Distributionally robust chance constrained optimal power flow with renewables: A conic reformulation

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Abstract

The uncertainty associated with renewable energy sources introduces significant challenges in optimal power flow (OPF) analysis. A variety of new approaches have been proposed that use chance constraints to limit line or bus overload risk in OPF models. Most existing formulations assume that the probability distributions associated with the uncertainty are known a priori or can be estimated accurately from empirical data, and/or use separate chance constraints for upper and lower line/bus limits. In this paper we propose a data driven distributionally robust chance constrained optimal power flow model (DRCC-OPF), which ensures that the worst-case probability of violating both the upper and lower limit of a line/bus capacity under a wide family of distributions is small. Assuming that we can estimate the first and second moments of the underlying distributions based on empirical data, we propose an exact reformulation of DRCC-OPF as a tractable convex program. The key theoretical result behind this reformulation is a second order cone programming (SOCP) reformulation of a general two-sided distributionally robust chance constrained set by lifting the set to a higher dimensional space. Our numerical study shows that the proposed SOCP formulation can be solved efficiently and that the results of our model are quite robust.

Nomenclature

Sets

$\mathcal{V}$ = set of all buses
$\mathcal{E}$ = set of all transmission lines linking two buses
$\mathcal{G}$ = subset of buses that house generators
$\mathcal{W}$ = subset of buses that holds uncertain power sources (wind farms)

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Parameters

μ_j = average generation at bus $j \in \mathcal{W}$ ($\mu_j = 0$, for all $j \in \mathcal{V} \setminus \mathcal{W}$)
$\boldsymbol{\omega}_j$ = random fluctuation of power generation at uncertain power source $j \in \mathcal{W}$ ($\boldsymbol{\omega}_i = 0$, if $i \in \mathcal{V} \setminus \mathcal{W}$)
Σ = known covariance matrix of random vector ω
d_i = demand at bus $i \in \mathcal{V}$
β_{ij} = the line susceptance between $(i,j) \in \mathcal{E}$ ($\beta_{ij} = \beta_{ji}$ by symmetry)
B = weighted Laplacian matrix defined as

$$B(i,j) = \begin{cases} -\beta_{ij}, & \text{if } (i,j) \in \mathcal{E} \\ \sum_{k:(k,j)\in\mathcal{E}} \beta_{kj}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

for each (i,j)
$\widehat{B}$ the submatrix of B by removing the last row and column
$\breve{\mathcal{B}}$ = pseudo-inverse of B defined as

$$\breve{\mathcal{B}} = \begin{bmatrix} \widehat{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$\breve{\mathcal{B}}^{\mathcal{W}} = |V| \times |W|$ submatrix of $\breve{\mathcal{B}}$ where its columns are from $\mathcal{W}$
$\breve{\mathcal{B}}^{\mathcal{W}}_{i\cdot}$ = ith row of $\breve{\mathcal{B}}^{\mathcal{W}}$
e = all one vector
c_i, r_i = quadratic cost coefficients of generator $i \in \mathcal{G}$
$\widehat{\epsilon}_{ij}$ = risk parameter of violating the capacity of transmission line $(i,j) \in \mathcal{E}$
$\bar{\epsilon}_i$ = risk parameter of violating the capacity of bus $i \in \mathcal{G}$
$f^{\max}_{ij}$ = max capacity of transmission line $(i,j) \in \mathcal{E}$
$p^{\min}_i$ = generation lower bound of bus $i \in \mathcal{G}$
$p^{\max}_i$ = generation upper bound of bus $i \in \mathcal{G}$

Decision Variables

$\bar{\theta}_j$ = be the phase of bus $j \in \mathcal{V}$
$\bar{p}_i$ = regular generation at generator $i \in \mathcal{G}$ ($\bar{p}_i = 0$, if $i \in \mathcal{V} \setminus \mathcal{G}$)
α_i = ith assignment of total renewables to generator $i \in \mathcal{G}$ ($\alpha_i = 0$, if $i \in \mathcal{V} \setminus \mathcal{G}$)

1 Introduction

Recently with growing interests in environmentally friendly power generation such as wind, solar, geothermal energy [21, 34, 36], optimal power flow (OPF) under uncertainty has attracted much attention from researchers [1, 5, 35, 38, 44, 45, 48]. A particular issue caused by renewables is the voltage fluctuations which can lead to severe issues, for example, overloaded transmission lines [8, 25]. To mitigate these issues, [1, 5] proposed a chance constrained optimal power flow model (CC-OPF) that constrains the overloading probability. This paper extends this work by enforcing power flow within lower and upper bounds simultaneously and show tractability results of such an approach under data driven distributionally robust setting.

There are many works on solving OPF, unit commitment (UC) problem or reserve scheduling via stochastic programming [28, 32, 37], robust optimization [3, 11, 15, 16, 39, 40, 43, 49], and chance constrained program approaches [5, 4, 19, 27, 31, 35, 38, 44] (see [23] for some discussions). Stochastic programming approaches highly rely on the underlying distribution, which could be unknown in many cases, and the performance of solution algorithms is usually very sensitive to the distribution used [33]. Robust optimization is often too conservative [2], while chance constrained programming is less conservative but is often NP-hard [24, 26]. Thus, to overcome the difficulties from these two approaches, here we adopt a distributionally robust chance constrained approach, which allows violation of uncertain constraints with a small probability for a large class of probability distributions and could be reformulated as a tractable convex program [1, 7, 41, 51].

