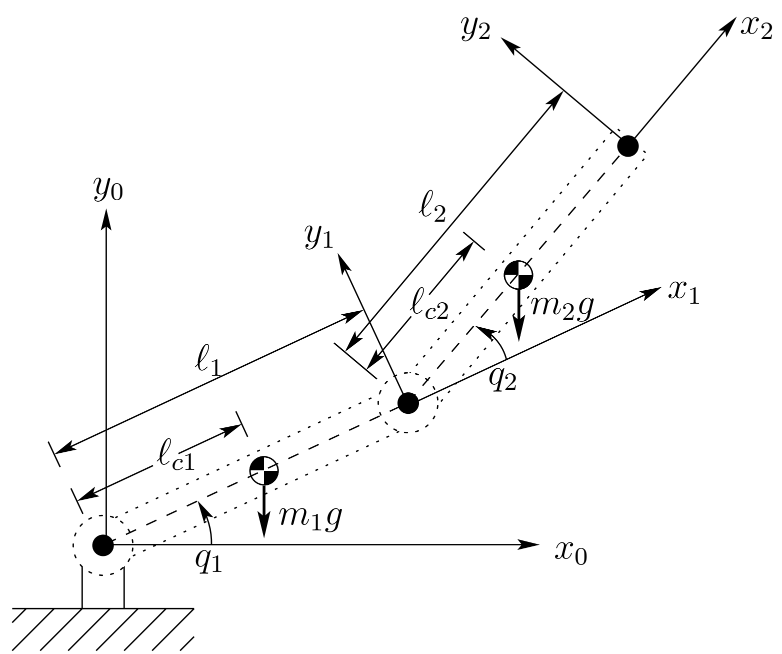
MCHA3900 – Dynamics of Robotic Manipulators



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Review of Lagrangian Models

Kinetic & Potential Energy of RB

Lagrangian Models of Robotic Manip

Review Lagrange Equations

The **Lagrange equations for holonomic systems**:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \triangleq \mathcal{T}^*(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{V}(\mathbf{q}, t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i', \qquad i = 1, 2, \ldots, n$$

Kinetic co-energy

- System of particles

$$\mathcal{T}^* = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{v}_i$$

- Single rigid body

$$\mathcal{T}^*_{RB} = \frac{1}{2}\int_{\mathcal{B}} \vec{v}_P \cdot \vec{v}_P \, \mathrm{d}m$$

$\vec{v}_P$ is the velocity of the differential mass element dm, and the integral is over the volume of the body.

Kinetic co-energy

$$\mathcal{T}^*_{RB} = \frac{1}{2}\int_{\mathcal{B}} \vec{v}_P \cdot \vec{v}_P \, \mathrm{d}m$$

Transport theorem: $\vec{v}_P = \vec{v}_C + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{P/C}$

$$\begin{aligned}
\mathcal{T}^*_{RB} &= \frac{1}{2}\int_{\mathcal{B}} (\vec{v}_C + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{P/C}) \cdot (\vec{v}_C + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{P/C}) \, \mathrm{d}m, \\
&= \frac{1}{2} m \, \vec{v}_C \cdot \vec{v}_C + \frac{1}{2}\int_{\mathcal{B}} (\vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{P/C}) \cdot (\vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{P/C}) \, \mathrm{d}m, \\
&= \frac{1}{2} m \, \vec{v}_C \cdot \vec{v}_C - \frac{1}{2} \vec{\omega}_{\mathcal{B}/\mathcal{N}} \cdot \int_{\mathcal{B}} \vec{r}_{P/C} \times \vec{r}_{P/C} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \, \mathrm{d}m
\end{aligned}$$

Therefore,

$$\mathcal{T}^*_{RB} = \frac{1}{2} m \, \vec{v}_C \cdot \vec{v}_C + \frac{1}{2} \vec{\omega}_{\mathcal{B}/\mathcal{N}} \cdot \vec{I}_C \cdot \vec{\omega}_{\mathcal{B}/\mathcal{N}}$$

