Lecture 6

Regular Simplex Method

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Standard Form

Introduction :

- A linear program (LP) that appears in a particular form where all constraints are equations and all variables are nonnegative is said to be in standard form.

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1- Nonnegativity Conditions.

1- all variables in the program must be nonnegative.

a) if any variable X_4 is negative

then we replace $X_5 = -X_4$; $X_5 \geq 0$

b) if any variable not already constrained to be nonnegative is replaced by the difference of two new variables which are nonnegative.

if x_i = is unrestricted

put $x_i = x_j - x_K$; x_j & $x_K \geq 0$

2- the right hand side of all constraints must be non-negative ; if this condition is not satisfied we multiply the constraint by (-1)

ex:

$2x_1 - 3x_2 + 4x_3 \leq -5$ $\quad \times(-1)$

$-2x_1 + 3x_2 - 4x_3 \geq 5$ $\quad$ which has non-negative R.H.S.

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2- Converting all inequalities into Equations.

(introduce slack and surplus Variable)

1- A linear constraint of the Form $\sum_j a_{ij} X_j \leq b_i$ Can be Converted into an equality by adding a new nonnegative Variable to the LHS of the inequality. such a variable is numerically equal to the difference between R.HS and L.H.S of the inequality and is known as Slack Variable..

ex:

Min $Z = 3x_1 + x_2 + 0S_1$

Sub to:

$5x_1 + 7x_2 \leq 4$

$5x_1 + 7x_2 + S_1 = 4$ $\quad$; S_1 slack

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2- A linear Constraint of the form $\sum a_{ij} x_j \geq b_i$ Can be Converted into an equality by Subtracting a new non-negative variable from the L.H.S of the Constraint. Such a variable is numerically equal to the differance between the L and R.H.S sides of the inequality. and is known as Surplus Variable.

ex:

Min: $Z = 3x + 4x_2 + 0S_2$

Sub To:

$$x_1 - x_2 \geq 5$$

$$x_1 - x_2 - S_2 = 5$$

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3- Introduce Artificial Variables:

we will add the artificial variable to all (=) and ($\geq$) Constraints To play the role of slacks at the First iteration and then dispose of them legitimately at a Later iteration. two Closely related methods are introduced
M-method and two phase method.

in M-method: the M-method starts will all Constraints ($\leq$)
if equation i doesnot have a slack, an artificial variable R_i is added to form a starting Soln
However, because the AV. are not part of the LP model. they are assigned to a very high penalty in the obj f^{n}
thus forcing them (eventually) To equal to zero in the optimum Solution, this will be always the Case if the problem has a feasible Soln,

- لذلك اذا كانت الـ artificial var بالنهاية = 0 هذا يعني ان هنا الـ system consistent والحل هو الذي حصلنا عليه بالنهاية
- اما اذا كانت نتيجة ≠ 0 هنا يعني ان الـ system بالنهاية كان inconsistent وليس له حل ...

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Penalty rule for Artificial Variables.

Given M: is a sufficiently large positive value ($M \to \infty$)
The objective Coefficient of the Artificial Variable is

$-M$: in maximization problem.
M: " minimization Problem

M: is a number three or four times larger in magnitude than any other number in the program

- لذلك اذا كانت الـ artificial var بنهاية = 0 هذا يعني أن هنا الـ system consistent والحل هو الذي حصلنا عليه بالنهاية
- اما اذا كانت قيمة ≠ 0 هذا يعني ان الـ system بالتالي كان inconsistent وليس له حل .....

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Standar Form:

A linear program is in standard form if the Constraints are all modeled as equalities, and if one feasible soln is Know. Standard Form is

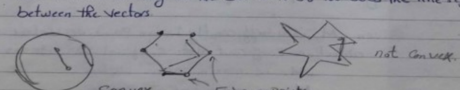
$$\begin{bmatrix} \text{optimize: } Z = C^T x \\ \text{Subject To:} \\ A x = B \\ \text{with: } X \geqslant 0 \end{bmatrix}$$

- In Regular Simplex Method we have to start the method by initial Basic feasible Soln:

So we have First under stand what we mean by Basic feasible Soln, To .. we have first To learn some important defi-

Convex Set:

A set of m-dimentional Vectors is Convex, if when ever two vectors belong To the Set then So Too does the line segment between the vectors



Extreme point A vector P is an extreme point of a Convex set if it Can't be expressed as a Convex Combination of two other two vectors in the set. that is an extreme point does not lie on the line segment between any other two vectors in the Set.

