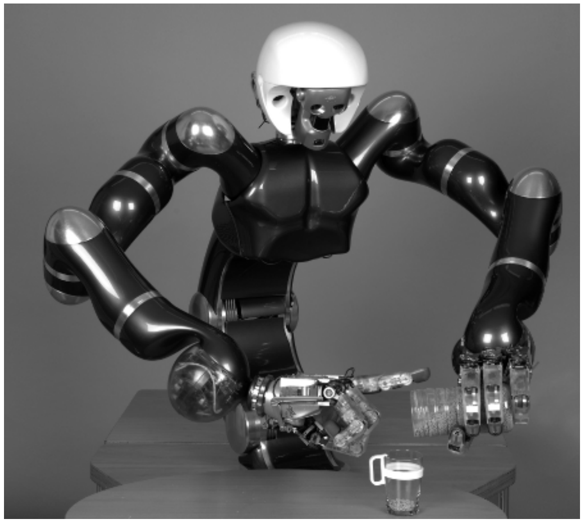
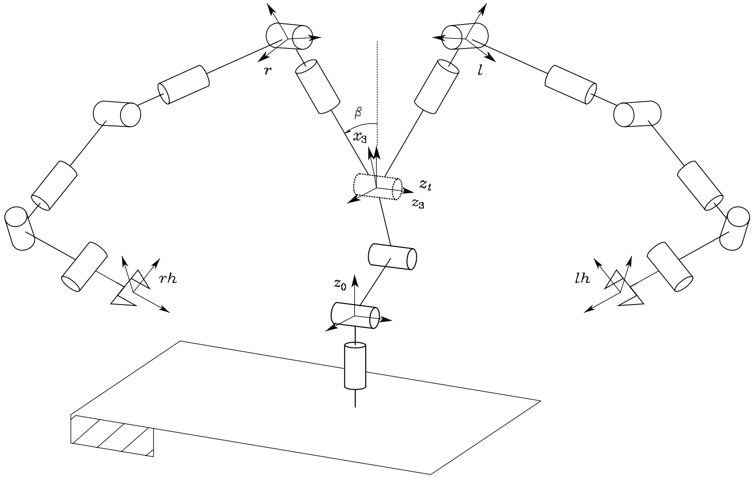
MCHA3900 – Robotic Manipulator Kinematics





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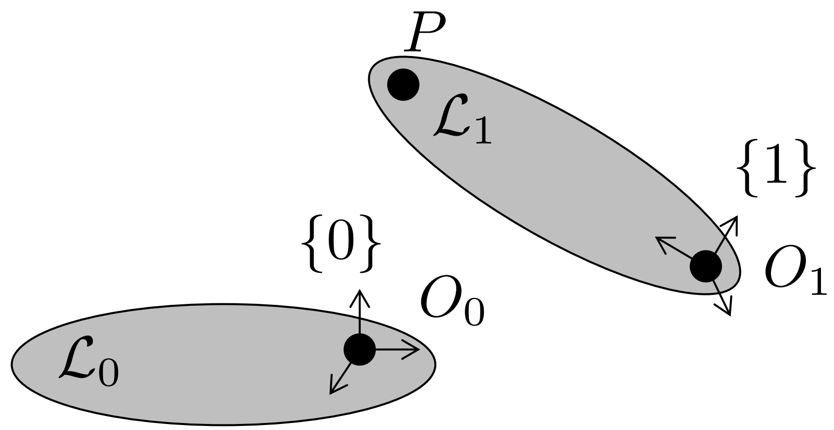
Rigid-body Motion in Robotics

The literature of robotics has favoured the matrix over the tensor notation, hence all kinematic relations are directly described using matrices and linear algebra and the vectors are intrinsically associated with particular coordinates.

$\mathcal{L}_i$ Link i (rigid body)
O_i Reference point in link i
$\{i\}$ Basis assoc. with link i with point O_i



Condensed notation:

$$\vec{r}_{O_j/O_i} \equiv \mathbf{r}^i_{j/i} \equiv \mathbf{r}^i_j$$

The right-upper index not only indicates the coordinate system, but also a point of reference in the reference frame.