Kinetic co-energy

$$\mathcal{T}^*_{RB} = \frac{1}{2} m\, \vec{v}_C \cdot \vec{v}_C + \frac{1}{2}\, \vec{\omega}_{\mathcal{B}/\mathcal{N}} \cdot \vec{I}_C \cdot \vec{\omega}_{\mathcal{B}/\mathcal{N}}$$

Recall that $\mathbf{L}^b_C = \mathbf{I}^b_C\, \boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} \quad \Leftrightarrow \quad \mathbf{L}^n_C = \mathbf{R}^n_b\, \mathbf{I}^b_C\, (\mathbf{R}^n_b)^\mathsf{T}\, \boldsymbol{\omega}^n_{\mathcal{B}/\mathcal{N}}$

In inertial coordinates:

$$\mathcal{T}^*_{RB} = \frac{1}{2} m\, (\mathbf{v}^n_C)^\mathsf{T}\, \mathbf{v}^n_C + \frac{1}{2}\, (\boldsymbol{\omega}^n_{\mathcal{B}/\mathcal{N}})^\mathsf{T}\, [\mathbf{R}(\mathbf{q}) \mathbf{I}^b_C \mathbf{R}^\mathsf{T}(\mathbf{q})] \boldsymbol{\omega}^n_{\mathcal{B}/\mathcal{N}}$$

$$\mathcal{T}^*_{RB} = \frac{1}{2} \begin{bmatrix} \mathbf{v}^n_C \\ \boldsymbol{\omega}^n_{\mathcal{B}/\mathcal{N}} \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(\mathbf{q})\, \mathbf{I}^b_C\, \mathbf{R}^\mathsf{T}(\mathbf{q}) \end{bmatrix}}_{\mathbf{M}_{RB}(\mathbf{q}) \text{ Mass matrix}} \begin{bmatrix} \mathbf{v}^n_C \\ \boldsymbol{\omega}^n_{\mathcal{B}/\mathcal{N}} \end{bmatrix}$$

