10 The Exponential and Logarithm Functions

Some texts define e^x to be the inverse of the function $\ln x = \int_1^x 1/t\,dt$. This approach enables one to give a quick definition of e^x and to overcome a number of technical difficulties, but it is an unnatural way to define exponentiation. Here we give a complete account of how to define $\exp_b(x) = b^x$ as a continuation of rational exponentiation. We prove that $\exp_b$ is differentiable and show how to introduce the number e.

Powers of a Number

If n is a positive integer and b is a real number, the power b^n is defined as the product of b with itself n times:

$$b^n = b \cdot b \cdot \ldots \cdot b \quad (n \text{ times})$$

If b is unequal to 0, so is b^n, and we define

$$b^{-n} = \frac{1}{b^n} = \frac{1}{b} \cdots \frac{1}{b} \quad (n \text{ times}).$$

We also set

$$b^0 = 1$$

If b is positive, we define $b^{1/2} = \sqrt{b}$, $b^{1/3} = \sqrt[3]{b}$, etc., since we know how to take roots of numbers. Recall that $\sqrt[n]{b}$ is the unique positive number such that $(\sqrt[n]{b})^n = b$; i.e., $\sqrt[n]{y}$ is the inverse function of x^n. Formally, for n a positive integer, we define

$$b^{1/n} = \sqrt[n]{b} \quad \text{(the positive } n\text{th root of } b)$$

and we define

$$b^{-1/n} = \frac{1}{b^{1/n}}$$

Worked Example 1 Express $9^{-1/2}$ and $625^{-1/4}$ as fractions.

Solution $9^{-1/2} = 1/9^{1/2} = 1/\sqrt{9} = \frac{1}{3}$ and $625^{-1/4} = 1/\sqrt[4]{625} = \frac{1}{5}$.

Worked Example 2 Show that, if we assume the rule $b^{x+y} = b^x b^y$, we are *forced* to define $b^0 = 1$ and $b^{-x} = 1/b^x$.

Solution If we set $x = 1$ and $y = 0$, we get $b^{1+0} = b^1 \cdot b^0$, i.e., $b = b \cdot b^0$ so $b^0 = 1$. Next, if we set $y = -x$, we get $b^{x-x} = b^x b^{-x}$, i.e., $1 = b^0 = b^x b^{-x}$, so $b^{-x} = 1/b^x$. (Notice that this is an argument for *defining* b^0, $b^{-1/n}$, and b^{-n} the way we did. It does not prove it. Once powers are defined, and only then, can we claim that rules like $b^{x+y} = b^x b^y$ are true.)

Finally, if r is a rational number, we define b^r by expressing r as a quotient m/n of positive integers and defining

$$b^r = (b^m)^{1/n}$$

We leave it to the reader (Exercise 8) to verify that the result is independent of the way in which r is expressed as a quotient of integers. Note that $b^{m/n}$ is always positive, even if m or n is negative.

Thus the laws of exponents,

$$b^n b^m = b^{n+m} \quad \text{and} \quad b^n/b^m = b^{n-m} \tag{i}$$

$$(b^n)^m = b^{nm} \tag{ii}$$

$$(bc)^n = b^n c^n \tag{iii}$$

which are easily seen for integer powers from the definition of power, may now be extended to rational powers.

Worked Example 3 Prove (i) for rational exponents, namely,

$$b^{m_1/n_1}\, b^{m_2/n_2} = b^{(m_1/n_1)+(m_2/n_2)} \tag{i'}$$

Solution From (iii) we get

$$(b^{m_1/n_1}\, b^{m_2/n_2})^{n_1 n_2} = (b^{m_1/n_1})^{n_1 n_2} (b^{m_2/n_2})^{n_1 n_2}$$

By (ii) this equals

$$((b^{m_1/n_1})^{n_1})^{n_2} ((b^{m_2/n_2})^{n_2})^{n_1}$$

By definition of $b^{m/n}$, we have $(b^{m/n})^n = b^m$, so the preceding expression is

$$(b^{m_1})^{n_2} (b^{m_2})^{n_1} = b^{m_1 n_2}\, b^{m_2 n_1}$$

again by (ii), which equals

$$b^{m_1 n_2 + m_2 n_1}$$

by (i).

Hence

$$(b^{m_1/n_1}\, b^{m_2/n_2})^{n_1 n_2} = b^{m_1 n_2 + m_2 n_1}$$

so

$$b^{m_1/n_1}\, b^{m_2/n_2} = (b^{(m_1 n_2 + m_2 n_1)})^{1/n_1 n_2}$$

By the definition $b^{m/n} = (b^m)^{1/n}$, this equals

$$b^{(m_1 n_2 + m_2 n_1)/n_1 n_2} = b^{(m_1/n_1) + (m_2/n_2)}$$

as required.

Similarly, we can prove (ii) and (iii) for rational exponents:

$$(b^{m_1/n_1})^{m_2/n_2} = b^{m_1 m_2/n_1 n_2} \qquad \textbf{(ii')}$$
$$(bc)^{m/n} = b^{m/n} c^{m/n} \qquad \textbf{(iii')}$$

Worked Example 4 Simplify $(x^{2/3}(x^{-3/2}))^{8/3}$.