There are two concerns about existing literatures on CC-OPF formulations. It is known that each transmission line as well as each bus (node) in general has lower and upper bound limits. However, most works [1, 5, 23, 47] treats the lower- and upper- bound overloading separately, which is an inexact approximation. To the best of our knowledge, [22] is the only known work treating lower and upper bounds simultaneously. However the results in [22] highly depend on the assumption of a Gaussian distribution on the underlying uncertainties. In this paper, we will consider incorporating power flow within lower and upper bounds simultaneously and our results are distribution-free.

Another issue regarding previous CC-OPF studies is that they assumed a particular distribution of renewables' output. For example, [5, 23, 30] assumed that the prior distributions of renewables are Gaussian while [20] assumed that it is Weibull. However, these assumptions might not be true in practice [1]. In general the underlying probability distributions of renewables are not known or are hard to estimate from empirical data. Thus, to hedge against the uncertainty of probability distributions we consider a data driven distributionally robust chance constrained optimal power flow model (DRCC-OPF) by considering the overload within the upper and lower bounds jointly with high probability. And the underlying probability distribution comes from a family of distributions (called "*ambiguity set*") that share the same mean and covariance matrix estimated from empirical data.

Distributionally robust chance constrained problems with multiple uncertain constraints (joint-DRCCP) are very challenging [14]. There are only few setting under which joint DRCCP can be equivalently reformulated into a convex program. For example, in [13], they assumed right-hand uncertainty with mean dispersion moment ambiguity set, and proposed convex reformulations. Recently, [41] explored several sufficient conditions under which joint DRCCP is convex and [42] showed that joint DRCCP with separable structure (the uncertainties are separable across the individual uncertain constraints and their corresponding distribution families) can be convex with small risk parameter. However none of known sufficient conditions for convexity can be directly applied to the two-sided DRCC-OPF here, where in two-sided chance constraint, the uncertain constraints are defined as lower and upper bounds of one uncertain affine function. This paper fills the gap in [41] by showing that joint DRCCP with two-sided constraint has a conic reformulation.

The remaining of the paper is organized as follows. Section 2 introduces the model formulation and Section 3 shows how to reformulate the model into a convex second order cone program. Section 4 numerically illustrates the strengths of the proposed model. Section 5 concludes the paper.

2 Model Formulation

In this section, we consider an extension of distributionally robust chance constrained optimal flow model (DRCC-OPF) proposed in [5].

In the optimal power flow problem, we suppose that there is a subset $\mathcal{W}$ of the buses with uncertain power sources (e.g., wind farms). For each $j \in \mathcal{W}$, we model the uncertain power generated by $\mu_j + \boldsymbol{\omega}_j$, where μ_j represents the mean of uncertain power generation and $\boldsymbol{\omega}_j$ is a random variable with zero mean and covariance matrix denoted by Σ. The net output of bus $i \in \mathcal{G}$ is fluctuated by the output of wind generators. Let α_i for each $i \in \mathcal{G}$ be the proportion of wind power allocated to bus i, i.e., the output of bus $i \in \mathcal{G}$ is $\bar{p}_i - (e^\top \boldsymbol{\omega})\alpha_i$ with nonnegative variables $\bar{p}_i, \alpha_i$ and $\sum_{i\in\mathcal{G}} \alpha_i = 1$. Each bus $i \in \mathcal{V}$ has demand d_i. For notational convenience, we extend vectors $\boldsymbol{\omega}, \mu, \alpha, \bar{p}$ to $\mathbb{R}^{|\mathcal{V}|}$ by letting $\boldsymbol{\omega}_j = 0, \mu_j = 0$ for each $j \in \mathcal{V} \setminus \mathcal{W}$ and $\alpha_i = 0, \bar{p}_i = 0$ for each $i \in \mathcal{V} \setminus \mathcal{G}$.

Let $\bar{\theta}$ be the phases of all the buses. To approximate nonconvex AC power flow equations, we use DC-approximation. Thus, the power flow between line (i,j) is approximated as $\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j)$ where $\beta_{ij} = \beta_{ji}$ denotes the line susceptance.

Following [5], a distributionally robust chance constrained optimal power flow problem (DRCC-OPF) is formulated as

$$
\begin{aligned}
v^* = \min_{\bar{p},\alpha,\bar{\theta}} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i\in\mathcal{G}} c_i(\bar{p}_i + \alpha_i(e^\top \boldsymbol{\omega}) + r_i)^2 \right] && \text{(1a)}\\
\text{s.t.} \sum_{i\in\mathcal{G}} \alpha_i = 1, && \text{(1b)}\\
\sum_{i\in\mathcal{V}} (\bar{p}_i + \mu_i - d_i) = 0, && \text{(1c)}\\
B\bar{\theta} = \bar{p} + \mu - d, && \text{(1d)}\\
\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\{\boldsymbol{\omega} : |\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j) + [\breve{\mathcal{B}}^{\mathcal{W}}(\boldsymbol{\omega} - (e^\top \boldsymbol{\omega})\alpha)]_i \\
- [\breve{\mathcal{B}}^{\mathcal{W}}(\boldsymbol{\omega} - (e^\top \boldsymbol{\omega})\alpha)]_j)| \le f_{ij}^{\max}\} \ge 1 - \widehat{\epsilon}_{ij}, \\
\forall (i,j) \in \mathcal{E}, && \text{(1e)}\\
\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\{\boldsymbol{\omega} : p_i^{\min} \le \bar{p}_i - (e^\top \boldsymbol{\omega})\alpha_i \le p_i^{\max}\} \\
\ge 1 - \bar{\epsilon}_i, \forall i \in \mathcal{G}, && \text{(1f)}\\
\bar{p} \ge 0, \alpha \ge 0, \bar{p}_i = 0, \alpha_i = 0, \forall i \in \mathcal{V} \setminus \mathcal{G}. && \text{(1g)}
\end{aligned}
$$