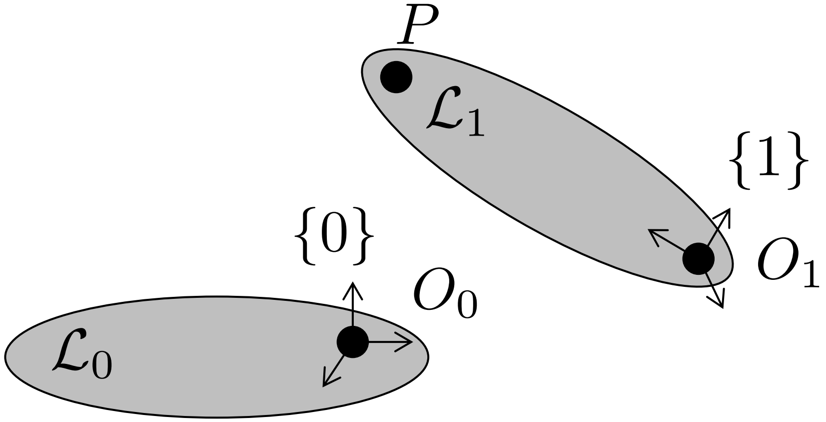
Example: $$\mathbf{r}^0_P = \mathbf{R}^0_1\,\mathbf{r}^1_P + \mathbf{r}^0_1$$

Due to reference points, coordinate transformations become affine, and position vectors are no longer tensors!

Rigid-body Motion in Robotics

The rigid body motion in space can be described by an ordered pair (**R**,**r**), where **R** is in SO(3) and **r** is a column vector in R^3. The group of all rigid motions is called SE(3) – **Special Euclidean group**.

The relative motion of two links can be expressed using a so called **Homogeneous Transformation matrix**:



$$\mathbf{A}_1^0 \triangleq \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{r}_1^0 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$\mathbf{r}_P^0 = \mathbf{R}_1^0\,\mathbf{r}_P^1 + \mathbf{r}_1^0$$

$$\Downarrow$$

$$\bar{\mathbf{r}}_P^0 = \mathbf{A}_1^0\,\bar{\mathbf{r}}_P^1 \qquad \bar{\mathbf{r}}_P^0 = \begin{bmatrix} \mathbf{r}_P^0 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{r}}_P^1 = \begin{bmatrix} \mathbf{r}_P^1 \\ 1 \end{bmatrix}$$

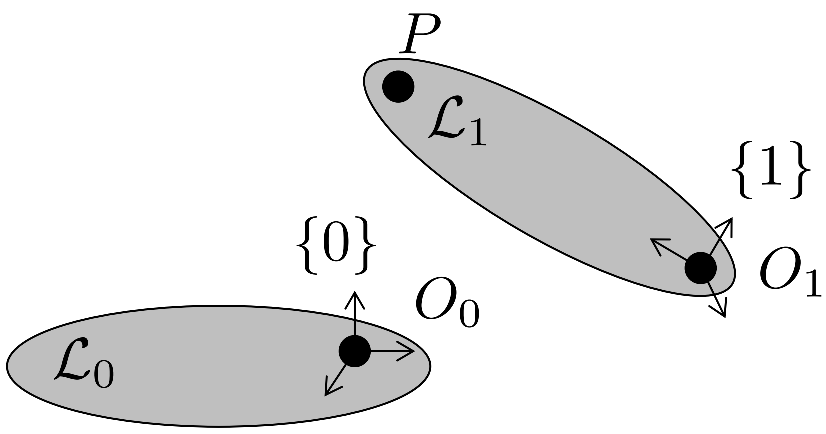
Rigid-body Motion in Robotics

The inverse of the Homogeneous Transformation matrix $\mathbf{A}_1^0 \triangleq \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{r}_1^0 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$

is given by $\mathbf{A}_0^1 = (\mathbf{A}_1^0)^{-1} = \begin{bmatrix} (\mathbf{R}_1^0)^\mathsf{T} & -(\mathbf{R}_1^0)^\mathsf{T}\mathbf{r}_1^0 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$