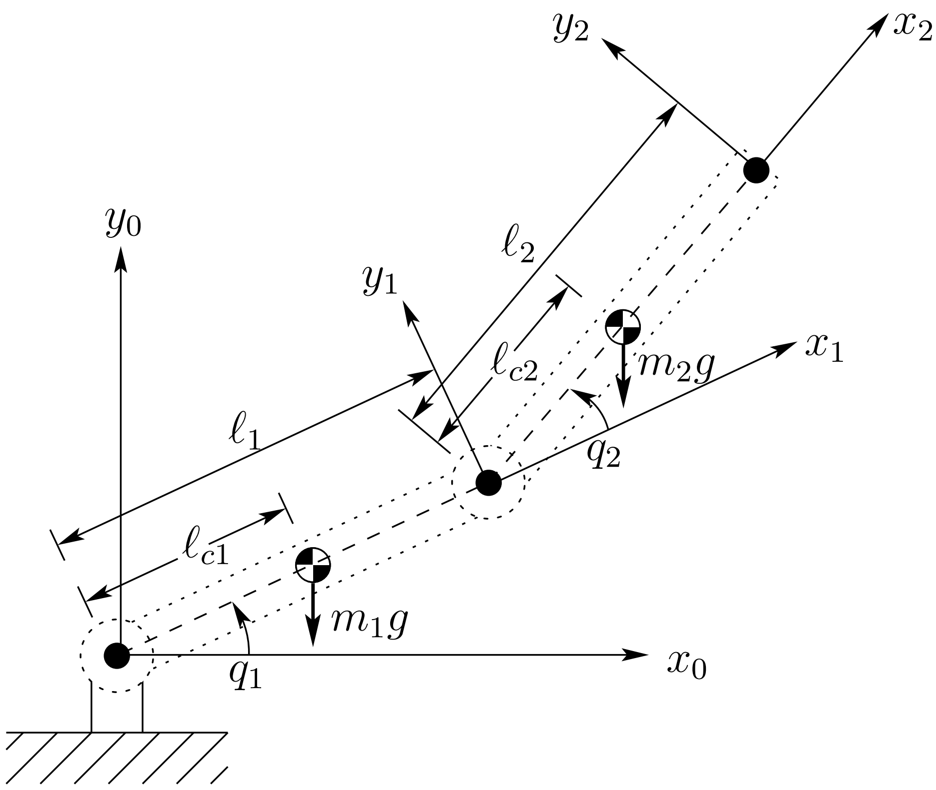
Potential energy of a Rigid Body

Using the position of the centre of mass, and assuming a gravitational field:

$$\mathcal{V}_{RB}(\mathbf{q}) = -(m\mathbf{g}^n)^\mathsf{T}\,\mathbf{r}^n_{C/N}(\mathbf{q})$$

Kinetic co-energy of an n-link Robot

- Each link has a mass m_i and an inertia tensor about its centre of mass



T & V Energies of an n-link Robot

We can use the geometric Jacobian to express the velocity of centre of mass and the angular velocity of the links as a function of the generalised coordinates and generalised velocities:

$$\boldsymbol{\nu}_{ci} = \mathbf{J}_{ci}(\mathbf{q})\,\dot{\mathbf{q}} \qquad\qquad \boldsymbol{\nu}_{ci} = \begin{bmatrix} \mathbf{v}_{ci}^{0} \\ \boldsymbol{\omega}_{i}^{0} \end{bmatrix}$$

$$\begin{aligned}
\mathcal{T}^{*} &= \sum_{i=1}^{n} \frac{1}{2} \begin{bmatrix} \mathbf{v}_{ci}^{0} \\ \boldsymbol{\omega}_{i}^{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} m_i \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{i}^{0}(\mathbf{q})\,\mathbf{I}_{ci}^{i}\,\mathbf{R}_{0}^{i}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \mathbf{v}_{ci}^{0} \\ \boldsymbol{\omega}_{i}^{0} \end{bmatrix} \\
&= \sum_{i=1}^{n} \frac{1}{2} \dot{\mathbf{q}}^{\mathsf{T}} \mathbf{J}_{ci}^{\mathsf{T}}(\mathbf{q}) \begin{bmatrix} m_i \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{i}^{0}(\mathbf{q})\,\mathbf{I}_{ci}^{i}\,\mathbf{R}_{0}^{i}(\mathbf{q}) \end{bmatrix} \mathbf{J}_{ci}(\mathbf{q})\dot{\mathbf{q}} \\
&= \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(\mathbf{q})\,\dot{q}_i\dot{q}_j
\end{aligned}$$

Potential energy:

$$\mathcal{V} = -\sum_{i=1}^{n} (m_i \mathbf{g}^{0})^{\mathsf{T}}\, \mathbf{r}_{ci}^{0}(\mathbf{q})$$

Lagrange Equations

The Lagrangian is $\mathcal{L} = \mathcal{T}^* - \mathcal{V} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(\mathbf{q})\,\dot{q}_i\dot{q}_j - \mathcal{V}(\mathbf{q})$

The partial derivatives are

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} M_{kj}(\mathbf{q})\,\dot{q}_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} M_{kj}(\mathbf{q})\,\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial M_{kj}(\mathbf{q})}{\partial q_i}\,\dot{q}_i\dot{q}_j$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\,\dot{q}_i\dot{q}_j - \frac{\partial \mathcal{V}}{\partial q_k}$$

Lagrange Equations

Putting it all together:

$$\sum_{j=1}^{n} M_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial \mathcal{V}}{\partial q_k} = \tau_k, \quad k = 1, 2, \ldots, n$$

Using symmetry properties

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ijk}\,\dot{q}_i\dot{q}_j$$

Christoffel symbols of the 1st kind:

$$\Gamma_{ijk} = \frac{1}{2}\left\{\frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} + \frac{\partial M_{ki}(\mathbf{q})}{\partial q_j} - \frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\right\}$$