where (1a) is to optimize cost function where $c > 0$ and $r \in \mathbb{R}^{|\mathcal{G}|}$ are constant, (1b) implies that the total assignment of power from wind is 1, (1c) means on average, the total generation equals to the total demand, (1d) is the DC-approximation equation with

$$
B(i,j) = \begin{cases} -\beta_{ij}, & \text{if } (i,j) \in \mathcal{E} \\ \sum_{k:(k,j)\in\mathcal{E}} \beta_{kj}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}
$$

for each (i,j), (1e) enforce that the worst case probability that the absolute flow on (i,j) does not exceed the maximum capacity $f_{ij}^{\max}$ should be no smaller than $1-\widehat{\epsilon}_{ij}$ with pseudo-inverse of B

$$\breve{\mathcal{B}} = \begin{bmatrix} \widehat{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and its submatrix $\breve{\mathcal{B}}^{\mathcal{W}}$ and $\widehat{B}$ the submatrix of B by removing the last row and column; and (1f) ensures that with probability at least $1-\bar{\epsilon}_i$, the generated power at i satisfied the lower bound $p_i^{\min}$ and the upper bound $p_i^{\max}$, (1g) defines the boundary of variables. Here we assume the all the risk parameters are within $(0,1)$.

Similar to [9, 17, 51], let us consider the ambiguity set defined by first and second moments as

$$\mathcal{P} = \{\mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{|\mathcal{V}|}) : \mathbb{E}_{\mathbb{P}}[\boldsymbol{\omega}] = 0, \mathbb{E}_{\mathbb{P}}[\boldsymbol{\omega}\boldsymbol{\omega}^\top] = \Sigma\} \tag{2}$$

where $\mathcal{P}_0\left(\mathbb{R}^{|\mathcal{V}|}\right)$ denotes the set of all of probability measures on $\mathbb{R}^{|\mathcal{V}|}$ with a sigma algebra $\mathcal{F}$, and $\Sigma \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ is a positive semi-definite matrix (i.e., $\Sigma \succeq 0$). We remark that various other works have studied different moment ambiguity sets, for example, mean-dispersion or mean deviation ambiguity set [13, 42], ambiguity set with known sum of variances and covariance [9, 46, 47], or ambiguity set with a bounded support [35]. The results for other types of ambiguity set might be different from the one of (2), however, the proof technique here is very general and may be applicable to these settings as well.

As we know the mean and covariance of $\boldsymbol{\omega}$, the cost function (1a) is equivalent to

$$\begin{aligned} &\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[\sum_{i\in\mathcal{G}} c_i(\bar{p}_i + \alpha_i(e^\top\boldsymbol{\omega}) + r_i)^2\right] \\ &= \sum_{i\in\mathcal{G}} \left(c_i(\bar{p}_i + r_i)^2 + c_i\alpha_i^2 e^\top \Sigma e\right). \end{aligned} \tag{3}$$

Thus, apart from the chance constraints (1e) and (1f), the DRCC-OPF formulation (1) is a convex quadratic optimization problem.

3 Convex Reformulation of Chance Constraints (1e) and (1f)

In this section, we will develop a deterministic convex formulation of (1) by reformulating the chance constraints (1e) and (1f) into equivalent convex constraints.

To reformulate the chance constraints (1e) and (1f), let us first consider a generic distributionally robust chance constrained set defined as follows:

$$Z := \left\{x : \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[|a(x)^\top\boldsymbol{\omega} + b(x)| \leq T] \geq 1-\epsilon\right\} \tag{4}$$

where $a(x), b(x)$ are affine mappings. This set is defined by a distributionally robust two-sided chance constraint.

Note that (1e) and (1f) are special cases of (4), where in (1e), we let $a(x) = \beta_{ij}(\breve{\mathcal{B}}_{i\cdot}^{\mathcal{W}} - \alpha_i e - \breve{\mathcal{B}}_{j\cdot}^{\mathcal{W}} + \alpha_j e)$, $b(x) = \beta_{ij}(\bar{\theta}_i - \bar{\theta}_j)$, $T = f_{ij}^{\max}$ and in (1f), we let $a(x) = -\alpha_i\mathbf{e}$, $b(x) = \bar{p}_i - \frac{p_i^{\max}+p_i^{\min}}{2}$, $T = \frac{p_i^{\max}-p_i^{\min}}{2}$.

3.1 Approximation by Two Single-sided Chance Constraints

Recently, including [1, 5, 45, 44], many studies tried to approximate the two-side chance constrained set (4) by two single-sided chance constraints. In particular, let

$$Z_A(\alpha) = \left\{ x : \begin{array}{ll} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[a(x)^\top \boldsymbol{\omega} + b(x) \le T] \ge 1-\alpha, & \quad (5a) \\ \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[a(x)^\top \boldsymbol{\omega} + b(x) \ge -T] \ge 1-\alpha. & \quad (5b) \end{array} \right\}$$

By choosing $\alpha \sim \epsilon$ existing works use $Z_A(\alpha)$ to approximate Z. It turns out that sets $Z_A(\epsilon)$ and $Z_A(\epsilon/2)$ are outer and inner approximations of set Z and can be formulated as second order cone programs (SOCP).

