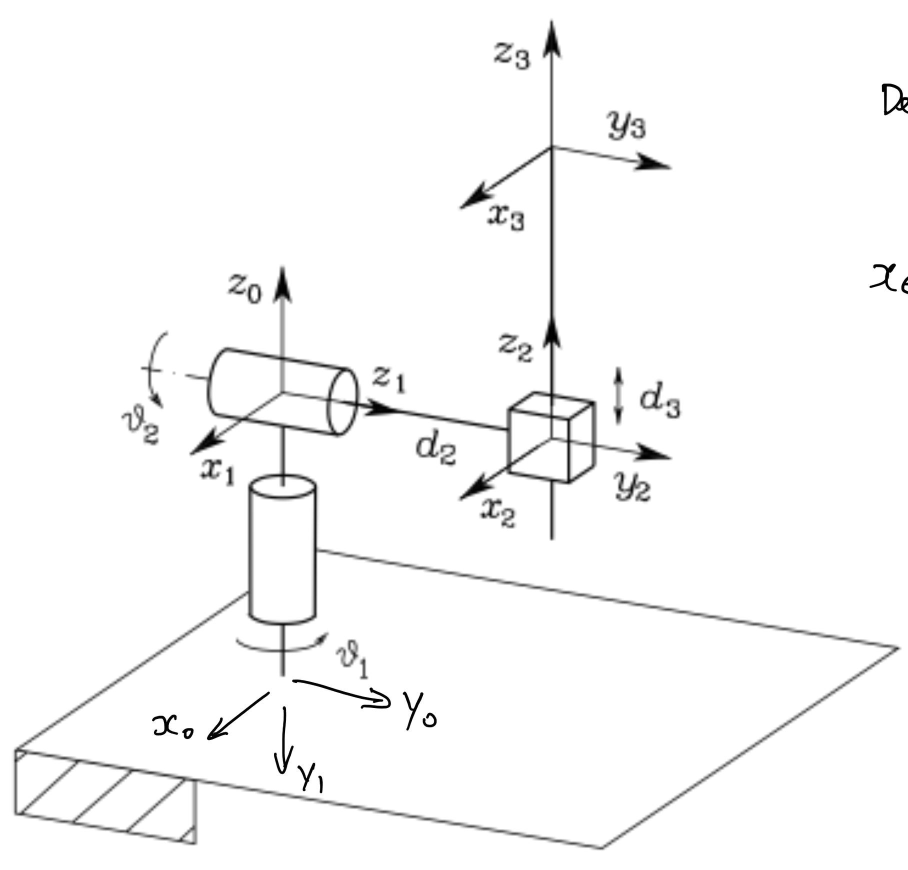
Example 1



Determine the forward kinematic model

$$x_e = \begin{bmatrix} r_n^0 \\ \Theta_n^0 \end{bmatrix} = k(q)$$

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ d_2 \\ d_3 \end{bmatrix}$$

Write the kinematic chain as a product of homogeneous transformations

$$T_n^0(q) = A_1^0(\theta_1)\, A_2^1(\theta_2, d_2)\, A_3^2(d_3)$$

$$\begin{bmatrix} R_n^0(q) & r_n^0(q) \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1^0(\theta_1) & r_1^0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_2^1(\theta_2) & r_2^1(d_2) \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_3^2 & r_3^2(d_3) \\ 0^T & 1 \end{bmatrix}$$

$R_1^0(\theta_1)$ can be written as a product of a rotation of θ_1 about $\vec{z}_0$ followed by a fixed coordinate transformation.

$$R_1^0(\theta_1) = R_z(\theta_1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix}$$

Note: The fixed transformation can be obtained by considering that $\theta_1 = 0$ and expressing the basis vectors $\{1\}$ in the coordinate system $\{0\}$ as columns.

i.e., $[x_1^0 \;\; y_1^0 \;\; z_1^0]$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note: We can also obtain $R_1^0(\theta_1)$ by doing the fixed transformation first, followed by a negative rotation of θ_1 about $\vec{y}_1$

i.e., $$R_1^0(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} R_y(-\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & c_1 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix}$$

There is no position offset between O_0 and O_1, therefore

$$r_1^0 = r_{O_1/O_0}^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assembling A_1^0 yields

$$A_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2^1(\theta_2)$ can be written as a product of a rotation of θ_2 about $\vec{z}_1$ followed by a fixed coordinate transformation.

$$R_2^1(\theta_2) = R_z(\theta_2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} c_2 & 0 & s_2 \\ s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2^1(0) = I \quad [\, x_2^1 \;\; y_2^1 \;\; z_2^1 \,]$$