Which follows from the relation

$$\mathbf{r}_P^0 = \mathbf{R}_1^0\,\mathbf{r}_P^1 + \mathbf{r}_1^0$$



Rigid-body Motion in Robotics

If we have three links

$$\mathbf{r}_P^0 = \mathbf{R}_2^0\,\mathbf{r}_P^2 + \mathbf{r}_2^0$$

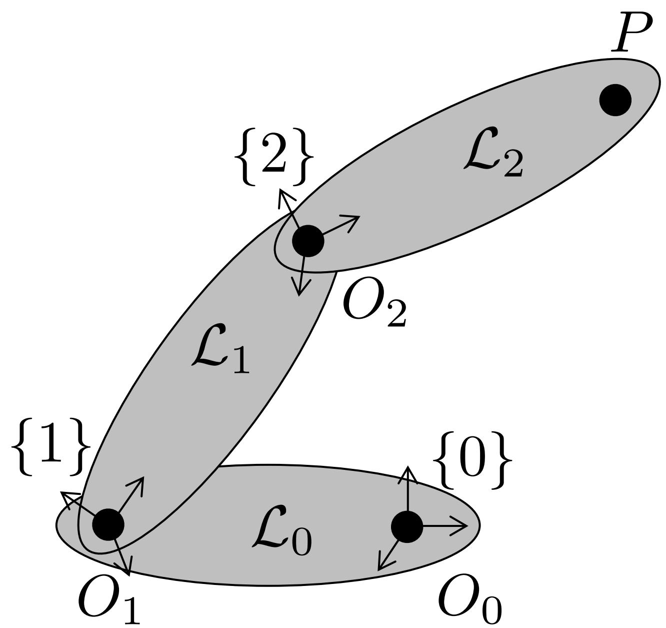
Using an intermediate link

$$\mathbf{r}_P^0 = \mathbf{R}_1^0\mathbf{R}_2^1\,\mathbf{r}_P^2 + \mathbf{R}_1^0\,\mathbf{r}_2^1 + \mathbf{r}_1^0$$

Kinematics in terms of
homogeneous transformation: $\bar{\mathbf{r}}_P^0 = \mathbf{A}_1^0\mathbf{A}_2^1\,\bar{\mathbf{r}}_P^2$

$$\mathbf{A}_1^0 \triangleq \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{r}_1^0 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \qquad \bar{\mathbf{r}}_P^0 = \begin{bmatrix} \mathbf{r}_P^0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_2^1 \triangleq \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{r}_2^1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \qquad \bar{\mathbf{r}}_P^1 = \begin{bmatrix} \mathbf{r}_P^1 \\ 1 \end{bmatrix}$$



Canonical Homogeneous Transformations

Most homogeneous transformations can be expressed as a product of 6 types of homogeneous transformations, which we call canonical:

$$\exp\colon \mathfrak{se}(3) \to \mathsf{SE}(3) \qquad \begin{bmatrix} \mathbf{S}(\boldsymbol{\omega}) & \mathbf{v} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \in \mathfrak{se}(3) \qquad \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \in \mathsf{SE}(3)$$

Translations

$$\exp\left(x \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp\left(y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp\left(z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations

$$\exp\left(\phi \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp\left(\theta \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

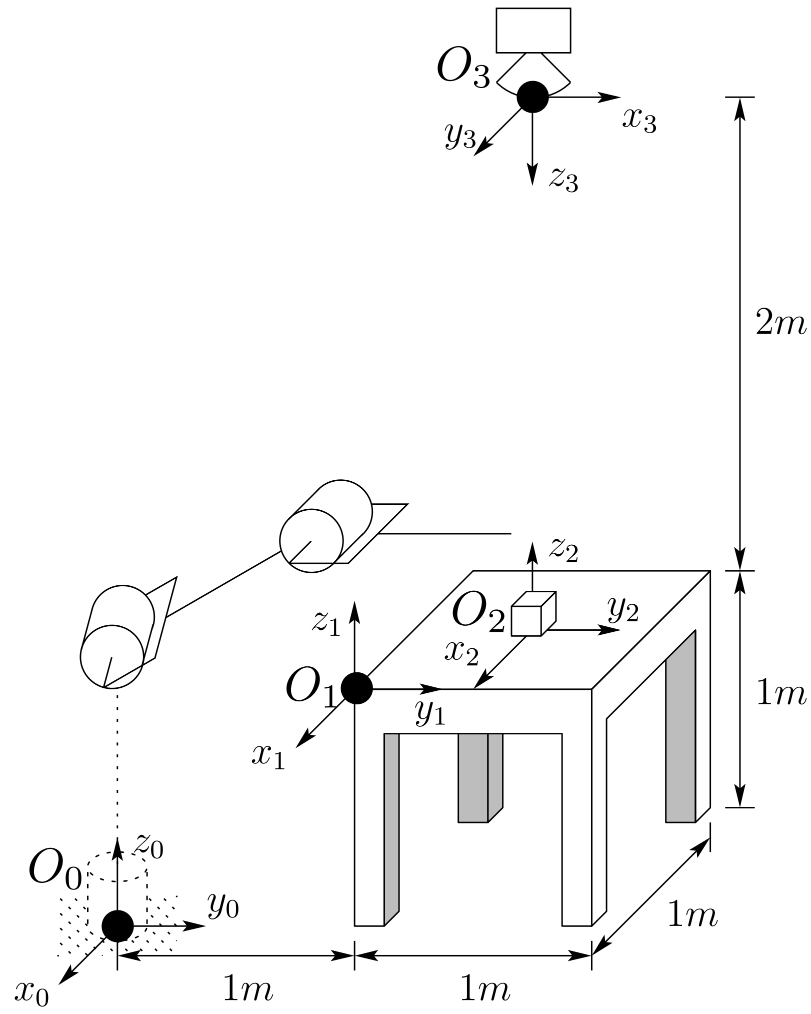
$$\exp\left(\psi \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