Theorem 1. *Suppose the ambiguity set $\mathcal{P}$ is defined as in* (2)*, then set Z_A is equivalent to the following SOCP*

$$Z_A(\alpha) = \left\{ x : \begin{array}{ll} b(x) + \sqrt{\frac{1-\alpha}{\alpha}}\sqrt{a(x)^\top \Sigma a(x)} \le T, & \quad (6a) \\ -b(x) + \sqrt{\frac{1-\alpha}{\alpha}}\sqrt{a(x)^\top \Sigma a(x)} \le T. & \quad (6b) \end{array} \right\}$$

and $Z_A(\epsilon/2) \subseteq Z \subseteq Z_A(\epsilon)$.

Proof. For $x \in Z$, clearly, we have

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[|a(x)^\top \boldsymbol{\omega} + b(x)| \le T] \le \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[a(x)^\top \boldsymbol{\omega} + b(x) \le T]$$

and

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[|a(x)^\top \boldsymbol{\omega} + b(x)| \le T] \le \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[a(x)^\top \boldsymbol{\omega} + b(x) \ge -T].$$

Clearly, $Z \subseteq Z_A(\epsilon)$.

The result that $Z_A(\epsilon/2) \subseteq Z$ follows by Bonferroni approximation of joint chance constrained set (c.f. [26]). The equivalent reformulation of $Z_A(\alpha)$ follows by Theorem 3.1 in [7]. □

As discussed in the sequel the approximations offered by $Z_A(\alpha)$ could be very crude, especially when the risk parameter ϵ is modest. In the next subsection, we will explore an exact convex reformulation of the set Z.

3.2 Exact Reformulation

Our main result is the following theorem which provides a convex reformulation of the two-sided chance constrained set (4) as an SOCP.

Theorem 2. *Suppose the ambiguity set $\mathcal{P}$ is defined in* (2)*, then set Z is equivalent to the following convex SOCP (involving two additional variables):*

$$Z = \left\{ x : \begin{array}{ll} y^2 + a(x)^\top \Sigma a(x) \le \epsilon (T-\pi)^2, & \quad (7a) \\ |b(x)| \le y + \pi, & \quad (7b) \\ T \ge \pi \ge 0, y \ge 0. & \quad (7c) \end{array} \right\}$$

Proof. Observe that $|b(x)| \leq T$ for each $x \in Z$. This is because by choosing $\boldsymbol{\omega}_1 = 0$ with probability $1-\epsilon$, $\boldsymbol{\omega}_{i+1} = \sqrt{\frac{|\mathcal{W}|}{\epsilon}}\lambda_i^{0.5}u_i, \boldsymbol{\omega}_{i+|\mathcal{W}|+1} = -\sqrt{\frac{|\mathcal{W}|}{\epsilon}}\lambda_i^{0.5}u_i$ with probability $\frac{\epsilon}{2|\mathcal{W}|}$ for each $i \in \mathcal{W}$, where λ_i and u_i are ith eigenvalue and eigenvector of Σ. Then, under this particular construction, we have $|b(x)| \leq T$.

Lemma 1.

$$Z = Z_1 \cup Z_2 \cup Z_3$$

where

$$Z_1 = \left\{ x : \begin{array}{l} \sqrt{\dfrac{1}{\epsilon}}\sqrt{a(x)^\top \Sigma a(x) + b(x)^2} \leq T \\ |b(x)| \leq \epsilon T, \end{array} \right\}$$

$$Z_2 = \left\{ x : \begin{array}{l} b(x) + \sqrt{\dfrac{1-\epsilon}{\epsilon}}\sqrt{a(x)^\top \Sigma a(x)} \leq T, \\ b(x) \geq \epsilon T, \end{array} \right\}$$

and

$$Z_3 = \left\{ x : \begin{array}{l} -b(x) + \sqrt{\dfrac{1-\epsilon}{\epsilon}}\sqrt{a(x)^\top \Sigma a(x)} \leq T, \\ b(x) \leq -\epsilon T. \end{array} \right\}$$

Proof. Suppose $x \in Z$, then by the standard random variable transformation (c.f. [10]) and Theorem 1 in [29] (see also in [18, 41]), set Z is equivalent to

$$Z := \left\{ x : \inf_{\mathbb{P} \in \mathcal{P}_1} \mathbb{P}[|\boldsymbol{\xi}| \leq T] \geq 1-\epsilon \right\}$$

where

$$\mathcal{P}_1 = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}) : \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}] = b(x), \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}^2] = a(x)^\top \Sigma a(x) + b(x)^2. \right\}$$

Let $\mathcal{M}(\mathbb{R})$ be the set of all the positive measures on $\mathbb{R}$. Then set $\mathcal{P}_1$ is equivalent to

$$\mathcal{P}_1 = \left\{ \mathbb{P} \in \mathcal{M}(\mathbb{R}) : \mathbb{E}_{\mathbb{P}}[1] = 1, \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}] = b(x), \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}^2] = a(x)^\top \Sigma a(x) + b(x)^2, \right\}$$

where the first equality is to guarantee that $\mathbb{P}$ is indeed a probability measure. Thus, the infimum in the set Z is equivalent to

$$\begin{aligned} \inf_{\mathbb{P} \in \mathcal{M}(\mathbb{R})} \quad & \mathbb{E}[\mathbb{I}(|\boldsymbol{\xi}| \leq T)] \\ \text{s.t.} \quad & \mathbb{E}_{\mathbb{P}}[1] = 1, \\ & \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}] = b(x), \\ & \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}^2] = a(x)^\top \Sigma a(x) + b(x)^2, \end{aligned}$$

