Graphical

Sensitivity Analysis

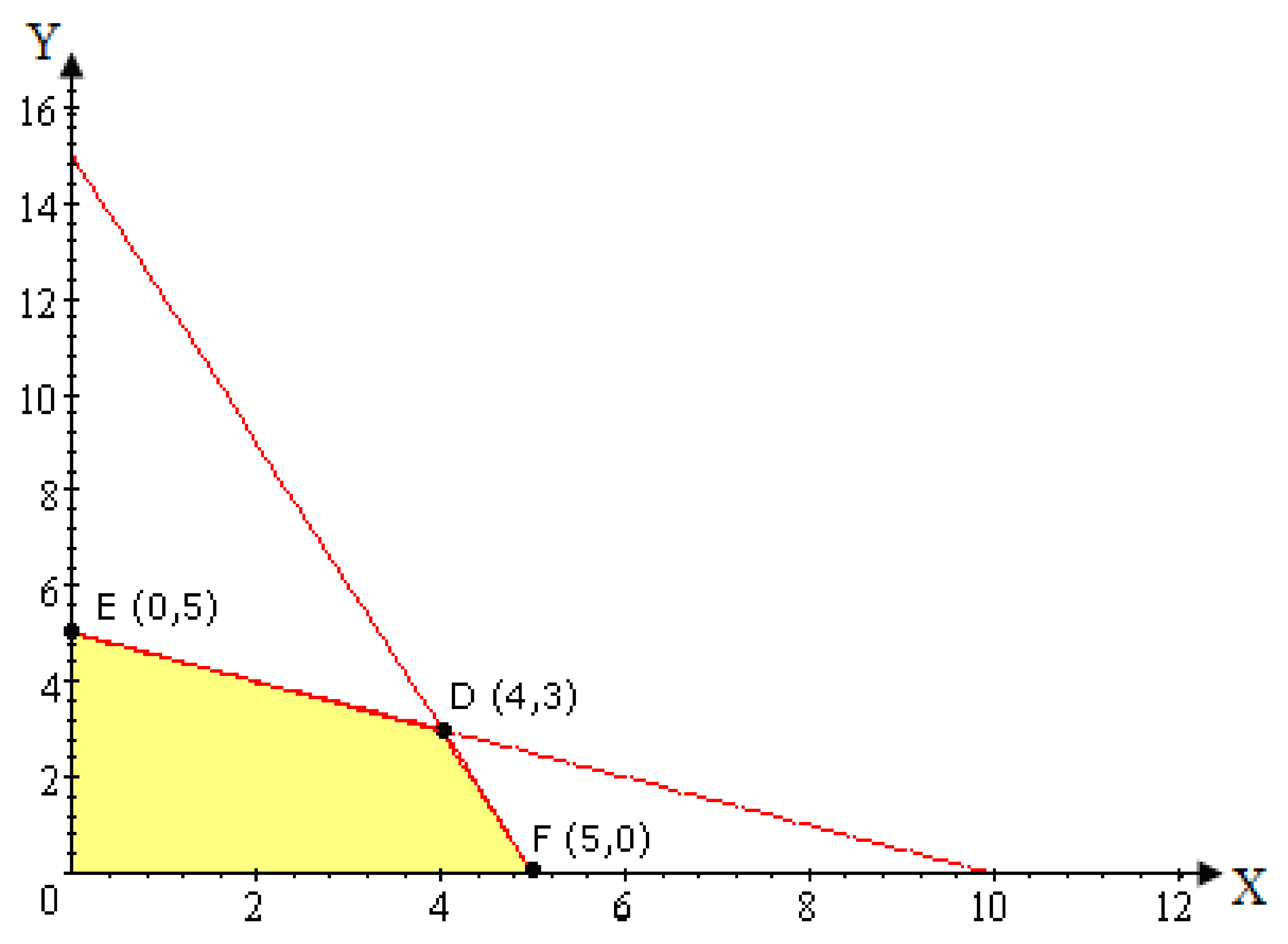
&

Special Cases

Maximize: $P = 10X + 6Y$

Subject to:

$$12X + 4Y \le 60$$
$$4X + 8Y \le 40$$
$$X, Y \ge 0$$



$\Rightarrow$ **The optimal solution is: $P = \$58$ at point D (4, 3).**

Sensitivity of the Coefficients of the Objective Function

$$P = 10\,X + 6\,Y$$

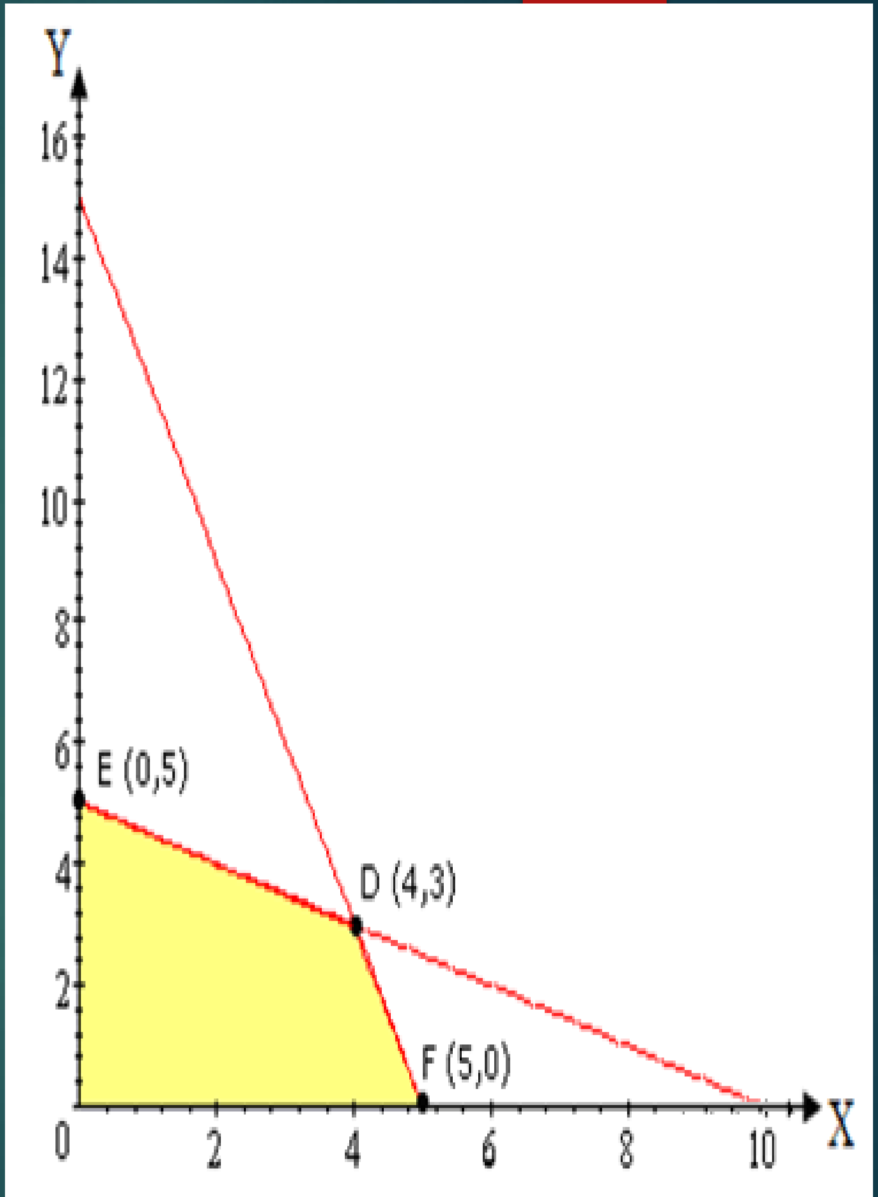
P = 10 X + 6 Y

$$Let\ P = c_1\, x + c_2\, y$$

$$c_2\, y = -c_1\, x + P \quad \Rightarrow \quad y = -\frac{c_1}{c_2}\, x + \frac{P}{c_2}$$

First binding constraint: $12\,x + 4\,y = 60 \quad \Rightarrow \quad y = -3\,x + 15$

Second binding constraint: $4\,x + 8\,y = 40 \quad \Rightarrow \quad y = -0.5\,x + 5$



The point D (4, 3) remains an optimal solution as long as the slope of the objective function $\left(-\frac{c_1}{c_2}\right)$ lies between the slopes of the two binding constraints (-3) *and* (-0.5).

