin{aligned}\tilde{q}_1 &= x \\ \tilde{q}_2 &= \theta\end{aligned}$$

$$\dot{\tilde{q}}_1 = r \dot{\tilde{q}}_2.$$

$$q_1 = x_P, \quad q_2 = y_P, \quad q_3 = \psi, \quad q_4 = \theta.$$



$$\dot{q}_1 - (r \dot{q}_4 \cos q_3) = 0,$$

$$\dot{q}_2 - (r \dot{q}_4 \sin q_3) = 0,$$

# Variational mechanics

Jean-Luc Picard: *I wish I could spare a few centuries to learn.*

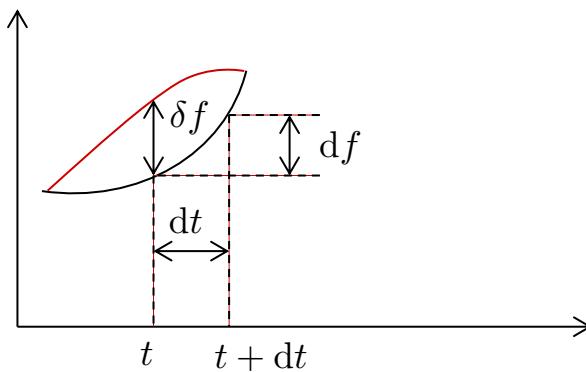
Anij: *It took us centuries to learn that it doesn't have to take centuries to learn.*

(Star Trek Insurrection, 1998)

# Virtual Displacements

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- Differential  $dq(t) = \dot{q}(t) dt$
- A **virtual displacement** or **variation**,  $\delta q(t)$ , is a change of  $q(t)$  in harmony with the constraints occurring at  $t$  and not due to  $dt$ . This displacement is something that we impose as a mathematical experiment and hence it is called virtual whereas  $dq(t)$  is an actual displacement.



# Virtual Displacements

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- Differential  $\mathrm{d}q(t) = \dot{q}(t) \mathrm{d}t$
- A **virtual displacement** or **variation**,  $\delta q(t)$ , is a change of  $q(t)$  in harmony with the constraints occurring at  $t$  and not due to  $\mathrm{d}t$ . This displacement is something that we impose as a mathematical experiment and hence it is called virtual whereas  $\mathrm{d}q(t)$  is an actual displacement.
- Consider a function  $r(\mathbf{q}, t)$ ,

$$\delta r = \frac{\partial r}{\partial q_1} \delta q_1 + \cdots + \frac{\partial r}{\partial q_n} \delta q_n.$$

once we choose  $\delta q_i(t)$ ,  $\delta r(t)$  is prescribed.

# Virtual Work and Generalised Forces

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The virtual work of a set of forces  $\vec{F}_j$  acting on a system at the locations  $\vec{r}_j$  is

$$\delta\mathcal{W} \triangleq \sum_j \vec{F}_j \cdot \delta\vec{r}_j$$

In terms of generalised coordinates:

$$\delta\mathcal{W} = \sum_j \vec{F}_j \cdot \left( \frac{\partial \vec{r}_j}{\partial q_1} \delta q_1 + \cdots + \frac{\partial \vec{r}_j}{\partial q_n} \delta q_n \right).$$

Generalised forces:

$$Q_j \triangleq \sum_{i=1}^n \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} \quad \delta\mathcal{W} = \sum_{i=1}^n Q_i \delta q_i$$

The units of each generalised forces are **not homogeneous**; they depend on the units of the associated generalised coordinates such that their product has units of energy.

# Principle of Virtual Work

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**Principle 1 (Principle Virtual Work - PVW)** *A mechanical system is in equilibrium if, and only if, the virtual work of the set of forces acting on the system is zero, namely,*

$$\delta\mathcal{W} = \sum_j \vec{F}_j \cdot \delta\vec{r}_j = \sum_{i=1}^n Q_i \delta q_i = \mathbf{Q}^\top \delta \mathbf{q} = 0$$

If there are no constraints, this implies the vanishing of  $\mathbf{Q}$

If there are constraints:  $\delta f_k = \sum_i \frac{\partial f_k}{\partial q_i} \delta q_i = 0$  (Holonomic),

$$\delta g_k = \sum_i G_{ki}(\mathbf{q}, t) \delta q_i = 0, \quad (\text{Nonholonomic-Pfaffian}).$$

the virtual displacements must be in harmony with the constraints.