A robot is set up 1 meter from a table. The table top is 1 metre high and 1 metre square. The table is the reference frame 1, with point of reference O_1 and basis $\{1\}$. A cube measuring 20 cm on each side is placed in the centre of the table, which is a reference frame 2 with point of reference O_2 at the centre of the cube. A camera is located directly above the centre of the block 2m above the table top, and it is considered a reference frame 3.

Find the homogeneous transformations relating each of these frames to the base frame 0 of the robot.

Find the homogeneous transformation relating the frame 2 to the camera frame 3.



Example

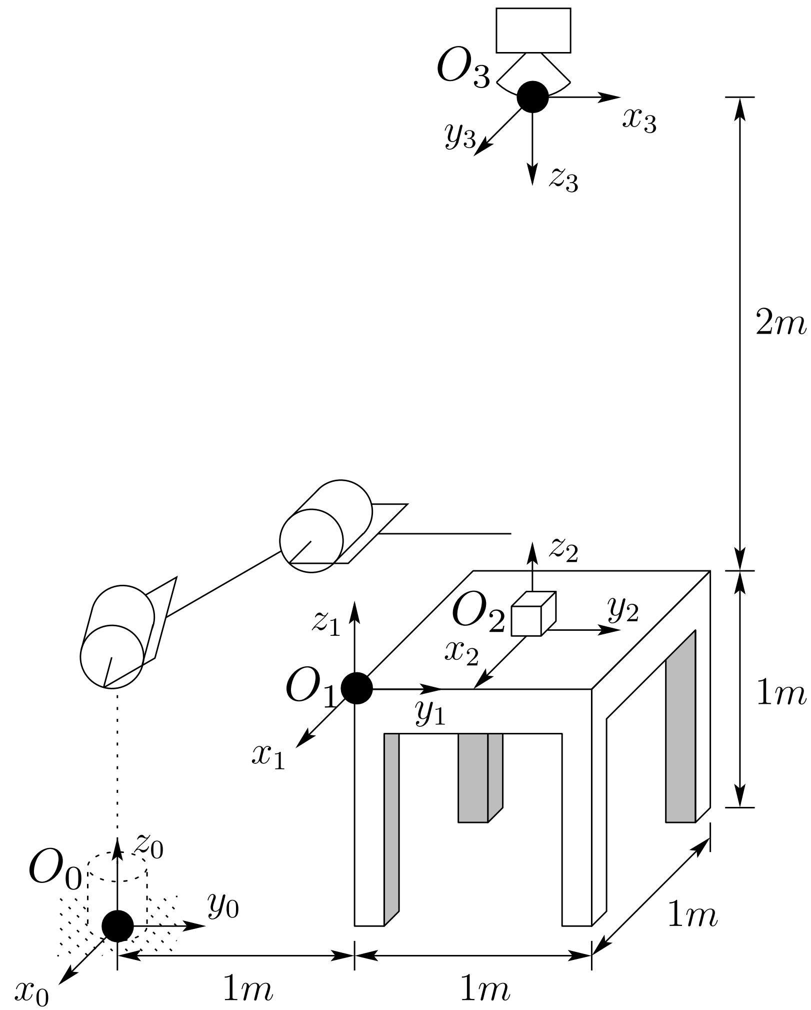
One possible solution:

$$A_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^0 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Trans} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot_{z,-\pi/2}} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot_{y,\pi}}$$

$$A_3^2 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Trans_{z,1.9}} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot_{z,-\pi/2}} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot_{y,\pi}}$$



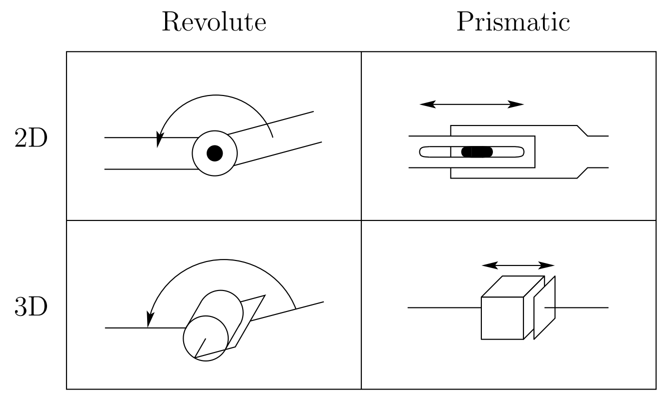
Forward and Inverse Kinematics

- The problem of forward kinematics is to determine the position and orientation of a robot effector for a given value of the joint variables of the robot.