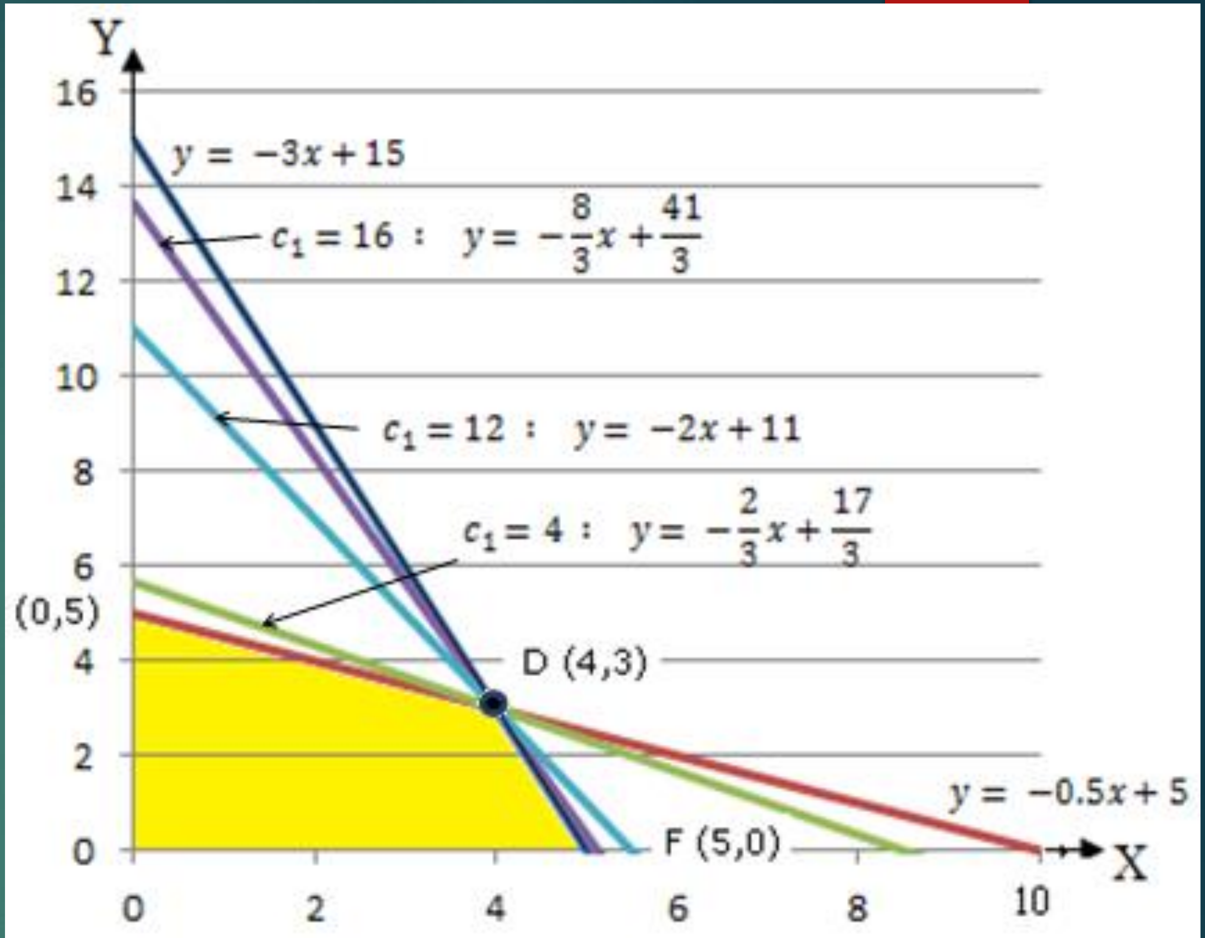
$$\Rightarrow \qquad -3 \le -\frac{c_1}{c_2} \le -0.5$$

$$* \quad Fix\ c_2 = 6: \quad \Rightarrow -3 \le -\frac{c_1}{6} \le -0.5$$

$$Multiply\ by\ (-6): \Rightarrow (-6)(-3) \ge c_1 \ge (-6)(-0.5)$$

$$\Rightarrow 3 \le c_1 \le 18 \qquad increase: 8\ , \qquad decrease: 7$$

$$Old\ optimal\ value:\ P = 58\ \ at\ point\ \ D(4,3)$$



$$For\ \ c_1 = 3\ , \quad new\ optimal\ value;\ \ P = (3)(4) + (6)(3) = 30\ \ at\ \ point\ \ D(4,3)$$

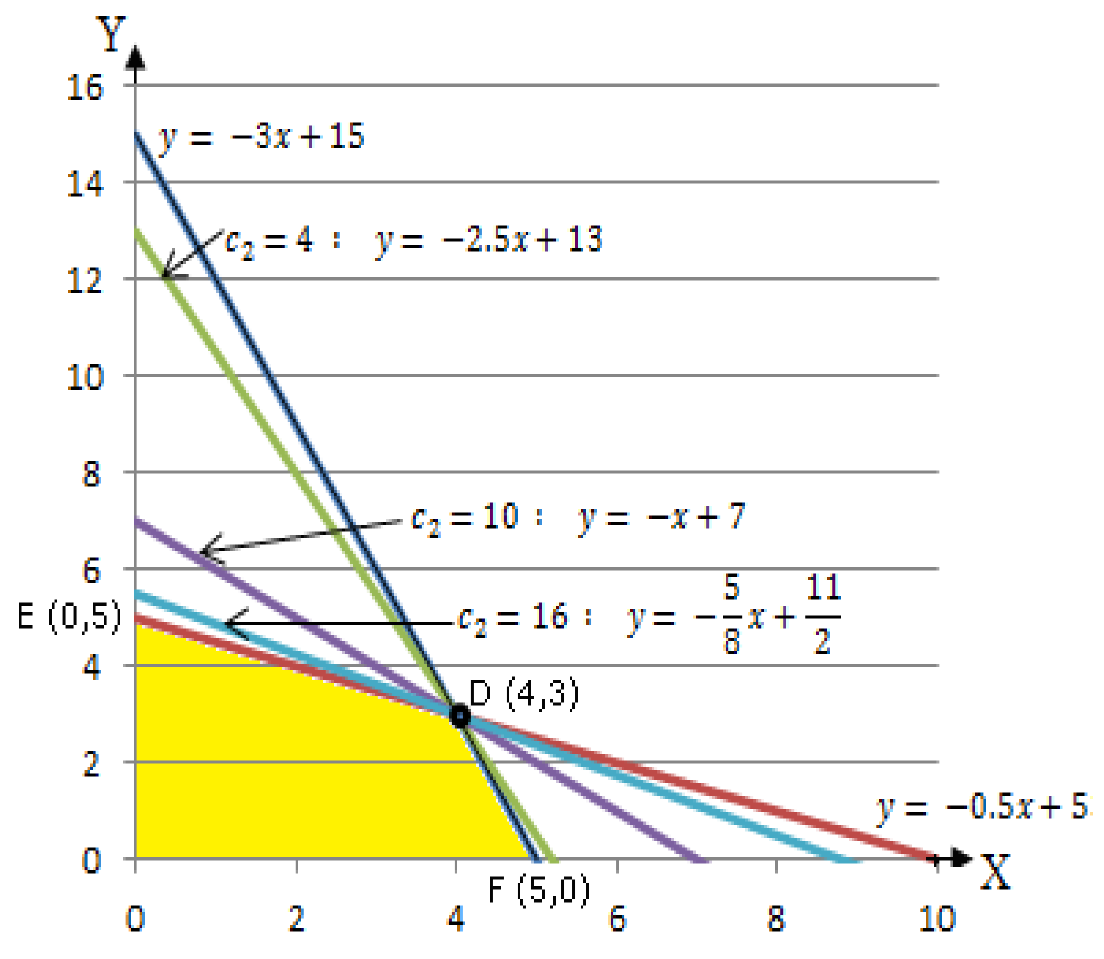
$$For\ \ c_1 = 18\ , \quad new\ optimal\ value;\ \ P = (18)(4) + (6)(3) = 90\ \ at\ \ point\ \ D(4,3)$$

