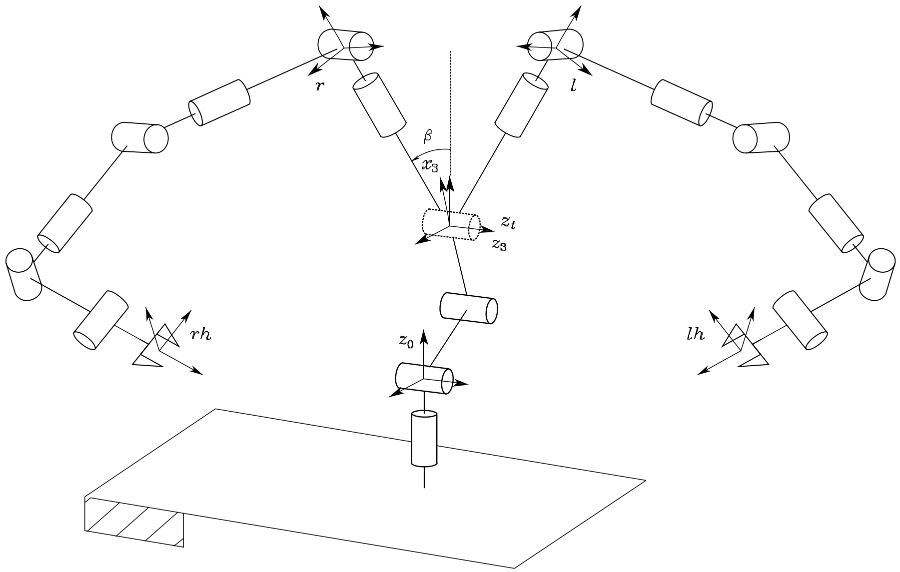
MCHA3900 – Robotic Manipulator Differential Kinematics

$$\boldsymbol{\nu} = \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}}$$

$$\dot{\mathbf{x}}_e = \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}}$$

$$\boldsymbol{\tau}_{\mathbf{q}} = \mathbf{J}^{\mathsf{T}}(\mathbf{q})\boldsymbol{\tau}$$



Dr. Chris Renton
School of Engineering

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

FACULTY OF ENGINEERING AND BUILT ENVIRONMENT

www.newcastle.edu.au

Differential Kinematic Model (DKM)

The Forward Kinematic Model maps the joint variables into the position and orientation of the of the manipulator effector.

$$\mathbf{T}_n^0(\mathbf{q}) = \mathbf{A}_1^0(q_1)\,\mathbf{A}_2^1(q_2)\,\cdots\,\mathbf{A}_n^{n-1}(q_n) = \begin{bmatrix}\mathbf{R}_n^0(\mathbf{q}) & \mathbf{r}_n^0(\mathbf{q}) \\ \mathbf{0}^\mathsf{T} & 1\end{bmatrix} \in \mathsf{SE}(3)$$

The differential kinematic model gives the relationship between the time derivative of the joint variables and the linear and angular velocity of the manipulator:

$$\mathbf{v}_n^0 = \mathbf{J}_v(\mathbf{q})\,\dot{\mathbf{q}} \qquad \mathbf{v}_n^0 \equiv {}^0\dot{\mathbf{r}}_{n/0}^0$$

$$\boldsymbol{\omega}_n^0 = \mathbf{J}_\omega(\mathbf{q})\,\dot{\mathbf{q}} \qquad \boldsymbol{\omega}_n^0 \equiv \boldsymbol{\omega}_{n/0}^0$$

This can be written as

$$\boldsymbol{\nu} = \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}} \qquad \boldsymbol{\nu} = \begin{bmatrix}\mathbf{v}_n^0 \\ \boldsymbol{\omega}_n^0\end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix}\mathbf{J}_v \\ \mathbf{J}_\omega\end{bmatrix}$$

