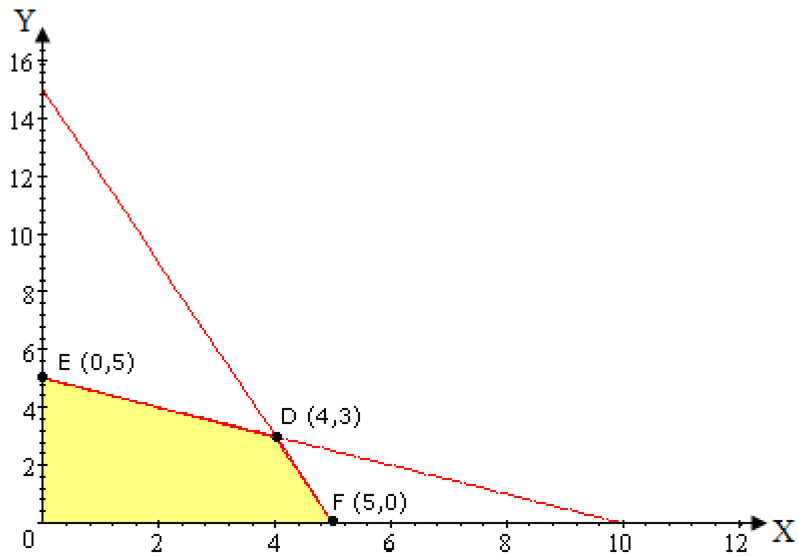
Fall 2022-2023

Operations Research

Solution of Assignment #3 - Graphical Sensitivity Analysis

1. Maximize: $P = 10x + 6y$

   Subject to:
   $12x + 4y \le 60$
   $4x + 8y \le 40$
   All variables ≥ 0



$\Rightarrow$ The optimal solution is: P = \$58 at point D(4,3).

$$Let\ P = c_1\, x + c_2\, y$$

$$c_2\, y = -c_1\, x + P \quad \Rightarrow \quad y = -\frac{c_1}{c_2}\, x + \frac{P}{c_2}$$

First binding constraint: $12\,x + 4\,y = 60 \quad \Rightarrow \quad y = -3\,x + 15$

Second binding constraint: $4\,x + 8\,y = 40 \quad \Rightarrow \quad y = -0.5\,x + 5$

The point D(4,3) remains an optimal solution as long as the slope of the objective function $\left(-\frac{c_1}{c_2}\right)$ lies between the slopes of the two binding constraints $(-3)\ and\ (-0.5)$.

$$\Rightarrow \qquad -3 \le -\frac{c_1}{c_2} \le -0.5$$

$$* \quad Fix \ c_2 = 6: \quad \Rightarrow -3 \le -\frac{c_1}{6} \le -0.5$$

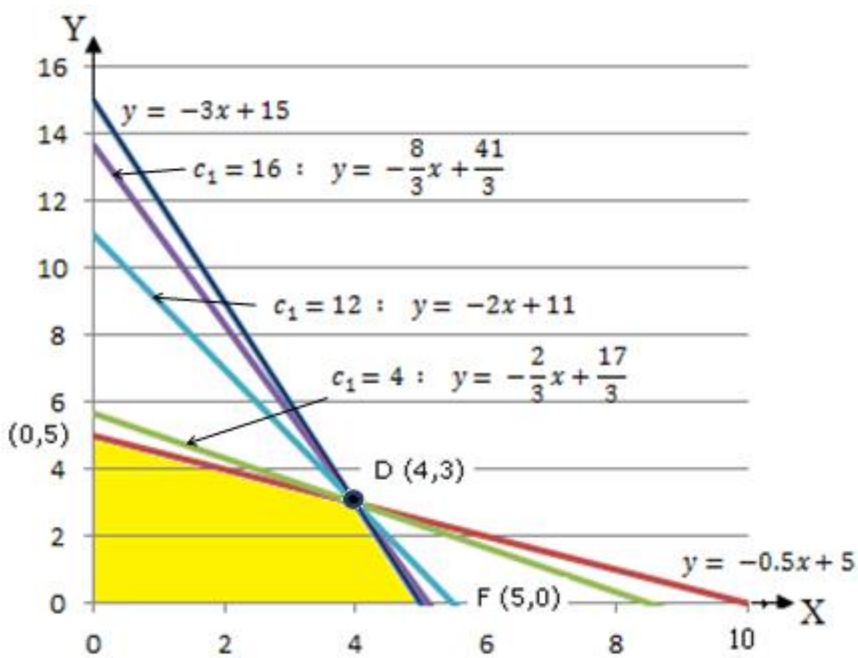
$$Multiply\ by\ (-6): \Rightarrow (-6)(-3) \ge c_1 \ge (-6)(-0.5)$$

$$\Rightarrow 3 \le c_1 \le 18 \qquad increase: 8 \ , \qquad decrease: 7$$

$$Old\ optimal\ value:\ P = 58 \ \ at\ point\ \ D(4,3)$$

$$For \ \ c_1 = 3\,, \ new\ optimal\ value; \ \ P = (3)(4) + (6)(3) = 30 \ \ at\ \ point\ \ D(4,3)$$

$$For \ \ c_1 = 18\,, \ new\ optimal\ value; \ \ P = (18)(4) + (6)(3) = 90 \ \ at\ \ point\ \ D(4,3)$$



$$* \quad Fix\ c_1 = 10: \quad \Rightarrow -3 \le -\frac{c_1}{c_2} \le -0.5 \quad \Rightarrow -3 \le -\frac{10}{c_2} \le -0.5$$

