Graphical Method

Mathematical Programming Terminology

- **Decision variables** are quantities you can control, which completely describe the set of decisions to be made.
- **Constraints** are limitations on the values of the decision variables.
- The **objective function** is a measure that can be used to rank alternative solutions (e.g., cost, production rate, travel time).
  - The goal is to either maximize or minimize its value.
- A **solution** is any combination of values for all decision variables.
- A **feasible solution** is a solution that satisfies all of the constraints.
  - An **infeasible solution** doesn't satisfy some constraint(s).
- An **optimal solution** is the best feasible solution.

Linear Programming

- General symbolic form

Maximize:

$$z = c_1x_1 + c_2x_2 + \ldots c_nx_n$$

Objective

Subject to:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \{\le, \ge, =\} \quad b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \{\le, \ge, =\} \quad b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \{\le, \ge, =\} \quad b_m \end{array}\right\}$$

Constraints

$$0 \le x_j, \quad j = 1, \ldots, n$$

Bounds

… where a_{ij}, b_j, c_j are the model **parameters**

and $x_1, x_2 \ldots x_n$ are the **decision variables**.

Linear Programming (2)

- **General restrictions**
  - All decision variables must be nonnegative
    - not strictly required anymore
  - Constant terms cannot appear on the LHS of a constraint
  - No variable can appear on the RHS of a constraint
  - No variable can appear more than once in the objective function or in any constraint

- **Steps for formulating LP models**
  - Construct a verbal model
  - Define the decision variables
  - Construct the symbolic model

Linear Programming (3)

- **Linear programming has several key assumptions:**
  - The **proportionality assumption** is the first point in the previous slide.
  - The **additivity assumption** is the second point in the previous slide.
  - The **divisibility assumption** says that each decision variable is permitted to assume fractional values.
  - The **certainty assumption** says that all parameters (i.e., coefficients) are known with certainty.

Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing *only two decision variables* (can be used with three variables but only with great difficulty).

- Graphical methods provide *visualization of how* a solution for a linear programming problem is obtained.

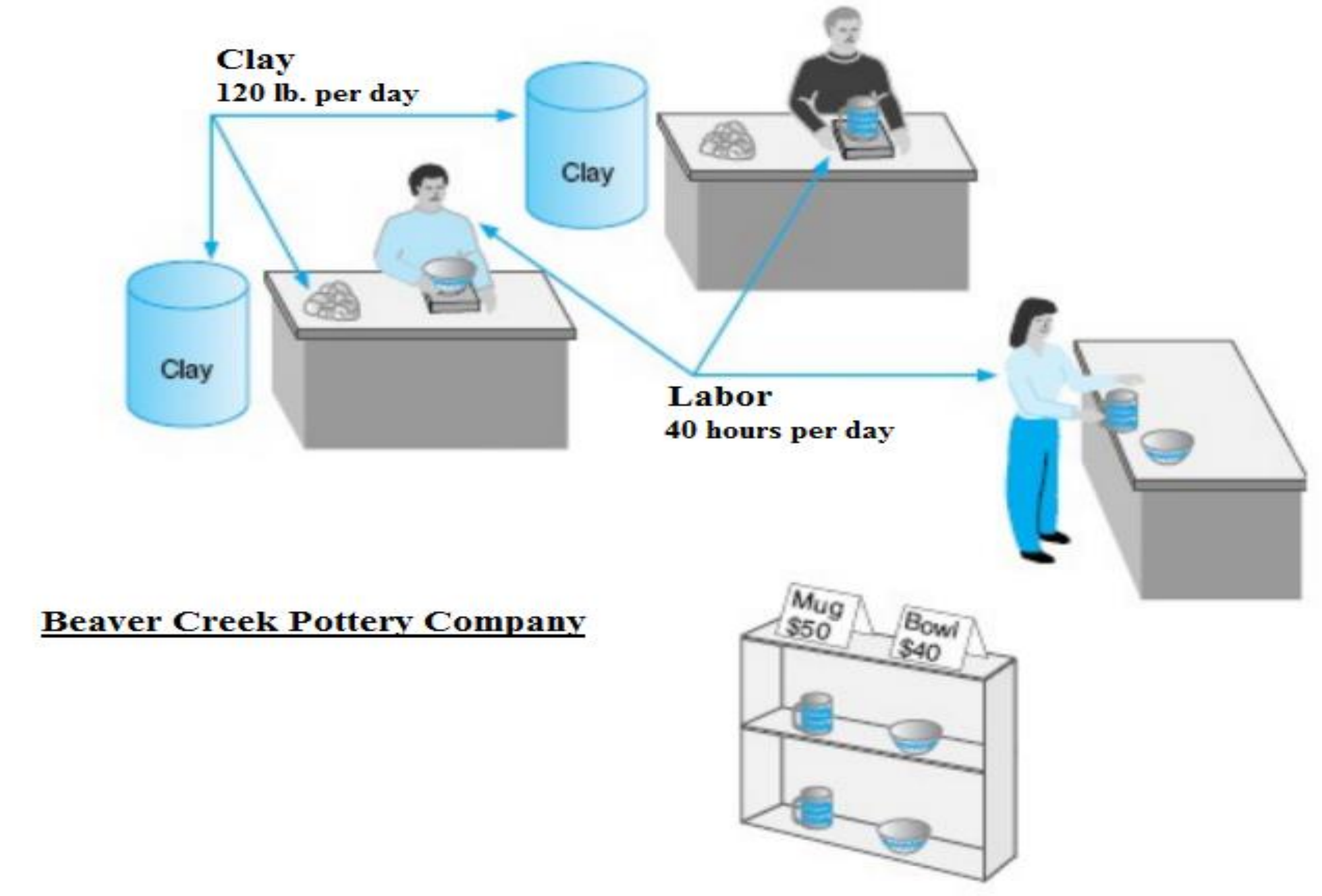
- Graphical methods can be classified under two categories:
  - 1. Iso-Profit(Cost) Line Method
  - 2. Extreme-point evaluation Method.

LP Model Formulation
A Maximization Example

- **Product mix problem - Beaver Creek Pottery Company**
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Product		
	Bowl	Mug	Available Resources
Labor (Hrs./Unit)	1	2	40
Clay (Lb./Unit)	4	3	120
Profit ($/Unit)	$40	$50	

LP Model Formulation
A Maximization Example



LP Model Formulation
A Maximization Example

- **Resource Availability:** 40 hrs of labor per day

  120 lbs of clay

- **Decision Variables:** x_1 = number of bowls to produce per day

  x_2 = number of mugs to produce per day

- **Objective Function:** Maximize $Z = \$40x_1 + \$50x_2$

  Where Z = profit per day

- **Resource Constraints:** $1x_1 + 2x_2 \leq 40$ hours of labor

  $4x_1 + 3x_2 \leq 120$ pounds of clay

- **Non-Negativity Constraints:** $x_1 \geq 0; \quad x_2 \geq 0$

Complete Linear Programming Model:

Maximize $Z = \$40\,x_1 + \$50\,x_2$

subject to:

$$1\,x_1 + 2\,x_2 \le 40$$
$$4\,x_1 + 3\,x_2 \le 120$$
$$x_1, x_2 \ge 0$$

	Product		
	Bowl	Mug	Available Resources
Labor (Hrs./Unit)	1	2	40
Clay (Lb./Unit)	4	3	120
Profit ($/Unit)	$40	$50	

Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$ **bowls**
$x_2 = 10$ **mugs**
$Z = \$40\, x_1 + \$50\, x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 < 40$ hours

Clay constraint check: $4(5) + 3(10) = 70 < 120$ pounds

Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: **x_1 = 10 bowls**

x_2 = 20 mugs

$Z = \$40\ x_1 + \$50\ x_2 = \$1400$

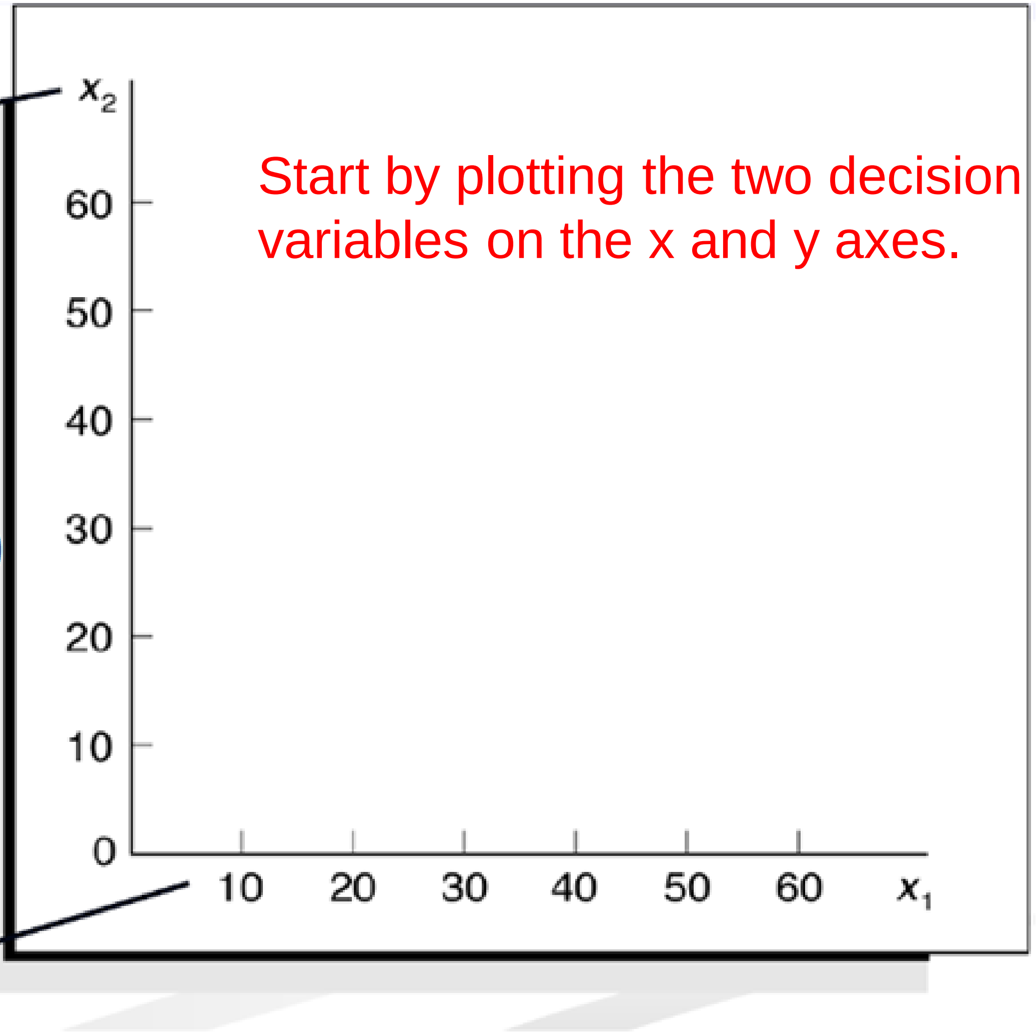
Labor constraint check: **$1(10) + 2(20) = 50 > 40$ hours**

Coordinate Axes
Graphical Solution of Maximization Model

X_2 is mugs

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\begin{aligned} \text{subject to:} \quad & 1x_1 + 2x_2 \le 40 \\ & 4x_1 + 3x_2 \le 120 \\ & x_1, x_2 \ge 0 \end{aligned}$$



X_1 is bowls

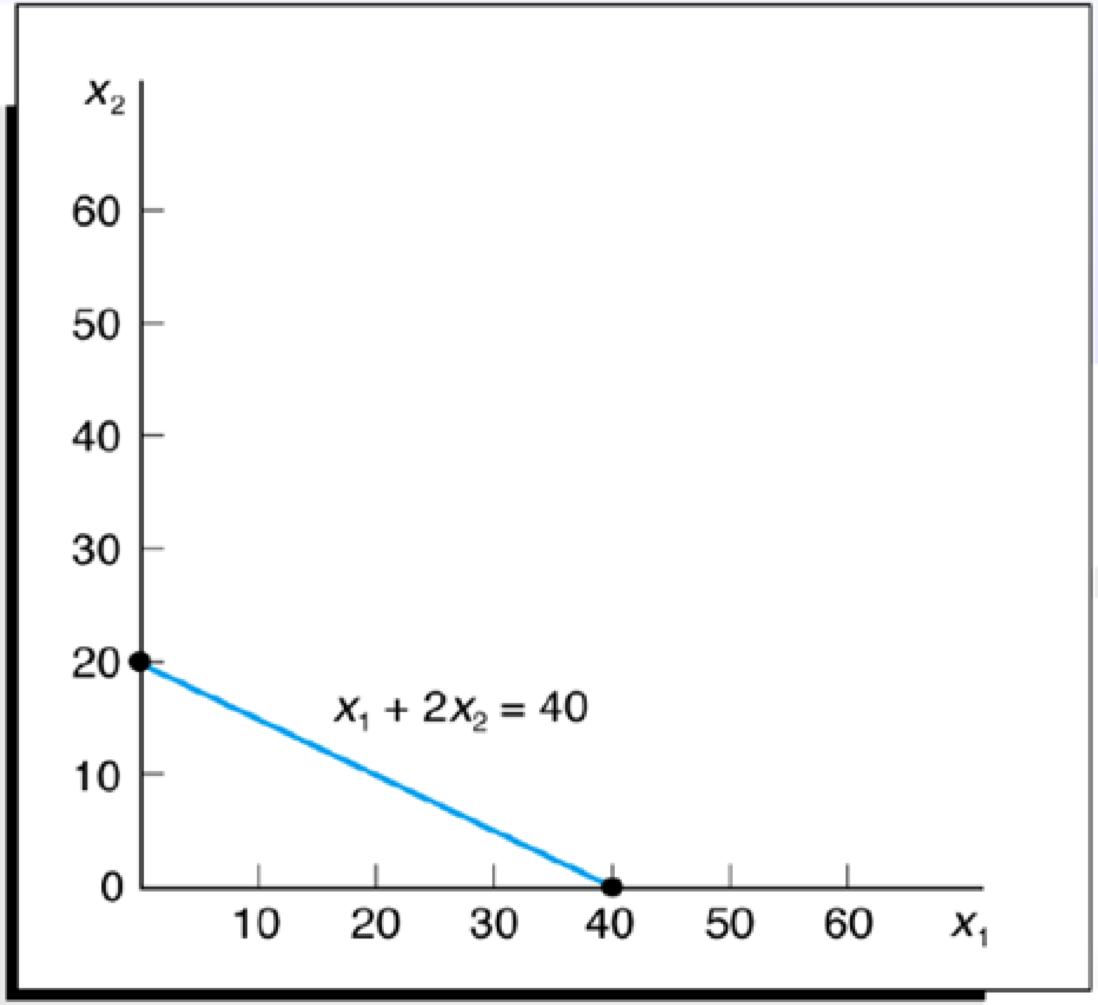
Labor Constraint
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\text{subject to:} \quad 1x_1 + 2x_2 \leq 40$$