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Basic Solution $(m<n)$

if we have m. Constraints and n. ...

if we set $n-m$ var equal to Zero and then Solve the remaining m Constraints for the remaining m-variables, the resulting soln, (if unique) is Basic Solution and must Corresponds To a (feasible or infeasible) Corner point of the soln Space, the max number of Corner point is

$$C_m^n = \frac{n!}{m!\,(n-m)!}$$

ex: $[x_1, x_2, x_3]$ $n=3$, $m=2$, $n-m=1$

$\frac{3!}{2!\,(n-m)!} = \frac{3\cdot 2!}{2!} = 3$ Basic Soln

$[0, x_2, x_3]$

$[x_1, 0, x_3]$

$[x_1, x_2, 0]$

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the Zero $n-m$ variables are known as (non-basic variables)

The remaining m-variables are (Basic Variables)

So we can identify the initial Basic soln by putting slack or artificial var $=$ R.H.S. these variable the number of these var $=$ the number of Constraints $= m$ and all the remaining Var (non-Basic) $=$ zero.

Example(1): Put the following program is standard Form

$$\begin{aligned} \max:\ & Z = x_1 + x_2 \\ \text{Sub To}:\ & x_1 + 5x_2 \leq 5 \\ & 2x_1 + x_2 \leq 4 \\ \text{with}\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & Z = x_1 + x_2 + 0S_1 + 0S_2 \\ \text{Sub To}\ & x_1 + 5x_2 + S_1 = 5 \\ & 2x_1 + x_2 + S_2 = 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

1 * the IBFS: $x_1 = x_2 = 0$, $S_1 = 5$, $S_2 = 4$

$X = [x_1, x_2, S_1, S_2]$ $\quad C = [1, 1, 0, 0]^t$

$A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = X_0 = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ B.V.

Example(2):

$$\begin{aligned} \max:\ & Z = 5x_1 + 2x_2 \\ \text{Sub To}:\ & 6x_1 + x_2 \geq 6 \quad \rightarrow (1) \\ & 4x_1 + 3x_2 \geq 12 \\ & x_1 + 2x_2 = 4 \\ & x_1 \geq 0. \end{aligned}$$

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1) non-negativity Condition

let $x_2 = x_3 - x_4$; $x_3, x_4 \geq 0$.

multiply eqn 1 by (-)

→ max: $Z = 5x_1 + 2x_3 - 2x_4$

→ Sub To:

$$-6x_1 - x_3 + x_4 \leq +6$$
$$4x_1 + 3x_3 - 3x_4 \geq 12$$
$$x_1 + 2x_3 - 2x_4 = 4$$
$$x_1, x_3, x_4 \geq 0$$

2) introduce Slack and Surplus Var.)

max: $Z = 5x_1 + 2x_3 - 2x_4 + 0S_1 + 0S_2 - MA_1 - MA_2$

Sub To:

$$-6x_1 - x_3 + x_4 + S_1 = 6$$
$$4x_1 + 3x_3 - 3x_4 - S_2 + A_1 = 12$$
$$x_1 + 2x_3 - 2x_4 + A_2 = 4$$
$$x_i \geq 0$$

IBFS

$S_1 = 6$, $A_1 = 12$, $A_2 = 4$

$x_1 = x_3 = x_4 = S_2 = 0$

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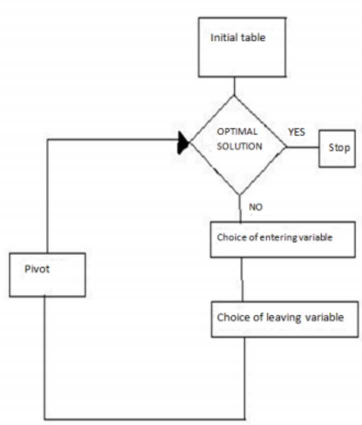
THE STEPS OF THE SIMPLEX ALGORITHM



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Optimize: $z = c^t x$

Subject To: $AX = B$

$X \geq 0$

where $B \geq 0$, BS X_0 is known

First we have SBFs

$n = 5$, $m = 3$

Non Basic $n - m = 2$

$[x_1, x_2, x_3, 0, 0]$

$[0, x_2, x_3, x_4, 0]$

L.V. N.B. E.V. Basic

- For minimization Problem:

		x^t	
		c^T	0
$\rightarrow$	X_0 C_0	A	B
		$c^T - C_0^T A$	$-C_0^T B$

max: $C_0^T A - C^T$

The Simplex method:

Step 1: Locate the most negative number in the bottom row of the Simplex tableau, excluding the Last Column. and Call the Column in which this number appear the Work Coloumn ...