Manipulator Geometric Jacobian

The Jacobian

The manipulator Jacobian $\mathbf{J} = \begin{bmatrix} \mathbf{J}_v \\ \mathbf{J}_\omega \end{bmatrix}$ $\quad \boldsymbol{\nu} = \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}}$ $\quad \boldsymbol{\nu} = \begin{bmatrix} \mathbf{v}_n^0 \\ \boldsymbol{\omega}_n^0 \end{bmatrix}$

is a very important descriptor of the manipulator because if the Jacobian is rank deficient, at a particular configuration $\mathbf{q}$, then

- The manipulator mobility is reduced, it loses DOF; thus it may not possible to impose the effector position and orientation independently.
- Generally there are infinite solutions of the IKP.
- Near a singular configuration, small velocities in the operational space (effector position and Euler angles) require large velocities in the configuration space.

A configuration in which the Jacobian is rank deficient is called a singular configuration.

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

Computing the Jacobian

Computing the Jacobian is simple, each column is related to a joint, the entries of the column depend on whether the joint is prismatic or revolute

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_v \\ \mathbf{J}_\omega \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{v1} & \mathbf{J}_{v2} & \cdots & \mathbf{J}_{vn} \\ \mathbf{J}_{\omega 1} & \mathbf{J}_{\omega 2} & \cdots & \mathbf{J}_{\omega n} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{vi} \\ \mathbf{J}_{\omega i} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}^0_{i-1} \\ \mathbf{0} \end{bmatrix} & \text{for prismatic} \\ \\ \begin{bmatrix} \mathbf{S}(\mathbf{z}^0_{i-1})(\mathbf{r}^0_{n/0} - \mathbf{r}^0_{i-1/0}) \\ \mathbf{z}^0_{i-1} \end{bmatrix} & \text{for revolute} \end{cases}$$

The unit vector of the basis {*i*-1} through the joint *i* axis (Denavit-Hartenberg convention):

$$\mathbf{z}^0_{i-1} = \mathbf{R}^0_1(q_1)\mathbf{R}^1_2(q_2)\cdots\mathbf{R}^{i-2}_{i-1}(q_{i-1})[0, 0, 1]^T$$

Kinematic redundancy

- Recall that the forward KM

$$\mathbf{T}_n^0(\mathbf{q}) = \mathbf{A}_1^0(q_1)\,\mathbf{A}_2^1(q_2)\,\cdots\,\mathbf{A}_n^{n-1}(q_n) = \begin{bmatrix} \mathbf{R}_n^0(\mathbf{q}) & \mathbf{r}_n^0(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix}$$

- Can be expressed in terms of the space of $\mathbf{x}_e$, called operational space:

$$\mathbf{x}_e = \mathbf{k}(\mathbf{q}) \qquad \mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \boldsymbol{\Theta}_n^0 \end{bmatrix}$$

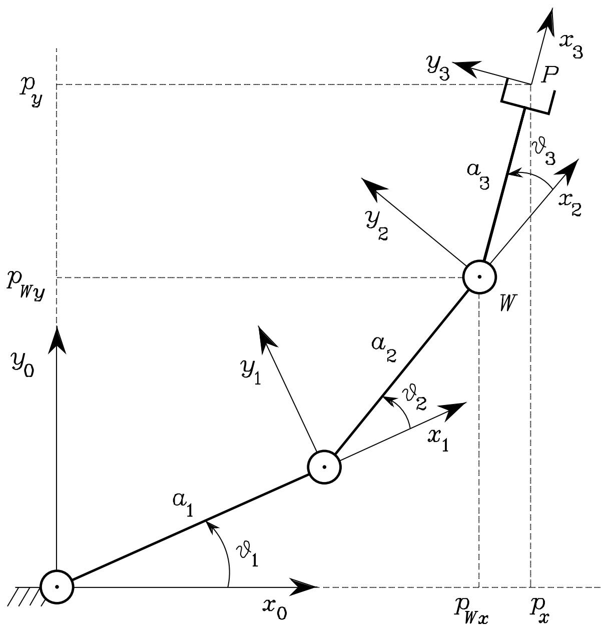
Effector Pose vector

A manipulator is said to be **kinematically redundant for a particular task** if

$$\dim \mathbf{x}_e < \dim \mathbf{q}$$

Kinematic redundancy

Three link planar manipulator:



