# Zero Padding: A Misnomer for Frequency Resolution

*Hassan Islam B.Eng in Electrical Engineering*
*An exploration into and expression of my fondness for Signal Processing*

## 1. INTRODUCTION

Frequency resolution measures the ability to discern nearby frequencies. When working with the DFT, bin spacing, and frequency resolution are often used interchangeably. In this article, we show why this equivalence is misleading, and how it can lead to incorrect conclusions.

$$\Delta \mathrm{f}_{\mathrm{res}} = \Delta \mathrm{f}_{\mathrm{bin}} = \frac{\mathrm{f}_{\mathrm{s}}}{\mathrm{N}_{\mathrm{samples}}}$$

Looking at this equation naively, we might conclude that increasing the number of samples will always increase our resolution, thus, will allow us to resolve frequency components that are nearby. And so, instead of increasing the number of samples measured by a sampling device, we are compelled to zero-pad a signal afterwards to increase N artificially.

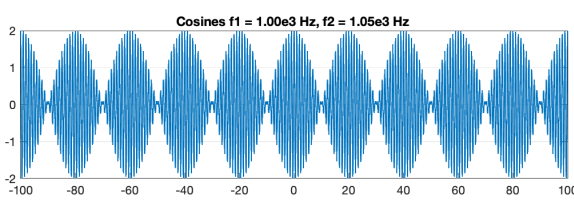
In this case study, we will observe how zero-padding does not represent an increase in true spectral resolution, but only offers extra graphical detail through interpolation.

To make the difference clear, we study a simple test case: a sum of two pure sinusoids whose frequencies are only 50 Hz apart.

## 2. SET UP

Consider a signal $x(t)$ composed of two similar frequencies $\mathrm{f}_1, \mathrm{f}_2$, where $f_2 = f_1 + \Delta f$ and $\Delta \mathrm{f} = 50$ Hz

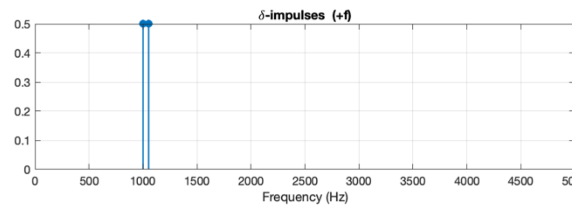
$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t),$$




The frequency transform of the signal above is as follows.

$$\mathcal{F}\{x(t)\} =$$
$$X(f) = \frac{1}{2}[\delta(f-f_1) + \delta(f+f_1) + \delta(f-f_2) + \delta(f+f_2)]$$

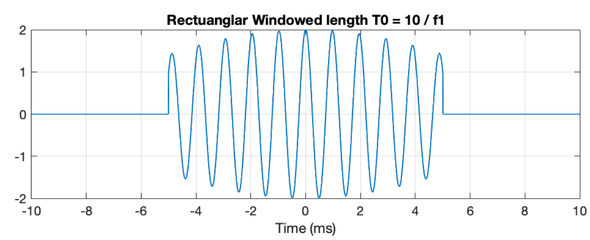
We only observe the +frequency side of the spectrum in the plot below.




The spectrum above assumes the sinusoid is infinite, but in real applications, we can only sample a certain length of a signal. This is mathematically represented by applying the rect function to the signal.

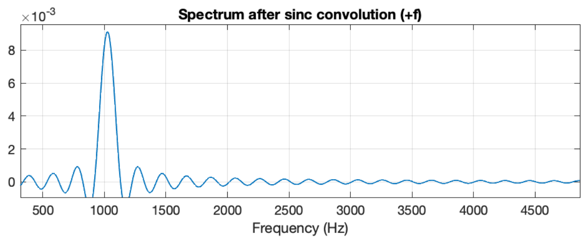
$$w(t) = rect(t/T) = \begin{cases} 1, |t| \leq T/2 \\ 0, \mathit{otherwise} \end{cases}$$

$$x_{frame}(t) = \ x(t) \cdot w(t)$$




Given the convolution theorem, we see that the effect of performing this operation in the time domain results in a spectrum convolved with a sinc.