$$* \quad Fix\ c_1 = 10: \quad \Rightarrow -3 \le -\frac{c_1}{c_2} \le -0.5 \quad \Rightarrow -3 \le -\frac{10}{c_2} \le -0.5$$

$$Take: \quad (-3) \le -\frac{10}{c_2} \Rightarrow Multiply\ by\ (-c_2): \quad \Rightarrow 3c_2 \ge 10 \quad \Rightarrow c_2 \ge \frac{10}{3}$$

$$Take: \quad -\frac{10}{c_2} \le (-0.5) \Rightarrow Multiply\ by\ (-c_2): \quad \Rightarrow 10 \ge 0.5c_2 \quad \Rightarrow c_2 \le 20$$

$$\Rightarrow \frac{10}{3} \le c_2 \le 20 \qquad increase: 14\ , \qquad decrease: \frac{8}{3}$$



$$For \quad c_2 = \frac{10}{3}, \quad new\ optimal\ value; \quad P = (10)(4) + \left(\frac{10}{3}\right)(3) = 50 \ at\ point\ D(4,3)$$

$$For \quad c_2 = 20, \quad new\ optimal\ value; \quad P = (10)(4) + (20)(3) = 100 \ at\ point\ D(4,3)$$

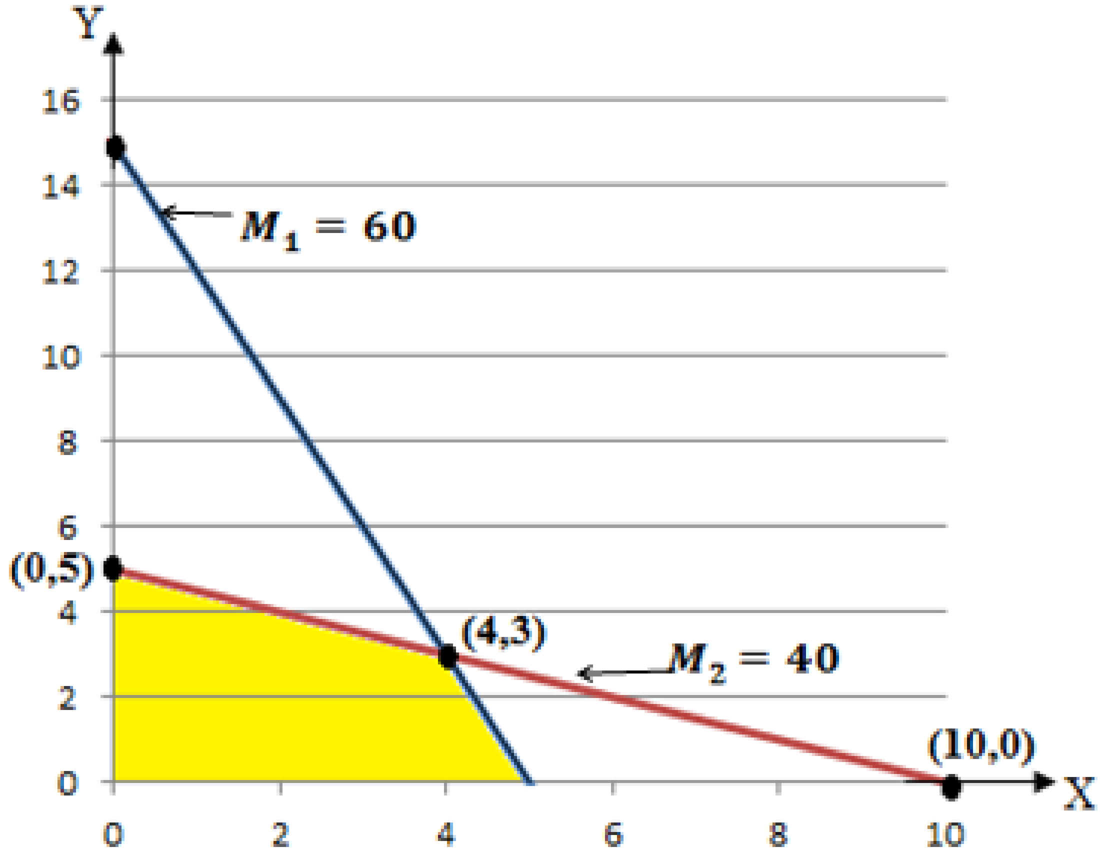
Sensitivity of the Resources

$$12\,X + 4\,Y \le 60$$

$$4\,X + 8\,Y \le 40$$

For the resources: $M_1 = 60$, $M_2 = 40$

$$Fix \quad M_2 = 40$$



$$The\ line\ M_1\ will\ slide\ between\ points\ (0,5)\ and\ (10,0)$$

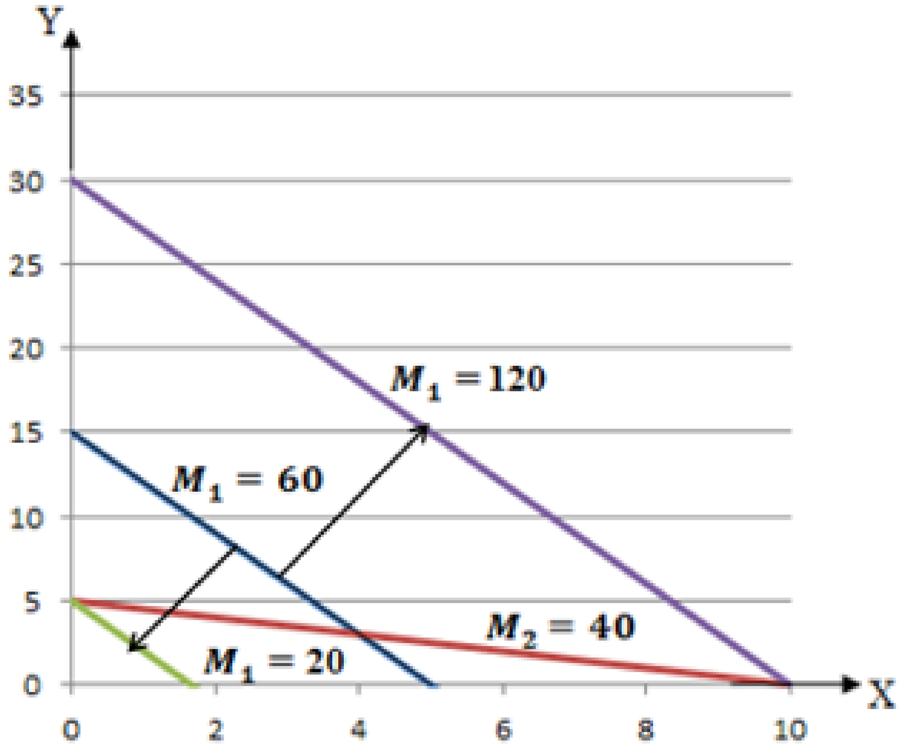
$$M_1\ at\ (0,5):\quad (12)(0) + (4)(5) = 20$$