$$\mathbf{x}_e = \mathbf{k}(\mathbf{q})$$

$$\mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \boldsymbol{\Theta}_n^0 \end{bmatrix}$$

Effector Pose vector

If the task is just positioning the point P, then the manipulator is kinematically redundant: $\dim \mathbf{x}_e = 2 < \dim \mathbf{q} = 3$

If the orientation of the tool is specified, we lose redundancy: $\dim \mathbf{x}_e = \dim \mathbf{q} = 3$

Inverse Differential Kinematics

- If we consider non-kinematically redundant manipulator, then

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\,\boldsymbol{\nu} \qquad\qquad \boldsymbol{\nu} = \begin{bmatrix} \mathbf{v}_n^0 \\ \boldsymbol{\omega}_n^0 \end{bmatrix}$$

Thus, we can use this to determine the IKP given a trajectory for **ν** and the initial configuration:

$$\mathbf{q}(t) = \mathbf{q}(0) + \int_0^t \mathbf{J}^{-1}(\mathbf{q}(t'))\,\boldsymbol{\nu}(t')\,\mathrm{d}t'$$

- For a kinematically redundant manipulator, one possible IDKM (with minimum-norm joint rates) is given by

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})\,\boldsymbol{\nu} \qquad\qquad \mathbf{J}^{\dagger} = \mathbf{J}^{\mathsf{T}}(\mathbf{J}\mathbf{J}^{\mathsf{T}})^{-1}$$

which can be integrated from **q**(0) as before to solve the IKM

Analytical Jacobian

Given a FKM in terms of the effector pose,

$$\mathbf{x}_e = \mathbf{k}(\mathbf{q}) \qquad\qquad \mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \boldsymbol{\Theta}_n^0 \end{bmatrix}$$

The **Analytical Jacobian** is defined as

$$\mathbf{J}_A(\mathbf{q}) = \frac{\partial \mathbf{k}(\mathbf{q})}{\partial \mathbf{q}} \qquad\qquad \dot{\mathbf{x}}_e = \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}}$$

This differs from the other Jacobian (geometric Jacobian) due to the relation between the angular velocity and Euler angles of the effector:

$$\mathbf{J}_A = \mathbf{T}_A(\boldsymbol{\Theta}_n^0)\,\mathbf{J}$$

$$\mathbf{T}_A(\boldsymbol{\Theta}_n^0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^{-1}(\boldsymbol{\Theta}_n^0) \end{bmatrix}$$

$$\mathbf{E}(\boldsymbol{\Theta}_n^0) = \begin{bmatrix} 0 & -s_\phi & c_\phi\, s_\theta \\ 0 & c_\phi & s_\phi\, s_\theta \\ 1 & 0 & c_\theta \end{bmatrix}$$

(ZYZ Euler angles)

$$\mathbf{E}(\boldsymbol{\Theta}_n^0) = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & s_\phi\, c_\theta \\ 0 & -s_\phi & c_\phi\, c_\theta \end{bmatrix}$$

(RPY Euler angles)

IKP using the Analytical Jacobian

The analytic Jacobian can be used to compute the solution of the IKP.