# Corollary of PVW

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- If a system is in equilibrium the external forces must be equal and opposed to the forces of constraints

$$\vec{F}_j = -\vec{F}_j^c$$

then the PVW reduces to

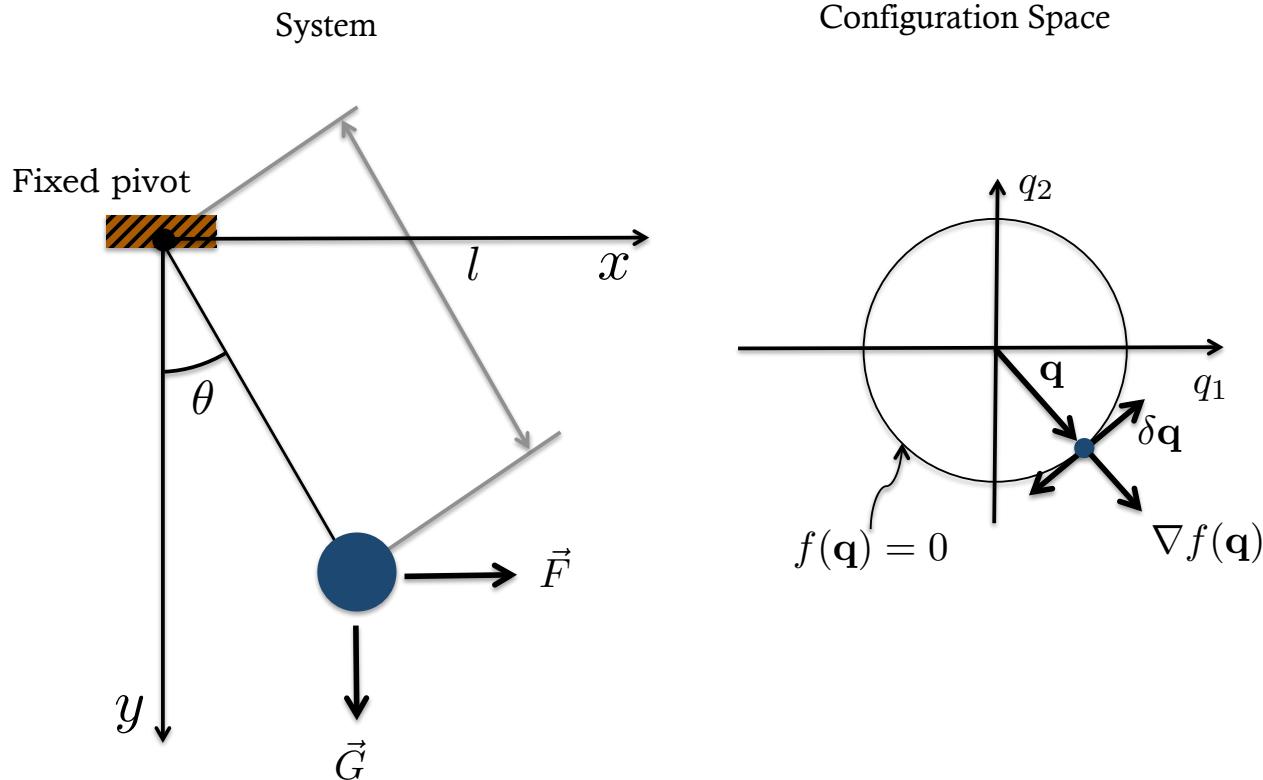
$$\sum_j \vec{F}_j^c \cdot \delta \vec{r}_j = 0$$

**Corollary 1 (Virtual work of constraint forces)** *The virtual work of the forces of constraint is always zero for virtual displacements in harmony with constraints; therefore, the virtual displacements are always perpendicular to the forces of constraints.*

This result extends beyond the realm of statics.