$$M_1\ at\ (10,0):\quad (12)(10) + (4)(0) = 120$$

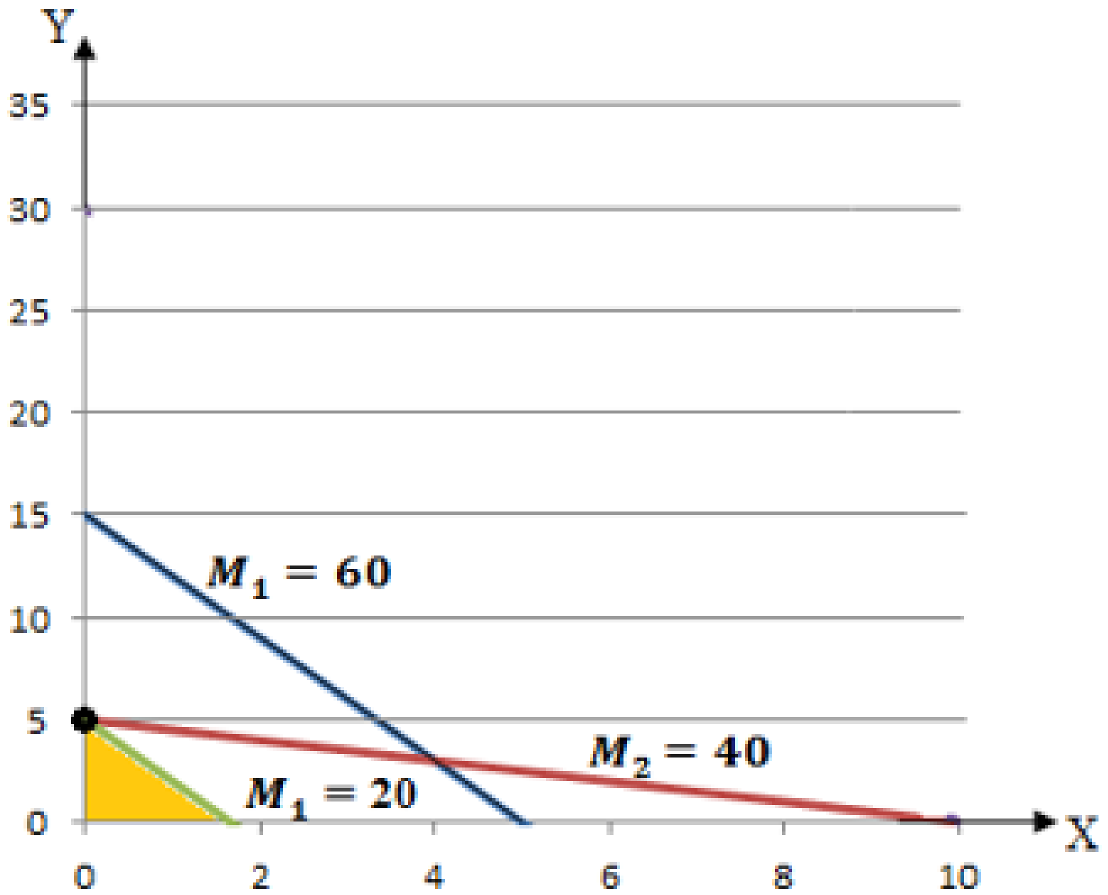
$$20 \le M_1 \le 120$$

$\therefore\ M1\ decreases\ by\ (60 - 20) = 40$

$\therefore\ M1\ increases\ by\ (120 - 60) = 60$

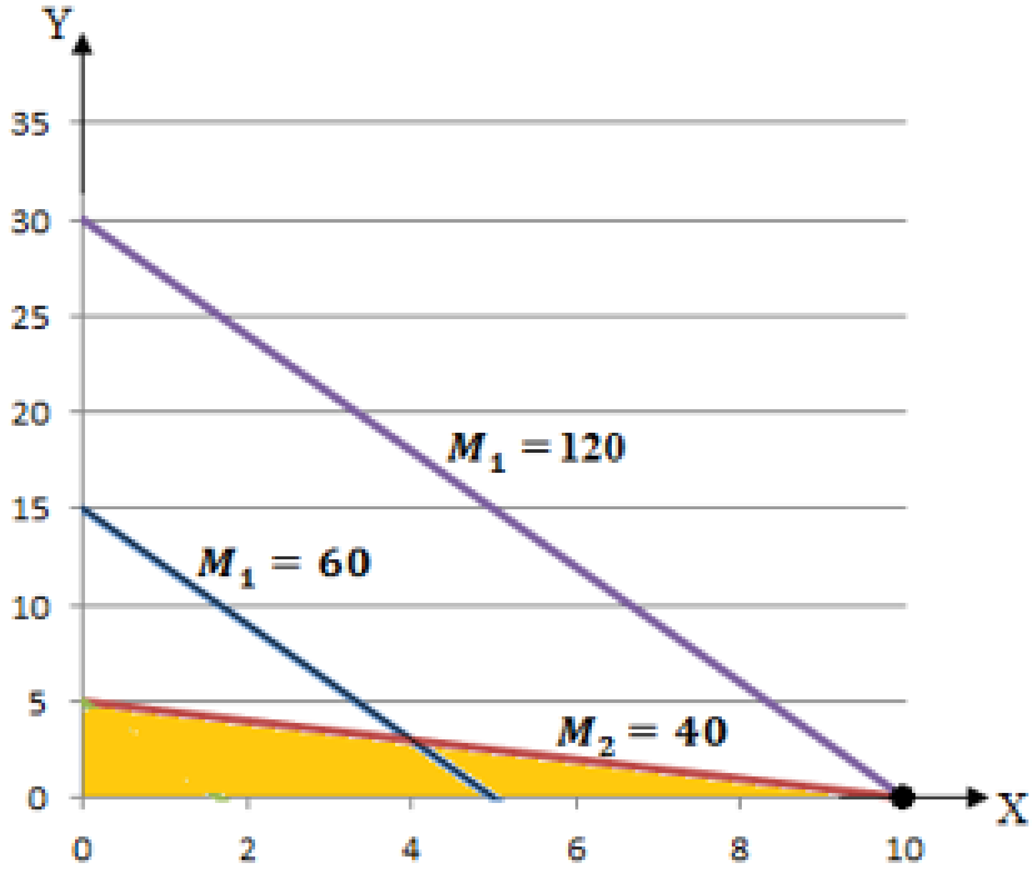


For $M1 = 20$



New optimal solution at point $(0,5)$: $P = (10)(0) + (6)(5) = 30$

For $M1 = 120$



$$New\ optimal\ solution\ at\ point\ (10,0):\ P = (10)(10) + (6)(0) = 100$$

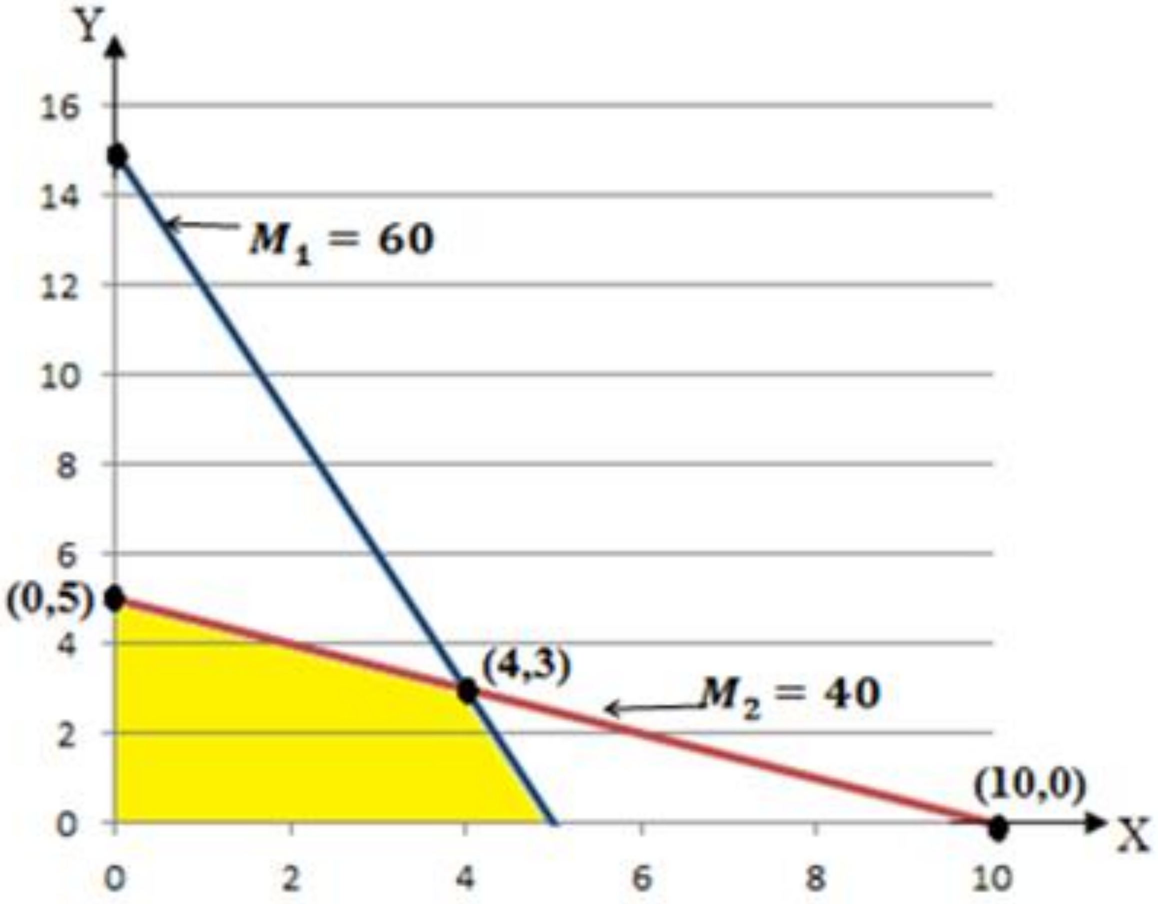