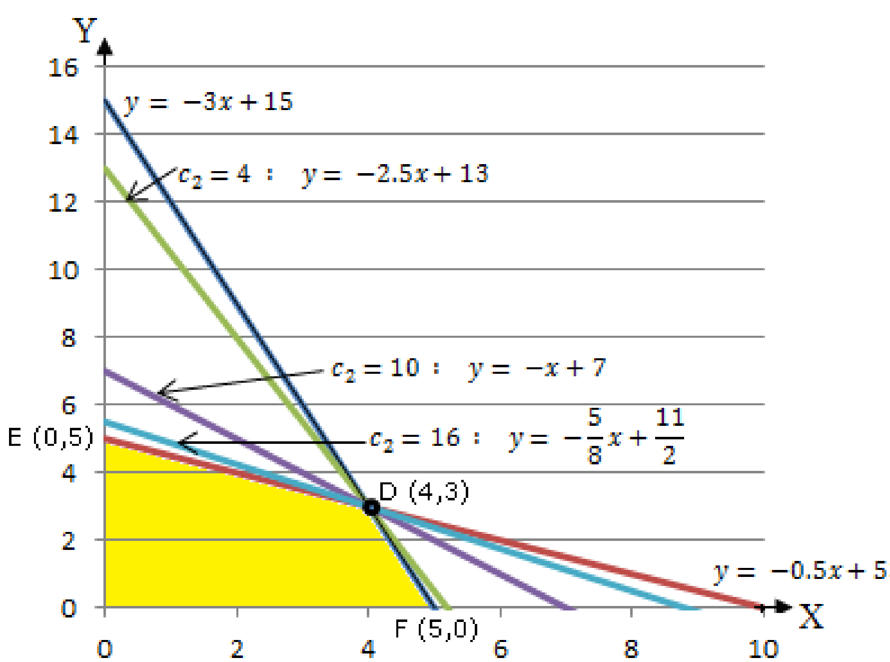
$$Take: \quad (-3) \le -\frac{10}{c_2} \ \Rightarrow \ Multiply\ by\ (-c_2): \quad \Rightarrow 3\,c_2 \ge 10 \quad \Rightarrow \quad c_2 \ge \frac{10}{3}$$

$$Take: \quad -\frac{10}{c_2} \le (-0.5) \ \Rightarrow \ Multiply\ by\ (-c_2): \quad \Rightarrow 10 \ge 0.5\,c_2 \quad \Rightarrow \quad c_2 \le 20$$

$$\Rightarrow \frac{10}{3} \le c_2 \le 20 \qquad increase: 14 \ , \qquad decrease: \frac{8}{3}$$

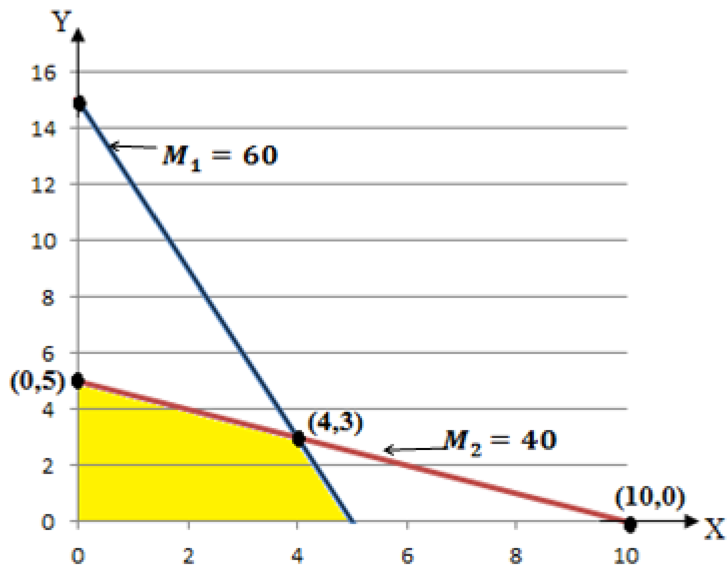
$$For \ \ c_2 = \frac{10}{3}\ , \ new\ optimal\ value; \ \ P = (10)(4) + \left(\frac{10}{3}\right)(3) = 50 \ \ at\ \ point\ \ D(4{,}3)$$

$$For \ \ c_2 = 20\ , \ new\ optimal\ value; \ \ P = (10)(4) + (20)(3) = 100 \ \ at\ \ point\ \ D(4{,}3)$$