# Example – Pendulum in Equilibrium

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# d'Alembert's Principle

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d'Alembert (1717–1785) re-arranged Newtons 2<sup>nd</sup> law and defined the force of inertia:

$$\vec{F} - m\vec{a} = \vec{0} \quad \vec{F}^I \triangleq -m\vec{a}$$

This established a condition of equilibrium to which he applied the PVW:

$$\delta\mathcal{W} = \sum_{j=1}^N (\vec{F}_j + \vec{F}_j^c + \vec{F}_j^I) \cdot \delta\vec{r}_i = 0$$

But the constraint forces do no virtual work.

# d'Alembert's Principle

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**Principle 2 (d'Alembert's Principle)** *The motion of a mechanical system made of  $N$  particles is such that the virtual work of the sum of the acting forces  $\vec{F}_j$  and the force of inertia  $\vec{F}_j^I = -m \ddot{\vec{r}}_j$  is zero:*

$$\delta\mathcal{W} = \sum_{j=1}^N (\vec{F}_j - m\ddot{\vec{r}}_j) \cdot \delta\vec{r}_j = 0$$

*where the accelerations  $\ddot{\vec{r}}_j$  are relative to an inertial frame.*

- If the particles are free of constraints, this reduces to Newton's 2<sup>nd</sup> law
- If the particles are constrained, d'Alembert's Principle provides a way to deal with such problems. In fact, the equations of motion of many mechanical systems can be derived directly from d'Alembert Principle.

# Lagrange Equations for Holonomic Systems

For a system of particles d'Alembert's Principle gives

$$\sum_{k=1}^{3N} (F_k - m_k \ddot{r}_k) \delta r_k = 0$$

If the system is holonomic, we can consider a set of independent generalised coordinates. This leads to **Lagrange's Principle**:

$$\sum_{i=1}^n \left[ \frac{d}{dt} \left( \frac{\partial \mathcal{T}^*}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{T}^*}{\partial q_i} - Q_i \right] \delta q_i = 0$$

Since the virtual displacements are in harmony with the constraints and independent, the terms in the bracket must vanish. This leads to **the holonomic form of the Lagrange Equations**:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}^*}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{T}^*}{\partial q_i} = Q_i, \quad i = 1, 2, \dots, n$$

# Lagrange Equations for Holonomic Systems

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Separate  $Q_i$  into conservative and nonconservative components:

$$Q_i = -\frac{\partial \mathcal{V}}{\partial q_i} + Q'_i$$

and define the Lagrangian:  $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \triangleq \mathcal{T}^*(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{V}(\mathbf{q}, t)$

Then we can obtain the **Lagrange equations for holonomic systems**:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q'_i, \quad i = 1, 2, \dots, n$$

# Lagrange Equations for Nonholonomic Systems

For a nonholonomic system, the generalised coordinates are not independent, and we need to consider constraints.

The generalised forces of constraints can be expressed as  $Q_i^c = \sum_{k=1}^{3N} F_k^c \frac{\partial r_k}{\partial q_i}$ ,

The PVW establishes that  $\sum_{i=1}^n Q_i^c \delta q_i = 0$

provided that the virtual displacements satisfy the nonholonomic constraints:

$$\sum_{i=1}^n G_{ji} \delta q_i = 0, \quad j = 1, 2, \dots, m$$

We can multiply by the Lagrange multiplier and sum over  $j$ :

$$\sum_{j=1}^m \sum_{i=1}^n \lambda_j G_{ji} \delta q_i = 0.$$

# Lagrange Equations for Nonholonomic Systems

Subtraction leads to

$$\sum_{i=1}^n Q_i^c \delta q_i - \sum_{j=1}^m \sum_{i=1}^n \lambda_j G_{ji} \delta q_i = 0 \quad \Rightarrow \quad \sum_{i=1}^n \left( Q_i^c - \sum_{j=1}^m \lambda_j G_{ji} \right) \delta q_i = 0$$