$$unit\ worth\ of\ M_1 = \frac{change\ in\ P\ from\ (0,5)\ to\ (10,0)}{change\ in\ M_1\ from\ (0,5)\ to\ (10,0)}$$

$$= \frac{100-30}{120-20} = 0.70$$

$\Rightarrow$ *An increase (decrease) in* M_1 ***by one unit in the range*** $20 \le M_1 \le 120$ *increases (decreases) the profit by* $\$0.70$.

<u>For the resources:</u> $M_1 = 60$, $M_2 = 40$

$$Fix \quad M_1 = 60$$



$$\textit{The line } M_2 \textit{ will slide between points } (0,15) \textit{ and } (5,0)$$

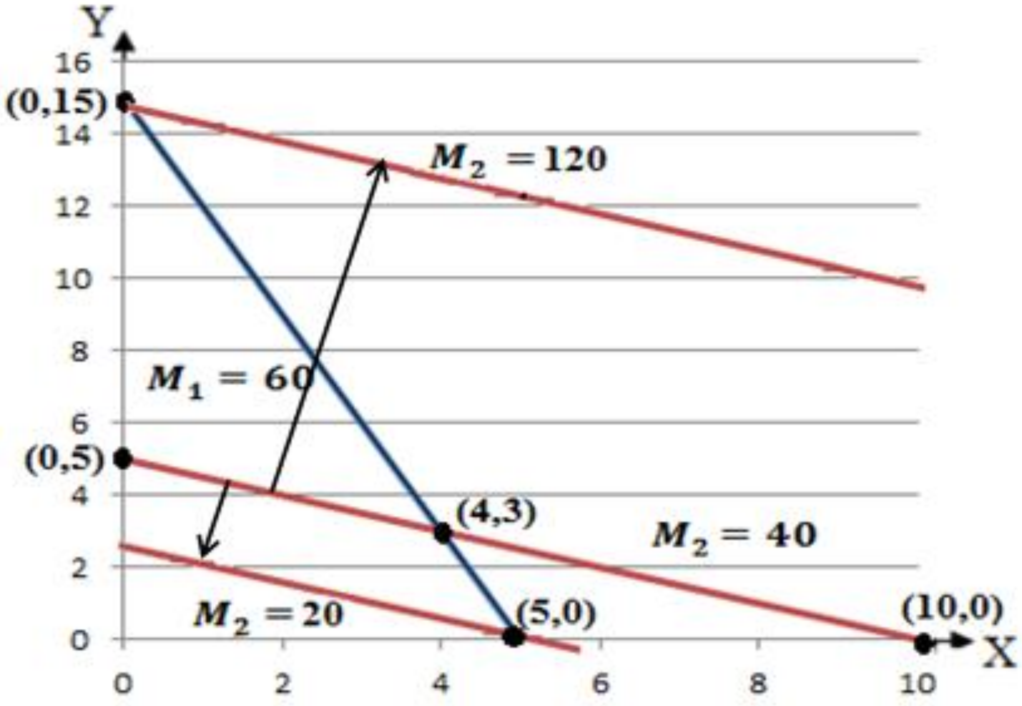
$$M_2 \; at \; (0,15): \quad (4)(0) + (8)(15) = 120$$