The position offset from O_1 to O_2 is given by

$$r_2^1 = r_{O_2/O_1}^1 = \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}$$

Assembling A_2^1 yields

$$A_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The $\{2\}$ and $\{3\}$ coordinate systems have the same basis vectors, therefore

$$R_3^2 = I_{3\times3}$$

The position offset from O_2 to O_3 is given by

$$r_3^2 = r_{O_3/O_2}^2 = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix}$$

Assembling A_3^2 yields

$$A_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying the homogeneous transformations yields

$$T_3^0 = A_1^0 A_2^1 A_3^2$$

$$= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & -s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & c_1 d_2 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} R_3^0 \end{bmatrix} & \begin{bmatrix} r_3^0 \end{bmatrix} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}$$

Therefore $$R_3^0 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 \\ s_1 c_2 & c_1 & s_1 s_2 \\ -s_2 & 0 & c_2 \end{bmatrix} \text{ and } r_3^0 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

To parameterise the end-effector pose, we can determine the set of Euler angles that produce R_3^0.

Let us choose ZYX Euler angles, which satisfy

$$R(\Theta) = R_z(\psi) R_y(\theta) R_x(\phi)$$

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\psi c_\theta & -s_\psi & c_\psi s_\theta \\ s_\psi c_\theta & c_\psi & s_\psi s_\theta \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & c_\psi c_\phi + s_\psi s_\theta s_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

To determine ϕ, θ and ψ, let us consider the following three ratios:

1) $$\frac{R_{32}}{R_{33}} = \frac{\cos\theta \sin\phi}{\cos\theta \cos\phi} = \frac{\sin\phi}{\cos\phi} \qquad (\text{if } \cos\theta \neq 0)$$

Therefore $\phi = \operatorname{atan2}(R_{32}, R_{33})$

2) $$\frac{R_{21}}{R_{11}} = \frac{\sin\psi \cos\theta}{\cos\psi \cos\theta} = \frac{\sin\psi}{\cos\psi} \qquad (\text{if } \cos\theta \neq 0)$$

Therefore $\psi = \operatorname{atan2}(R_{21}, R_{11})$

3) $$\frac{-R_{31}}{\sqrt{R_{32}^2 + R_{33}^2}} = \frac{\sin\theta}{\sqrt{\cos^2\theta \sin^2\phi + \cos^2\theta \cos^2\phi}} = \frac{\sin\theta}{\sqrt{\cos^2\theta \, \underbrace{(\sin^2\phi + \cos^2\phi)}_{=1}}}$$

since $\det(\underbrace{\begin{bmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{bmatrix}}_{\in SO(2)}) = +1$

$$= \frac{\sin\theta}{\cos\theta} \qquad \left(\text{if } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\right)$$

Therefore $\theta = \operatorname{atan2}\left(-R_{31}, \sqrt{R_{32}^2 + R_{33}^2}\right)$

Note: We could do $\theta = -\sin^{-1}(R_{31})$, but this will fail if R_{31} is slightly below -1 or slightly above $+1$ due to numerical issues.

Note: In computing $\sqrt{R_{32}^2 + R_{33}^2}$, we should use the HYPOT function available in most languages (e.g., C/Matlab), since it can correctly deal with numerical underflow or overflow.

For our particular example, there are some simplifications that can be made

$$\phi = \operatorname{atan2}(R_{32}, R_{33}) = \operatorname{atan2}(0, c_2) = 0$$

$$\theta = \operatorname{atan2}\left(-R_{31}, \sqrt{R_{32}^2 + R_{33}^2}\right) = \operatorname{atan2}\left(s_2, \sqrt{0^2 + c_2^2}\right) = \theta_2$$

$$\psi = \operatorname{atan2}(R_{21}, R_{11}) = \operatorname{atan2}(s_1 c_2, c_1 c_2) = \theta_1$$

Therefore $$\Theta_3^0 = \begin{bmatrix} 0 \\ \theta_2 \\ \theta_1 \end{bmatrix}.$$

Finally, we can write the parameters of the end-effector pose, x_e as a function of the configuration variables, q.

$$x_e = \begin{bmatrix} r_3^0 \\ \Theta_3^0 \end{bmatrix} = \begin{bmatrix} d_3 \cos\theta_1 \sin\theta_2 - d_2 \sin\theta_1 \\ d_3 \sin\theta_1 \sin\theta_2 + d_2 \cos\theta_1 \\ d_3 \cos\theta_2 \\ 0 \\ \theta_2 \\ \theta_1 \end{bmatrix} = k(q)$$