We can choose the Lagrange multipliers such that

$$\lambda_j : \quad Q_i^c = \sum_{j=1}^m \lambda_j G_{ji}$$

We can now add the forces of constraint to the Lagrange equations, which leads to the **nonholonomic form of Lagrange's Equations**:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}^*}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{T}^*}{\partial q_i} = Q_i + \sum_{j=1}^m \lambda_j G_{ji}, \quad i = 1, 2, \dots, n$$

# Lagrange Equations for Nonholonomic Systems

Separate  $Q_i$  into conservative and nonconservative components:

$$Q_i = -\frac{\partial \mathcal{V}}{\partial q_i} + Q'_i$$

and define the Lagrangian:  $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \triangleq \mathcal{T}^*(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{V}(\mathbf{q}, t)$

Then we can obtain the **Lagrange equations for nonholonomic systems**:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q'_i + \sum_{j=1}^m \lambda_j G_{ji}, \quad i = 1, 2, \dots, n$$

Note that these equations are also applicable to holonomic systems when we choose to work with a surplus of configuration variables (dependent coordinates). In this case,

$$G_{ji} = \frac{\partial f_j(\mathbf{q}, t)}{\partial q_i}$$

# Example—dependent coordinates

$$q_1 \triangleq x, \quad q_2 \triangleq y.$$

$$f(q_1, q_2) = \sqrt{q_1^2 + q_2^2} - \ell = 0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + mgq_2$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = \lambda \frac{\partial f}{\partial q_i}, \quad i = 1, 2$$

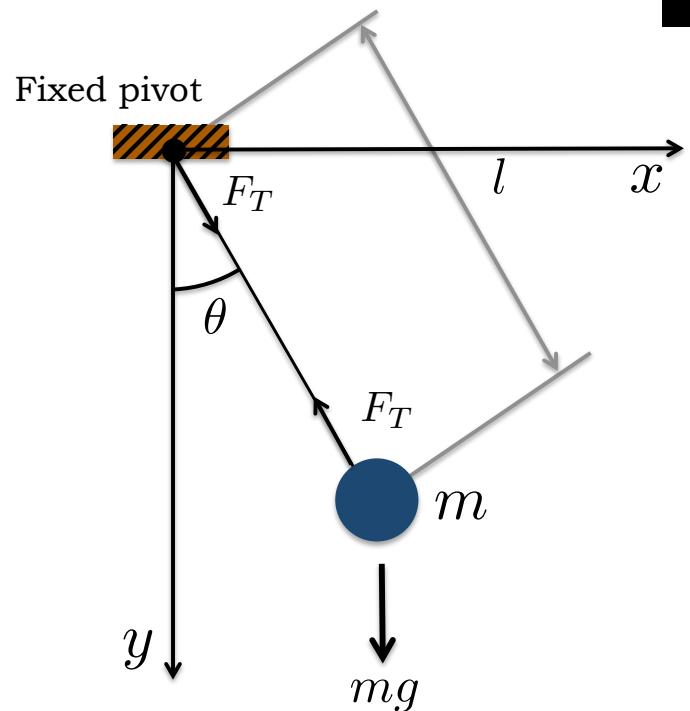
$$m\ddot{q}_1 - \lambda \frac{q_1}{\ell} = 0$$

$$m\ddot{q}_2 - mg - \lambda \frac{q_2}{\ell} = 0$$

2<sup>nd</sup> time derivative of the constraint:  $\dot{q}_1^2 + q_1\ddot{q}_1 + \dot{q}_2^2 + q_2\ddot{q}_2 = 0$

