Optimization Project Report

Portfolio Optimization with Risk Constraints

1. Problem Statement

We constructed an optimal investment portfolio across 5 distinct stocks (Tech_A, Finance_B, Energy_C, Healthcare_D, Consumer_E). The objective was to maximize expected returns while strictly adhering to defined risk management limits and budget constraints.

1.1 Mathematical Formulation

Objective Function:
Minimize the risk-adjusted negative return:
$f(w) = -\mu^T w + (\lambda/2)w^T \Sigma w$
Constraints:

1. **Budget:** $\Sigma w_i = 1$ (Full investment).
2. **Non-Negativity:** $w_i \geq 0$ (No short selling).
3. **Risk Limit:** $w^T \Sigma w \leq 0.025$ (Portfolio variance limit).

Parameters:

- **Returns (μ):** $[0.12, 0.10, 0.08, 0.11, 0.09]^T$ for Tech_A, Finance_B, Energy_C, Healthcare_D, Consumer_E respectively.
- **Risk (Σ):** A 5×5 covariance matrix representing volatility.
- **Risk Aversion (λ):** 1.0.

Note: We implemented the Penalty Method (required) and Projected Gradient Descent for comparison.

2. Methodology

2.1 Penalty Method

This approach converts hard constraints into soft penalty terms added to the objective.

Penalty Function:
$L(w, \rho) = f(w) + (\rho/2) [(\Sigma w_i - 1)^2 + \Sigma \max(0, -w_i)^2 + \max(0, w^T \Sigma w - 0.025)^2]$ Algorithm Strategy:
Initialize weights $w = [0.2]^5$ and $\rho = 1.0$. We utilize an outer loop that increases the penalty parameter ($\rho \leftarrow 5\rho$) after every inner optimization step. The process stops when the total constraint violation is $< 10^{-6}$.

2.2 Projected Gradient Descent

This approach enforces constraints geometrically via projection.

Algorithm Strategy:
Update weights using $w \leftarrow w - \alpha \nabla f(w)$ with a learning rate $\alpha = 0.1$. After each step, the weights are projected onto the probability simplex to ensure $\Sigma w_i = 1$ and $w_i \geq 0$. The process stops when $||\nabla f|| < 10^{-6}$ or after 2000 iterations.

2.3 Mathematical Derivations

Objective Gradient:

Given $f(w) = -\mu^T w + (\lambda/2)w^T \Sigma w$:

$\nabla f(w) = -\mu + \lambda \Sigma w$

(Using $\partial(w^T \Sigma w)/\partial w = 2\Sigma w$ for symmetric Σ)

Penalty Function:

$L(w, \rho) = f(w) + (\rho/2)[(\Sigma w_i - 1)^2 + \Sigma \max(0, -w_i)^2 + \max(0, w^T \Sigma w - 0.025)^2]$

Penalty Gradient:

$\nabla L(w,\rho) = -\mu + \lambda \Sigma w + \rho[(\Sigma w_i - 1)\cdot\mathbb{1} + \min(0,w) + 2\max(0, w^T \Sigma w - 0.025)\cdot\Sigma w]$

Update Rules:

- Inner: $w^{k+1} = w^k - \alpha\nabla L(w^k, \rho)$ where $\alpha = 0.01/(1+\rho/10)$
- Outer: $\rho^{j+1} = 5\rho^j$, starting from $\rho^0 = 1.0$
- Stop when constraint violation $< 10^{-6}$

Projected GD:

- Update: $\tilde{w}^{k+1} = w^k - 0.1\nabla f(w^k)$
- Project: $w^{k+1} = \text{ProjectSimplex}(\tilde{w}^{k+1})$ to enforce $\Sigma w_i = 1$, $w_i \geq 0$
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3. Results

3.1 Portfolio Allocation & Performance

Constraint Satisfaction:

Metric	Gradient Descent	Penalty Method
Tech_A	59.38%	33.38%
Finance_B	6.52%	18.83%
Energy_C	0.00%	6.89%
Healthcare_D	34.10%	27.69%
Consumer_E	0.00%	13.21%
Expected Return	**11.53%**	10.67%
Portfolio Variance	**2.31%**	1.23%
Iterations	2000	9 (Outer)

Both methods successfully satisfied all constraints: $\Sigma w_i = 1.000$, $w_i \geq 0$, and Variance $\leq 2.5\%$.