$$4x_1 + 3x_2 \leq 120$$

$$x_1, x_2 \geq 0$$



Graph of Labor Constraint

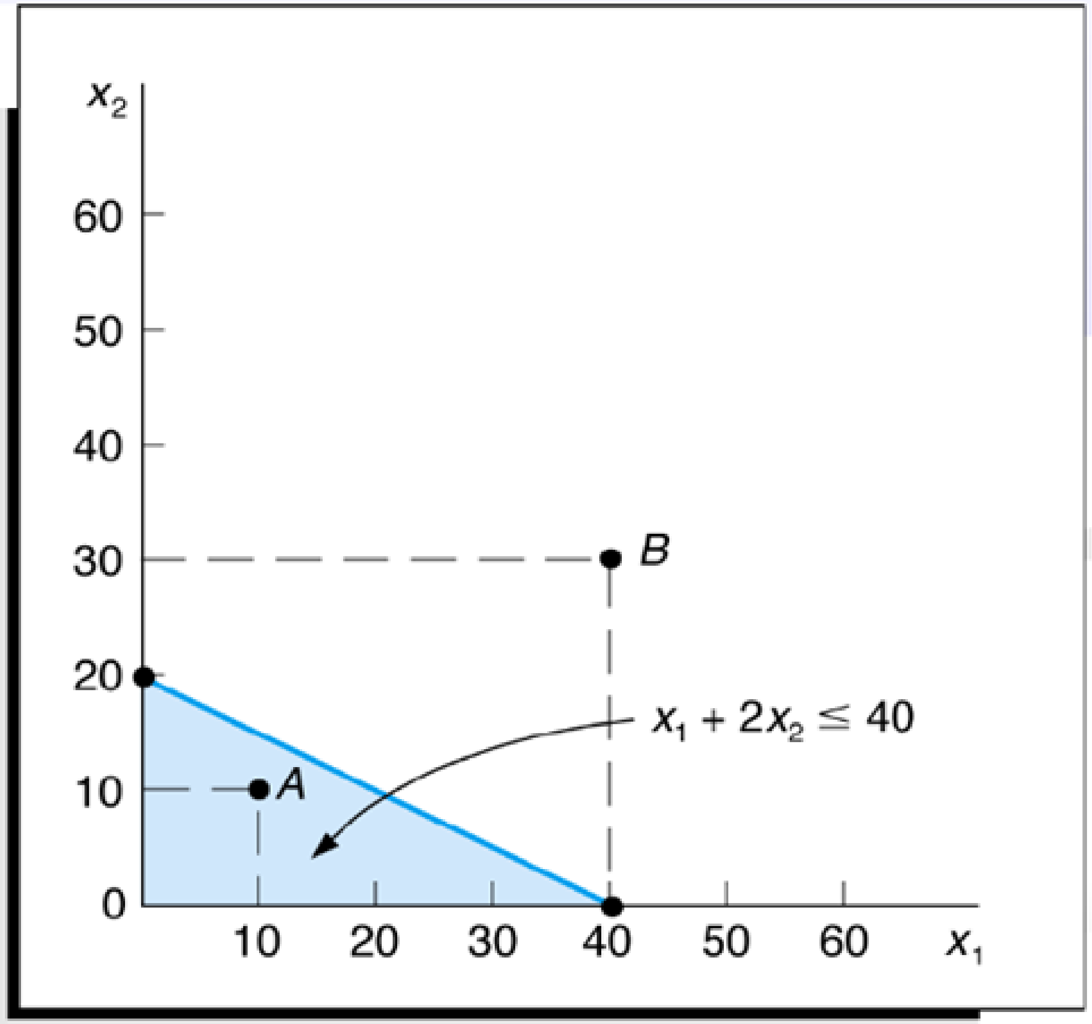
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\text{subject to:} \quad 1x_1 + 2x_2 \le 40$$

$$4x_1 + 3x_2 \le 120$$

$$x_1, x_2 \ge 0$$



Labor Constraint Area

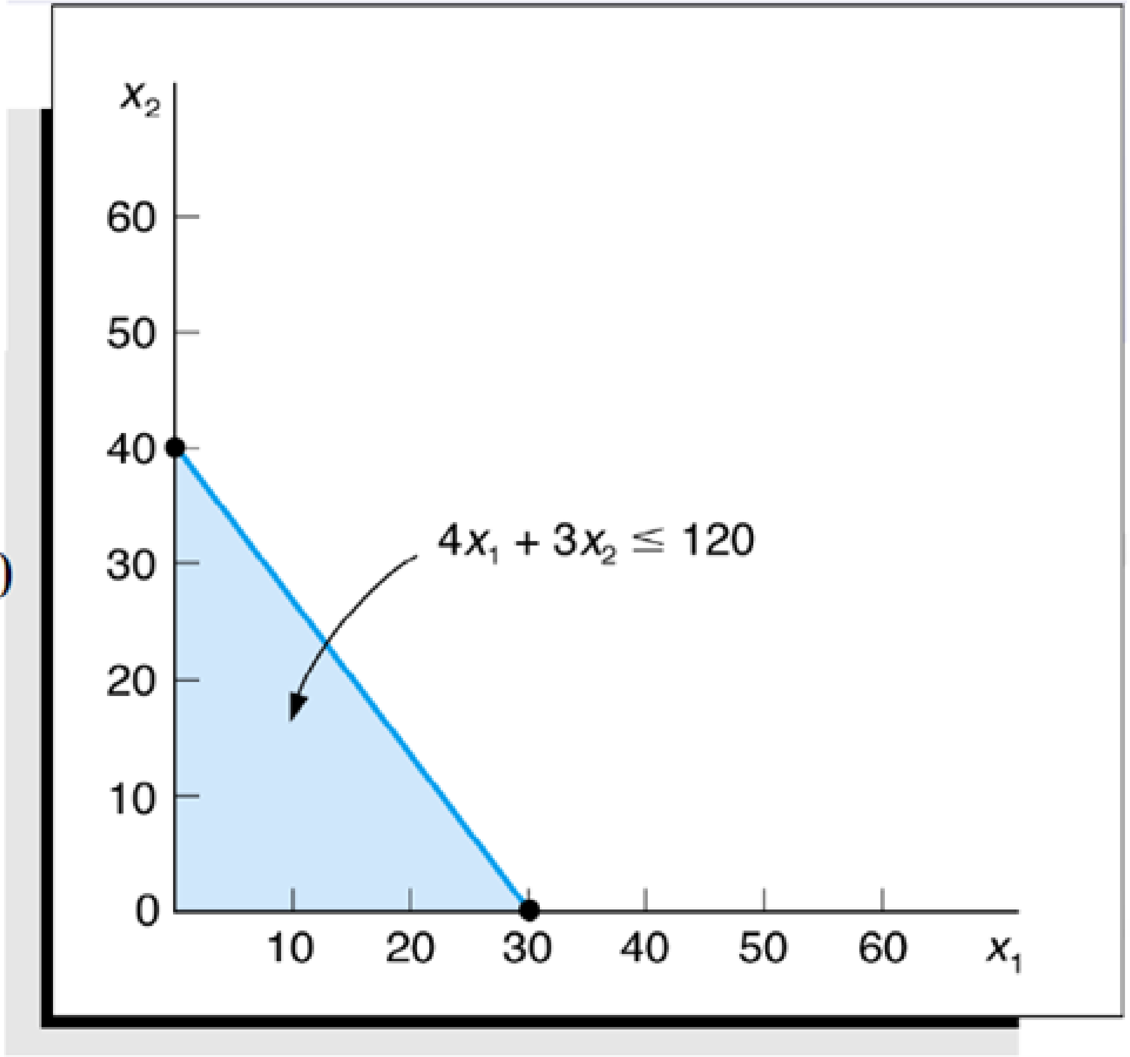
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Clay Constraint Area

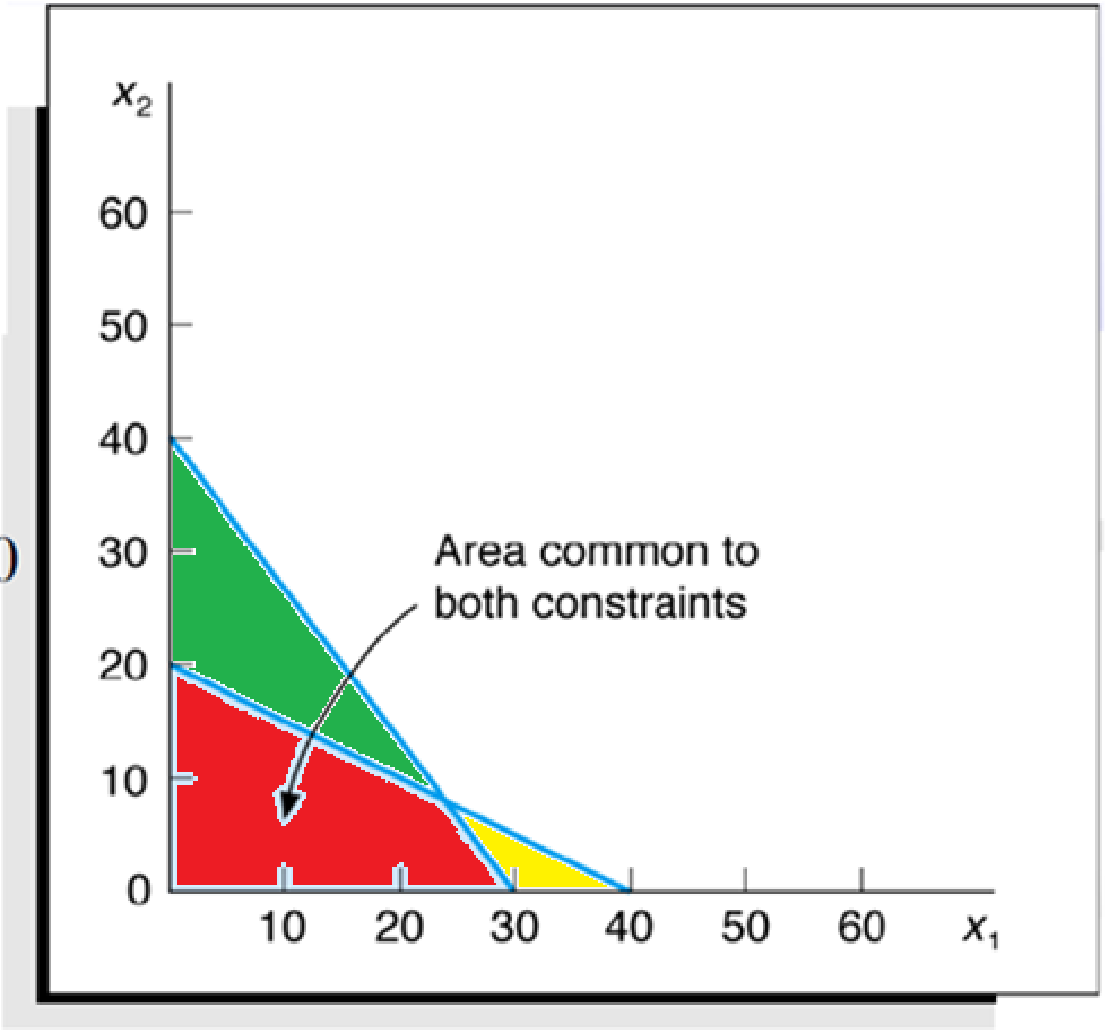
Both Constraints
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\text{subject to:} \quad 1x_1 + 2x_2 \le 40$$