where $\mathbb{I}(\mathcal{R})$ is 1 if event $\mathcal{R}$ is true, 0, otherwise. By dualizing three equality constraints above with dual multipliers λ, γ, β, Theorem 5.99 in [6] implies that for any $x \in Z$, $\inf_{\mathbb{P} \in \mathcal{P}_1} \mathbb{P}[|\boldsymbol{\xi}| \leq T] = \inf_{\mathbb{P} \in \mathcal{P}_1} \mathbb{E}[\mathbb{I}(|\boldsymbol{\xi}| \leq T)]$ can be reformulated as

$$\max_{\lambda,\gamma,\beta} \lambda + b(x)\gamma + (a(x)^\top \Sigma a(x) + b(x)^2)\beta \tag{11a}$$

$$
\begin{aligned}
\text{s.t. } & \lambda + \xi\gamma + \xi^2\beta \le 1, \forall \xi \in \mathbb{R} && \text{(11b)}\\
& \lambda + \xi\gamma + \xi^2\beta \le 0, \forall \xi : \xi > T && \text{(11c)}\\
& \lambda + \xi\gamma + \xi^2\beta \le 0, \forall \xi : \xi < -T. && \text{(11d)}
\end{aligned}
$$

Note that in (11), we must have $\beta < 0$ and $|\frac{\gamma}{2\beta}| \le T$. Otherwise, $\sup_\xi \lambda + \xi\gamma + \xi^2\beta \le 0$ which implies that for any probability measure $\mathbb{P} \in \mathcal{P}_1$, we have

$$\lambda + b(x)\gamma + (a(x)^\top \Sigma a(x) + b(x)^2)\beta = \lambda + \mathbb{E}_{\mathbb{P}}[\gamma \boldsymbol{\xi} + \boldsymbol{\xi}^2 \beta] \le 0$$

contradiction that $x \in Z$.

Since $\beta < 0$ and $|\frac{\gamma}{2\beta}| \le T$, (11) is equal to

$$
\begin{aligned}
\max_{\lambda,\gamma,\beta}\ & \lambda + b(x)\gamma + (a(x)^\top \Sigma a(x) + b(x)^2)\beta && \text{(12a)}\\
\text{s.t. } & \lambda - \frac{\gamma^2}{4\beta} \le 1, && \text{(12b)}\\
& \lambda + T|\gamma| + T^2\beta \le 0, && \text{(12c)}\\
& \beta < 0. && \text{(12d)}
\end{aligned}
$$

Note that in (12), the optimal γ must have the same sign as $b(x)$ so as to maximize the objective function. Thus, set Z is equivalent to

$$
Z = \left\{ x : \begin{array}{l}
\lambda + |b(x)||\gamma| + b(x)^2\beta + a(x)^\top \Sigma a(x)\beta \ge 1 - \epsilon,\\
\lambda - \dfrac{\gamma^2}{4\beta} \le 1,\\
\lambda + T|\gamma| + T^2\beta \le 0,\\
\beta < 0.
\end{array} \right\} \tag{13}
$$

Now in (13), let $\pi := -\frac{1}{\beta}, \widehat{\gamma} := -|\gamma|/\beta, \widehat{\lambda} := -\lambda/\beta$ and define a new set below

$$
\bar{Z} = \left\{ x : \begin{array}{ll}
\widehat{\lambda} + |b(x)|\widehat{\gamma} - b(x)^2 - a(x)^\top \Sigma a(x) \ge (1-\epsilon)\pi, & \text{(14a)}\\
\widehat{\lambda} + \dfrac{\widehat{\gamma}^2}{4} \le \pi, & \text{(14b)}\\
\widehat{\lambda} + T\widehat{\gamma} - T^2 \le 0, & \text{(14c)}\\
\widehat{\gamma} \ge 0. & \text{(14d)}
\end{array} \right\}
$$

Now we claim that $\bar{Z} = Z$. To prove this claim, we first show that $\bar{Z} \subseteq Z$. Given $x \in \bar{Z}$, there exists $(\widehat{\lambda}, \widehat{\gamma}, \pi)$ such that $(\widehat{\lambda}, \widehat{\gamma}, \pi, x)$ satisfy (14). First of all, by letting (14b) minus (14a), we have

$$\left(\frac{\widehat{\gamma}}{2} - |b(x)|\right)^2 + a(x)^\top \Sigma a(x) \le \epsilon\pi$$

thus, $\pi \ge 0$. There are two case:

Case 1. If $\pi = 0$, then by (14b) and (14a), we have $\widehat{\lambda} = \widehat{\gamma} = 0$ and $a(x)^\top \Sigma a(x) = 0, b(x)^2 = 0$. Next, choose $\lambda = 1, \gamma = 0, \beta = -T^{-2}$ and we have $(\lambda, \gamma, \beta, x)$ satisfies constraints in (13). Hence, $x \in Z$.

Case 2. If $\pi > 0$, now define $(\bar{\lambda}, \bar{\gamma}) = (\widehat{\lambda}, \widehat{\gamma})/\pi$ and $\beta = -1/\pi < 0$. Clearly, $(\bar{\lambda}, \bar{\gamma}, \beta, x)$ satisfy (13). Thus, $x \in Z$.
Thus $\bar{Z} \subseteq Z$. Next we show that $\bar{Z} \supseteq Z$. Given $x \in Z$, there exists (λ, γ, β) such that $(\lambda, \gamma, \beta, x)$ satisfy (13). As $\beta < 0$, let $(\widehat{\lambda}, \widehat{\gamma}) = -(\lambda, \gamma)/\beta$ and $\pi = -1/\beta$, then $(\widehat{\lambda}, \widehat{\gamma}, \pi, x)$ satisfy (14). Hence, $x \in \bar{Z}$. Thus $\bar{Z} = Z$.