- The problem of inverse kinematics is to determine the value of the joint variables given the position and orientation of the end effector.

- Joint variables are

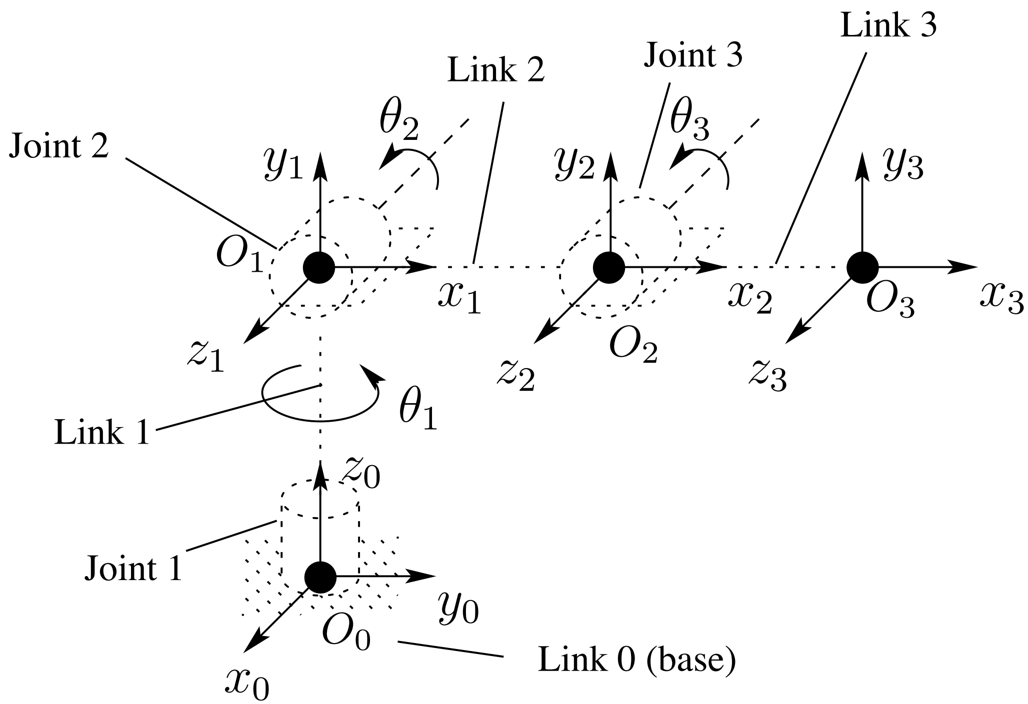
$$q_i = \begin{cases} \theta_i & \text{for revolute joints} \\ d_i & \text{for prismatic joints} \end{cases}$$



Without loss of generality, we will consider that all joints have a single degree of freedom. Joints with multiple DOF like a ball-socket, can be thought of as a succession of 1DOF joints with links of length zero.

Manipulator Description

- A manipulator with n joints will have n+1 links,
- We number joints from 1 to n and the links from 0 (base) to n (effector),
- The location of the joint i is fixed with respect to the location of the link i-1,
- The joint i connects the link i to the link i-1,
- When the joint i is actuated the link i moves,
- The link 0 is fixed.

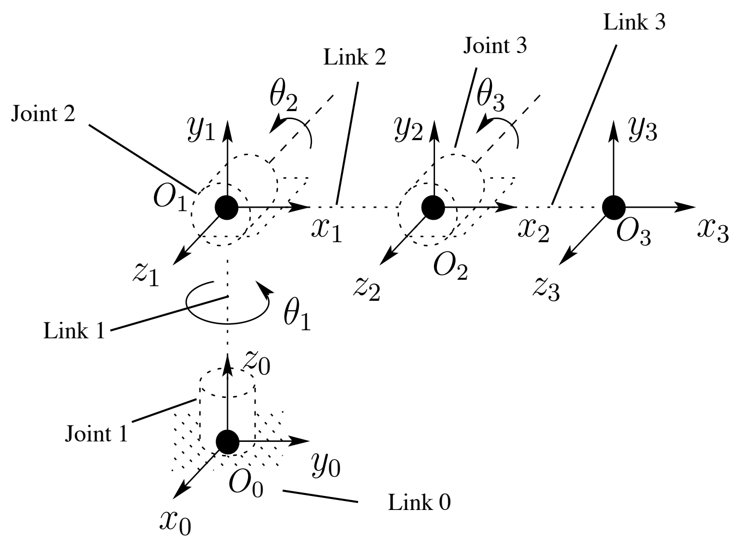


Forward Kinematic Model (FKM)