Let the desired pose be $\mathbf{x}_d$ and the pose be $\mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \boldsymbol{\Theta}_n^0 \end{bmatrix}$

Let us now define the error $\mathbf{e} = \mathbf{x}_d - \mathbf{x}_e$

Then, $\dot{\mathbf{e}} = \dot{\mathbf{x}}_d - \dot{\mathbf{x}}_e = \dot{\mathbf{x}}_d - \mathbf{J}_A(\mathbf{q})\,\dot{\mathbf{q}}$

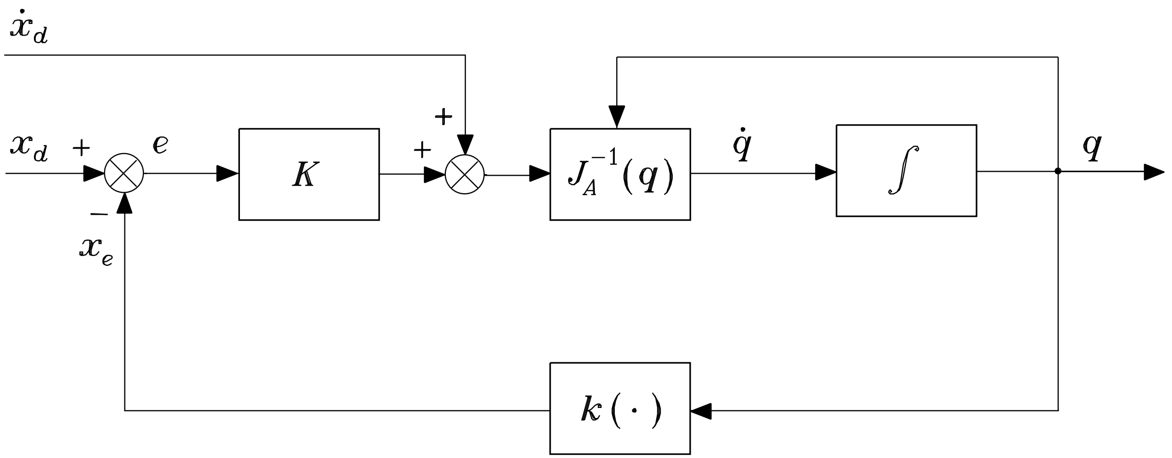
Let us also consider the homogeneous ODE

$$\dot{\mathbf{e}} + \mathbf{K}\mathbf{e} = \mathbf{0}, \qquad \mathbf{K} > \mathbf{0}$$

IKP using the Analytical Jacobian

Then we can use the following block diagram to obtain the solution of the IKP:

$$\dot{\mathbf{e}} + \mathbf{K}\mathbf{e} = \mathbf{0} \qquad \dot{\mathbf{e}} = \dot{\mathbf{x}}_d - \mathbf{J}_A(\mathbf{q})\,\dot{\mathbf{q}}$$



Given an initial configuration $\mathbf{q}_0$, we can simulate this system to obtain the solution of the IKP. This is a way to solve the optimisation problem we posed last lecture.

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

Force kinematics

Express effector pose velocity as a function of the joint angles and joint rates

$$\boldsymbol{\nu} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

The rate of work done by the joints is given by the inner product of joint torques and joint velocities

$$P = \langle \boldsymbol{\tau}_\mathbf{q}, \dot{\mathbf{q}} \rangle = \boldsymbol{\tau}_\mathbf{q}^\mathsf{T}\dot{\mathbf{q}}$$

The rate of work done by the end effector is given by the inner product of end-effector generalised forces and velocities

$$P = \langle \boldsymbol{\tau}, \boldsymbol{\nu} \rangle = \boldsymbol{\tau}^\mathsf{T}\boldsymbol{\nu} = \boldsymbol{\tau}^\mathsf{T}\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \left(\mathbf{J}^\mathsf{T}(\mathbf{q})\boldsymbol{\tau}\right)^\mathsf{T}\dot{\mathbf{q}}$$

Equating the power yields the joint torques as a function of the joint angles and effector generalised forces

$$\boldsymbol{\tau}_\mathbf{q} = \mathbf{J}^\mathsf{T}(\mathbf{q})\boldsymbol{\tau}$$