For the resources: $M_1 = 60$, $M_2 = 40$

$$Fix \quad M_2 = 40$$



$$The\ line\ M_1\ will\ slide\ between\ points\ (0,5)\ and\ (10,0)$$

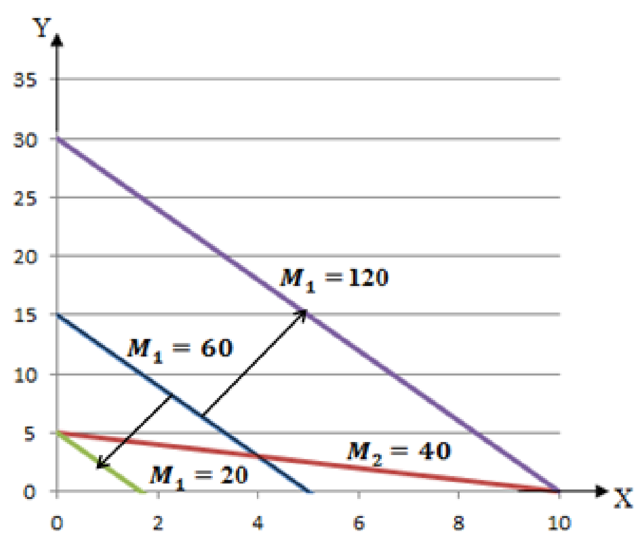
$$M_1\ at\ (0,5):\quad (12)(0) + (4)(5) = 20$$

$$M_1\ at\ (10,0):\quad (12)(10) + (4)(0) = 120$$

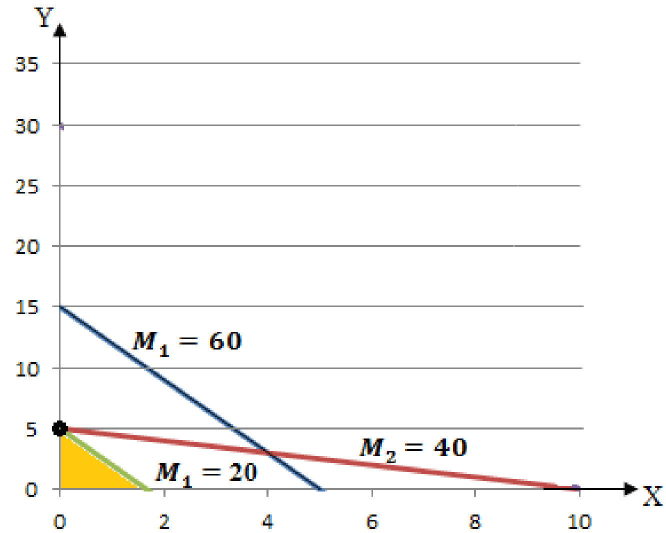
$$20 \leq M_1 \leq 120$$

$\therefore M1\ decreases\ by\ (60 - 20) = 40$

$\therefore M1\ increases\ by\ (120 - 60) = 60$

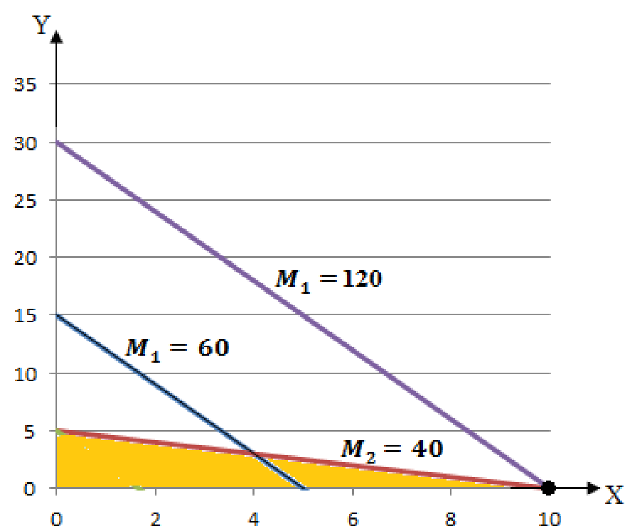


For $M1 = 20$



$$New\ optimal\ solution\ at\ point\ (0{,}5){:}\ \ P = (10)(0) + (6)(5) = 30$$

For $M1 = 120$



$$New\ optimal\ solution\ at\ point\ (10{,}0){:}\ \ P = (10)(10) + (6)(0) = 100$$

$$\boldsymbol{unit\ worth\ of\ M_1} = \frac{change\ in\ P\ from\ (0{,}5)\ to\ (10{,}0)}{change\ in\ M_1\ from\ (0{,}5)\ to\ (10{,}0)} = \frac{100-30}{120-20} = 0.70$$

$$\Rightarrow An\ increase\ (decrease) in\ M_1\ by\ one\ unit\ in\ the\ range\ \ 20 \le M_1 \le 120$$