$$M_2 \; at \; (5,0): \quad (4)(5) + (8)(0) = 20$$

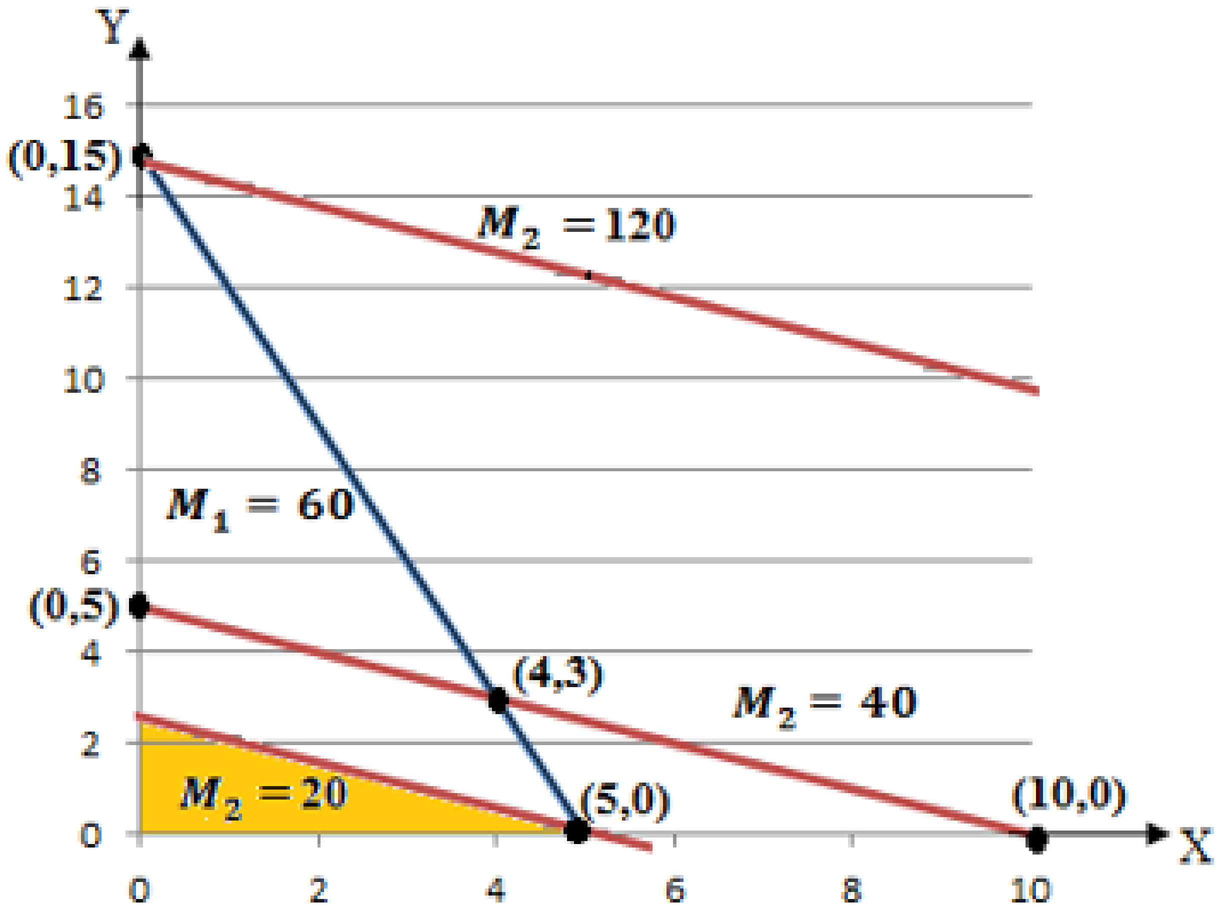
$$20 \leq M_2 \leq 120$$

$\therefore$ *M2 decreases by* $(40 - 20) = 20$

$\therefore$ *M2 increases by* $(120 - 40) = 80$

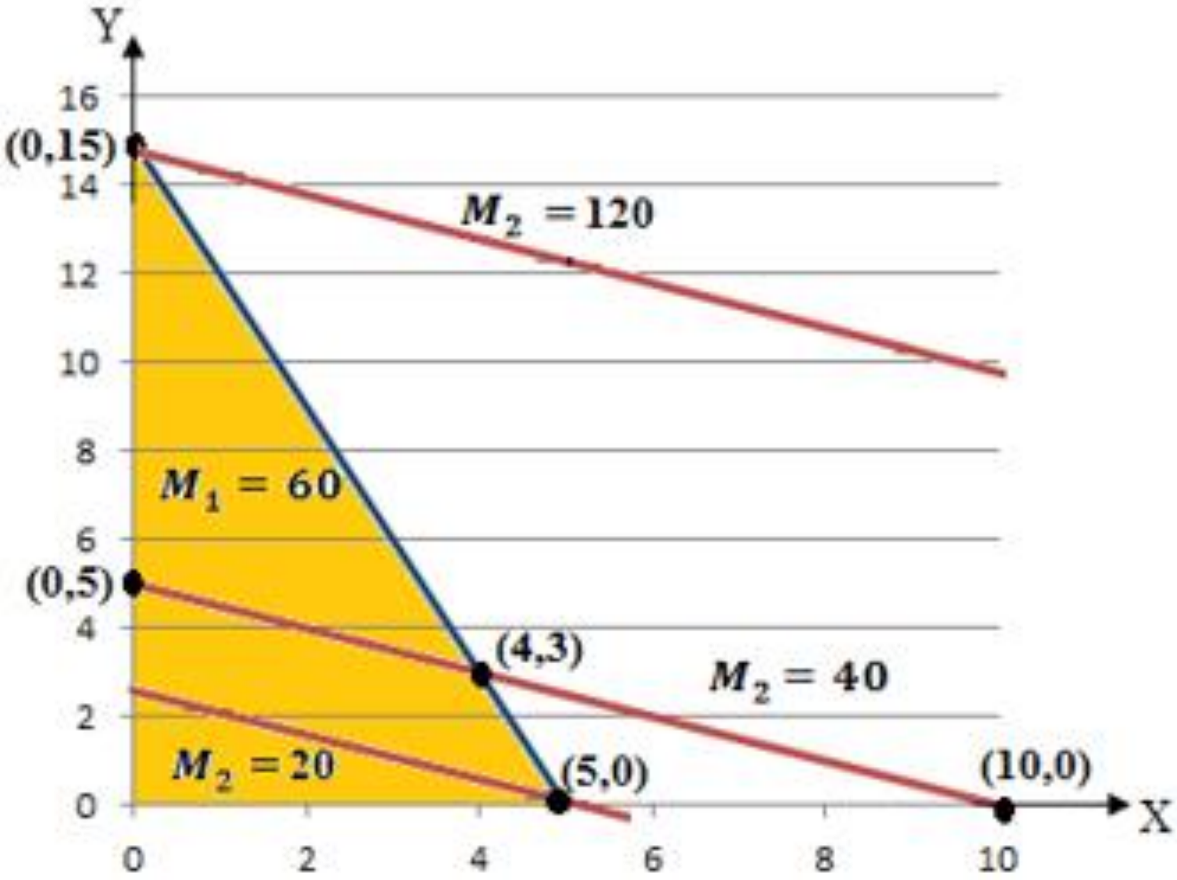


For M2 = 20



New optimal solution at point $(5,0)$: $P = (10)(5) + (6)(0) = 50$

For $M2 = 120$



New optimal solution at point $(0{,}15)$: $P = (10)(0) + (6)(15) = 90$

$$unit\ worth\ of\ M_2 = \frac{change\ in\ P\ from\ (0,15)\ to\ (5,0)}{change\ in\ M_2\ from\ (0,15)\ to\ (5,0)}$$

$$= \frac{90 - 50}{120 - 20} = 0.40$$

$$\Rightarrow An\ increase\ (decrease) in\ M_2\ by\ one\ unit\ in\ the\ range\ \ 20 \le M_1 \le 120$$