$$4x_1 + 3x_2 \le 120$$

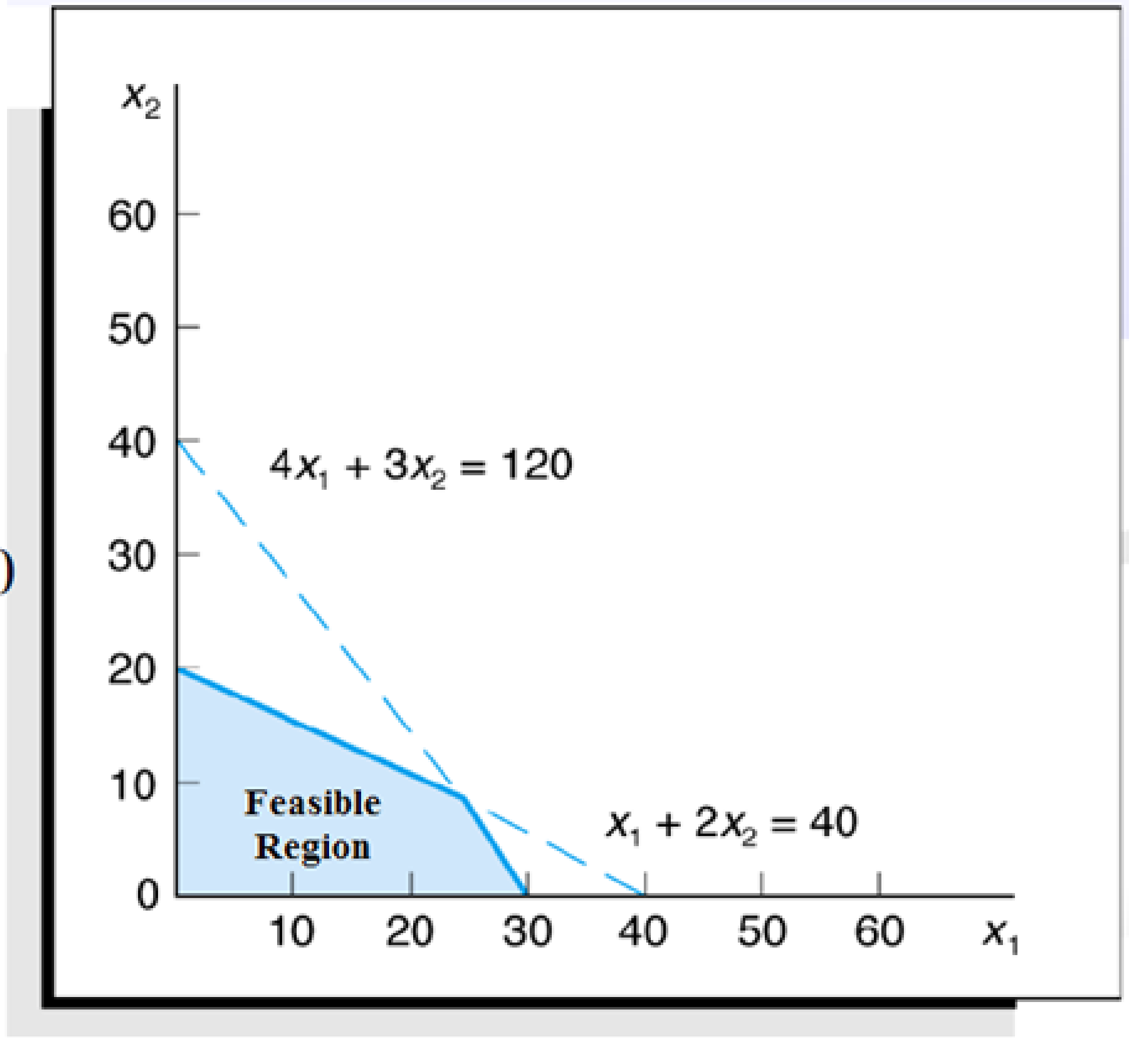
$$x_1, x_2 \ge 0$$



Graph of Both Model Constraints

Feasible Solution Area
Graphical Solution of Maximization Model

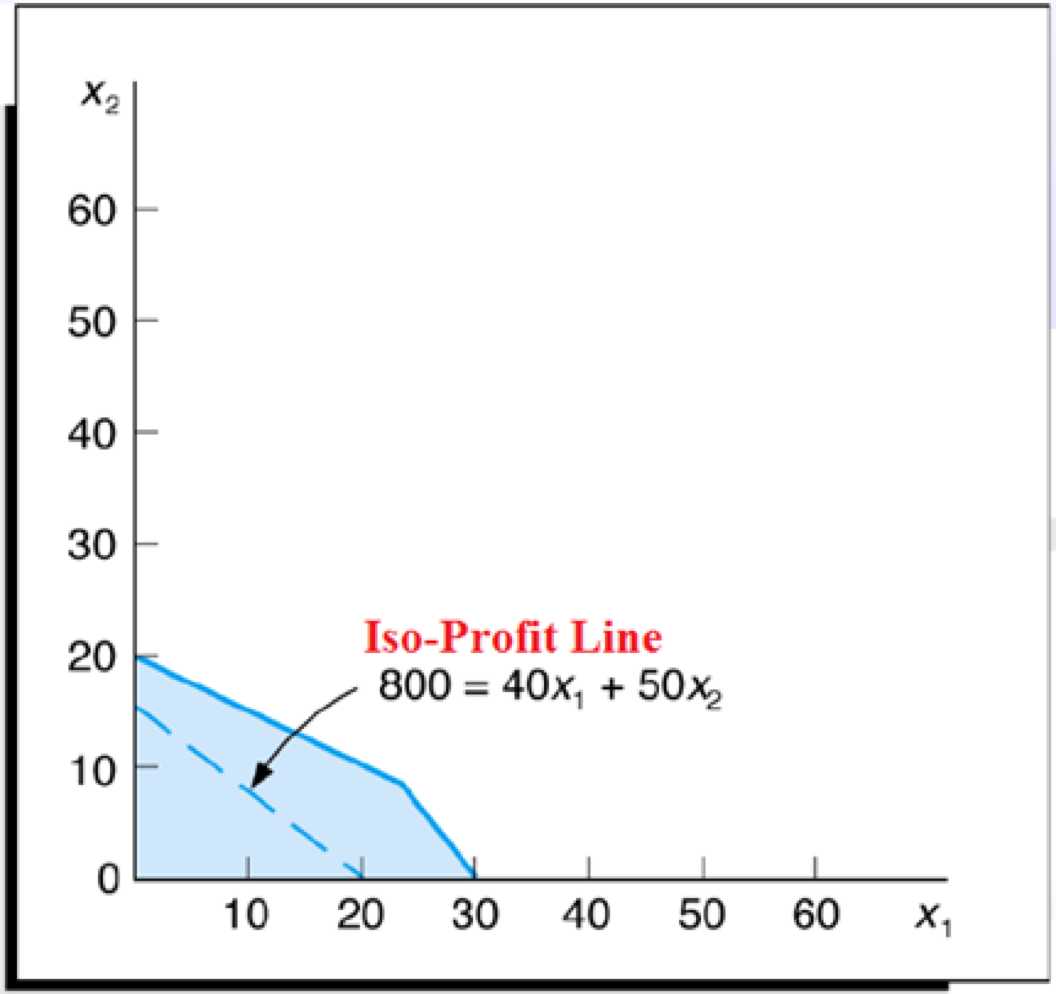
$$\text{Maximize } Z = \$40x_1 + \$50x_2$$
$$\text{subject to:} \quad 1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$



Feasible Solution Area

Objective Function Solution = $800
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$
$$\text{subject to: } \quad 1x_1 + 2x_2 \le 40$$
$$4x_1 + 3x_2 \le 120$$
$$x_1, x_2 \ge 0$$



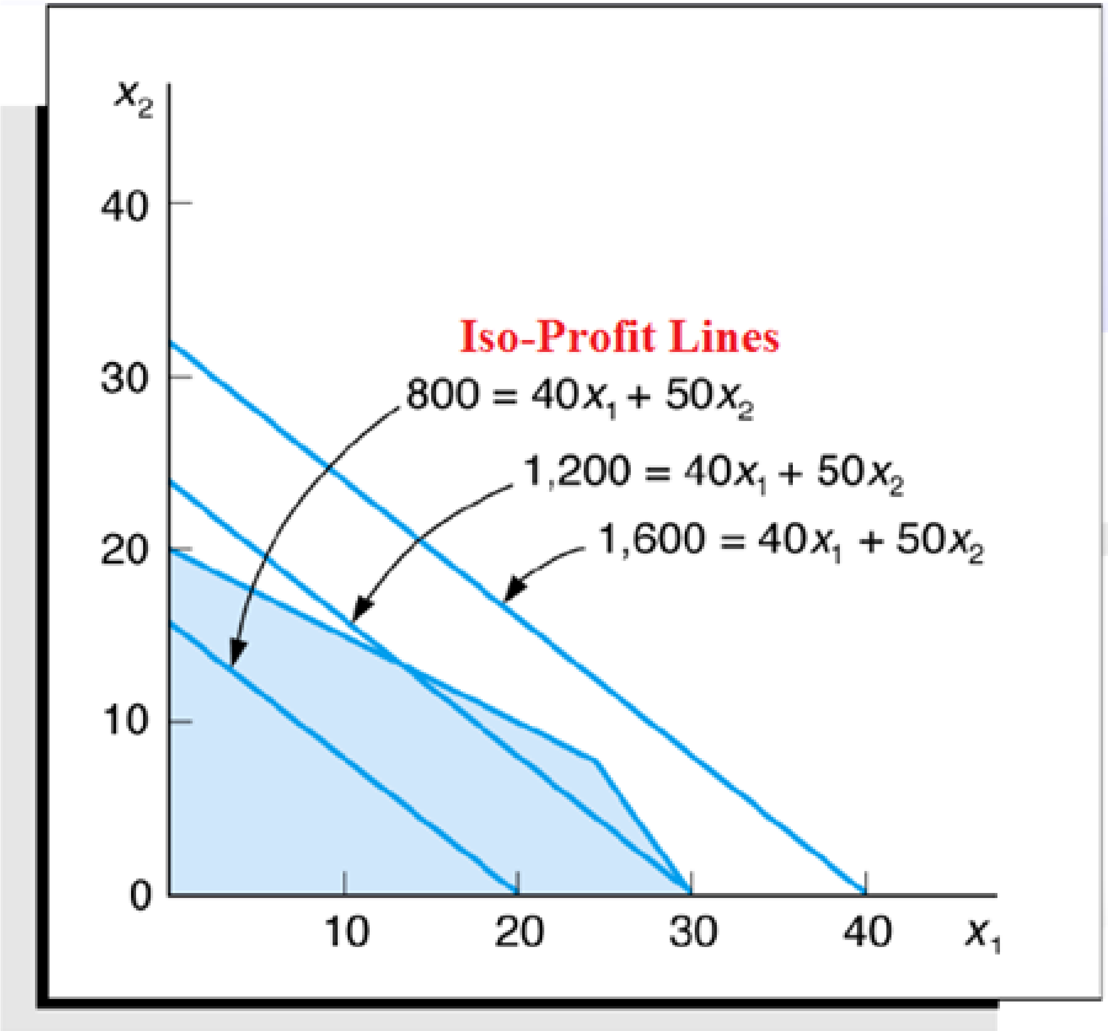
Objective Function Line for Z = $800

Alternative Objective Function Solution Lines
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

subject to:

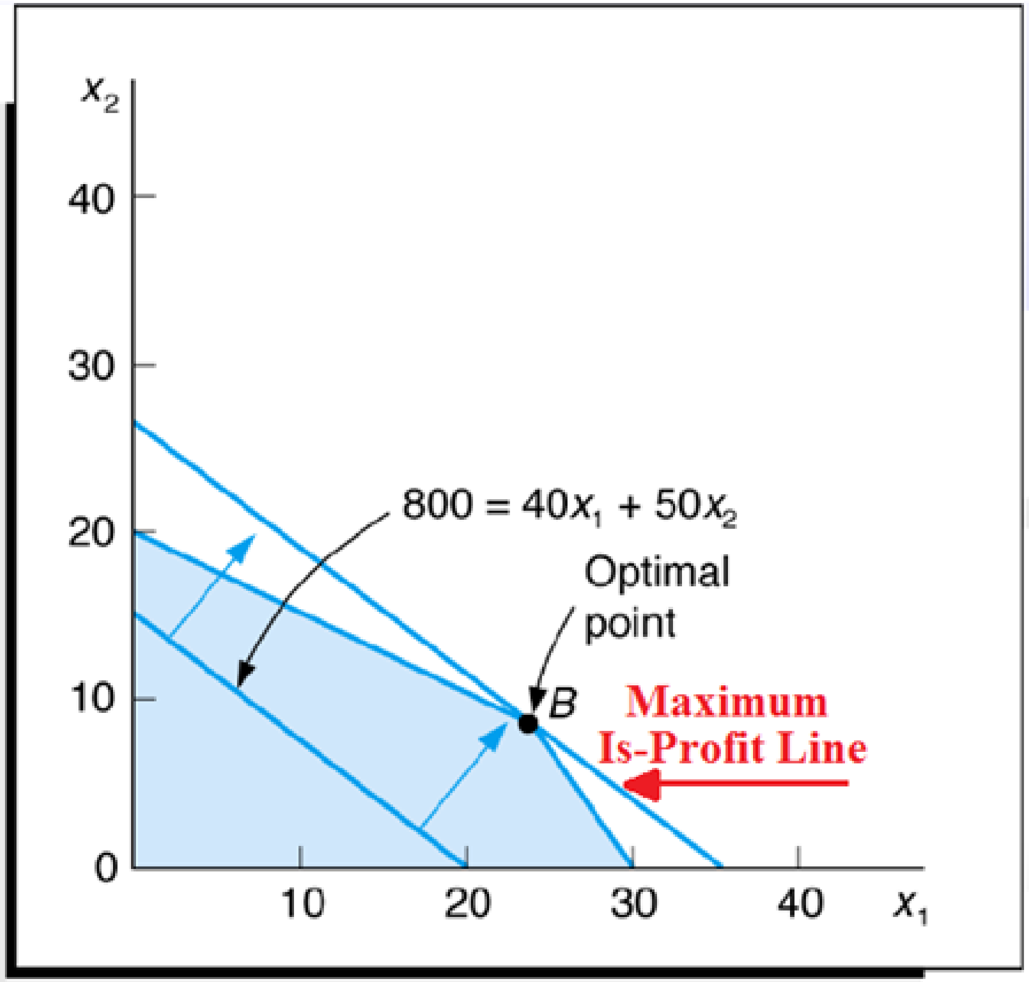
$$1x_1 + 2x_2 \le 40$$
$$4x_1 + 3x_2 \le 120$$
$$x_1, x_2 \ge 0$$



Alternative Objective Function Lines

Optimal Solution
Graphical Solution of Maximization Model

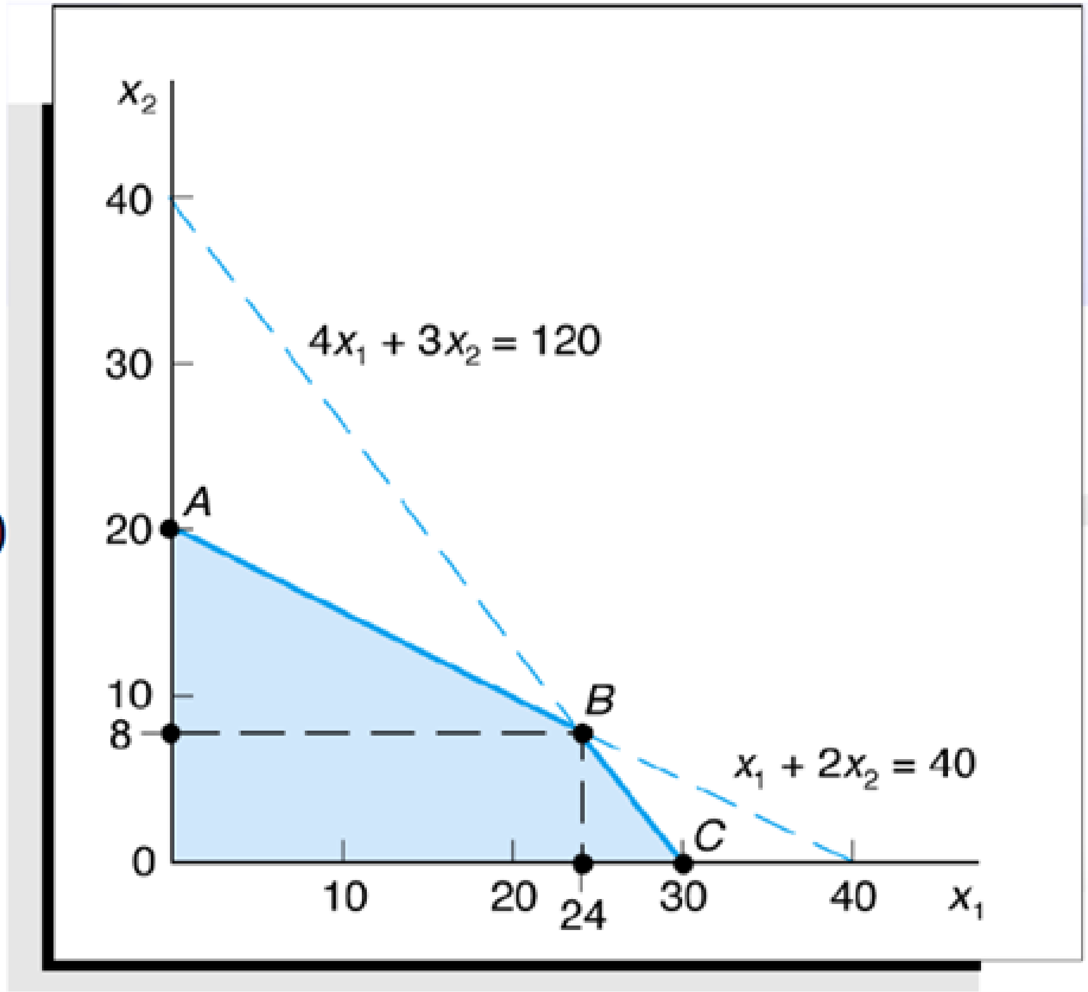
$$\begin{aligned}
\text{Maximize } Z &= \$40x_1 + \$50x_2 \\
\text{subject to:} \quad 1x_1 + 2x_2 &\le 40 \\
4x_1 + 3x_2 &\le 120 \\
x_1, x_2 &\ge 0
\end{aligned}$$