$$increases\ (decreases\ )\ the\ profit\ \ by\ \$0.70.$$

$$Fix \quad M_1 = 60$$

The line M_2 will slide between points $(0, 15)$ and $(5, 0)$

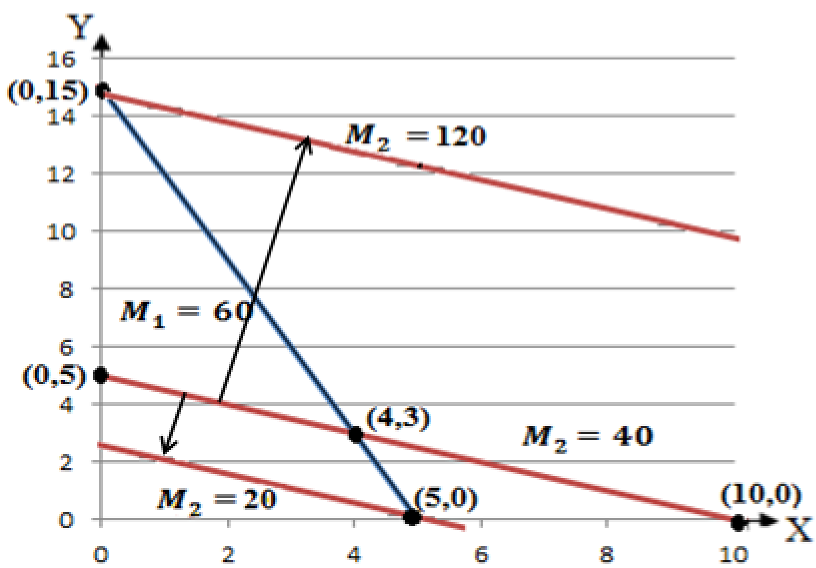
$$M_2 \ at \ (0, 15): \quad (4)(0) + (8)(15) = 120$$

$$M_2 \ at \ (5, 0): \quad (4)(5) + (8)(0) = 20$$

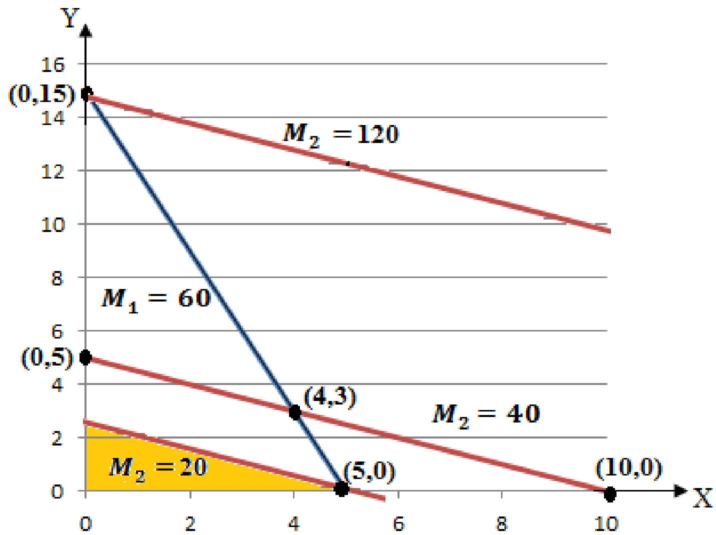
$$20 \leq M_2 \leq 120$$

$$\therefore M2 \ decreases \ by \ (40 - 20) = 20$$

$$\therefore M2 \ increases \ by \ (120 - 40) = 80$$

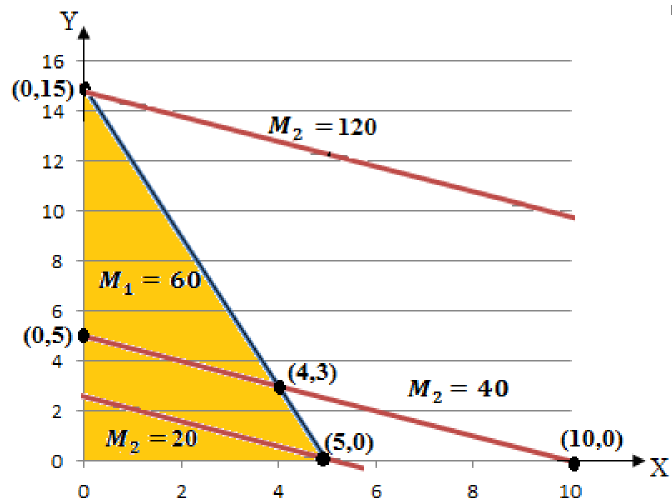


For $M2 = 20$



$$New\ optimal\ solution\ at\ point\ (5{,}0):\ P = (10)(5) + (6)(0) = 50$$

For $M2 = 120$



$$New\ optimal\ solution\ at\ point\ (0{,}15):\ P = (10)(0) + (6)(15) = 90$$

$$\boldsymbol{unit\ worth\ of\ M_2} = \frac{change\ in\ P\ from\ (0{,}15)\ to\ (5{,}0)}{change\ in\ M_2\ from\ (0{,}15)\ to\ (5{,}0)} = \frac{90 - 50}{120 - 20} = 0.40$$

$\Rightarrow An\ increase\ (decrease) in\ M_2\ by\ one\ unit\ in\ the\ range\ 20 \le M_2 \le 120$

