1. Introduction

A polynomial function is a function such as a quadratic, a cubic, a quartic, and so on, involving only non-negative integer powers of x. We can give a general defintion of a polynomial, and define its degree.

2. What is a polynomial?

A polynomial of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where the a's are real numbers (sometimes called the *coefficients* of the polynomial). Although this general formula might look quite complicated, particular examples are much simpler. For example,

$$f(x) = 4x^3 - 3x^2 + 2$$

is a polynomial of degree 3, as 3 is the highest power of x in the formula. This is called a cubic polynomial, or just a *cubic*. And

$$f(x) = x^7 - 4x^5 + 1$$

is a polynomial of degree 7, as 7 is the highest power of x. Notice here that we don't need every power of x up to 7: we need to know only the *highest* power of x to find out the degree. An example of a kind you may be familiar with is

$$f(x) = 4x^2 - 2x - 4$$

which is a polynomial of degree 2, as 2 is the highest power of x. This is called a *quadratic*.

Functions containing other operations, such as square roots, are not polynomials. For example,

$$f(x) = 4x^3 + \sqrt{x} - 1$$

is not a polynomial as it contains a square root. And

$$f(x) = 5x^4 - 2x^2 + 3/x$$

is not a polynomial as it contains a 'divide by x'.

Key Point

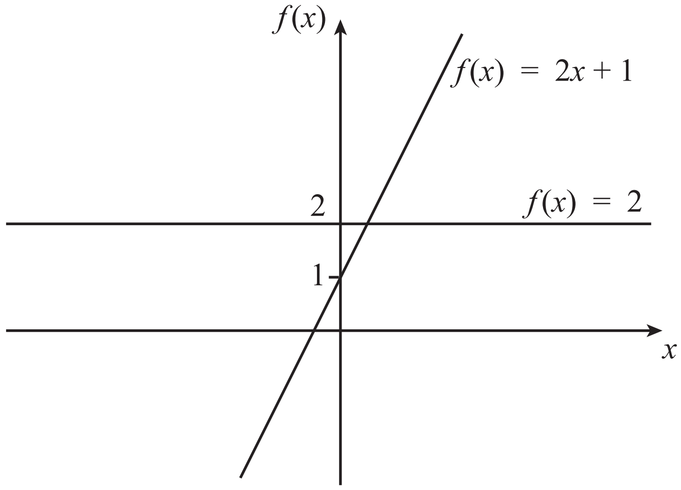
A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\,.$$

The *degree* of a polynomial is the highest power of x in its expression. Constant (non-zero) polynomials, linear polynomials, quadratics, cubics and quartics are polynomials of degree 0, 1, 2 , 3 and 4 respectively. The function $f(x) = 0$ is also a polynomial, but we say that its degree is 'undefined'.

3. Graphs of polynomial functions

We have met some of the basic polynomials already. For example, $f(x) = 2$ is a constant function and $f(x) = 2x + 1$ is a linear function.

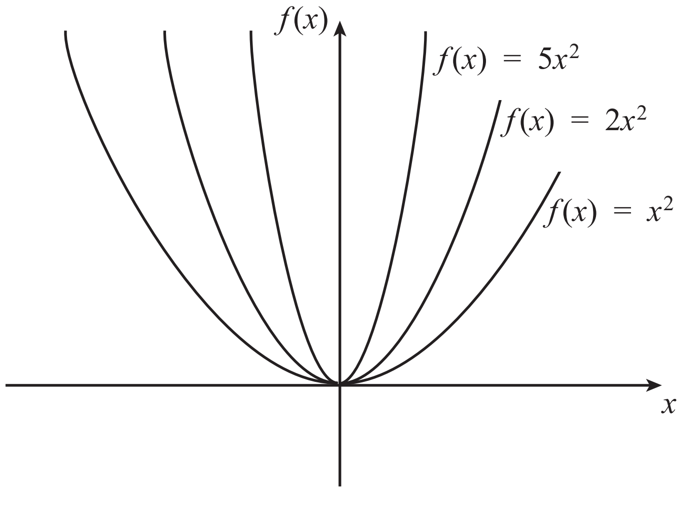


It is important to notice that the graphs of constant functions and linear functions are always straight lines.