Substitution leads to

$$\lambda = -\frac{m}{\ell}(\dot{q}_1^2 + \dot{q}_2^2 + gq_2)$$



# Example—Independent coordinates

$$q \triangleq \theta$$

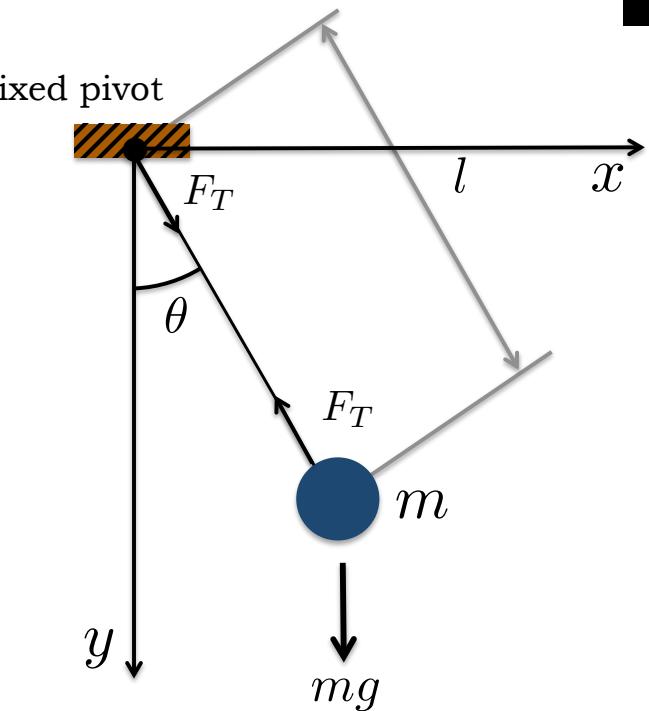
$$\mathcal{T}^* = \frac{1}{2}m\vec{v} \cdot \vec{v} = \frac{1}{2}m\ell^2\dot{\theta}^2, \quad \mathcal{V} = mgl(1 - \cos q)$$

$$\mathcal{L} = \frac{1}{2}m\ell^2\dot{\theta}^2 - mgl(1 - \cos q)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Partial differentiation leads to

$$\ddot{q} = -\frac{g}{\ell} \sin q$$



# Principle of Least Action



Euler and Lagrange's principle of least action:

- Assume motion along a path  $\mathbf{q}(t)$  is such that total energy remains constant (scleronomic systems).
- Fixing energy determines velocity at each point along the path, and hence the traversal time.
- The path taken minimises the time-integral of kinetic co-energy:

$$\frac{\delta}{\delta \mathbf{q}} \int \mathcal{T}^*(\mathbf{q}, \dot{\mathbf{q}}) dt = \mathbf{0}$$

Hamilton's principle of least action:

- Energy may be explicitly time-varying (rheonomic systems)
- Specify path traversal time
- The path taken minimises the time-integral of the Lagrangian

$$\frac{\delta}{\delta \mathbf{q}} \int \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt = \mathbf{0}$$

# Hamilton's principle

The action functional is defined by

$$\mathcal{S}[\mathbf{q}] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt$$

Let us consider variations in harmony with the constraints  $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = \mathbf{0}$

$$\delta\mathcal{S}[\mathbf{q}] = \mathcal{S}[\mathbf{q} + \delta\mathbf{q}] - \mathcal{S}[\mathbf{q}]$$

$$= \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t) + \delta\mathbf{q}(t), \dot{\mathbf{q}}(t) + \delta\dot{\mathbf{q}}(t), t) dt - \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt$$

Expanding Lagrangian variation in first-order terms (i.e., chain rule)

$$\begin{aligned}\delta\mathcal{S}[\mathbf{q}] &= \int_{t_1}^{t_2} \left( \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) + \frac{\partial\mathcal{L}}{\partial\mathbf{q}}\delta\mathbf{q}(t) + \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\delta\dot{\mathbf{q}}(t) \right) dt - \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt \\ &= \int_{t_1}^{t_2} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{q}}\delta\mathbf{q}(t) + \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\delta\dot{\mathbf{q}}(t) \right) dt & \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\delta\dot{\mathbf{q}} \right) dt = \left[ \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\delta\dot{\mathbf{q}} \right]_{t=t_1}^{t=t_2} \\ &= \int_{t_1}^{t_2} \left( \frac{\partial\mathcal{L}}{\partial\mathbf{q}} - \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} \right) \delta\mathbf{q}(t) dt & = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}(\delta\mathbf{q}(t_2) - \delta\mathbf{q}(t_1)) = 0 \\ && \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} \delta\mathbf{q} + \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} \delta\dot{\mathbf{q}} dt = 0\end{aligned}$$