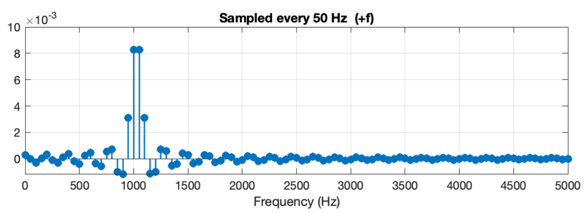
$$Y(f) = X(f) * W(f) =$$
$$\frac{T}{2}[\mathrm{sinc}[(f-f_1)T] + \mathrm{sinc}[(f+f_1)T] + \mathrm{sinc}[(f-f_2)T]$$
$$+ \mathrm{sinc}[(f+f_2)T]$$




Again, in real applications, we observe the frequency spectrum in a discrete form. This can be mathematically represented as the spectrum multiplied by an impulse train, or Dirac comb function.

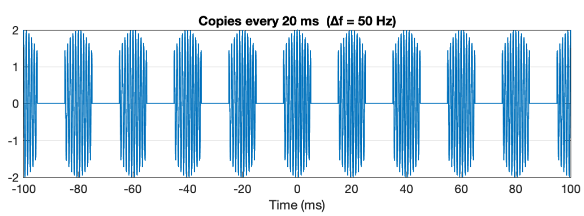
$$\Omega(f) = \sum_{k=-\infty}^{\infty} \delta(f - k\Delta f_{bin})$$

In the plot below, $\Delta f_{bin} = 50$ Hz




Applying the convolution theorem again, we see that the time domain signal is convolved with a Dirac comb function as well.

$$\Omega(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\frac{1}{\Delta f_{bin}})$$

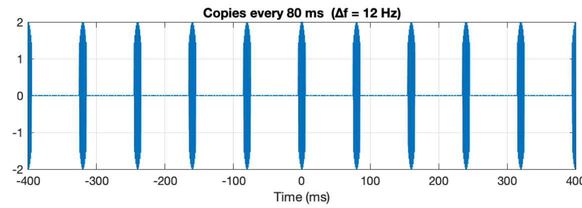



The signal is has become periodic; it is inherently zero-padded to match the periodicity of $\frac{1}{\Delta f_{bin}}$.

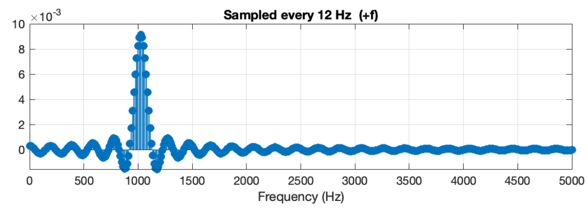
We can now evaluate whether increasing the amount of zero-padding will resolve the two frequencies in the signal.

### 3. OBSERVATIONS

Here we decrease the bin spacing to $\Delta f_{bin} = 12$ Hz and observe its time domain dual.




We see that we have increased the zero-padding, but when we observe the frequency dual of the signal, we cannot resolve the two frequencies.

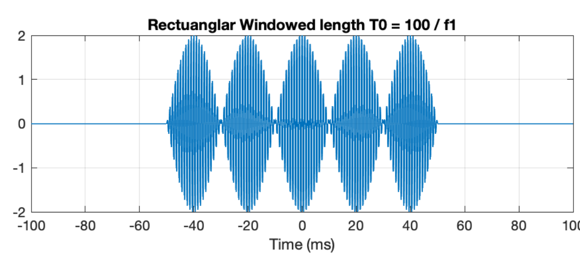


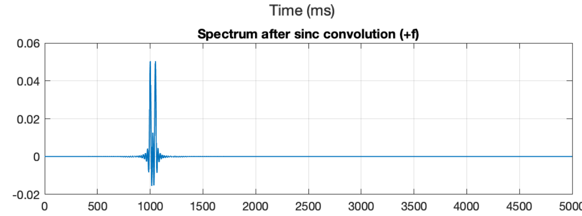

In fact, we will never be able to resolve the two frequencies. As $\Delta f_{bin} \rightarrow 0$ we converge on the continuous spectrum we started with, where the sinc lobes of the two components still overlap.

Zero-padding only adds interpolated points between the frequencies. It increases the smoothness of the plot, but it cannot improve the underlying frequency resolution set by the finite observation time.

The two sinusoids are too close, and their sinc lobes interfere with each other. As such, the only way to resolve them would be to reduce the width of these lobes. This can be done by increasing T.

We can observe the effect of increasing T (increasing our observation window and thereby recording more samples) and can now discern the two frequencies.



## 4. CONCLUSION

Zero-padding reduces the DFT bin spacing, but as the study reveals, it does not shrink the main-lobe width set by the observation window length T, which is what governs the ability to resolve nearby frequencies. Extra zeros only interpolate between existing spectral samples; only a longer record of actual samples provides genuine gains in frequency resolution.