Identification of Optimal Solution Point

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$
$$\text{subject to:} \quad 1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

A (0 , 20)
B (24 , 8)
C (30 , 0)



Optimal Solution Coordinates

Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\text{subject to: } \quad 1x_1 + 2x_2 \le 40$$

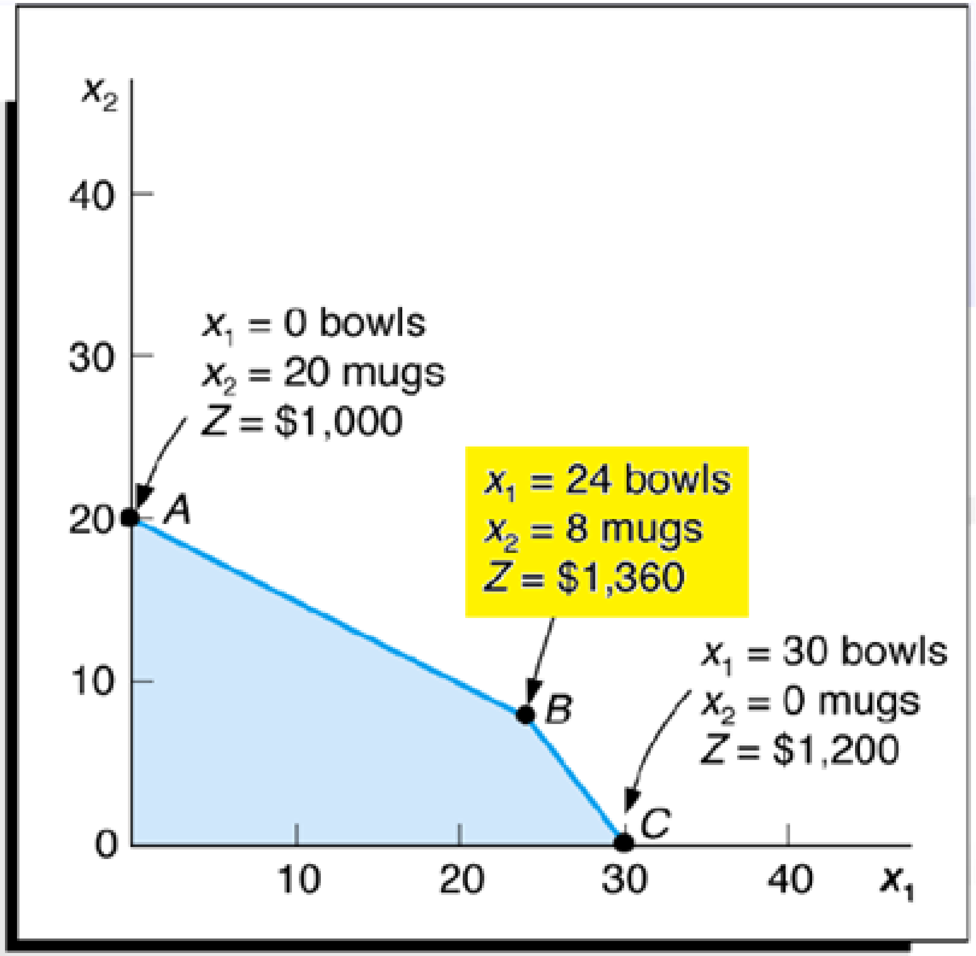
$$4x_1 + 3x_2 \le 120$$

$$x_1, x_2 \ge 0$$

At A(0,20) : Z = (40)(0)+(50)(20) = $1,000

At B(24,8) : Z = (40)(24)+(50)(8) = $1,360 ← **Maximum**

At C(30,0) : Z = (40)(30)+(50)(0) = $1,200



Solutions at All Corner Points

LP Optimal Solutions

- If an optimal solution exists, it occurs at a **corner point** of the feasible region.
- In two dimensions with all inequality constraints plotted, a corner point is a solution at which two (or more) constraints are **binding**.
- If a feasible solution exists, there is always an optimal solution that is a corner point solution.
- An optimal solution isn't necessarily unique.

Another Example

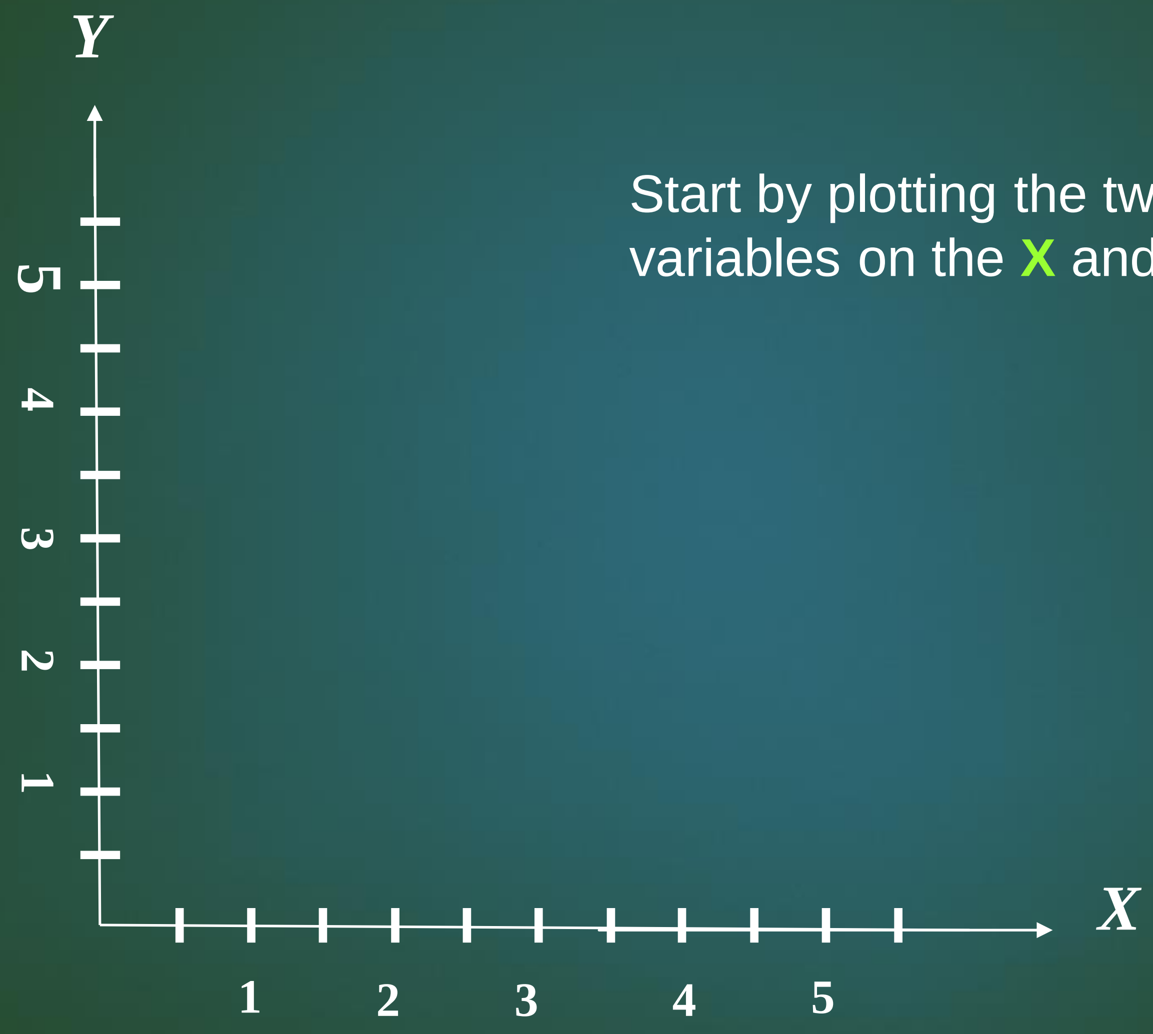
LP Model Formulation
A Minimization Example

	Product		Available Resources
	X	Y	
Machine A/Unit	4	3	12
Machine B/Unit	3	6	12
Machine C/Unit	5	12	10
Cost/Unit	$6	$10	

Minimize: $C = 6X + 10Y$

Subject to:

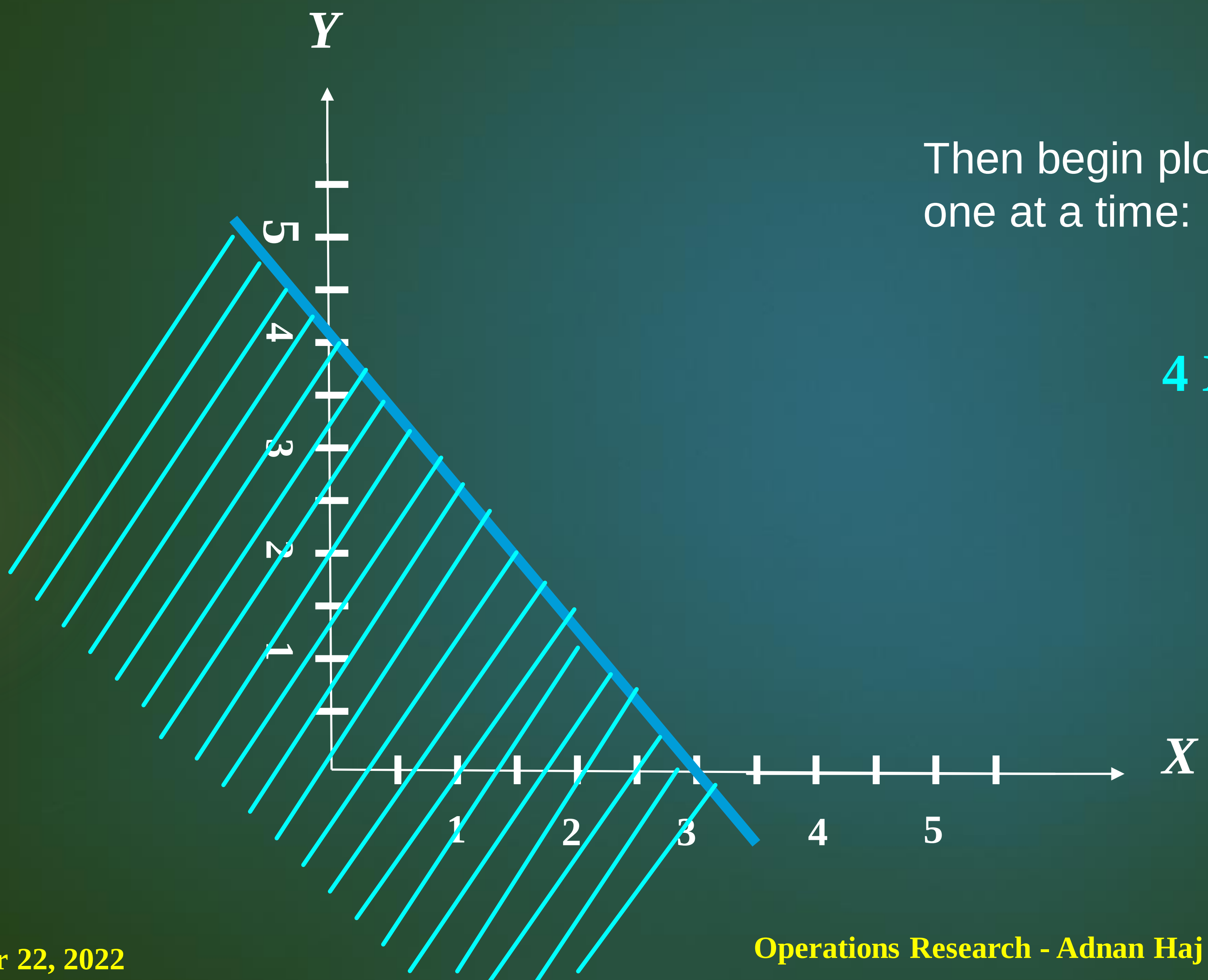
$$4X + 3Y \geq 12$$
$$3X + 6Y \geq 12$$
$$5X + 12Y \geq 10$$
$$X, Y \geq 0$$



Start by plotting the two decision variables on the **X** and **Y** axes.

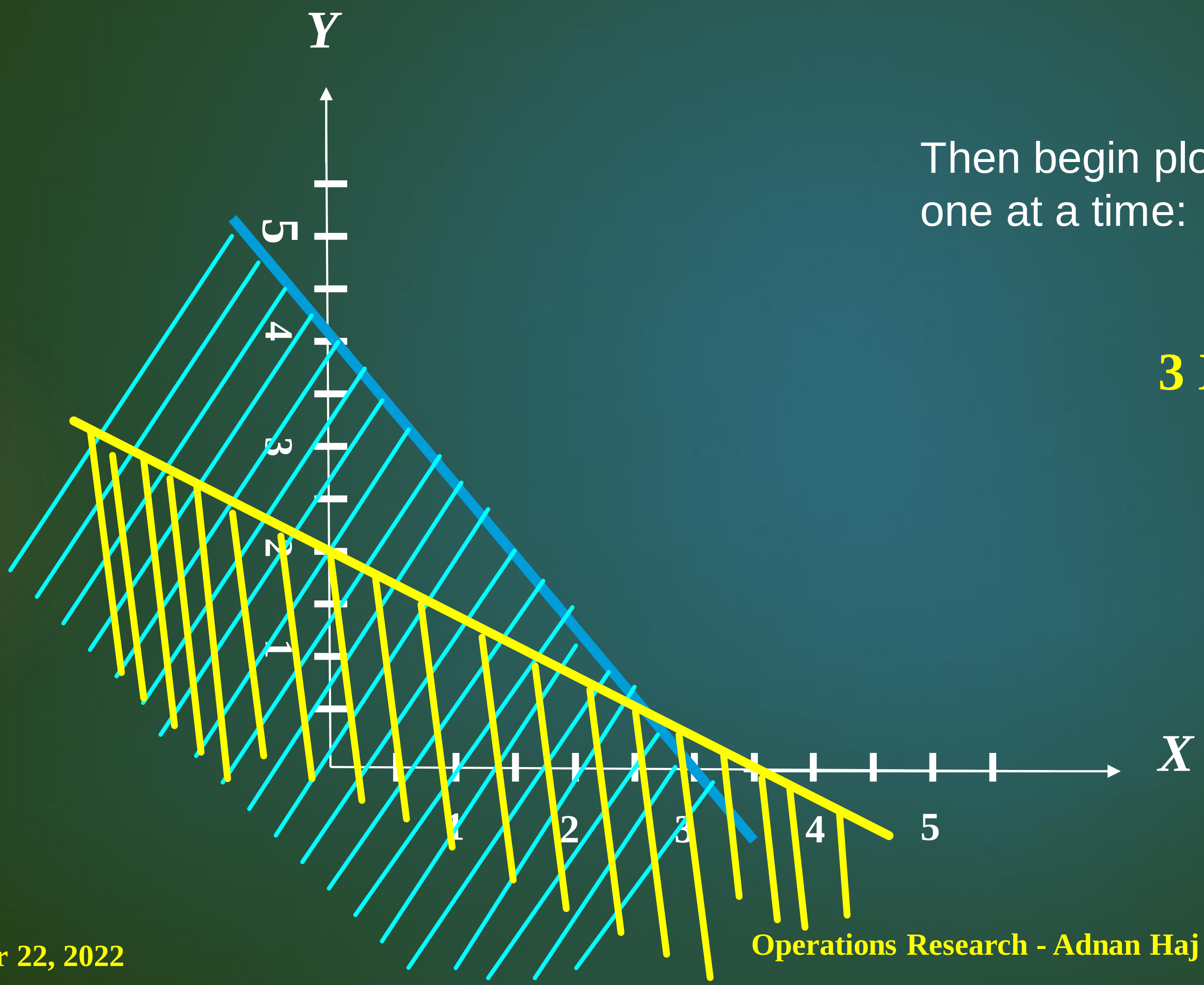
Then begin plotting the constraints, one at a time:

$$4X + 3Y \geq 12$$

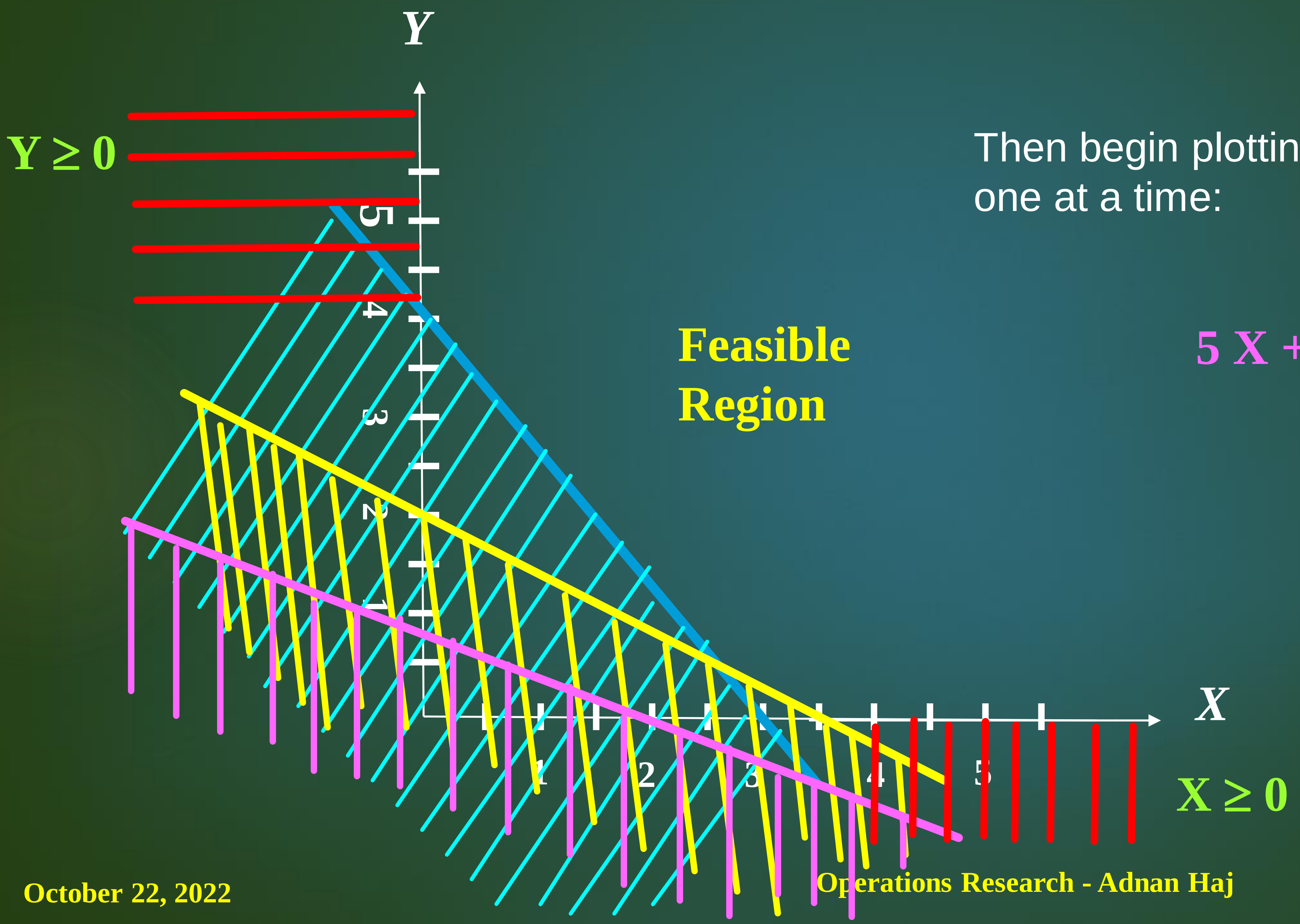


Then begin plotting the constraints, one at a time:

$$3X + 6Y \geq 12$$



Then begin plotting the constraints, one at a time:

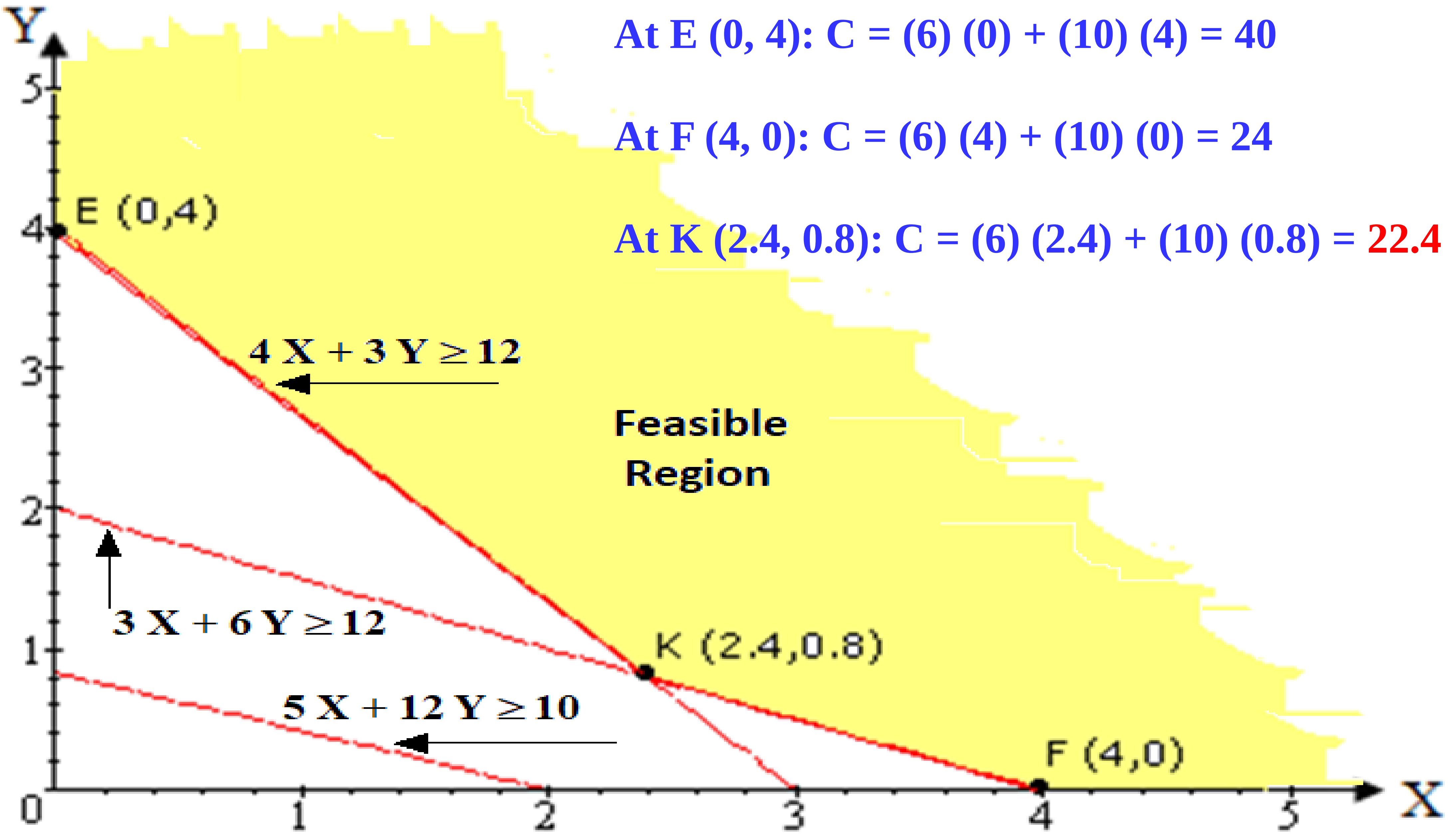


$5X + 12Y \geq 10$

At E (0, 4): C = (6) (0) + (10) (4) = 40

At F (4, 0): C = (6) (4) + (10) (0) = 24

At K (2.4, 0.8): C = (6) (2.4) + (10) (0.8) = 22.4

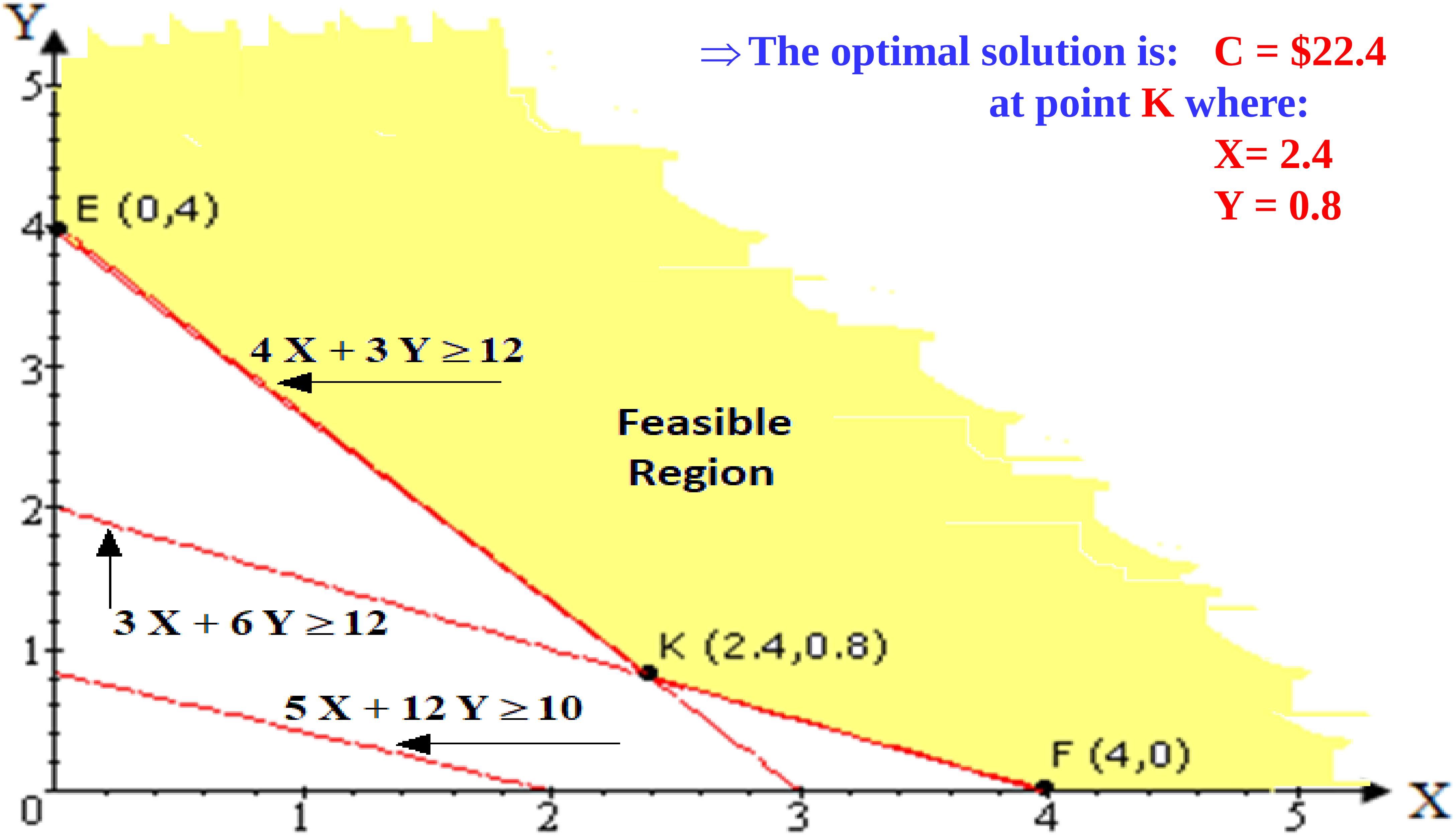


$\Rightarrow$ The optimal solution is: C = \$22.4

at point K where:

X= 2.4

Y = 0.8



One more Example

A workshop has three (3) types of machines A, B and C; it can manufacture two (2) products W and P, and all products have to go to each machine and each one goes in the same order; First to the machine A, then to B and then to C. The following table shows:

The hours needed at each machine, per product unit

The total available hours for each machine, per week

The profit of each product per unit sold

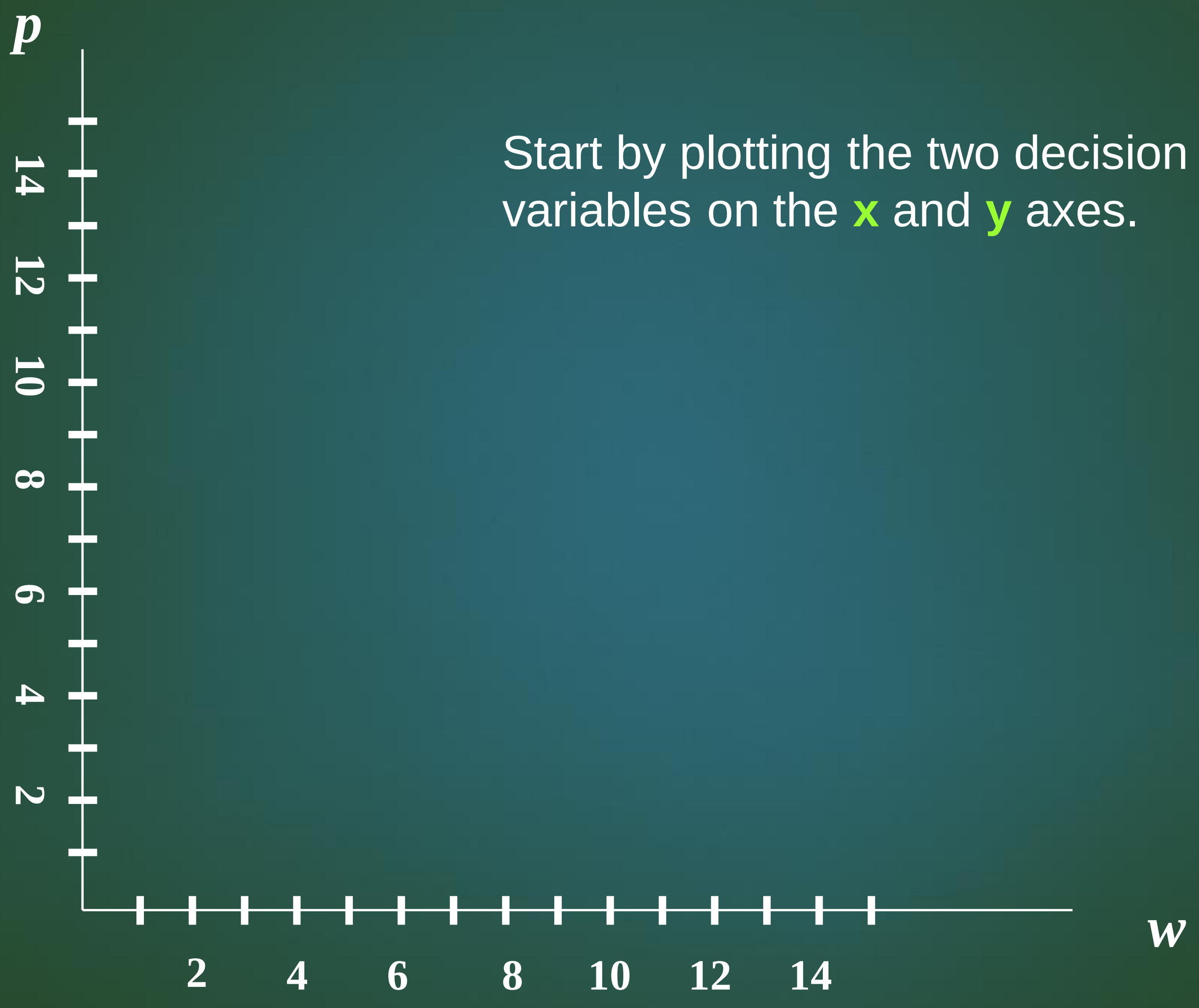
	Product		
	W	P	Available Resources
Machine A	1.5	1	15 hrs.
Machine B	1.0	1.0	12 hrs.
Machine C	0.4	0.5	5 hrs.
Profit ($/Unit)	$400	$300	

- **Objective Function** $\max: \quad profit = 400 \times w + 300 \times p$

- **Subject to**
  - Machine A: $1.5 \times w + 1.0 \times p \le 15$
  - Machine B: $1.0 \times w + 1.0 \times p \le 12$
  - Machine C: $0.4 \times w + 0.5 \times p \le 5$
  - Non-negativity: $p, w \ge 0$

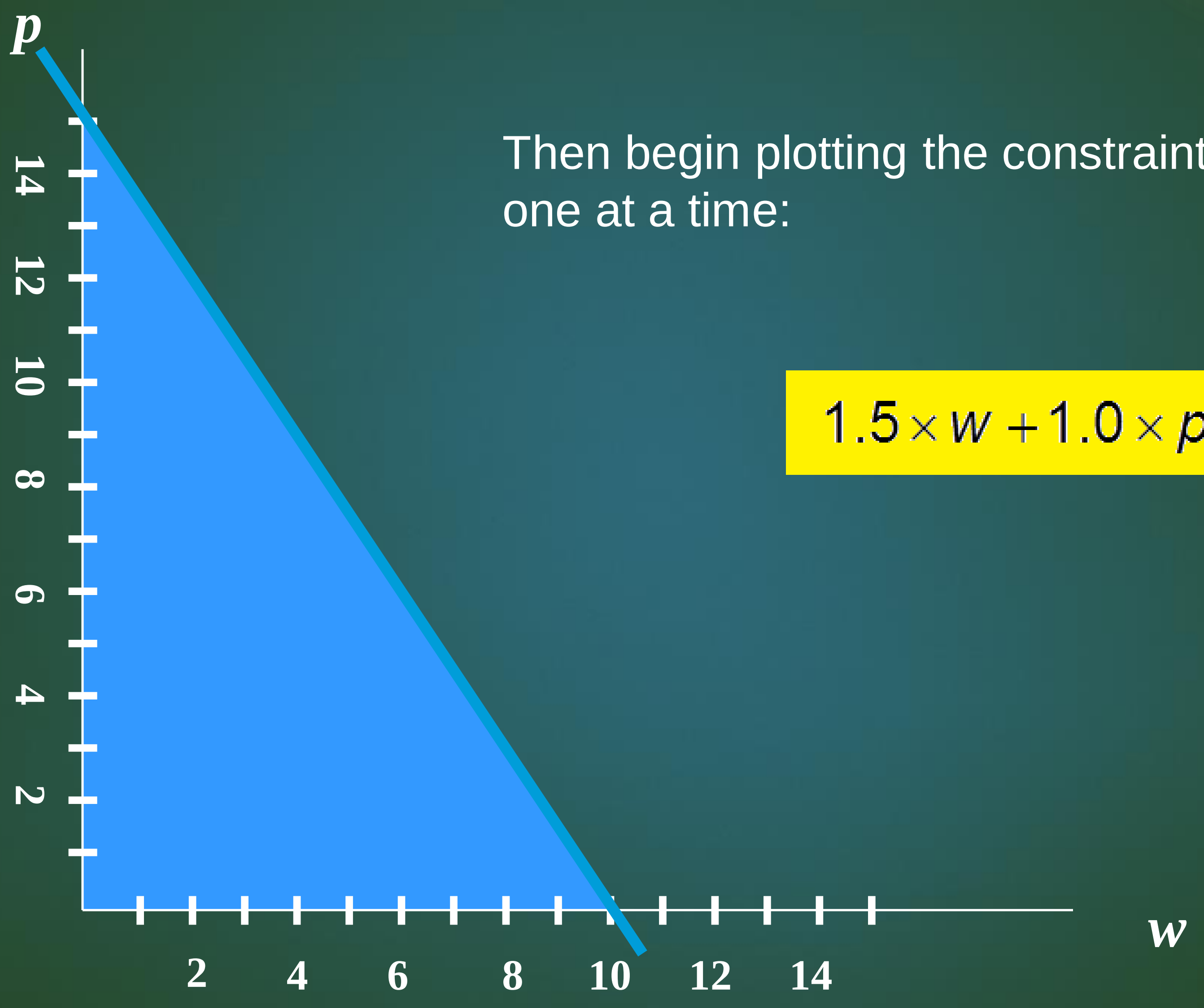
	Product		
	W	P	Available Resources
Machine A	1.5	1	15 hrs.
Machine B	1.0	1.0	12 hrs.
Machine C	0.4	0.5	5 hrs.
Profit ($/Unit)	$400	$300	

Solving an LP Graphically



Start by plotting the two decision variables on the x and y axes.

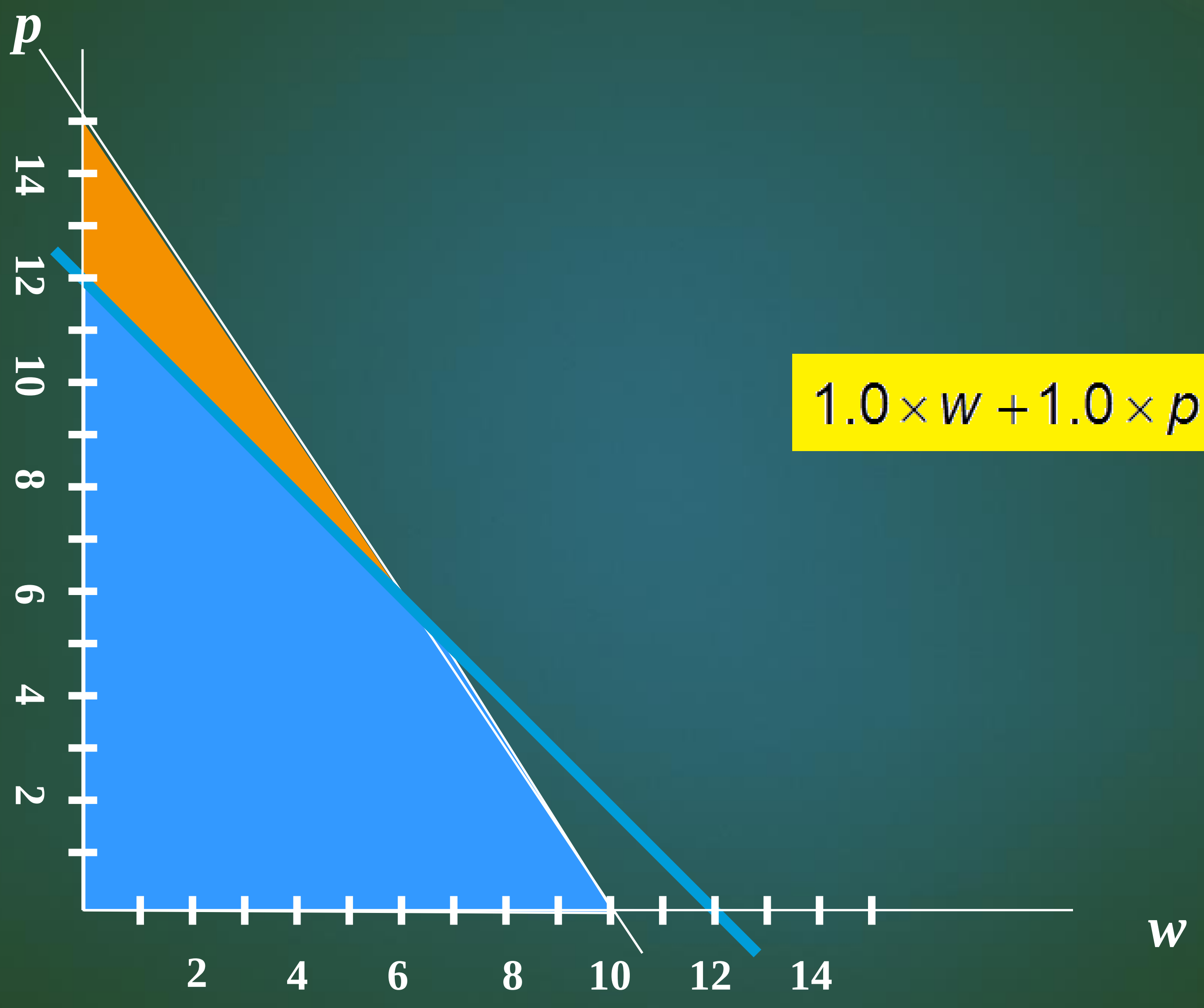
Solving an LP Graphically (2)



Then begin plotting the constraints, one at a time:

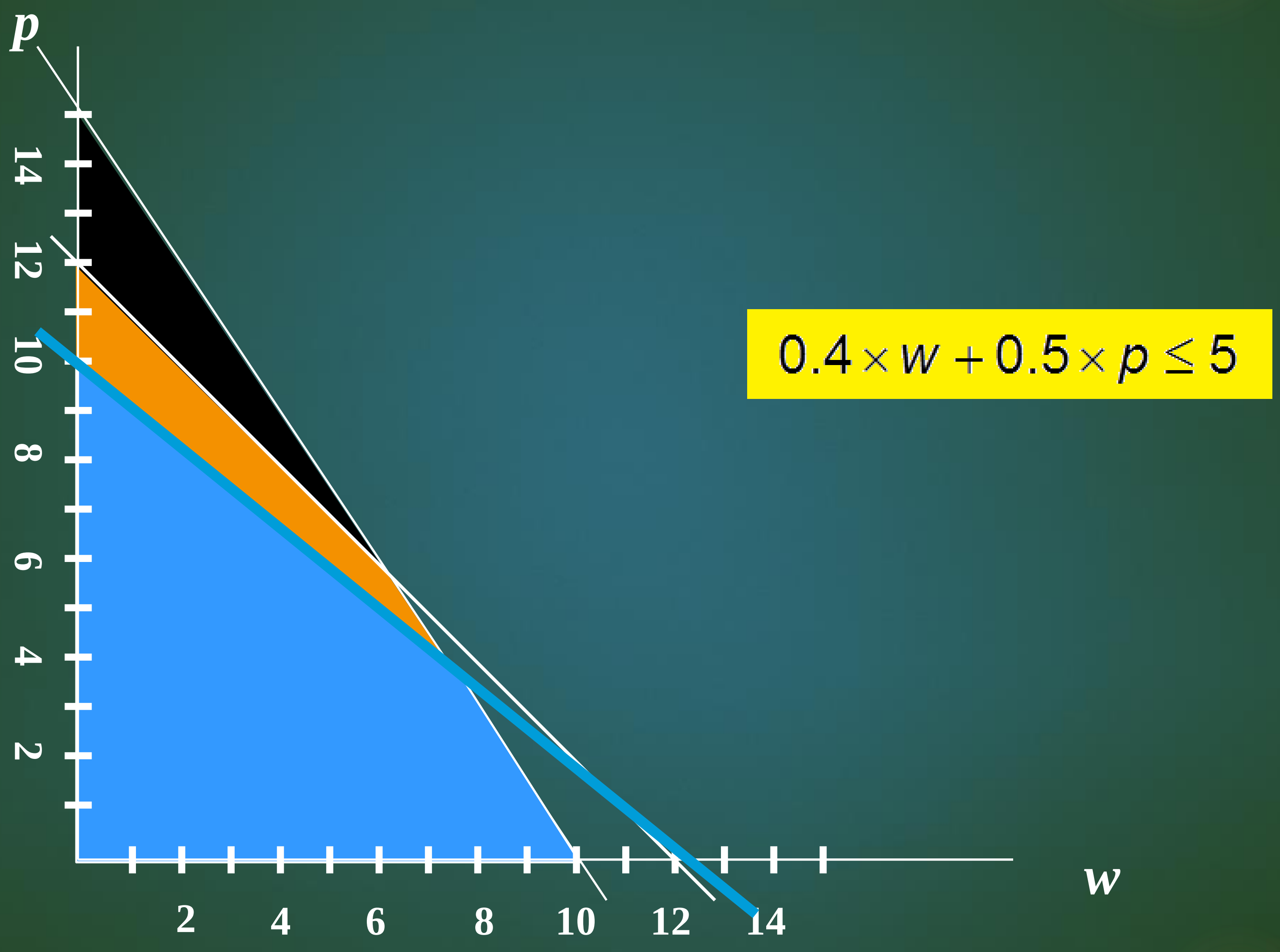
$$1.5 \times w + 1.0 \times p \leq 15$$

Solving an LP Graphically (3)