By eliminating variables λ, π by Fourier-Motzkin procedure, (14) yields

$$Z = \left\{ x : \begin{array}{l} (|b(x)| - \gamma/2)^2 + a(x)^\top \Sigma a(x) \le \epsilon (T - \gamma/2)^2, \\ \gamma \ge 0. \end{array} \right\} \quad \begin{array}{r} (15a) \\ (15b) \end{array}$$

Optimizing γ in (15) by distinguishing the cases $\epsilon T \ge |b(x)|$ ($\gamma^* = 0$), $b(x) \le -\epsilon T$ ($\gamma^* = \frac{2}{1-\epsilon}(|b(x)| - \epsilon T)$) and $b(x) \ge \epsilon T$ ($\gamma^* = \frac{2}{1-\epsilon}(|b(x)| - \epsilon T)$), set Z can be reformulated as a disjunction of three sets Z_1, Z_2 and Z_3. □

We are now ready to prove Theorem 2. From the proof of Lemma 1, we observe that the best γ in (15) must be no larger than $2|b(x)|$; otherwise, we will arrive at a smaller set. Thus, (15) is equivalent to

$$Z = \left\{ x : \begin{array}{l} (|b(x)| - \pi)^2 + a(x)^\top \Sigma a(x) \le \epsilon (T - \pi)^2, \\ 0 \le \pi \le |b(x)| \end{array} \right\} \quad \begin{array}{r} (16a) \\ (16b) \end{array}$$

where $\pi := \frac{\gamma}{2}$.

Now let $\tilde{Z}$ denote the right-hand side in (7). We claim. $\tilde{Z} = Z$. Given $x \in Z$, there exists a π such that (π, x) satisfy (16). Now let $y = |b(x)| - \pi$, then (y, π, x) satisfy (7). Hence, $x \in \tilde{Z}$. Thus $\tilde{Z} \supseteq Z$. Next we show that $\tilde{Z} \subseteq Z$. Given $x \in \tilde{Z}$, there exists (y, π) such that (y, π, x) satisfy (7). There are two cases:
Case 1. if $|b(x)| \le \pi \le T$, then by (7), we have

$$a(x)^\top \Sigma a(x) \le y^2 + a(x)^\top \Sigma a(x) \le \epsilon (T - \pi)^2 \le \epsilon (T - |b(x)|)^2 \quad (17)$$

where the first inequality is due to $y^2 \ge 0$, the second inequality is because of (7a) and the third inequality is because of $|b(x)| \le \pi \le T$. Now we distinguish whether $|b(x)| \le \epsilon T$ or not.

a) if $|b(x)| \le \epsilon T$, then by (7), we have

$$a(x)^\top \Sigma a(x) \le \epsilon (T - \pi)^2 \le \epsilon (T - |b(x)|)^2 \le \epsilon T^2 - (2 - \epsilon) b(x)^2 \le \epsilon T^2 - b(x)^2$$

where the first inequality is due to (7a), the second inequality is due to $|b(x)| \le \pi \le T$, the third inequality is due to $|b(x)| \le \epsilon T$, and the last one is because of $\epsilon \in (0, 1)$. This leads to $b(x)^2 + a(x)^\top \Sigma a(x) \le \epsilon T^2$, i.e., $x \in Z_1 \subseteq Z$.

b) if $|b(x)| \ge \epsilon T$, then (17) implies that

$$|b(x)| + \sqrt{\frac{1}{\epsilon}} \sqrt{a(x)^\top \Sigma a(x)} \le T.$$

Since $\sqrt{\frac{1-\epsilon}{\epsilon}} \le \sqrt{\frac{1}{\epsilon}}$, we must have $x \in Z_2 \cup Z_3 \subseteq Z$.

Case 2. if $0 \le \pi \le |b(x)|$, then $y \ge |b(x)| - \pi \ge 0$. Hence, (7) implies that

$$(|b(x)| - \pi)^2 + a(x)^\top \Sigma a(x) \le y^2 + a(x)^\top \Sigma a(x) \le \epsilon (T - \pi)^2$$

where the first inequality is due to $y \ge |b(x)| - \pi \ge 0$, second inequality is because of (7a) and thus (π, x) satisfies (16), i.e., $x \in Z$.

This completes the proof. $\square$

Using the above result we can now provide an exact SOCP formulation of DRCC-OPF (1) as follows:

$$v^* = \min_{\bar{p},\alpha,\bar{\theta}} \sum_{i \in \mathcal{G}} \left(c_i(\bar{p}_i + r_i)^2 + c_i \alpha_i^2 e^\top \Sigma e \right) \tag{18a}$$

$$\text{s.t.}(1\text{b}) - (1\text{d}), (1\text{g})$$

$$\widehat{y}_{ij}^2 + \beta_{ij}^2 \left(\breve{\mathcal{B}}_{i\cdot}^{\mathcal{W}} - \alpha_i e - \breve{\mathcal{B}}_{j\cdot}^{\mathcal{W}} + \alpha_j e \right)^\top \Sigma \left(\breve{\mathcal{B}}_{i\cdot}^{\mathcal{W}} - \alpha_i e - \breve{\mathcal{B}}_{j\cdot}^{\mathcal{W}} + \alpha_j e \right)$$

$$\le \widehat{\epsilon}_{ij} (f_{ij}^{\max} - \widehat{\pi}_{ij})^2, \forall (i,j) \in \mathcal{E}, \tag{18b}$$

$$\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j) \le \widehat{y}_{ij} + \widehat{\pi}_{ij}, \forall (i,j) \in \mathcal{E}, \tag{18c}$$

