1.1 Linear Equations, Formulas, and Problem Solving

Learning Objectives

In Section 1.1 you will learn how to:

- ❑ **A.** Solve linear equations using properties of equality
- ❑ **B.** Recognize equations that are identities or contradictions
- ❑ **C.** Solve for a specified variable in a formula or literal equation
- ❑ **D.** Use the problem-solving guide to solve various problem types

In a study of algebra, you will encounter many **families of equations,** or groups of equations that share common characteristics. Of interest to us here is the family of **linear equations in one variable,** a study that lays the foundation for understanding more advanced families. In addition to *solving* linear equations, we'll use the skills we develop to *solve for a specified variable* in a formula, a practice widely used in science, business, industry, and research.

A. Solving Linear Equations Using Properties of Equality

An **equation** is *a statement that two expressions are equal*. From the expressions $3(x - 1) + x$ and $-x + 7$, we can form the equation

$$3(x - 1) + x = -x + 7,$$

which is a **linear equation in one variable.** To solve an equation, we attempt to find a specific input or x-value that will make the equation true, meaning the left-hand expression will be equal to the right. Using Table 1.1, we find that $3(x - 1) + x = -x + 7$ is a true equation when x is replaced by 2, and is a false equation otherwise. Replacement values that make the equation true are called **solutions** or **roots** of the equation.

Table 1.1

x	$3(x - 1) + x$	$-x + 7$
-2	-11	9
-1	-7	8
0	-3	7
1	1	6
2	**5**	**5**
3	9	4
4	13	3

CAUTION ► From Section R.6, an **algebraic *expression*** is a sum or difference of algebraic terms. Algebraic expressions can be simplified, evaluated or written in an equivalent form, but cannot be *"solved,"* since we're not seeking a specific value of the unknown.

Solving equations using a table is too time consuming to be practical. Instead we attempt to write a sequence of **equivalent equations,** each one simpler than the one before, until we reach a point where the solution is obvious. Equivalent equations are those that have the same solution set, and are obtained by using the distributive property to simplify the expressions on each side of the equation, and the additive and multiplicative properties of equality to obtain an equation of the form $x =$ constant.

The Additive Property of Equality	The Multiplicative Property of Equality
If A, B, and C represent algebraic expressions and $A = B$, then $A + C = B + C$	If A, B, and C represent algebraic expressions and $A = B$, then $AC = BC$ and $\frac{A}{C} = \frac{B}{C}$, $(C \neq 0)$

In words, the additive property says that like quantities, numbers or terms can be added to both sides of an equation. A similar statement can be made for the multiplicative property. These properties are combined into a general guide for solving linear equations, which you've likely encountered in your previous studies. Note that not all steps in the guide are required to solve every equation.

Guide to Solving Linear Equations in One Variable

- Eliminate parentheses using the distributive property, then combine any like terms.
- Use the additive property of equality to write the equation with all variable terms on one side, and all constants on the other. Simplify each side.
- Use the multiplicative property of equality to obtain an equation of the form $x =$ constant.
- For applications, answer in a complete sentence and include any units of measure indicated.

For our first example, we'll use the equation $3(x - 1) + x = -x + 7$ from our initial discussion.

EXAMPLE 1 ▶ **Solving a Linear Equation Using Properties of Equality**

Solve for x: $3(x - 1) + x = -x + 7$.

Solution ▶

$$\begin{aligned} 3(x-1)+x &= -x+7 && \text{original equation}\\ 3x-3+x &= -x+7 && \text{distributive property}\\ 4x-3 &= -x+7 && \text{combine like terms}\\ 5x-3 &= 7 && \text{add } x \text{ to both sides (additive property of equality)}\\ 5x &= 10 && \text{add 3 to both sides (additive property of equality)}\\ x &= 2 && \text{multiply both sides by } \tfrac{1}{5} \text{ or divide both sides by 5 (multiplicative property of equality)} \end{aligned}$$

As we noted in Table 1.1, the solution is $x = 2$.