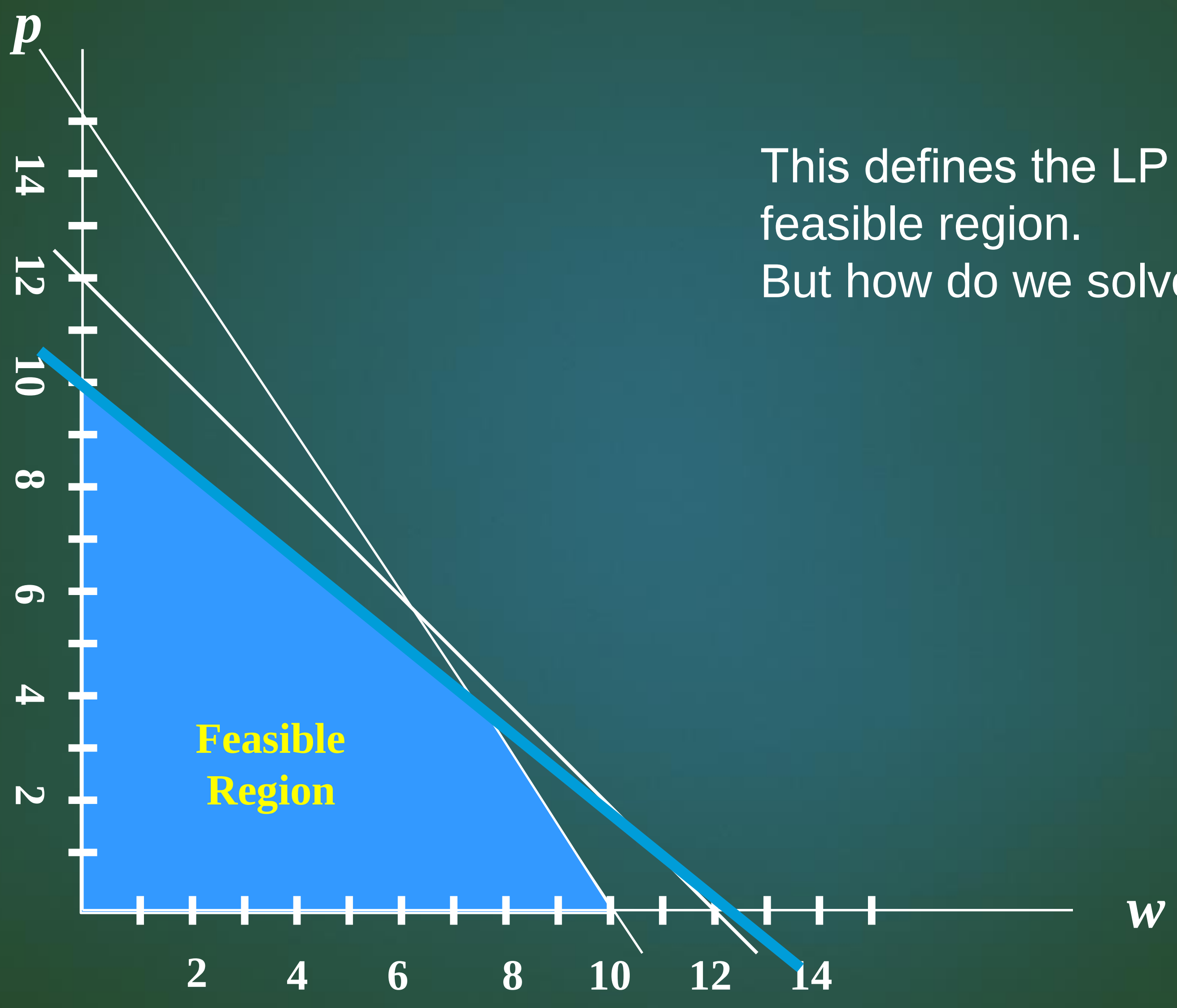


$$1.0 \times w + 1.0 \times p \leq 12$$

Solving an LP Graphically (4)

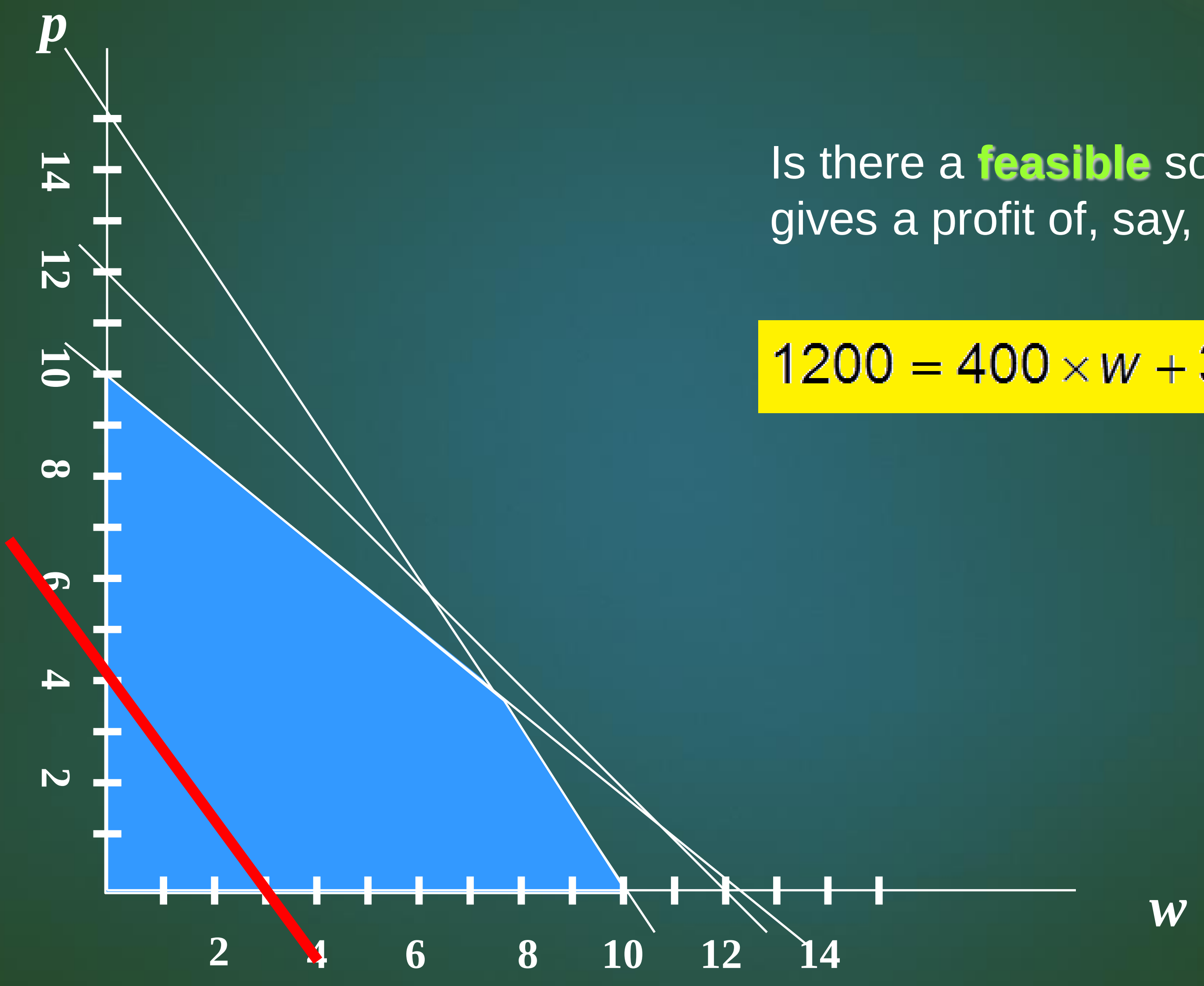


Solving an LP Graphically (4)



This defines the LP problem's feasible region.
But how do we solve it?

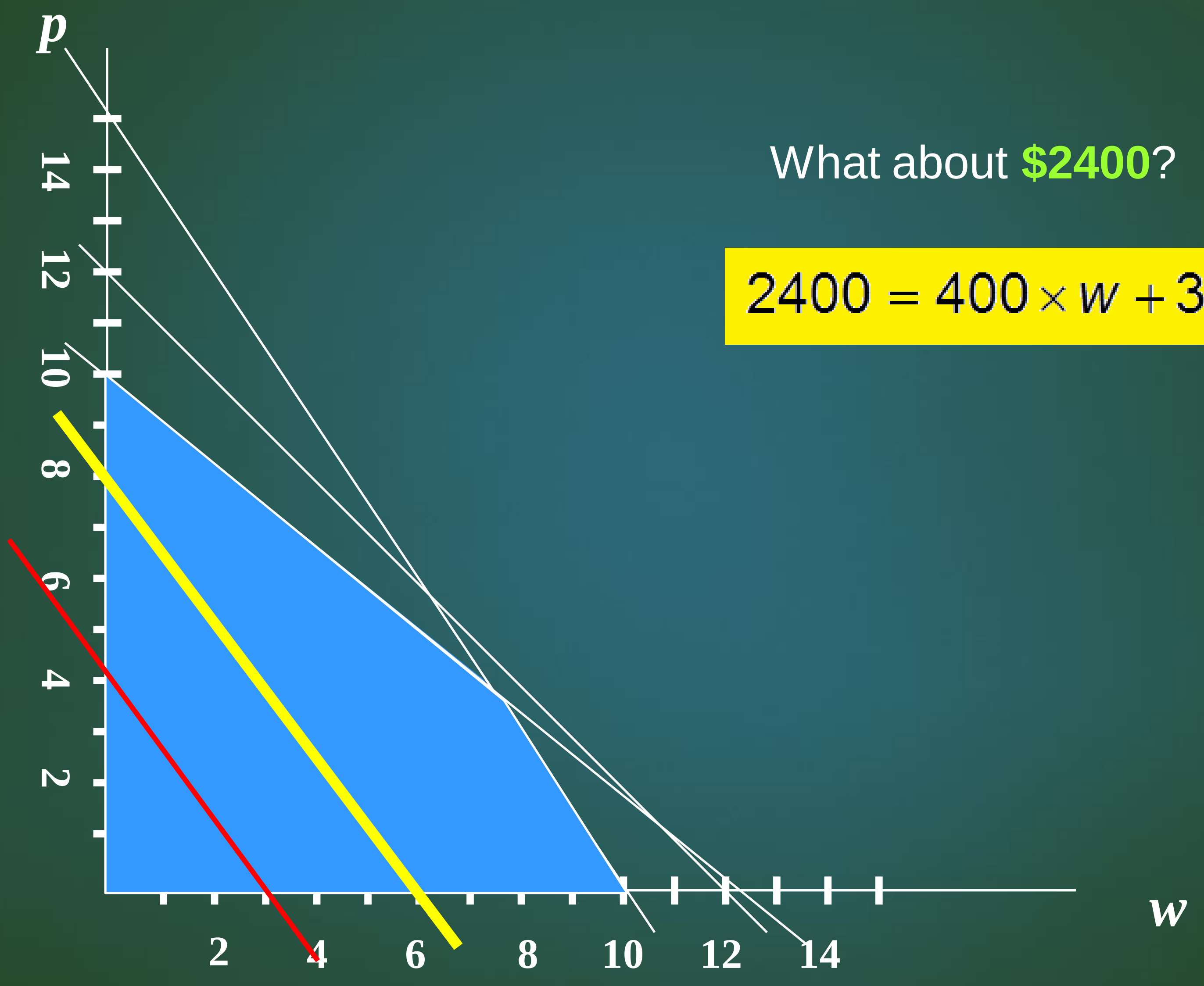
Solving an LP Graphically (7)



Is there a **feasible** solution that gives a profit of, say, **$1200**?

$$1200 = 400 \times w + 300 \times p$$

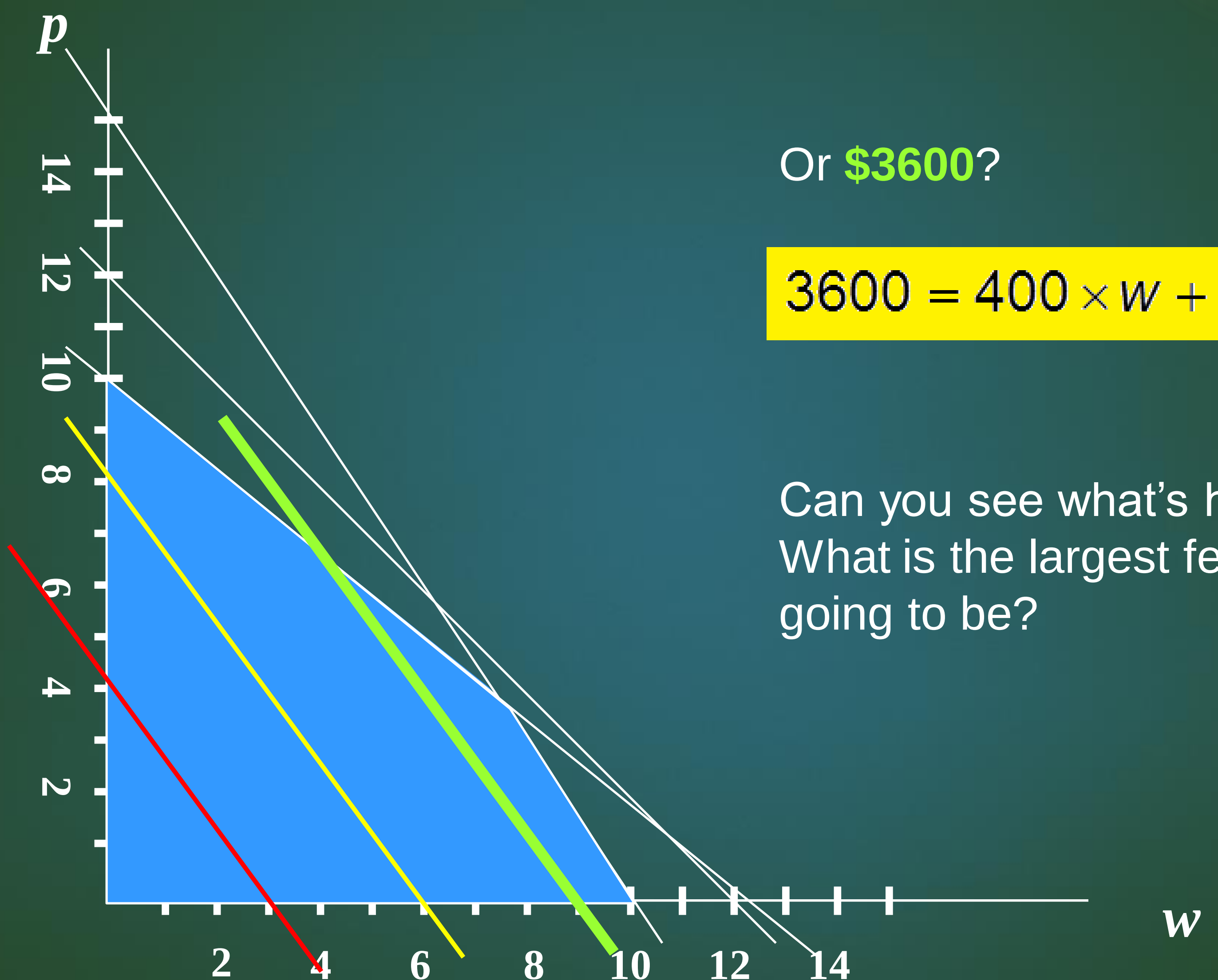
Solving an LP Graphically (8)