$increases\ (decreases\ )\ the\ profit\ \ by\ \$0.40.$

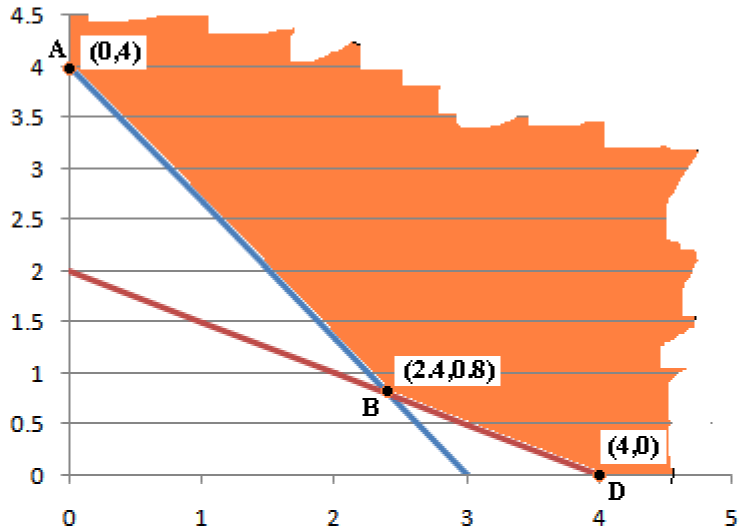
2. Minimize: $C = 6x + 10y$

Subject to:

$4x + 3y \geq 12$

$3x + 6y \geq 12$

All variables ≥ 0



$\Rightarrow$ The optimal solution is: P = $22.4 at point B(2.4,0.8).

$$Let\ P = c_1\, x + c_2\, y$$

$$c_2\, y = -c_1\, x + P \quad \Rightarrow \quad y = -\frac{c_1}{c_2}\, x + \frac{P}{c_2}$$

First binding constraint: $4\,x + 3\,y = 12 \quad \Rightarrow \quad y = -\frac{4}{3}\,x + 4$

Second binding constraint: $3\,x + 6\,y = 12 \quad \Rightarrow \quad y = -0.5\,x + 2$

The point B(2.4,0.8) remains an optimal solution as long as the slope of the objective function $\left(-\frac{c_1}{c_2}\right)$ lies between the slopes of the two binding constraints $\left(-\frac{4}{3}\right)$ and (-0.5).

$$\Rightarrow \qquad -\frac{4}{3} \leq -\frac{c_1}{c_2} \leq -0.5$$

$$* \quad Fix\ c_2 = 10: \quad \Rightarrow -\frac{4}{3} \le -\frac{c_1}{10} \le -0.5$$

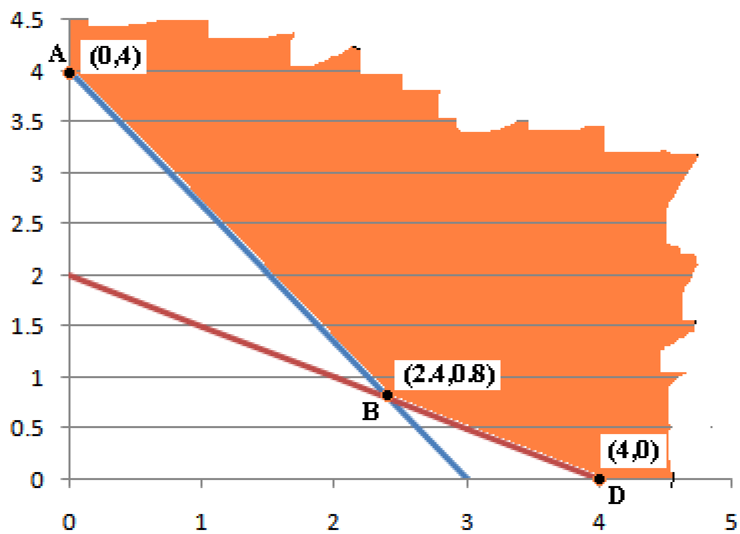
$$Multiply\ by\ (-10): \Rightarrow \left(\frac{40}{3}\right) \ge c_1 \ge (5)$$

$$\Rightarrow 5 \le c_1 \le \frac{40}{3} \qquad increase: \frac{22}{3} \ , \qquad decrease: 1$$

$$Old\ optimal\ value:\ P = 22.4 \ \ at\ point\ \ B(2.4,0.8)$$

$$For\ \ c_1 = 5\ ,\ new\ optimal\ value;\ \ P = (5)(2.4) + (10)(0.8) = 20\ \ at\ \ point\ \ B(2.4,0.8)$$

$$For\ \ c_1 = \frac{40}{3}\ ,\ new\ optimal\ value;\ P = \left(\frac{40}{3}\right)(2.4) + (10)(0.8) = 40\ \ at\ \ point\ \ D(2.4,0.8)$$



$$* \quad Fix\ c_1 = 6: \quad \Rightarrow -\frac{4}{3} \leq -\frac{c_1}{c_2} \leq -0.5 \quad \Rightarrow -\frac{4}{3} \leq -\frac{6}{c_2} \leq -0.5$$