Now try Exercises 7 through 12 ▶

To check a solution by substitution means we substitute the solution back into the original equation (this is sometimes called **back-substitution**), and verify the left-hand side is equal to the right. For Example 1 we have:

$$\begin{aligned} 3(x-1)+x &= -x+7 && \text{original equation}\\ 3(2-1)+2 &= -2+7 && \text{substitute 2 for } x\\ 3(1)+2 &= 5 && \text{simplify}\\ 5 &= 5\checkmark && \text{solution checks} \end{aligned}$$

If any coefficients in an equation are fractional, multiply both sides by the least common denominator (LCD) to *clear the fractions*. Since any decimal number can be written in fraction form, the same idea can be applied to decimal coefficients.

EXAMPLE 2 ▶ **Solving a Linear Equation with Fractional Coefficients**

Solve for n: $\frac{1}{4}(n + 8) - 2 = \frac{1}{2}(n - 6)$.

Solution ▶

$$\begin{aligned} \tfrac{1}{4}(n+8)-2 &= \tfrac{1}{2}(n-6) && \text{original equation}\\ \tfrac{1}{4}n+2-2 &= \tfrac{1}{2}n-3 && \text{distributive property}\\ \tfrac{1}{4}n &= \tfrac{1}{2}n-3 && \text{combine like terms}\\ 4(\tfrac{1}{4}n) &= 4(\tfrac{1}{2}n-3) && \text{multiply both sides by LCD} = 4\\ n &= 2n-12 && \text{distributive property}\\ -n &= -12 && \text{subtract } 2n\\ n &= 12 && \text{multiply by } -1 \end{aligned}$$

Verify the solution is $n = 12$ using back-substitution.

Now try Exercises 13 through 30 ▶

☑ **A.** You've just learned how to solve linear equations using properties of equality

B. Identities and Contradictions

Example 1 illustrates what is called a **conditional equation,** since the equation is true for $x = 2$, but false for all other values of x. The equation in Example 2 is also conditional. An **identity** is an equation that is *always true,* no matter what value is substituted for the variable. For instance, $2(x + 3) = 2x + 6$ is an identity with a solution set of all real numbers, written as $\{x|x \in \mathbb{R}\}$, or $x \in (-\infty, \infty)$ in interval notation. **Contradictions** are equations that are *never true,* no matter what real number is substituted for the variable. The equations $x - 3 = x + 1$ and $-3 = 1$ are contradictions. To state the solution set for a contradiction, we use the symbol "$\emptyset$" (the null set) or "{ }" (the empty set). Recognizing these special equations will prevent some surprise and indecision in later chapters.

EXAMPLE 3 ▶ **Solving an Equation That Is a Contradiction**

Solve for x: $2(x - 4) + 10x = 8 + 4(3x + 1)$, and state the solution set.

Solution ▶

$$
\begin{aligned}
2(x - 4) + 10x &= 8 + 4(3x + 1) && \text{original equation}\\
2x - 8 + 10x &= 8 + 12x + 4 && \text{distributive property}\\
12x - 8 &= 12x + 12 && \text{combine like terms}\\
-8 &= 12 && \text{subtract } 12x
\end{aligned}
$$

Since -8 is never equal to 12, the original equation is a contradiction. The solution is the empty set { }.

Now try Exercises 31 through 36 ▶

In Example 3, our attempt to solve for x ended with all variables being eliminated, leaving an equation that is *always false*—a contradiction (-8 is never equal to 12). There is nothing wrong with the solution process, the result is simply telling us the original equation has *no solution.* In other equations, the variables may once again be eliminated, but leave a result that is *always true*—an identity.