Lagrange Equations n-link manipulator

In summary:

$$\sum_{j=1}^{n} M_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_{ijk}(\mathbf{q})\,\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = \tau_k, \qquad k = 1, 2, \ldots, n$$

In matrix form:

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

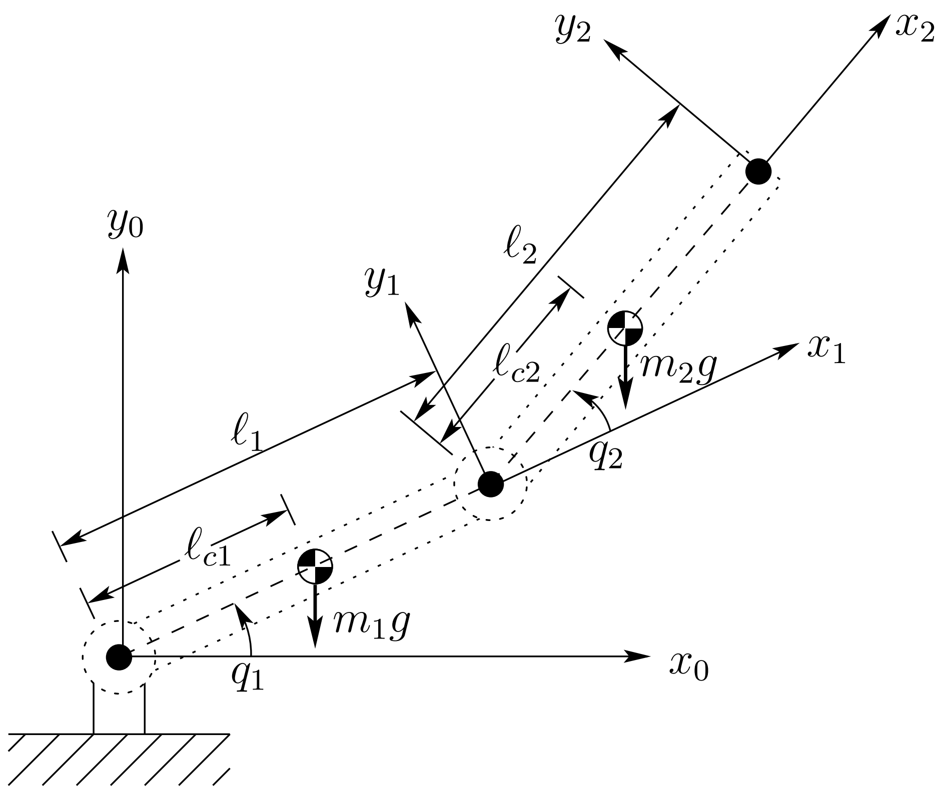
Where the entries

$$C_{kj}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{n} \Gamma_{ijk}(\mathbf{q})\,\dot{q}_i$$

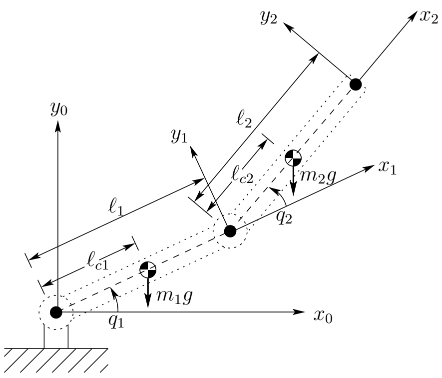
and

$$\mathbf{g}(\mathbf{q}) = [g_1(\mathbf{q}), \ldots, g_n(\mathbf{q})]^{\mathsf{T}} \qquad g_k(\mathbf{q}) = \frac{\partial \mathcal{V}(\mathbf{q})}{\partial q_k}$$