The homogeneous transformation between the base and the end effector is

$$\mathbf{T}_n^0(\mathbf{q}) = \mathbf{A}_1^0(q_1)\,\mathbf{A}_2^1(q_2)\,\cdots\,\mathbf{A}_n^{n-1}(q_n) = \begin{bmatrix} \mathbf{R}_n^0(\mathbf{q}) & \mathbf{r}_n^0(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix}$$

This is all there is to raw forward kinematics. However, there is a possibility to introduce further simplifications by introducing further conventions.

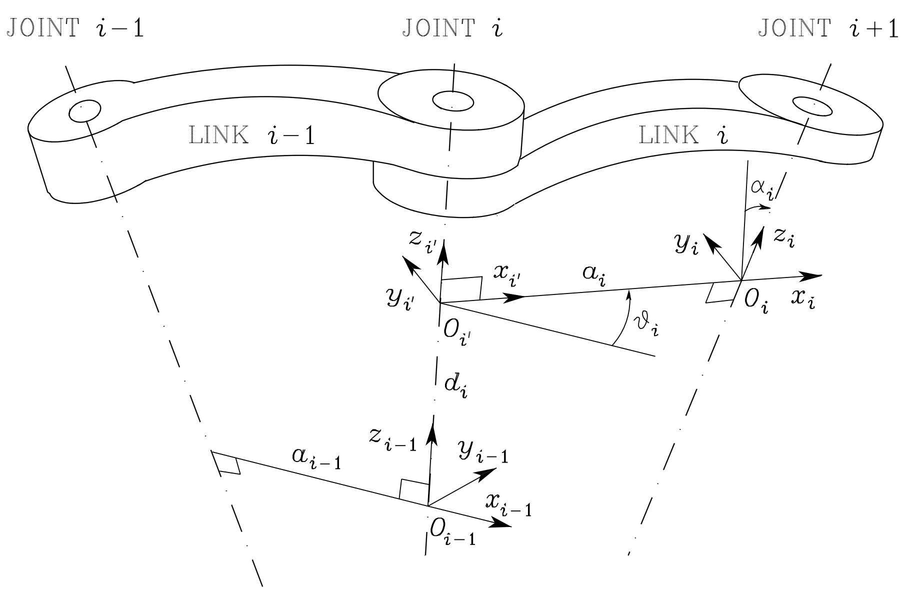


Denavit-Hartenberg Convention

- The DH convention can be used to set the coordinate systems and reference points of the different links.

- This convention can simplify the computations of the forward kinematic model.

- We will not review this here, but this covered in any standard robotics book; including the recommended readings.



Joint Space and Operational Space

- The Direct Kinematic Model (DKM) maps the joint variables into the position and orientation of the effector:

$$\mathbf{T}_n^0(\mathbf{q}) = \mathbf{A}_1^0(q_1)\,\mathbf{A}_2^1(q_2)\,\cdots\,\mathbf{A}_n^{n-1}(q_n) = \begin{bmatrix} \mathbf{R}_n^0(\mathbf{q}) & \mathbf{r}_n^0(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix}$$

- To plan a task for the robot effector, we need to assign a particular $\mathbf{T}_n^0$ as a function of time. This requires the specification of a general rotation matrix and a position vector.

- This task can be simplified if we express the rotation matrix in terms of Euler angles. Hence, it is possible to describe the end effector by

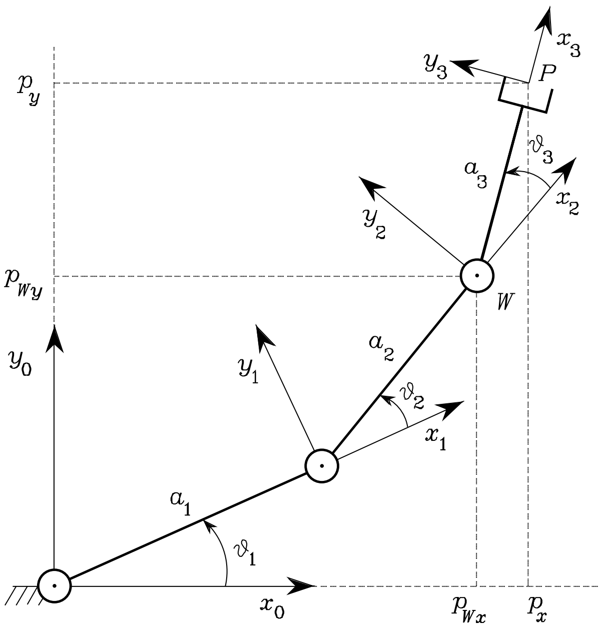
$$\mathbf{x}_e = \mathbf{k}(\mathbf{q}) \qquad\qquad \mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \boldsymbol{\Theta}_n^0 \end{bmatrix}$$