$$\beta_{ij}(\bar{\theta}_j - \bar{\theta}_i) \le \widehat{y}_{ij} + \widehat{\pi}_{ij}, \forall (i,j) \in \mathcal{E}, \tag{18d}$$

$$\widehat{y}_{ij} \ge 0, 0 \le \widehat{\pi}_{ij} \le f_{ij}^{\max}, \forall (i,j) \in \mathcal{E}, \tag{18e}$$

$$\bar{y}_i^2 + \alpha_i^2 e^\top \Sigma e \le \bar{\epsilon}_i \left(\frac{p_i^{\max} - p_i^{\min}}{2} - \bar{\pi}_i \right)^2, \forall i \in \mathcal{G}, \tag{18f}$$

$$\bar{p}_i - \frac{p_i^{\max} + p_i^{\min}}{2} \le \bar{y}_i + \bar{\pi}_i, \forall i \in \mathcal{G}, \tag{18g}$$

$$-\bar{p}_i + \frac{p_i^{\max} + p_i^{\min}}{2} \le \bar{y}_i + \bar{\pi}_i, \forall i \in \mathcal{G}, \tag{18h}$$

$$0 \le \bar{y}_i, 0 \le \bar{\pi}_i \le \frac{p_i^{\max} - p_i^{\min}}{2}, \forall i \in \mathcal{G}, \tag{18i}$$

where $\widehat{\pi}, \widehat{y}, \bar{\pi}, \bar{y}$ are auxiliary nonnegative variables.

3.3 Quality of approximation of Z by $Z_A(\epsilon/2), Z_A(\epsilon)$

We know that from Theorem 1, we have $Z_A(\epsilon/2) \subseteq Z \subseteq Z_A(\epsilon)$ and usually the inclusion is strict. The following example shows that the distances between set Z and $Z_A(\epsilon)$ and between set Z and $Z_A(\epsilon/2)$ can be large.

Example 1. Let $b(x) = 0, \Sigma = I$, then

$$Z_A(\epsilon/2) = \left\{ x : \|a(x)\|_2^2 \le \frac{\epsilon}{2-\epsilon} T^2 \right\}$$

$$Z_A(\epsilon) = \left\{ x : \|a(x)\|_2^2 \le \frac{\epsilon}{1-\epsilon} T^2 \right\}$$

$$Z = \left\{ x : \|a(x)\|_2^2 \le \epsilon T^2 \right\}$$

Clearly, when $\epsilon \to 1$, $Z_A(\epsilon) \to \mathbb{R}^n$ but Z is close to a ball $\{x : \|a(x)\|_2^2 \le T^2\}$. Hence the distance between Z and $Z_A(\epsilon)$ tends to be infinity.

On the other hand, we know that $\frac{\epsilon}{2-\epsilon} \approx \frac{\epsilon}{2}$ when ϵ is small. Thus, in this case, the radius of ball Z could be almost $\sqrt{2}$ larger than $Z_A(\epsilon/2)$. This inner approximation could easily lead the feasible region of a DRCC-OPF to be infeasible. For example, if there is an additional constraint $S = \{x : a(x) \geq T\sqrt{\frac{2\epsilon}{3m}}e\}$ where m is the dimension of $a(x)$, then clearly $S \cap Z_A(\epsilon/2) = \emptyset$ when $\epsilon < 0.5$, but $S \cap Z$ even has a nonempty interior.

4 Numerical Illustration

We test the DRCC-OPF model (18) with an example used in [5]: case39 of MATPOWER data originally from [50]. The case is available at `http://www.pserc.cornell.edu/matpower/`. In this data set, there are 39 buses (set $\mathcal{V}$), 46 lines (set $\mathcal{E}$) and 10 generators (set $\mathcal{G}$). We assume that renewable power can be generated from buses 1 to 4 (set $\mathcal{W}$) with mean $\mu_i = 40$ (MW) for each $i \in \mathcal{W}$ and its covariance matrix Σ is diagonal with $\Sigma(i,i) = 400$ for each $i \in \mathcal{W}$. All of the instances are solved by CVX [12].

In our first test, we let $\widehat{\epsilon}_{ij} = \bar{\epsilon}_i = 0.2$. We compare our method with a "risk neutral" model by assuming there is no uncertainty in (1e) and (1f), i.e., reformulating these constraints as

$$|\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j)| \leq f_{ij}^{\max}, \forall (i,j) \in \mathcal{E}, \tag{1e'}$$

$$p_i^{\min} \leq \bar{p}_i \leq p_i^{\max}, \forall i \in \mathcal{G}; \tag{1f'}$$

and to the model in [5] (we call it "*BCH model*") where they assume the underlying distribution is Gaussian. All three models can be solved within a second, with total costs 36059.1, 36448.6, 37885.3 for the risk neutral model, BCH model, and our model (18), respectively. Thus, there is no significant difference (within 5%) of total costs among all the three models.

We also test the reliability of models by simulating different distributions of renewables' output, i.e., Gaussian, student, Laplace, Logistic and uniform distributions. We generate 100,000 samples from each distribution and check the violation of line flow capacity and bus capacity for each transmission line and bus. In Table 1, we compute the maximum probability of violations across all the lines and buses under each distribution. It can be seen that even when the risk parameters are all equal to 0.2, our model is quite robust and the chance that a line or bus capacity will be violated is close to zero for most of distributions. However, in the risk neutral model, there is a 50% chance that a line or bus capacity is violated almost for each distribution. The BCH model does slightly better, but still under some distributions (e.g., Logistic), the probability of failure is relatively high (31%).