What about **$2400**?

$$2400 = 400 \times w + 300 \times p$$

Solving an LP Graphically (9)

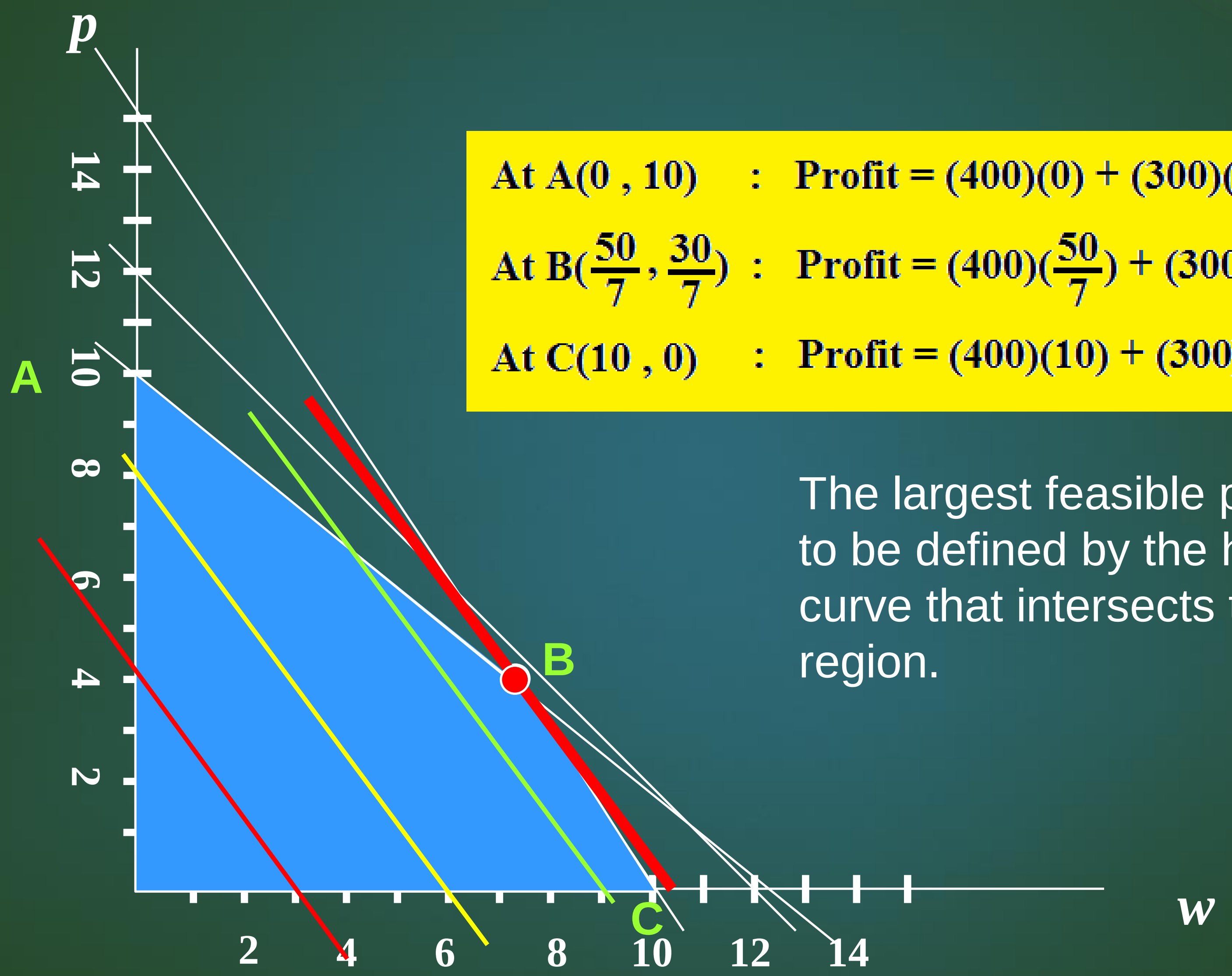


Or **$3600**?

$$3600 = 400 \times w + 300 \times p$$

Can you see what's happening? What is the largest feasible profit going to be?

Solving an LP Graphically (10)



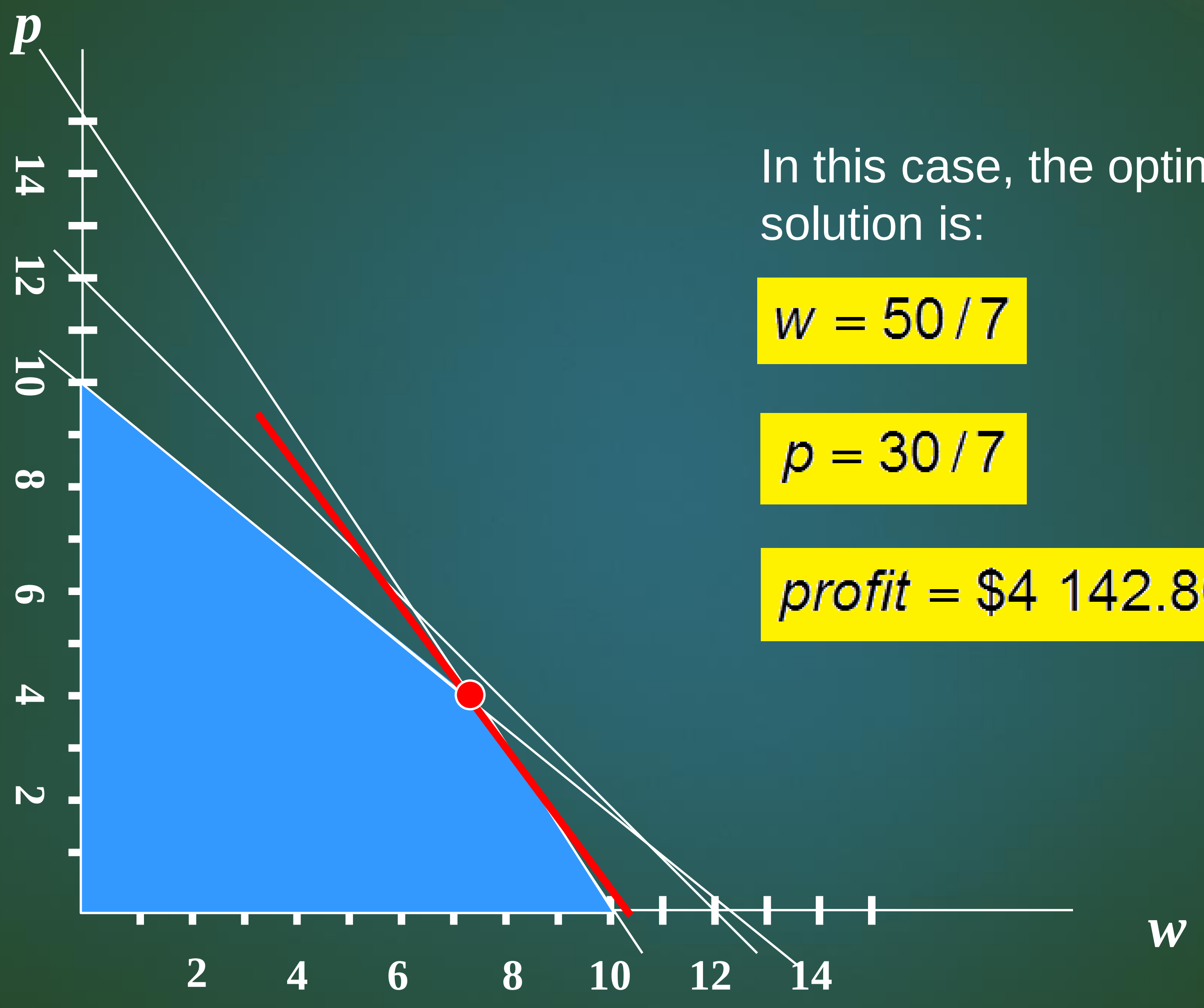
At A(0 , 10) : Profit = (400)(0) + (300)(10) = \$3,000

At B($\frac{50}{7}$, $\frac{30}{7}$) : Profit = (400)($\frac{50}{7}$) + (300)($\frac{30}{7}$) = \$4,142.86 ← Maximum

At C(10 , 0) : Profit = (400)(10) + (300)(0) = \$4,000

The largest feasible profit is going to be defined by the highest profit curve that intersects the feasible region.

Solving an LP Graphically (11)



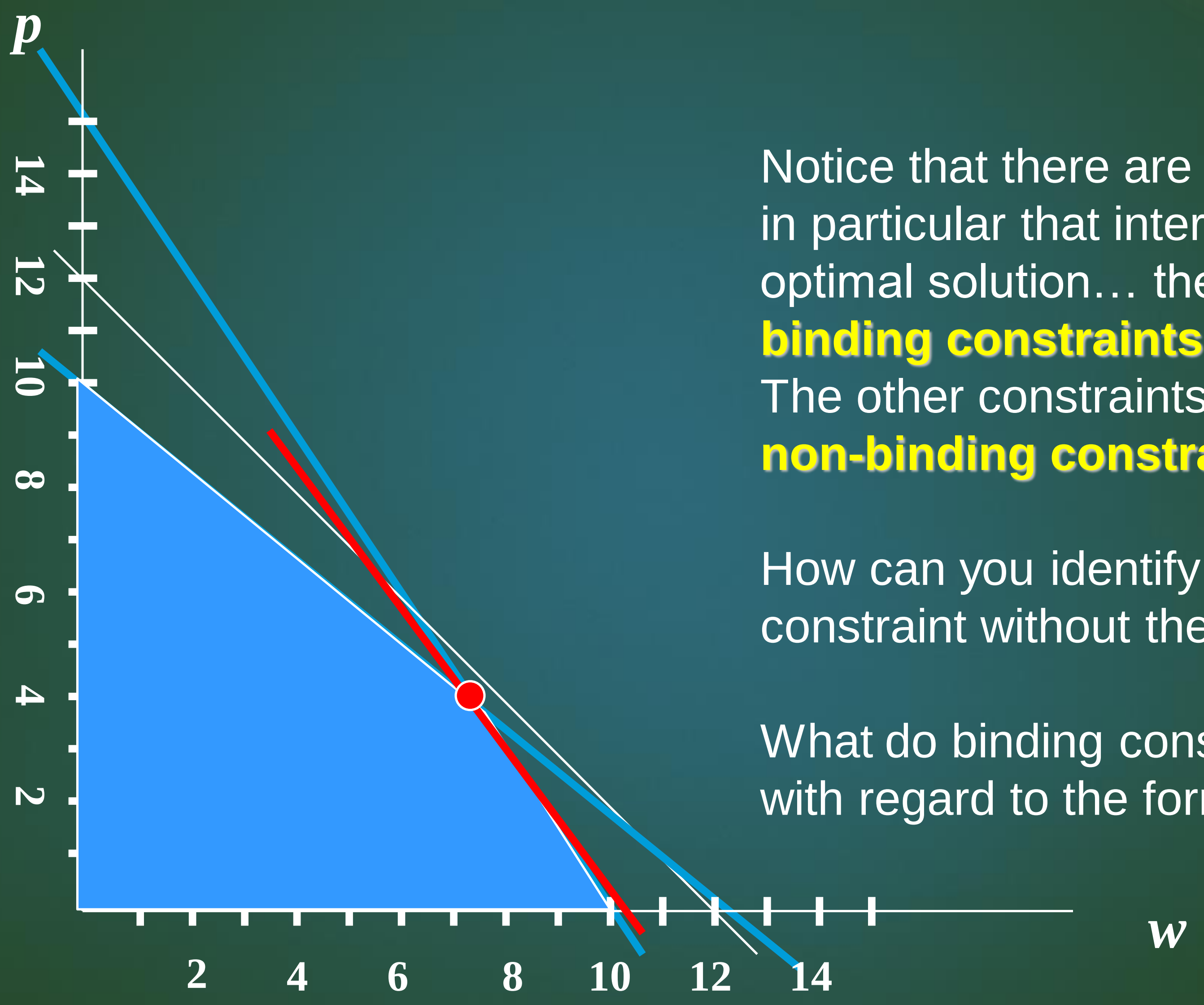
In this case, the optimal solution is:

$w = 50/7$

$p = 30/7$

$profit = \$4\ 142.86$

Solving an LP Graphically (12)



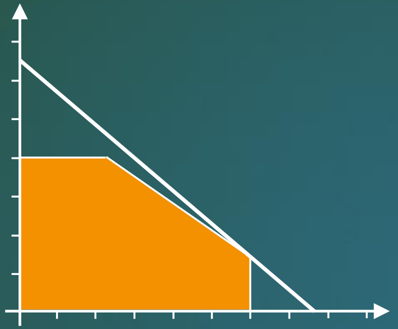
Notice that there are two constraints in particular that intersect with the optimal solution… these are called **binding constraints**.
The other constraints are called **non-binding constraints**.

How can you identify a binding constraint without the graph?

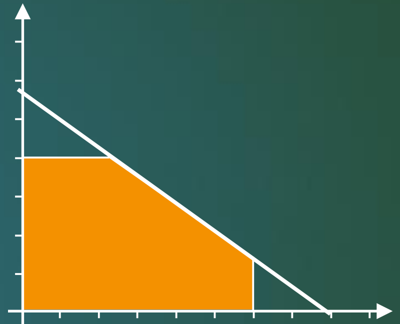
What do binding constraints mean with regard to the formulation?

Solving LP Problems Graphically – Outcomes

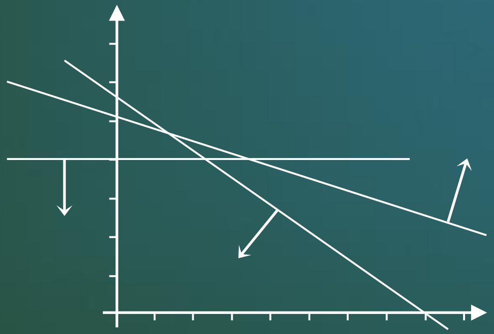
- 4 possible outcomes:



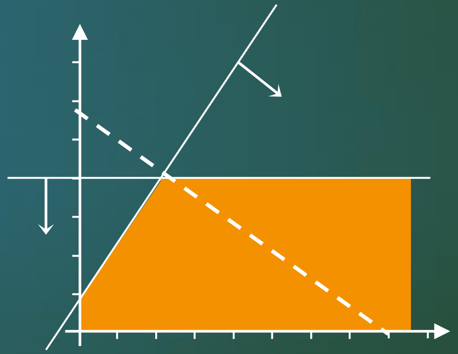
Unique Optimal Solution



Alternate Optimal Solutions



No Feasible Solution



Unbounded Optimal Solution

Till next lecture,

take care.