Paul's Online Notes

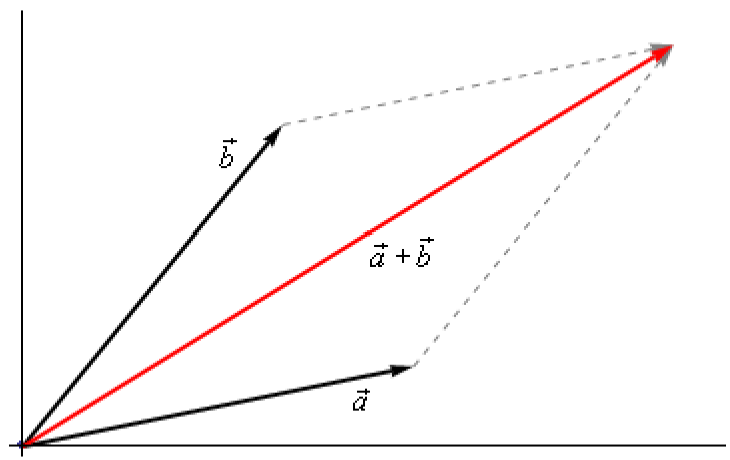
Section 5-2 : Vector Arithmetic

In this section we need to have a brief discussion of vector arithmetic.

We'll start with **addition** of two vectors. So, given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the addition of the two vectors is given by the following formula.

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The following figure gives the geometric interpretation of the addition of two vectors.

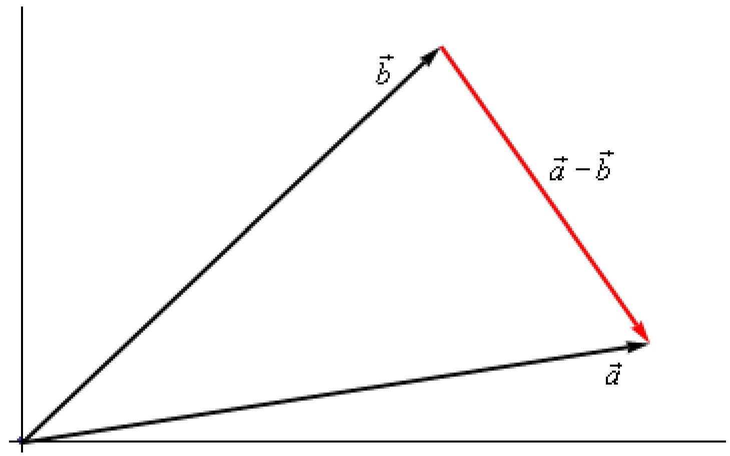


This is sometimes called the **parallelogram law** or **triangle law**.

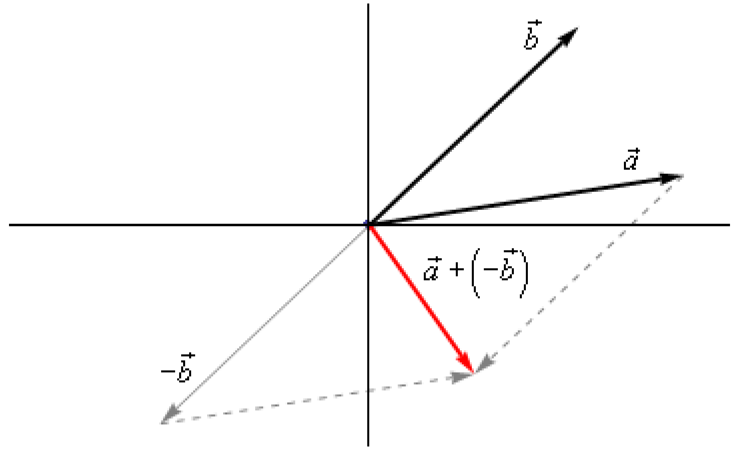
Computationally, **subtraction** is very similar. Given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the difference of the two vectors is given by,

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Here is the geometric interpretation of the difference of two vectors.



It is a little harder to see this geometric interpretation. To help see this let's instead think of subtraction as the addition of $\vec{a}$ and $-\vec{b}$. First, as we'll see in a bit $-\vec{b}$ is the same vector as $\vec{b}$ with opposite signs on all the components. In other words, $-\vec{b}$ goes in the opposite direction as $\vec{b}$. Here is the vector set up for $\vec{a} + \left(-\vec{b}\right)$.



As we can see from this figure we can move the vector representing $\vec{a}+\left(-\vec{b}\right)$ to the position we've got in the first figure showing the difference of the two vectors.

Note that we can't add or subtract two vectors unless they have the same number of components. If they don't have the same number of components then addition and subtraction can't be done.

The next arithmetic operation that we want to look at is **scalar multiplication**. Given the vector $\vec{a}=\langle a_1,a_2,a_3\rangle$ and any number c the scalar multiplication is,

$$c\vec{a}=\langle ca_1,ca_2,ca_3\rangle$$

So, we multiply all the components by the constant c. To see the geometric interpretation of scalar multiplication let's take a look at an example.