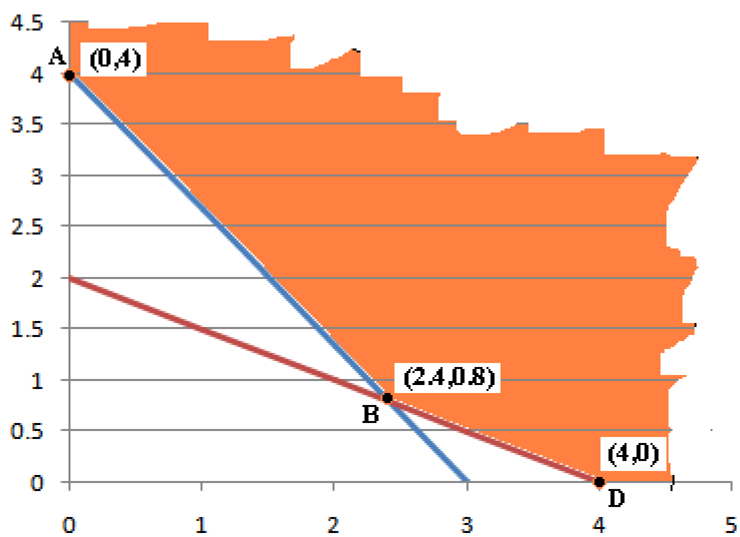
$$Take: \quad \left(-\frac{4}{3}\right) \leq -\frac{6}{c_2} \quad \Rightarrow \ Multiply\ by\ (-c_2): \quad \Rightarrow 4\,c_2 \geq 18 \quad \Rightarrow \quad c_2 \geq \frac{9}{2}$$

$$Take: \quad -\frac{6}{c_2} \leq (-0.5) \Rightarrow \ Multiply\ by\ (-c_2): \quad \Rightarrow 6 \geq 0.5\,c_2 \quad \Rightarrow \quad c_2 \leq 12$$

$$\Rightarrow \frac{9}{2} \leq c_2 \leq 12 \qquad increase: 2 \ , \qquad decrease: \frac{11}{2}$$

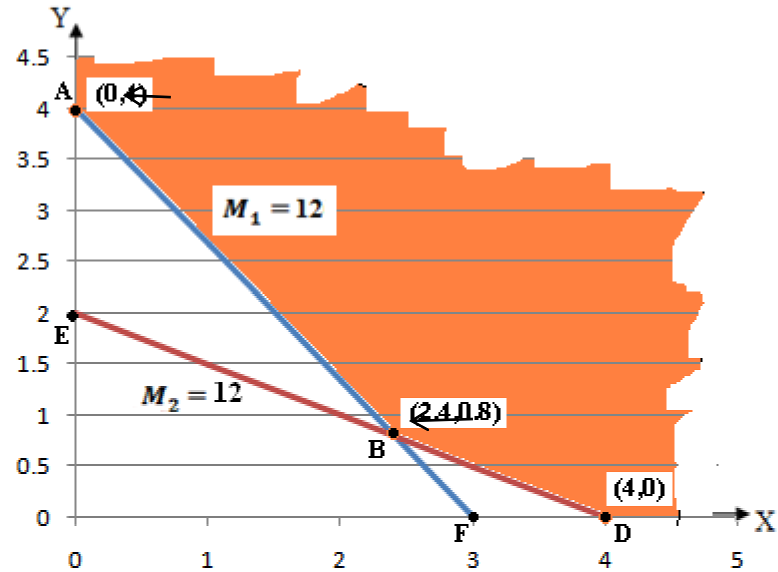
$$For \ \ c_2 = \frac{9}{2}, \ new\ optimal\ value; \ \ P = (6)(2.4) + \left(\frac{9}{2}\right)(0.8) = 18 \ \ at\ \ point\ \ B(2.4, 0.8)$$

$$For \ \ c_2 = 12, new\ optimal\ value; \ \ P = (6)(2.4) + (12)(0.8) = 24 \ \ at\ \ point\ \ B(2.4, 0.8)$$



<u>For the resources:</u> $M_1 = 12$, $M_2 = 12$

$$Fix \quad M_2 = 12$$



$$The\ line\ M_1\ will\ slide\ between\ points\ (0,2)\ and\ (4,0)$$

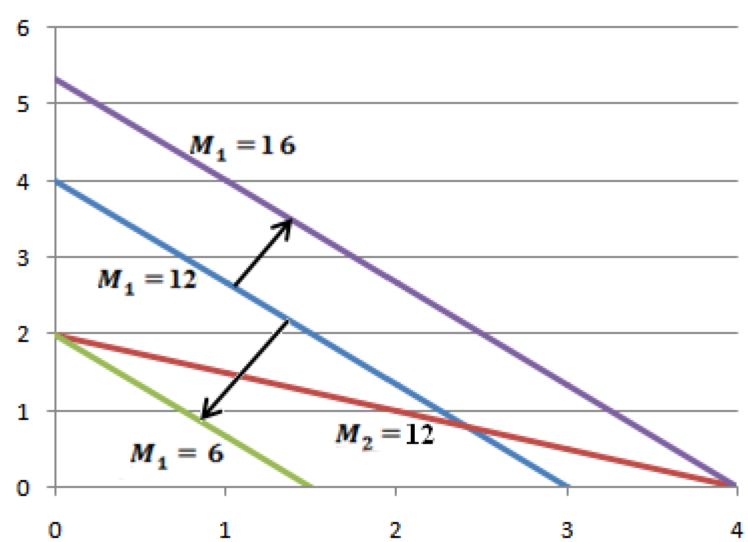
$$M_1\ at\ (0,2):\quad (4)(0) + (3)(2) = 6$$