$$increases\ (decreases)\ the\ profit\ \ by\ \$0.40.$$

Special Cases In Graphical Method

1) Degenerate Solution

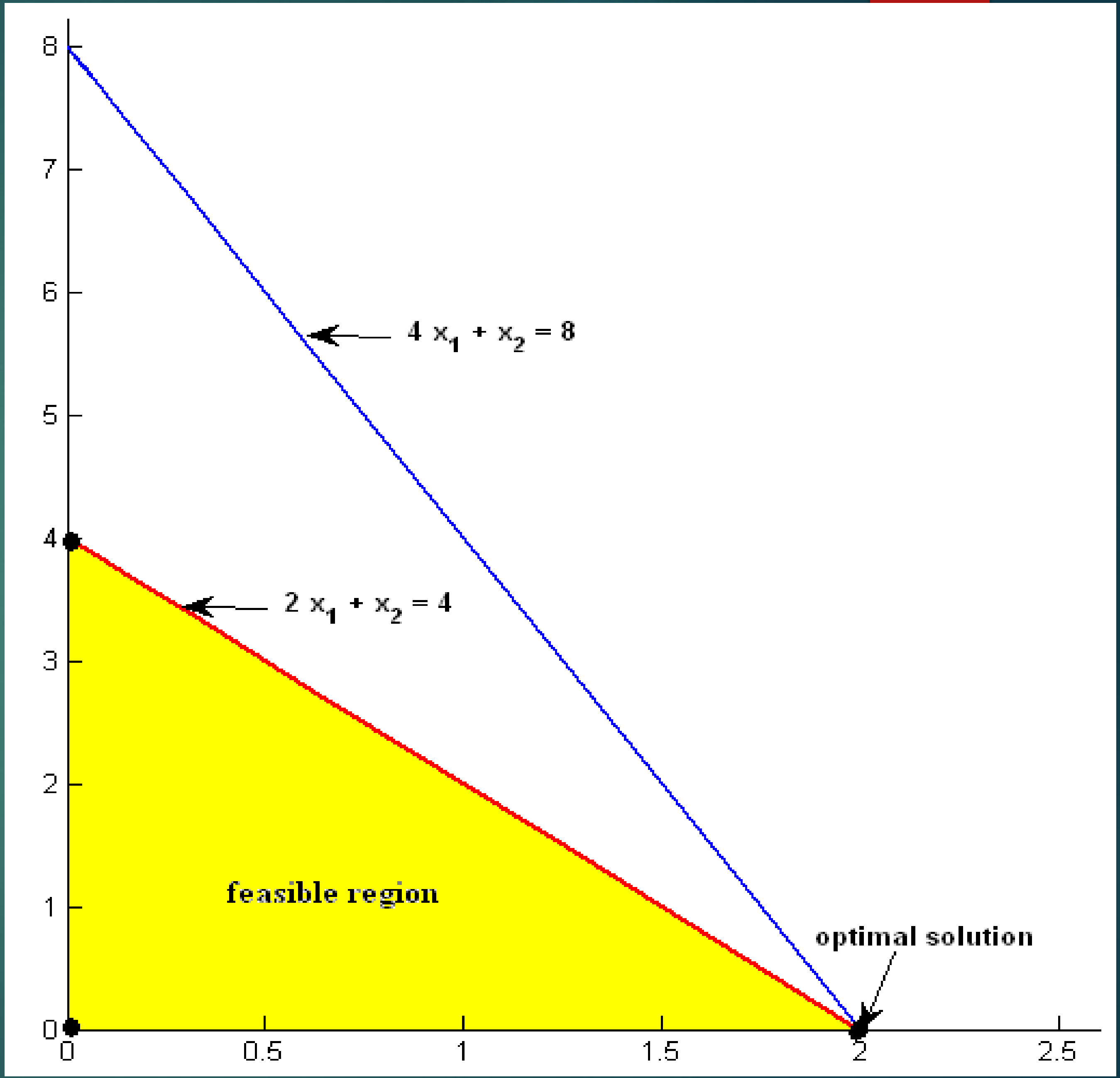
Maximize: $z = 2\,x_1 + 3\,x_2$

subject to:

$$4\,x_1 + x_2 \le 8$$

$$2\,x_1 + x_2 \le 4$$

$$x_1\,, x_2 \ge 0$$



Point (2 , 0) is over determined, one of the constraints is redundant.

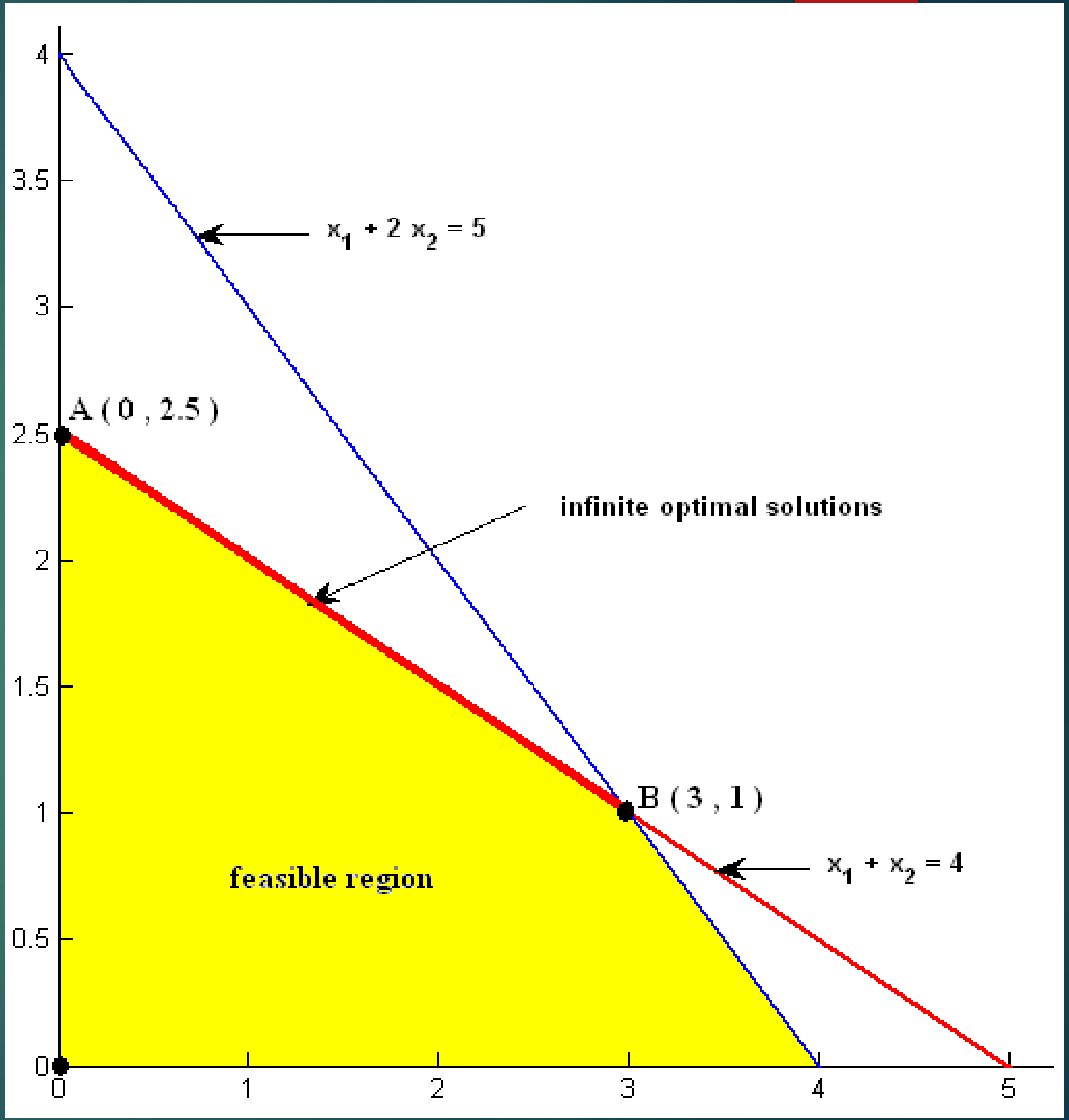
2) Infinity of Solutions

Maximize: $z = 2\,x_1 + 4\,x_2$

subject to: $x_1 + 2\,x_2 \le 5$

$x_1 + x_2 \le 4$

$x_1\ , x_2 \ge 0$



At point A (0, 2.5) : z = 10

At point B (3 , 1) : z = 10

Every point on the line segment AB is an optimum solution.