Effector Pose vector

- The space of $\mathbf{q}$ is called joint space, and the space of $\mathbf{x}_e$ is called operational space.

Example

Three link planar Manipulator:



$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3.$$

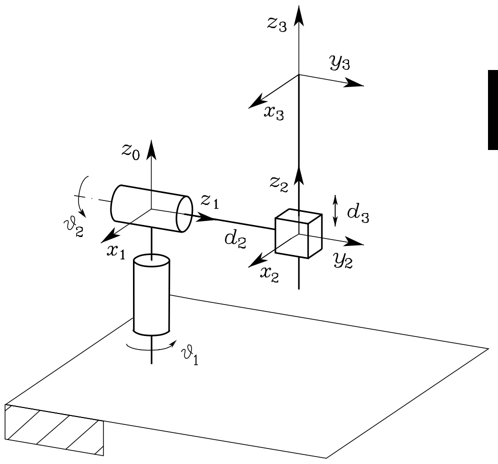
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_{ijk} \equiv \cos(\vartheta_i + \vartheta_j + \vartheta_k)$$

$$\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

Example

To obtain $\underbrace{\begin{bmatrix} \mathbf{r}_n^0 \\ \mathbf{\Theta}_n^0 \end{bmatrix}}_{\mathbf{x}_e} = \mathbf{k}(\mathbf{q})$



for more complex manipulators, the rotation matrix in $\mathbf{T}$ needs to be mapped to a standard rotation matrix in terms of yaw, pitch and roll, and then compute the Euler angles.

$$\underbrace{\begin{bmatrix} \mathbf{R}_3^0 & \mathbf{r}_{3/0}^0 \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}}_{\mathbf{T}_3^0(\mathbf{q})} = \underbrace{\begin{bmatrix} \mathbf{R}_1^0(\vartheta_1) & \mathbf{r}_{1/0}^0 \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}}_{\mathbf{A}_1^0(\vartheta_1)} \underbrace{\begin{bmatrix} \mathbf{R}_2^1(\vartheta_2) & \mathbf{r}_{2/1}^1(d_2) \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}}_{\mathbf{A}_2^1(\vartheta_2, d_2)} \underbrace{\begin{bmatrix} \mathbf{R}_3^2 & \mathbf{r}_{3/2}^2(d_3) \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}}_{\mathbf{A}_3^2(d_3)}$$

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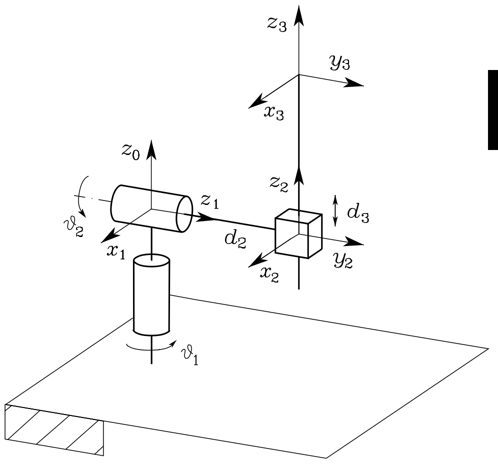
Example

To obtain

$$\underbrace{\begin{bmatrix}\mathbf{r}_n^0\\ \mathbf{\Theta}_n^0\end{bmatrix}}_{\mathbf{x}_e} = \mathbf{k}(\mathbf{q})$$



for more complex manipulators, the rotation matrix in $\mathbf{T}$ needs to be mapped to a standard rotation matrix in terms of yaw, pitch and roll, and then compute the Euler angles.