$$M_1\ at\ (4,0):\quad (4)(4) + (3)(0) = 16$$

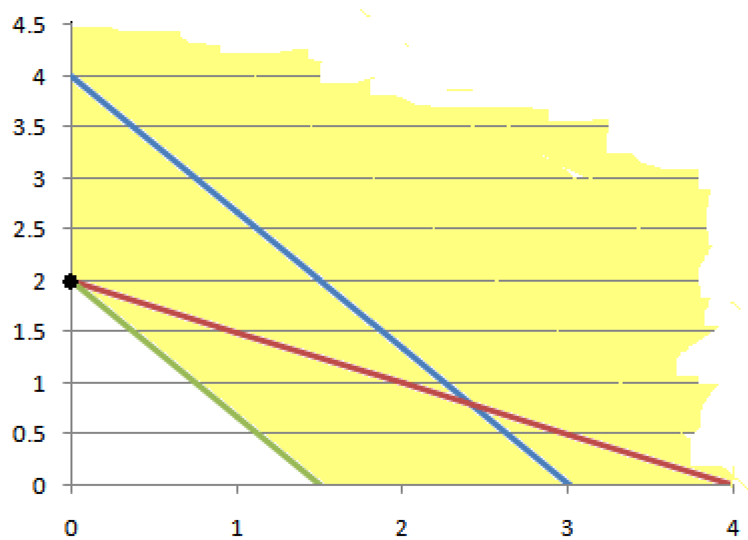
$$6 \le M_1 \le 16$$

$\therefore$ ***M1 decreases by*** $(12 - 6) = 6$

$\therefore$ ***M1 increases by*** $(16 - 12) = 4$

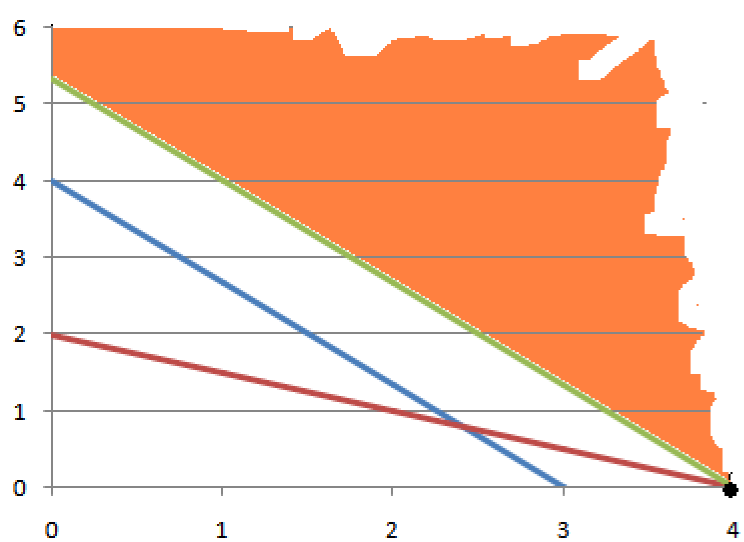


For $\quad$ ***M1 = 6***



$$New\ optimal\ solution\ at\ point\ \ (0,2):\ \ P = (6)(0) + (10)(2) = 20$$

For $\quad$ ***M1 = 16***



$$New\ optimal\ solution\ at\ point\ \ (4,0):\ \ P = (6)(4) + (10)(0) = 24$$

$$\boldsymbol{unit\ worth\ of\ M_1} = \frac{change\ in\ P\ from\ (0,2)\ to\ (4,0)}{change\ in\ M_1\ from\ (0,2)\ to\ (4,0)} = \frac{24-20}{16-6} = 0.40$$

$$\Rightarrow An\ increase\ (decrease) in\ M_1\ by\ one\ unit\ in\ the\ range\ \ 6 \le M_1 \le 16$$