Solution $(x^{2/3}x^{-3/2})^{8/3} = (x^{(2/3)-(3/2)})^{8/3} = (x^{-5/6})^{8/3} = x^{-20/9} = 1/\sqrt[9]{x^{20}}$.

Worked Example 5 If $b > 1$ and p and q are rational numbers with $p < q$, prove that $b^p < b^q$.

Solution By the laws of exponents, $b^q/b^p = b^{q-p}$. Let $z = q - p > 0$. We shall show that $b^z > 1$, so $b^q/b^p > 1$ and thus $b^q > b^p$.

Suppose that $z = m/n$. Then $b^z = (b^m)^{1/n}$. However, $b^m = b \cdot b \cdot \ldots \cdot b$ (m times) > 1 since $b > 1$, and $(b^m)^{1/n} > 1$ since $b^m > 1$. (The nth root $c^{1/n}$ of a number $c > 1$ is also greater than 1, since, if $c^{1/n} \leqslant 1$, then $(c^{1/n})^n = c \leqslant 1$ also.) Thus $b^z > 1$ if $z > 0$, and the solution is complete.

As a consequence, we can say that if $b \geqslant 1$ and $p \leqslant q$, then $b^p \leqslant b^q$.

Solved Exercises*

1. Find $8^{-3/2}$ and $8^{1/2}$.

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*Solutions appear in the Appendix.

2. Find $9^{3/2}$.
3. Simplify $(x^{2/3})^{5/2}/x^{1/4}$.
4. Verify **(ii)** if either p or q is zero.

Exercises

1. Simplify by writing with rational exponents:

   (a) $\left[\dfrac{\sqrt[4]{ab^3}}{\sqrt{b}}\right]^6$ (b) $\sqrt[3]{\dfrac{\sqrt{a^3b^9}}{\sqrt[4]{a^6b^6}}}$

2. Factor (i.e., write in the form $(x^a \pm y^b)(x^c \pm y^d)$, a, b, c, d rational numbers):

   (a) $x - \sqrt{xy} - 2y$ (b) $x - y$
   (c) $\sqrt[3]{xy^2} + \sqrt[3]{yx^2} + x + y$ (d) $x - 2\sqrt{x} - 8$
   (e) $x + 2\sqrt{3x} + 3$

3. Solve for x:

   (a) $10^x = 0.001$ (b) $5^x = 1$
   (c) $2^x = 0$ (d) $x - 2\sqrt{x} - 3 = 0$ (factor)

4. Do the following define the same function on (a) $(-1, 1)$, (b) $(0, 3)$?

   $f_1(x) = x^{1/2}$
   $f_2(x) = \sqrt[4]{x^2}$
   $f_3(x) = (\sqrt[4]{x})^2$ (which, if any, are the same?)

5. Based on the laws of exponents which we want to hold true, what would be your choice for the value of 0^0? Discuss.

6. Using rational exponents and the laws of exponents, verify the following root formulas.

   (a) $\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}$ (b) $\sqrt[ac]{x^{ab}} = \sqrt[c]{x^b}$

7. Find all real numbers x which satisfy the following inequalities.

   (a) $x^{1/3} > x^{1/2}$ (b) $x^{1/2} > x^{1/3}$
   (c) $x^{1/p} > x^{1/q}$, p, q positive odd integers, $p > q$
   (d) $x^{1/q} > x^{1/p}$, p, q positive odd integers, $p > q$

8. Suppose that $b > 0$ and that $p = m/n = m'/n'$. Show, using the definition of rational powers, that $b^{m/n} = b^{m'/n'}$; i.e., b^p is unambiguously defined.

The Function $f(x) = b^x$

Having defined $f(x) = b^x$ if x is rational, we wish to extend the definition to allow x to range through all real numbers. If we take, for example, $b = 2$ and compute some values, we get:

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2^x	0.25	0.354...	0.5	0.707...	1	1.414...	2	2.828...	4

These values may be plotted to get an impression of the graph (Fig. 10-1). It seems natural to conjecture that the graph can be filled in with a smooth curve, i.e., that b^x makes sense for all x.

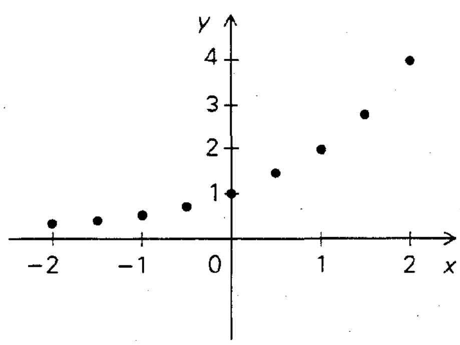

Fig. 10-1 The plot of some points $(x, 2^x)$ for rational x.

To calculate a number like $2^{\sqrt{3}}$, we should be able to take a decimal approximation to $\sqrt{3} \approx 1.732050808 \ldots$, say, 1.7320, calculate the rational power $2^{1.7320} = 2^{17320/10000}$, and hope to get an approximation to $2^{\sqrt{3}}$. Experimentally, this leads to reasonable data. On a calculator, one finds the following:

x	2^x
1	2
1.7	3.24900958
1.73	3.31727818
1.732	3.32188010
1.73205	3.32199523
1.7320508	3.32199707
1.732050808	3.32199708