We have already said that a quadratic function is a polynomial of degree 2. Here are some examples of quadratic functions:

$$f(x) = x^2, \qquad f(x) = 2x^2, \qquad f(x) = 5x^2.$$

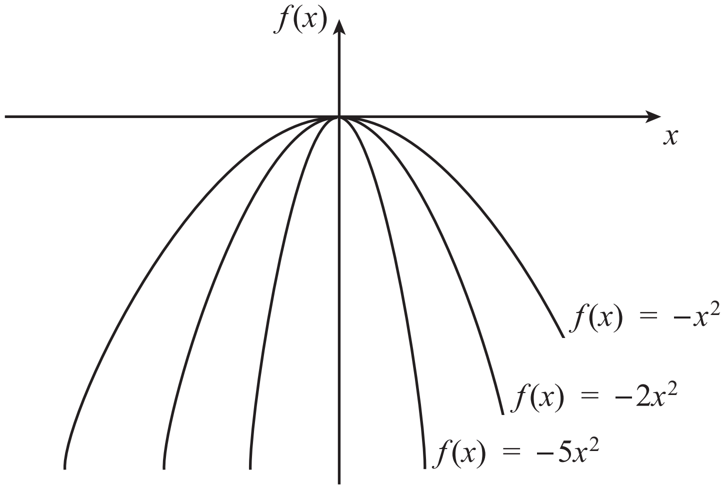
What is the impact of changing the coefficient of x^2 as we have done in these examples? One way to find out is to sketch the graphs of the functions.



You can see from the graph that, as the coefficient of x^2 is increased, the graph is stretched vertically (that is, in the y direction).

What will happen if the coefficient is negative? This will mean that all of the positive $f(x)$ values will now become negative. So what will the graphs of the functions look like? The functions are now

$$f(x) = -x^2, \qquad f(x) = -2x^2, \qquad f(x) = -5x^2.$$

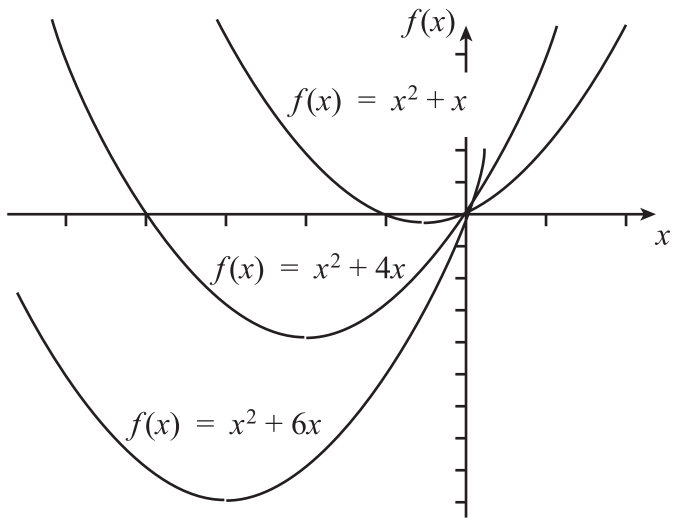


Notice here that all of these graphs have actually been reflected in the x-axis. This will always happen for functions of any degree if they are multiplied by -1.

Now let us look at some other quadratic functions to see what happens when we vary the coefficient of x, rather than the coefficient of x^2. We shall use a table of values in order to plot the graphs, but we shall fill in only those values near the turning points of the functions.

x	-5	-4	-3	-2	-1	0	1	2
x^2+x			6	2	0	0	2	6
x^2+4x		0	-3	-4	-3	0		
x^2+6x	-5	-8	-9	-8	-5			

You can see the symmetry in each row of the table, demonstrating that we have concentrated on the region around the turning point of each function. We can now use these values to plot the graphs.