☑ **B.** You've just learned how to recognize equations that are identities or contradictions

C. Solving for a Specified Variable in Literal Equations

A **formula** is an equation that models a known relationship between two or more quantities. A **literal equation** is simply one that has two or more variables. Formulas are a type of literal equation, but not every literal equation is a formula. For example, the formula $A = P + PRT$ models the growth of money in an account earning simple interest, where A represents the total amount accumulated, P is the initial deposit, R is the annual interest rate, and T is the number of years the money is left on deposit. To *describe* $A = P + PRT$, we might say the formula has been "solved for A" or that "A is written in terms of P, R, and T." In some cases, before using a formula it may be convenient to solve for one of the other variables, say P. In this case, P is called the **object variable.**

EXAMPLE 4 ▶ **Solving for Specified Variable**

Given $A = P + PRT$, write P in terms of A, R, and T (solve for P).

Solution ► Since the object variable occurs in more than one term, we first apply the distributive property.

$$A = \mathbf{P} + \mathbf{P}RT \qquad \text{focus on } \mathbf{P}\text{—the object variable}$$
$$A = \mathbf{P}(1 + RT) \qquad \text{factor out } \mathbf{P}$$
$$\frac{A}{1 + RT} = \frac{\mathbf{P}(1 + RT)}{(1 + RT)} \qquad \text{solve for } \mathbf{P} \text{ [divide by } (1 + RT)]$$
$$\frac{A}{1 + RT} = \mathbf{P} \qquad \text{result}$$

Now try Exercises 37 through 48 ►

We solve literal equations for a specified variable using the same methods we used for other equations and formulas. Remember that it's good practice to *focus on the object variable* to help guide you through the solution process, as again shown in Example 5.

EXAMPLE 5 ► Solving for a Specified Variable

Given $2x + 3y = 15$, write y in terms of x (solve for y).

Solution ►

$$2x + 3\mathbf{y} = 15 \qquad \text{focus on the object variable}$$
$$3\mathbf{y} = -2x + 15 \qquad \text{subtract } 2x \text{ (isolate term with } y)$$
$$\tfrac{1}{3}(3\mathbf{y}) = \tfrac{1}{3}(-2x + 15) \qquad \text{multiply by } \tfrac{1}{3} \text{ (solve for } y)$$
$$\mathbf{y} = \tfrac{-2}{3}x + 5 \qquad \text{distribute and simplify}$$

Now try Exercises 49 through 54 ►

WORTHY OF NOTE

In Example 5, notice that in the second step we wrote the subtraction of $2x$ as $-2x + 15$ instead of $15 - 2x$. For reasons that become clear later in this chapter, we generally write variable terms before constant terms.

Literal Equations and General Solutions

Solving literal equations for a specified variable can help us develop the general solution for an entire family of equations. This is demonstrated here for the family of linear equations written in the form $ax + b = c$. A side-by-side comparison with a specific linear equation demonstrates that identical ideas are used.

Specific Equation		Literal Equation
$2\mathbf{x} + 3 = 15$	focus on object variable	$a\mathbf{x} + b = c$
$2\mathbf{x} = 15 - 3$	subtract constant	$a\mathbf{x} = c - b$
$\mathbf{x} = \dfrac{15 - 3}{2}$	divide by coefficient	$\mathbf{x} = \dfrac{c - b}{a}$

Of course the solution on the left would be written as $x = 6$ and checked in the original equation. On the right we now have a general formula for all equations of the form $ax + b = c$.

EXAMPLE 6 ► Solving Equations of the Form $ax + b = c$ Using the General Formula

Solve $6x - 1 = -25$ using the formula just developed, and check your solution in the original equation.

Solution ▶ For this equation, $a = 6$, $b = -1$, and $c = -25$, this gives

$$\begin{aligned} x &= \frac{c - b}{a} \\ &= \frac{-25 - (-1)}{6} \\ &= \frac{-24}{6} \\ &= -4 \end{aligned}$$

$$\text{Check:} \quad \begin{aligned} 6x - 1 &= -25 \\ 6(-4) - 1 &= -25 \\ -24 - 1 &= -25 \\ -25 &= -25 \checkmark \end{aligned}$$

Now try Exercises 55 through 60 ▶

WORTHY OF NOTE

Developing a general solution for the linear equation $ax + b = c$ seems to have little practical use. But in Section 1.5 we'll use this idea to develop a general solution for *quadratic equations,* a result with much greater significance.