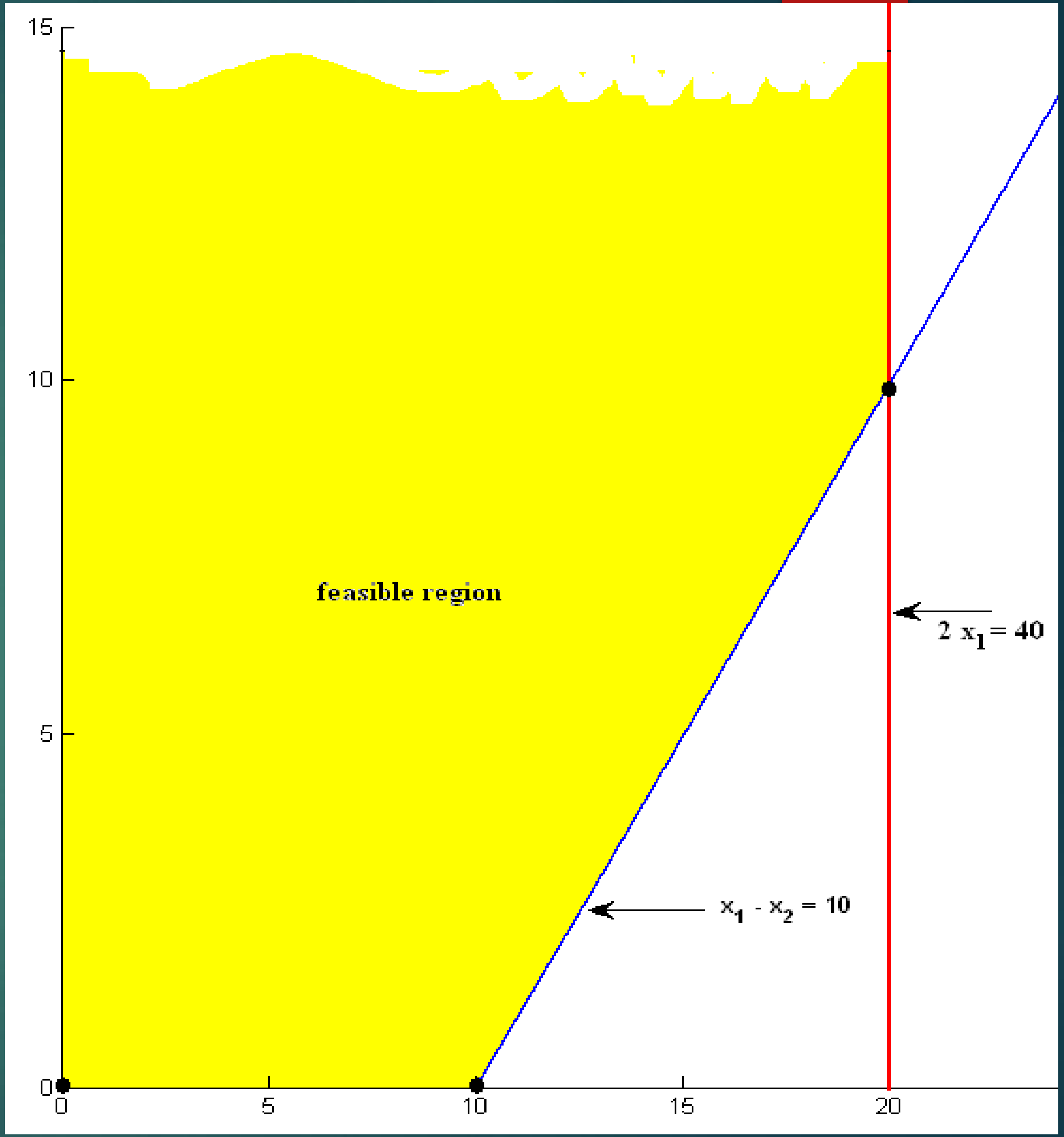
3) Unbounded Solution

Maximize: $z = 2\,x_1 + x_2$

subject to: $x_1 - x_2 \leq 10$

$2\,x_1 \leq 40$

$x_1\ ,x_2 \geq 0$



X_2 can increase indefinitely without violating any of the constraints.

Z can increase indefinitely.

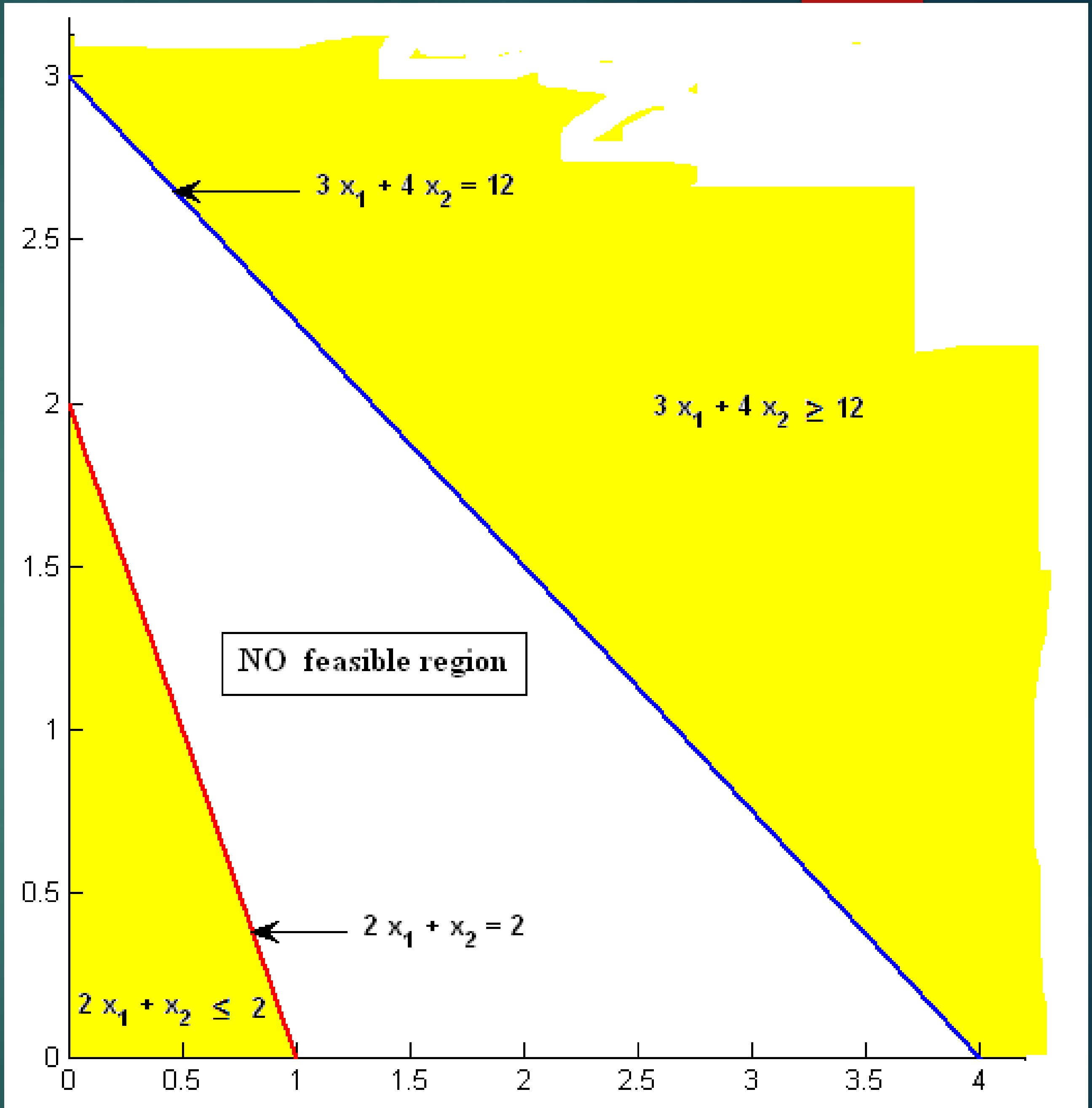
4) Infeasible Solution

$$\textbf{Maximize:} \quad z = 3x_1 + 2x_2$$

$$\textbf{subject to:} \quad 2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1 \;, x_2 \geq 0$$



There is no feasible region.

No solution.

Till next lecture,

take care.