In the second test, we let the risk parameters $\widehat{\epsilon}_{ij}$ and $\bar{\epsilon}_i$ range from 0.15 to 0.5 and observe how this affects the solutions. We compare our results on maximum probability of violations with the ones of BCH model through generating 100,000 samples from Logistic distribution. In Figure 1(a), we see that the results from BCH model are quite sensitive to the risk parameters, i.e., the probability of violating line capacity or bus capacity increases almost linearly as the risk parameters grows. Since the probability of violation curve is always above the neutral line which tells whether the probability of violations is larger than the prespecified risk parameter ϵ or not, hence the solution of BCH is not robust at all. Therefore, in the BCH model, one might need to stick to small risk parameters. Our model (18) turns out to be quite robust with the risk parameters. Even when all of the risk parameters are equal to 0.5, the chance of capacity violation is still quite small (around 28%). We also observe that in Figure 1(b), cost difference between two models

Table 1: Maximum probability of violations and total costs among model (18), risk neutral model and BCH model

Distribution	Model (18)	Risk Neutral	BCH
Total Cost	37885.3	36059.1	36448.6
Gaussian	0.02279	0.50149	0.2022
Student	1.00E-05	0.50128	5.00E-05
Laplace	0.0274	0.50366	0.18197
Logisitic	0.12856	0.5022	0.31184
Uniform	0.0211	0.5016	0.21614

reduces when the risk parameter increases. Another observation is that the total cost of our model (18) is the most costly due to its conservativeness, but the difference between ours and risk-free model is small (at most 6%) . This could be because in the objective function (1a), there is only production cost of regular generators but no cost on renewables. Hence influence of renewables to the total costs is small but to the system reliability is dramatic. On the other hand, if the operators would like to reduce the total costs and can tolerate a relatively high risk, they can increase the risk parameter.

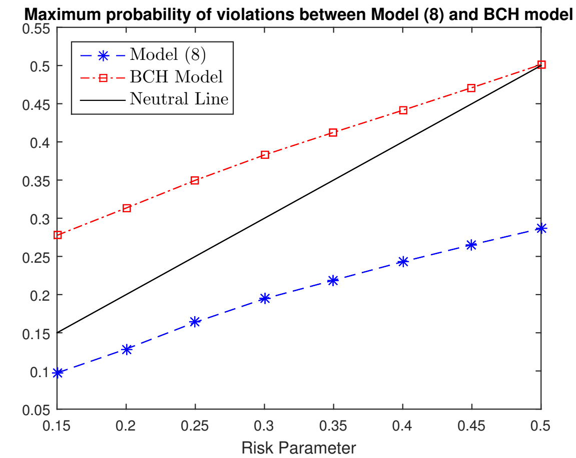
Finally, we compare the computational time for model (18) with that of the risk neutral model and the BCH model, by solving different MATPOWER cases: case30, case39, case57, case118 and case145. The sizes of these instances and the associated run times are shown in Table 2. The results in Table 2 show that even for the large-size power network (case145), all three models can be solved efficiently (i.e., within 4 seconds). We also observe that even though model (18) requires more variables – at most $2\mathcal{E} + 2\mathcal{G}$ additional variables – its computational time is similar to the other two models.

Table 2: Computational time comparison among model (18), risk neutral model and BCH model

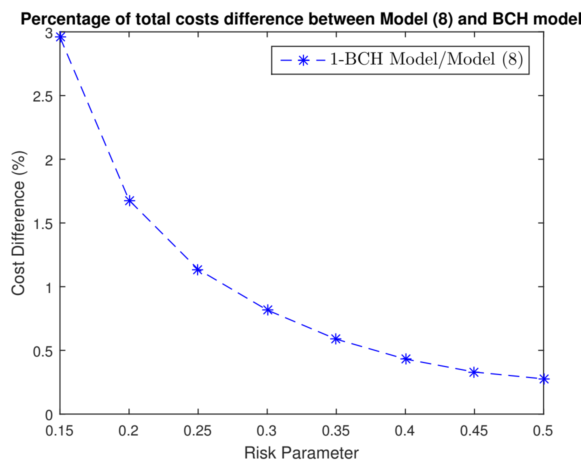
Cases		case30	case39	case57	case118	case145
Data	Buses ($\lvert\mathcal{V}\rvert$)	30	39	57	118	145
	Lines ($\lvert\mathcal{E}\rvert$)	41	46	80	186	453
	Generators ($\lvert\mathcal{G}\rvert$)	6	10	7	54	50
	Renewables ($\lvert\mathcal{W}\rvert$)	3	3	5	11	14
Results	Model (18)	1.48	1.13	1.02	2.08	3.35
	Risk Neutral	0.83	0.57	1.36	1.71	1.63
	BCH	0.83	0.92	1.63	1.60	3.41

5 Conclusion

This paper studies a distributionally robust chance constrained optimal power flow problem (DRCC-OPF) with known first and second moments. We propose an exact second order cone programming reformulation of DRCC-OPF. Our numerical study shows that the proposed model can be solved efficiently and the results are quite robust even for large risk parameters. We note that the same uncertain parameters appear on multiple chance constraints associated with different lines and buses. Therefore, it is interesting as a future study to explore the tractability of considering the uncertain constraints of lines and buses jointly (i.e., a joint chance constraint) in-



(a) Max. Prob. of violations



(b) Total costs

Figure 1: Comparison between model (18) and BCH model

stead of individual ones. Another future direction is to incorporate more structural information into the ambiguity set, for example, symmetry [14] or unimodality [14, 18].

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