C. You've just learned how to solve for a specified variable in a formula or literal equation

D. Using the Problem-Solving Guide

Becoming a good problem solver is an evolutionary process. Over time and with continued effort, your problem-solving skills grow, as will your ability to solve a wider range of applications. Most good problem solvers develop the following characteristics:

- A positive attitude
- A mastery of basic facts
- Strong mental arithmetic skills
- Good mental-visual skills
- Good estimation skills
- A willingness to persevere

These characteristics form a solid basis for applying what we call the **Problem-Solving Guide,** which simply organizes the basic elements of good problem solving. Using this guide will help save you from two common stumbling blocks—indecision and not knowing where to start.

Problem-Solving Guide

- **Gather and organize information.**
  Read the problem several times, forming a mental picture as you read. *Highlight key phrases.* List given information, including any related formulas. *Clearly identify what you are asked to find.*
- **Make the problem visual.**
  Draw and label a diagram or create a table of values, as appropriate. This will help you see how different parts of the problem fit together.
- **Develop an equation model.**
  Assign a variable to represent what you are asked to find and build any related expressions referred to in the exercise. Write an equation model from the information given in the exercise. *Carefully reread the exercise to double-check your equation model.*
- **Use the model and given information to solve the problem.**
  Substitute given values, then simplify and solve. State the answer in sentence form, and check that the answer is reasonable. Include any units of measure indicated.

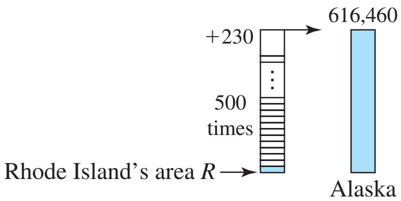
General Modeling Exercises

In Section R.2, we learned to translate word phrases into symbols. This skill is used to build equations from information given in paragraph form. Sometimes the variable *occurs more than once* in the equation, because two different items in the same exercise are related. If the relationship involves a comparison of size, we often use line segments or bar graphs to model the relative sizes.

EXAMPLE 7 ▶ **Solving an Application Using the Problem-Solving Guide**

The largest state in the United States is Alaska (AK), which covers an area that is 230 square miles (mi^2) more than 500 times that of the smallest state, Rhode Island (RI). If they have a combined area of 616,460 mi^2, how many square miles does each cover?

Solution ▶ Combined area is 616,460 mi^2, AK covers 230 more than 500 times the area of RI. — gather and organize information; highlight any key phrases



make the problem visual

Let R represent the area of Rhode Island. — assign a variable

Then $500R + 230$ represents Alaska's area. — build related expressions

Rhode Island's area + Alaska's area = Total

$R + (500R + 230) = 616{,}460$ — write the equation model

$501R = 616{,}230$ — combine like terms, subtract 230

$R = 1230$ — divide by 501

Rhode Island covers an area of 1230 mi^2, while Alaska covers an area of $500(1230) + 230 = 615{,}230$ mi^2.

Now try Exercises 63 through 68 ▶

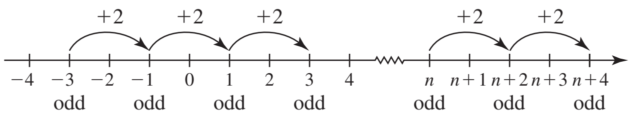
Consecutive Integer Exercises

Exercises involving **consecutive integers** offer excellent practice in assigning variables to unknown quantities, building related expressions, and the problem-solving process in general. We sometimes work with consecutive **odd** integers or consecutive **even** integers as well.

EXAMPLE 8 ▶ **Solving a Problem Involving Consecutive Odd Integers**

The sum of three consecutive *odd* integers is 69. What are the integers?

