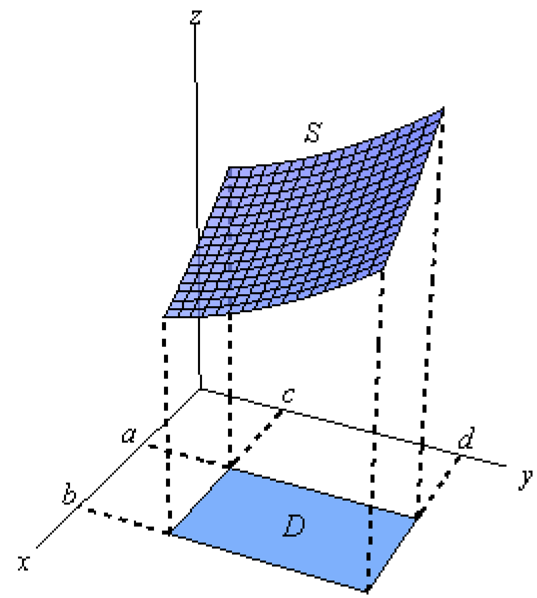
Paul's Online Notes

Section 6-3 : Surface Integrals

It is now time to think about integrating functions over some surface, S, in three-dimensional space. Let's start off with a sketch of the surface S since the notation can get a little confusing once we get into it. Here is a sketch of some surface S.



The region S will lie above (in this case) some region D that lies in the xy-plane. We used a rectangle here, but it doesn't have to be of course. Also note that we could just as easily looked at a surface S that was in front of some region D in the yz-plane or the xz-plane. Do not get so locked into the xy-plane that you can't do problems that have regions in the other two planes.

Now, how we evaluate the surface integral will depend upon how the surface is given to us. There are essentially two separate methods here, although as we will see they are really the same.

First, let's look at the surface integral in which the surface S is given by $z = g(x, y)$. In this case the surface integral is,

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA$$

Now, we need to be careful here as both of these look like standard double integrals. In fact the integral on the right is a standard double integral. The integral on the left however is a surface integral. The way to tell them apart is by looking at the differentials. The surface integral will have a dS while the standard double integral will have a dA.

In order to evaluate a surface integral we will substitute the equation of the surface in for z in the integrand and then add on the often messy square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

Note as well that there are similar formulas for surfaces given by $y = g(x, z)$ (with D in the xz-plane) and $x = g(y, z)$ (with D in the yz-plane). We will see one of these formulas in the examples and we'll leave the other to you to write down.

The second method for evaluating a surface integral is for those surfaces that are given by the parameterization,

$$\vec{r}(u, v) = x(u, v)\,\vec{i} + y(u, v)\,\vec{j} + z(u, v)\,\vec{k}$$

In these cases the surface integral is,

$$\iint_S f(x, y, z)\, dS = \iint_D f(\vec{r}(u, v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

where D is the range of the parameters that trace out the surface S.

Before we work some examples let's notice that since we can parameterize a surface given by $z = g(x, y)$ as,

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + g(x, y)\vec{k}$$

we can always use this form for these kinds of surfaces as well. In fact, it can be shown that,

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

for these kinds of surfaces. You might want to verify this for the practice of computing these cross products.

Let's work some examples.

Example 1 Evaluate $\displaystyle\iint\limits_S 6xy\,dS$ where S is the portion of the plane $x + y + z = 1$ that lies in the 1st octant and is in front of the yz-plane.

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Example 2 Evaluate $\displaystyle\iint\limits_S z\,dS$ where S is the upper half of a sphere of radius 2.

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Example 3 Evaluate $\displaystyle\iint\limits_S y\,dS$ where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.

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Example 4 Evaluate $\displaystyle\iint\limits_S y + z\,dS$ where S is the surface whose side is the cylinder $x^2 + y^2 = 3$, whose bottom is the disk $x^2 + y^2 \le 3$ in the xy-plane and whose top is the plane $z = 4 - y$.

Show Solution

 Page Last Modified : 5/26/2020