Example 1 For the vector $\vec{a}=\langle 2,4\rangle$ compute $3\vec{a}$, $\frac{1}{2}\vec{a}$ and $-2\vec{a}$. Graph all four vectors on the same axis system.

Show Solution

In the previous example we can see that if c is positive all scalar multiplication will do is stretch (if $c > 1$) or shrink (if $c < 1$) the original vector, but it won't change the direction. Likewise, if c is negative scalar multiplication will switch the direction so that the vector will point in exactly the opposite direction and it will again stretch or shrink the magnitude of the vector depending upon the size of c.

There are several nice applications of scalar multiplication that we should now take a look at.

The first is parallel vectors. This is a concept that we will see quite a bit over the next couple of sections. Two vectors are parallel if they have the same direction or are in exactly opposite directions. Now, recall again the geometric interpretation of scalar multiplication. When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).

So, let's suppose that $\vec{a}$ and $\vec{b}$ are parallel vectors. If they are parallel then there must be a number c so that,

$$\vec{a} = c\vec{b}$$

So, two vectors are parallel if one is a scalar multiple of the other.

Example 2 Determine if the sets of vectors are parallel or not.

(a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

(b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

(a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$ ***Show Solution***

(b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$ ***Show Solution***

The next application is best seen in an example.

Example 3 Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Show Solution

So, in general, given a vector $\vec{w}$, $\vec{u} = \frac{\vec{w}}{\|\vec{w}\|}$ will be a unit vector that points in the same direction as $\vec{w}$.

Standard Basis Vectors Revisited

In the previous section we introduced the idea of standard basis vectors without really discussing why they were important. We can now do that. Let's start with the vector

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

We can use the addition of vectors to break this up as follows,

$$\begin{aligned} \vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \end{aligned}$$

Using scalar multiplication we can further rewrite the vector as,

$$\begin{aligned} \vec{a} &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \end{aligned}$$

Finally, notice that these three new vectors are simply the three standard basis vectors for three dimensional space.

$$\langle a_1, a_2, a_3 \rangle = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

So, we can take any vector and write it in terms of the standard basis vectors. From this point on we will use the two notations interchangeably so make sure that you can deal with both notations.

Example 4 If $\vec{a} = \langle 3, -9, 1 \rangle$ and $\vec{w} = -\vec{i} + 8\vec{k}$ compute $2\vec{a} - 3\vec{w}$.

Show Solution

We will leave this section with some basic properties of vector arithmetic.

Properties

If $\vec{v}$, $\vec{w}$ and $\vec{u}$ are vectors (each with the same number of components) and a and b are two numbers then we have the following properties.

$$\begin{aligned} &\vec{v}+\vec{w}=\vec{w}+\vec{v} && \vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w} \\ &\vec{v}+\vec{0}=\vec{v} && 1\vec{v}=\vec{v} \\ &a\left(\vec{v}+\vec{w}\right)=a\vec{v}+a\vec{w} && \left(a+b\right)\vec{v}=a\vec{v}+b\vec{v} \end{aligned}$$

The proofs of these are pretty much just "computation" proofs so we'll prove one of them and leave the others to you to prove.

Proof of $a\left(\vec{v}+\vec{w}\right)=a\vec{v}+a\vec{w}$

We'll start with the two vectors, $\vec{v}=\langle v_1, v_2, \ldots, v_n\rangle$ and $\vec{w}=\langle w_1, w_2, \ldots, w_n\rangle$ and yes we did mean for these to each have n components. The theorem works for general vectors so we may as well do the proof for general vectors.

Now, as noted above this is pretty much just a "computational" proof. What that means is that we'll compute the left side and then do some basic arithmetic on the result to show that we can make the left side look like the right side. Here is the work.

$$\begin{aligned} a\left(\vec{v}+\vec{w}\right) &= a\left(\langle v_1, v_2, \ldots, v_n\rangle + \langle w_1, w_2, \ldots, w_n\rangle\right) \\ &= a\left\langle v_1+w_1, v_2+w_2, \ldots, v_n+w_n\right\rangle \\ &= \left\langle a\left(v_1+w_1\right), a\left(v_2+w_2\right), \ldots, a\left(v_n+w_n\right)\right\rangle \\ &= \left\langle av_1+aw_1, av_2+aw_2, \ldots, av_n+aw_n\right\rangle \\ &= \left\langle av_1, av_2, \ldots, av_n\right\rangle + \left\langle aw_1, aw_2, \ldots, aw_n\right\rangle \\ &= a\left\langle v_1, v_2, \ldots, v_n\right\rangle + a\left\langle w_1, w_2, \ldots, w_n\right\rangle = a\vec{v}+a\vec{w} \end{aligned}$$

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