Solution ▶ The sum of three consecutive odd integers . . . — gather/organize information; highlight any key phrases



make the problem visual

Let n represent the smallest consecutive odd integer, then $n + 2$ represents the second odd integer and $(n + 2) + 2 = n + 4$ represents the third. — assign a variable; build related expressions

In words: first + second + third odd integer = 69 — write the equation model

$n + (n + 2) + (n + 4) = 69$ — equation model

$3n + 6 = 69$ — combine like terms

$3n = 63$ — subtract 6

$n = 21$ — divide by 3

The odd integers are $n = 21$, $n + 2 = 23$, and $n + 4 = 25$.

$21 + 23 + 25 = 69$ ✓

Now try Exercises 69 through 72 ▶

WORTHY OF NOTE

The number line illustration in Example 8 shows that consecutive odd integers are *two units* apart and the related expressions were built accordingly: n, $n + 2$, $n + 4$, and so on. In particular, *we cannot use* n, $n + 1$, $n + 3$, . . . because n and $n + 1$ are *not two units apart*. If we know the exercise involves *even* integers instead, the same model is used, since even integers are also two units apart. For *consecutive* integers, the labels are n, $n + 1$, $n + 2$, and so on.

Solve for the specified variable in each formula or literal equation.

8. $V = \pi r^2 h$ for h

9. $P = 2L + 2W$ for L

10. $ax + b = c$ for x

11. $2x - 3y = 6$ for y

Use the problem-solving guidelines (page 6) to solve the following applications.

12. At a large family reunion, two kegs of lemonade are available. One is 2% sugar (too sour) and the second is 7% sugar (too sweet). How many gallons of the 2% keg, must be mixed with 12 gallons of the 7% keg to get a 5% mix?

13. A rectangular window with a width of 3 ft and a height of 4 ft is topped by a semi-circular window. Find the total area of the window.

14. Two cyclists start from the same location and ride in opposite directions, one riding at 15 mph and the other at 18 mph. If their radio phones have a range of 22 mi, how many minutes will they be able to communicate?

SECTION 1.2 Linear Inequalities in One Variable

KEY CONCEPTS

- Inequalities are solved using properties similar to those for solving equalities (see page 15). The one exception is the multiplicative property of inequality, since the truth of the resulting statement depends on whether a positive or negative quantity is used.
- Solutions to an inequality can be graphed on a number line, stated using a simple inequality, or expressed using set or interval notation.
- For two sets A and B: A intersect B $(A \cap B)$ is the set of elements in both A **and** B (i.e., *elements common to both sets*). A union B $(A \cup B)$ is the set of elements in either A **or** B (i.e., *all elements from either set*).
- Compound inequalities are formed using the conjunctions "and"/"or." These can be either a joint inequality as in $-3 < x \le 5$, or a disjoint inequality, as in $x < -2$ or $x > 7$.

EXERCISES

Use inequality symbols to write a mathematical model for each statement.

15. You must be 35 yr old or older to run for president of the United States.

16. A child must be under 2 yr of age to be admitted free.

17. The speed limit on many interstate highways is 65 mph.

18. Our caloric intake should not be less than 1200 calories per day.

Solve the inequality and write the solution using interval notation.

19. $7x > 35$

20. $-\frac{3}{5}m < 6$

21. $2(3m - 2) \le 8$

22. $-1 < \frac{1}{3}x + 2 \le 5$

23. $-4 < 2b + 8$ and $3b - 5 > -32$

24. $-5(x + 3) > -7$ or $x - 5.2 > -2.9$

25. Find the allowable values for each of the following. Write your answer in interval notation.

a. $\dfrac{7}{n - 3}$ **b.** $\dfrac{5}{2x - 3}$ **c.** $\sqrt{x + 5}$ **d.** $\sqrt{-3n + 18}$

26. Latoya has earned grades of 72%, 95%, 83%, and 79% on her first four exams. What grade must she make on her fifth and last exam so that her average is 85% or more?