Example – Planar Manipulator in a Gravity Field



Example – Planar Manipulator in a Gravity Field



Link centre of mass geometric Jacobians:

$$\boldsymbol{\nu}_{c1} = \begin{bmatrix} \mathbf{v}^0_{c1} \\ \boldsymbol{\omega}^0_1 \end{bmatrix} = \mathbf{J}_{c1}(\mathbf{q})\,\dot{\mathbf{q}} \qquad \mathbf{J}_{c1}(\mathbf{q}) = \begin{bmatrix} -\ell_{c1}\sin q_1 & 0 \\ \ell_{c1}\cos q_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{\nu}_{c2} = \begin{bmatrix} \mathbf{v}^0_{c2} \\ \boldsymbol{\omega}^0_2 \end{bmatrix} = \mathbf{J}_{c2}(\mathbf{q})\,\dot{\mathbf{q}} \qquad \mathbf{J}_{c2}(\mathbf{q}) = \begin{bmatrix} -\ell_1\sin q_1 - \ell_{c2}\sin(q_1+q_2) & -\ell_{c2}\sin(q_1+q_2) \\ \ell_1\cos q_1 + \ell_{c2}\cos(q_1+q_2) & \ell_{c2}\cos(q_1+q_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Example – Planar Manipulator in a Gravity Field

Mass Matrix: $$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} M_{11}(\mathbf{q}) & M_{12}(\mathbf{q}) \\ M_{21}(\mathbf{q}) & M_{22}(\mathbf{q}) \end{bmatrix}$$

$$\begin{aligned} M_{11}(\mathbf{q}) &= m_1\,\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_{c1} + I_{c2}, \\ M_{12}(\mathbf{q}) &= m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_{c2}, \\ M_{22}(\mathbf{q}) &= m_2\ell_{c2}^2 + I_{c2}, \end{aligned}$$

Note: Recall that the mass matrix is symmetric: $M_{12} = M_{21}$

Non-zero Christoffel symbols:

$$\begin{aligned} \Gamma_{121}(\mathbf{q}) &= -m_2\ell_1\ell_{c2}\sin q_2 \triangleq h, \\ \Gamma_{211}(\mathbf{q}) &= h, \\ \Gamma_{221}(\mathbf{q}) &= h, \\ \Gamma_{112}(\mathbf{q}) &= -h, \end{aligned}$$

Example – Planar Manipulator in a Gravity Field

Potential Energy:

$$\mathcal{V}(\mathbf{q}) = \mathcal{V}_1(\mathbf{q}) + \mathcal{V}_2(\mathbf{q}) = (m_1\ell_{c1} + m_2\ell_1)g\sin q_1 + m_2\ell_{c2}g\sin(q_1 + q_2)$$

Partial derivatives:

$$\begin{aligned} g_1(\mathbf{q}) &= (m_1\ell_{c1} + m_2\ell_1)g\cos q_1 + m_2\ell_{c2}g\cos(q_1 + q_2), \\ g_2(\mathbf{q}) &= m_2\ell_{c2}g\cos(q_1 + q_2). \end{aligned}$$

Example – Planar Manipulator in a Gravity Field

Dynamic model including joint torques and damping:

$$
\begin{aligned}
M_{11}(\mathbf{q})\,\ddot{q}_1 + M_{12}(\mathbf{q})\,\ddot{q}_2 + \Gamma_{121}(\mathbf{q})\,\dot{q}_1\dot{q}_2 + \Gamma_{211}(\mathbf{q})\,\dot{q}_2\dot{q}_1 + \Gamma_{221}(\mathbf{q})\,\dot{q}_2^2 + d_1\dot{q}_1 + g_1(\mathbf{q}) &= \tau_1, \\
M_{21}(\mathbf{q})\ddot{q}_1 + M_{22}(\mathbf{q})\ddot{q}_2 + \Gamma_{112}(\mathbf{q})\dot{q}_1^2 + d_2\dot{q}_2 + g_2(\mathbf{q}) &= \tau_2,
\end{aligned}
$$

Example – Planar Manip. in a Gravity Field