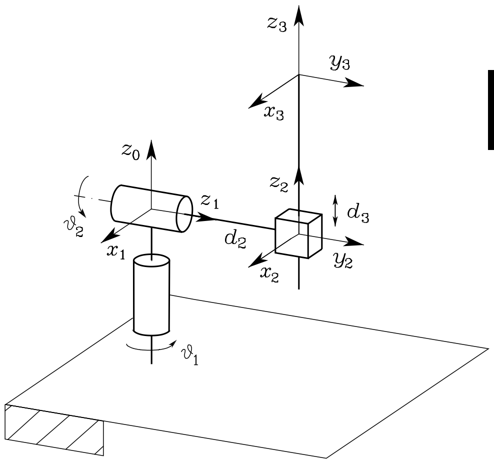
$$\mathbf{R}_1^0(\vartheta_1) = \underbrace{\begin{bmatrix} c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{e^{\vartheta_1 \mathbf{S}(\mathbf{e}_3)}} \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix} = \begin{bmatrix} c_1 & 0 & -s_1\\ s_1 & 0 & c_1\\ 0 & -1 & 0\end{bmatrix} \qquad \mathbf{r}_{1/0}^0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\mathbf{R}_2^1(\vartheta_2) = \underbrace{\begin{bmatrix} c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}}_{e^{\vartheta_2 \mathbf{S}(\mathbf{e}_3)}} \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} = \begin{bmatrix} c_2 & 0 & s_2\\ s_2 & 0 & -c_2\\ 0 & 1 & 0\end{bmatrix} \qquad \mathbf{r}_{2/1}^1(d_2) = \begin{bmatrix}0\\0\\d_2\end{bmatrix}$$

$$\mathbf{R}_3^2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \mathbf{r}_{3/2}^2(d_3) = \begin{bmatrix}0\\0\\d_3\end{bmatrix}$$

Example

Assembling the transformations



$$\mathbf{T}_3^0(\mathbf{q}) = \underbrace{\begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}_1^0(\vartheta_1)} \underbrace{\begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}_2^1(\vartheta_2, d_2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}_3^2(d_3)}$$

$$= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_3^0 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & c_\psi c_\phi + s_\psi s_\theta s_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}}_{e^{\psi \mathbf{S}(\mathbf{e}_3)} e^{\theta \mathbf{S}(\mathbf{e}_2)} e^{\phi \mathbf{S}(\mathbf{e}_1)}}$$

$$\mathbf{x}_e = \begin{bmatrix} \mathbf{r}_n^0 \\ \mathbf{\Theta}_n^0 \end{bmatrix}$$

$$\mathbf{\Theta}_3^0 = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \operatorname{atan2}(R_{32}, R_{33}) \\ \operatorname{atan2}\left(-R_{31}, \sqrt{R_{32}^2 + R_{33}^2}\right) \\ \operatorname{atan2}(R_{21}, R_{11}) \end{bmatrix}$$

Inverse Kinematic Models (IKM)

- The inverse kinematic model is a mapping from the effector position and orientation to the space of joint variables.

- The mapping can be obtained as a solution of the following problem:

$$\text{Given } (\mathbf{R}_e, \mathbf{r}_e), \text{ find } \mathbf{q}: \quad \begin{aligned} \mathbf{R}_e - \mathbf{R}_n^0(\mathbf{q}) &= \mathbf{0}, \\ \mathbf{r}_e - \mathbf{r}_n^0(\mathbf{q}) &= \mathbf{0} \end{aligned}$$

- This can be solved analytically only for some simple manipulators with a lot of algebraic intuition.

- The problem is complex because
  - The equations that need to be solved are nonlinear.
  - There may exist unique, multiple (finite or infinite), or no solutions.

General IKM solution as a Nonlinear Program

- One way to solve for the IKM is to use numerical optimisation and minimise a scalar quadratic cost function subject to nonlinear constraints. This is known in optimisation as a Nonlinear Program:

$$
\begin{aligned}
(\mathbf{q}^\star, \mathbf{x}^\star) = \arg\min_{\mathbf{q},\mathbf{x}} \, & \mathbf{q}^\top \mathbf{W} \mathbf{q} + (\mathbf{x} - \mathbf{x}_e)^\top \mathbf{K} (\mathbf{x} - \mathbf{x}_e) \\
\text{s.t.} \quad & \mathbf{x} - \mathbf{k}(\mathbf{q}) = \mathbf{0}
\end{aligned}
$$

which can be initialised with the current configuration and pose.

An alternative to the above optimisation problem is to use the Analytical Jacobian of the manipulator and run a simulation with a dynamic kinematic model—we'll see this next lecture.

Resources

- B. Siciliano, L. Sciavicco, L. Villani, G. Oriolo, *Robotics: Modelling, Planning and Control*
  - Section 2.7 – Homogeneous Transformations
  - Section 2.8 & 2.81 – Direct Kinematics (open-chain only)
  - Section 2.10 – Joint Space and Operational Space