# Hamilton's principle

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Set the variational derivative to zero

$$\begin{aligned}\frac{\delta \mathcal{S}}{\delta \mathbf{q}(t)} &= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt'} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \frac{\delta \mathbf{q}(t')}{\delta \mathbf{q}(t)} dt' \\ &= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt'} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \delta(t' - t) dt' \\ &= \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}\end{aligned}$$

This leads to the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{0}$$

# Conservation of momentum

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Recall

$$\dot{\mathbf{p}} = -\mathbf{e} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

If the Lagrangian is independent of a coordinate  $q_i$ , then its conjugate momentum is conserved

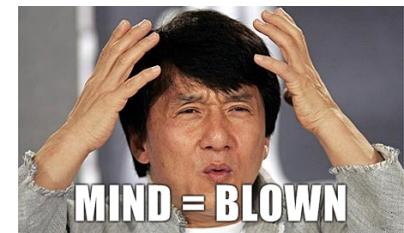
$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad p_i(t) = p_i(0), \quad \forall t$$

We call these  $q_i$  **kinosthenic** or **ignorable** coordinates.

Translational symmetry  $\rightarrow$  Conservation of momentum

Rotational symmetry  $\rightarrow$  Conservation of angular momentum

Time symmetry  $\rightarrow$  Conservation of energy



Noether's theorem: *Every differentiable symmetry of the action of a physical system has a corresponding conservation law.*

YOU'RE NOT THINKING

# Conservation of energy

Conservative action:  $\mathcal{S}[\mathbf{q}] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) dt$

Let *time* be an extra coordinate, dependent on  $\tau$

$$t = t(\tau) \quad t'(\tau) = \frac{dt}{d\tau} \quad \mathbf{q}'(\tau) = \frac{d\mathbf{q}}{d\tau}$$

Integral change of variable:

$$\begin{aligned} \mathcal{S}[\mathbf{q}] &= \int_{\tau_1}^{\tau_2} \mathcal{L}(\mathbf{q}(\tau), \dot{\mathbf{q}}(\tau)) \frac{dt}{d\tau} d\tau \\ &= \int_{\tau_1}^{\tau_2} \mathcal{L}\left(\mathbf{q}(\tau), \frac{d\mathbf{q}}{d\tau} \frac{d\tau}{dt}\right) \frac{dt}{d\tau} d\tau \\ &= \int_{\tau_1}^{\tau_2} \underbrace{\mathcal{L}\left(\mathbf{q}(\tau), \frac{\mathbf{q}'(\tau)}{t'(\tau)}\right)}_{\mathcal{L}_\tau(\mathbf{q}(\tau), \mathbf{q}'(\tau), \tau)} t'(\tau) d\tau \end{aligned}$$

*If time is an ignorable coordinate, energy is conserved.*



$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}_\tau}{\partial q'_i}$$

The conjugate momentum of time is

$$\begin{aligned} p_t &= \frac{\partial(\mathcal{L}t')}{\partial t'} = \frac{\partial \mathcal{L}}{\partial t'} t' + \mathcal{L} \\ &= \frac{\partial^\top \mathcal{L}}{\partial \dot{\mathbf{q}}} \frac{d}{dt'} \left( \frac{\mathbf{q}'}{t'} \right) t' + \mathcal{L} \\ &= \mathbf{p}^\top \left( -\frac{\mathbf{q}'}{t'^2} \right) t' + \mathcal{L} = -\mathbf{p}^\top \dot{\mathbf{q}} + \mathcal{L} \end{aligned}$$

$$p_t = -\mathcal{H}(\mathbf{p}, \mathbf{q})$$

the negative Hamiltonian!