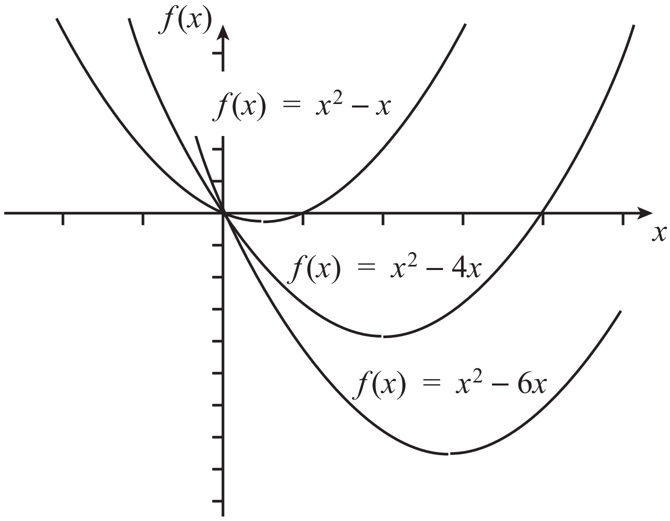


As you can see, increasing the positive coefficient of x in this polynomial moves the graph down and to the left.

What happens if the coefficient of x is negative?

x	-2	-1	0	1	2	3	4	5
$x^2 - x$	6	2	0	0	2	6		
$x^2 - 4x$			0	-3	-4	-3	0	
$x^2 - 6x$				-5	-8	-9	-8	-5

Again we can use these tables of values to plot the graphs of the functions.

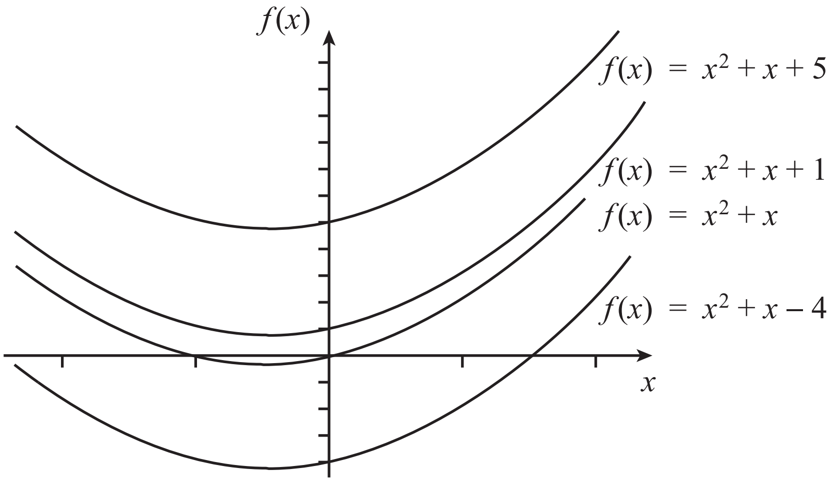


As you can see, increasing the negative coefficient of x (in absolute terms) moves the graph down and to the right.

So now we know what happens when we vary the x^2 coefficient, and what happens when we vary the x coefficient. But what happens when we vary the constant term at the end of our polynomial? We already know what the graph of the function $f(x) = x^2 + x$ looks like, so how does this differ from the graph of the functions $f(x) = x^2 + x + 1$, or $f(x) = x^2 + x + 5$, or $f(x) = x^2 + x - 4$? As usual, a table of values is a good place to start.

x	-2	-1	0	1	2
$x^2 + x$	2	0	0	2	6
$x^2 + x + 1$	3	1	1	3	7
$x^2 + x + 5$	7	5	5	7	11
$x^2 + x - 4$	-2	-4	-4	-2	2

Our table of values is particularly easy to complete since we can use our answers from the $x^2 + x$ column to find everything else. We can use these tables of values to plot the graphs of the functions.

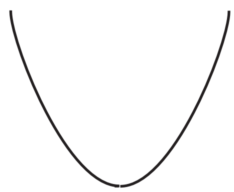


As we can see straight away, varying the constant term translates the $x^2 + x$ curve vertically. Furthermore, the value of the constant is the point at which the graph crosses the $f(x)$ axis.

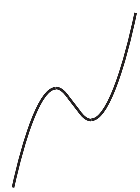
4. Turning points of polynomial functions

A turning point of a function is a point where the graph of the function changes from sloping downwards to sloping upwards, or vice versa. So the gradient changes from negative to positive, or from positive to negative. Generally speaking, curves of degree n can have up to $(n-1)$ turning points.

For instance, a quadratic has only one turning point.



A cubic could have up to two turning points, and so would look something like this.



However, some cubics have fewer turning points: for example $f(x) = x^3$. But no cubic has more than two turning points.