$$increases\ (decreases\ )\ the\ cost\ \ by\ \$0.40.$$

$$Fix \quad M_1 = 12$$

$$The\ line\ M_2\ will\ slide\ between\ points\ (0,4)\ and\ (3,0)$$

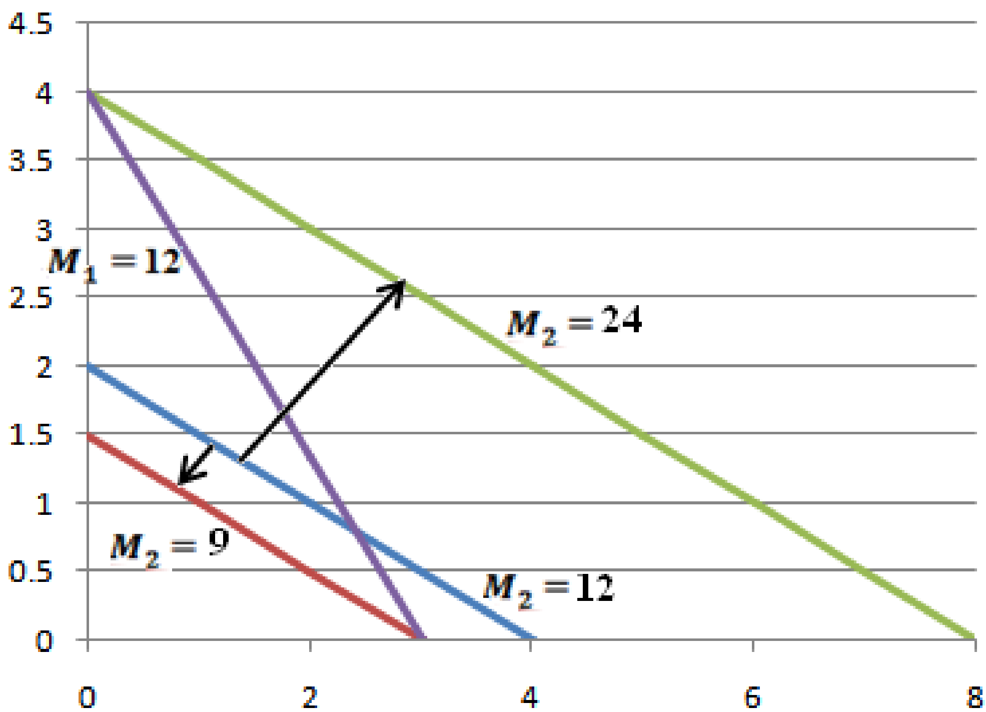
$$M_2\ at\ (0,4):\quad (3)(0) + (6)(4) = 24$$

$$M_2\ at\ (3,0):\quad (3)(3) + (6)(0) = 9$$

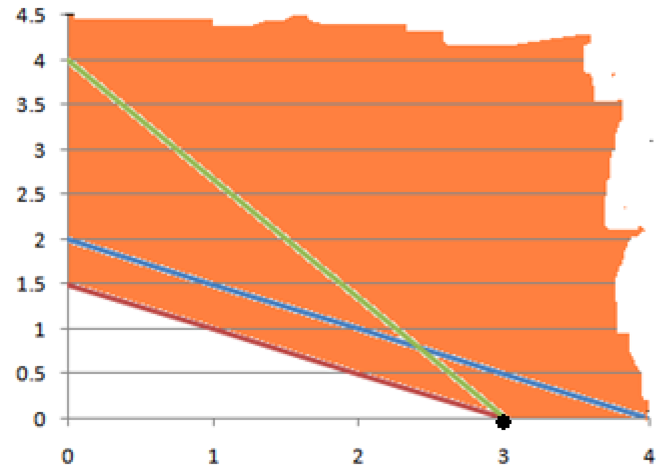
$$9 \le M_2 \le 24$$

$$\therefore M2\ decreases\ by\ (12-9) = 3$$

$$\therefore M2\ increases\ by\ (24-12) = 12$$

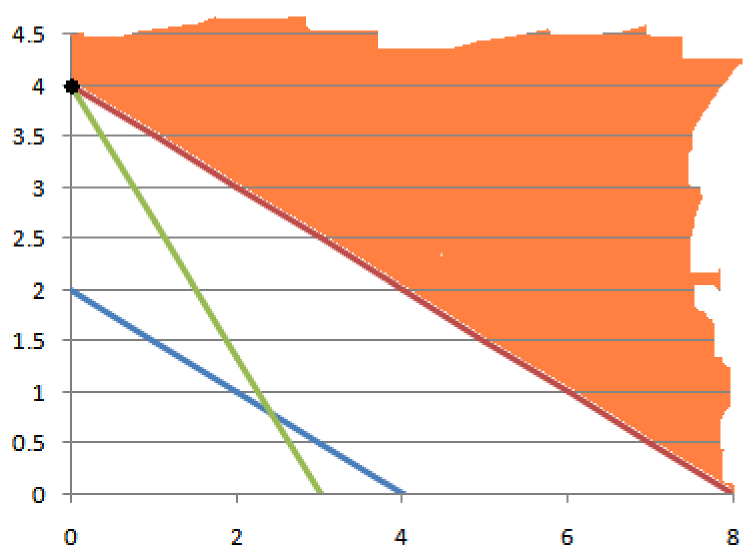


For $\quad M2 = 9$



$$New\ optimal\ solution\ at\ point\ \ (3,0):\ \ P = (6)(3) + (10)(0) = 18$$

For $\quad M2 = 24$



$$New\ optimal\ solution\ at\ point\ \ (0,4):\ \ P = (6)(0) + (10)(4) = 40$$

$$\boldsymbol{unit\ worth\ of\ M_2} = \frac{change\ in\ P\ from\ (0,4)\ to\ (3,0)}{change\ in\ M_2\ from\ (0,4)\ to\ (3,0)} = \frac{40-18}{24-9} = 1.467$$

$$\Rightarrow An\ increase\ (decrease) in\ M_2\ by\ one\ unit\ in\ the\ range\ \ 9 \le M_2 \le 24$$

$$increases\ (decreases\ )\ the\ cost\ \ by\ \$1.47.$$