3.2 Visual Analysis
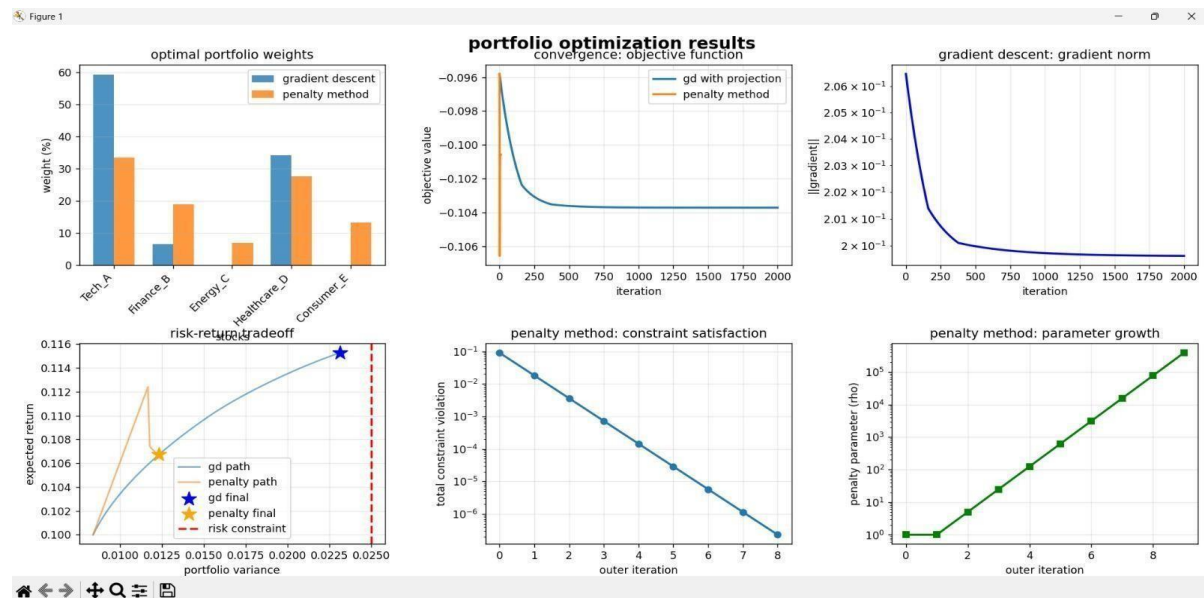
4. Discussion

4.1 Solution Divergence

The methods converged to different solutions due to the **approximation nature of the Penalty Method versus the geometric precision of Projected Gradient Descent**.

- **Gradient Descent** found a **boundary-optimal** portfolio (93.5% invested in Tech and Healthcare). It achieved a higher return (11.53%) with a variance of 2.31%, pushing effectively against the 2.5% risk limit to maximize utility.
- **Penalty Method** found a **conservative interior** portfolio (invested in all 5 stocks). It achieved a 10.67% return with a much lower variance of 1.23%.

4.2 Algorithm Behavior

- **Penalty Method:** Showed smooth constraint satisfaction (violations dropped from 0.09 $\rightarrow 10^{-6}$). However, because the penalty parameter ρ stabilizes, the method often converges to a safe solution inside the feasibility region rather than pushing aggressively to the boundary.
- **Gradient Descent:** Converged by immediately enforcing constraints. It effectively "slid" along the feasible boundary to find the high-return corner defined by the risk limit.
- GD reached iteration limit but satisfies all constraints—typical when optimal solution lies on constraint boundary.

4.3 Investment Implications & Trade-offs

- **Investor Profile:** The Gradient Descent portfolio optimizes the full risk budget (suitable for maximizing returns), while the Penalty Method portfolio offers a "safety buffer" (suitable for stability-focused investors).
- **Technical Trade-off:** Penalty methods are flexible for any constraint type but require careful tuning of ρ. Projection methods are efficient but require specific geometric solutions.

5. Conclusion

We successfully solved the portfolio optimization problem using penalty and projection methods. Both satisfied constraints but found different efficient portfolios, demonstrating how **penalty approximations can yield conservative interior solutions compared to projection methods.**

Key Findings:

1. Different valid solutions were found depending on algorithm precision.
2. Algorithm choice impacts solution characteristics (Aggressive vs. Conservative).
3. Penalty methods require gradual ρ increases to function correctly.
4. The aggressive portfolio (11.53% return) maximizes risk budget utility, while the diversified portfolio offers stability.