Remark: if more than one Condidate for most negative exist Choose one.

Step 2: Form ratios by dividing each positive number in the work Coloumn, excluding the last raw, into the element in the ~~last~~ same row and Last Column. Designate the element in the work Coloumn that yield the Smallest ratio as the Pivot element.

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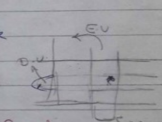
- if more than one element yield the Same Smallest ratio $\Rightarrow$ Choose one.
- if no element in the work column is positive. $\Rightarrow$ the program has no Solution. indefinite Sol$\underline{n}$

Step3: Use elementary raw operations to Convert the pivot element to $\underline{\underline{1}}$ and then reduce all other elements in the work column To $\underline{\underline{0}}$

E.V. and D.V.

Step4: replace the x-variable in the pivot row and First Column by the x-var in the First row and pivot Column.

This new First Column is the Current Set of basic variables



Step5: Repeat step 1→4 until there are no negative number in the Last row, excluding the last Column.

Step6: the optimal Solution z^*, is the number of in the Last row and Last Column for a maximization program, but the negative of this number for minimization program.

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Example:

maximize: $Z = x_1 + 9x_2 + x_3$

Sub To :

$$x_1 + 2x_2 + 3x_3 \leq 9$$
$$3x_1 + 2x_2 + 2x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0$$

max: $Z = x_1 + 9x_2 + x_3 + 0S_1 + 0S_2$

SubTo:

$$x_1 + 2x_2 + 3x_3 + S_1 = 9$$
$$3x_1 + 2x_2 + 2x_3 + S_2 = 15$$
$$x_i, s_i \geq 0$$

E.V → x_2

		x_1	x_2	x_3	S_1	S_2		ratio
c^t		1	9	1	0	0		
D.V ← S_1	0	1	2^* (P)	3	1	0	9	$9/2 = 4.5$ ← min
S_2	0	3	2	2	0	1	15	$\frac{15}{2} = 7.5$
max ← $c_0^t A - c^t$ →		-1	-9	-1	0	0	0	

X_0 C_0

$-9 + \frac{2}{9} * R_1$

most -ve negative

$\exists$ -ve number ⇒ not optimal

Z+2

		x_1	x_2	x_3	S_1	S_2	
$R_2 \leftarrow R_2 - R_1$	x_2 9	$1/2$	1	$3/2$	$1/2$	0	$9/2$
	S_2 0	2	0	-1	-1	1	6
$R_3 \leftarrow R_3 + \frac{9}{2}(R_1)$ →		$\frac{7}{2}$	0	$\frac{25}{2}$	$9/2$	0	$\frac{81}{2} = Z^*$

all ≥ 0 ⇒ optimal

$\frac{\text{element} * \text{pivot} - P_i P_i}{\text{pivot}}$

$x_2 = 9/2$, $x_1 = 0$, $x_3 = 0$

$Z^* = x_1 + 9x_2 + x_3$

$= 0 + \frac{9 * 9}{2} + 0$

$= \frac{81}{2}$ #

$S_2 = 6$ ⇒ this mean that the second constraint is strictly <

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Unbounded Solution

Unbounded Solution

in Some LP models, the values of the variables may be increased indefinitely without violating any of the constraints meaning that the solution space is unbounded in at least one variable. , As a result, the objective value may increase (max case) or decrease (min. case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

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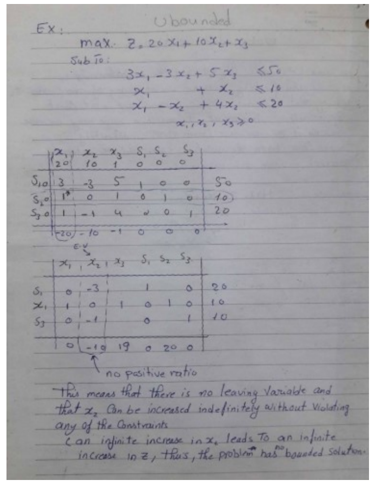
Example



Ubounded

Ex:

max. $Z = 20x_1 + 10x_2 + x_3$

Sub To:

$3x_1 - 3x_2 + 5x_3 \leq 50$

$x_1 + x_2 \leq 10$

$x_1 - x_2 + 4x_2 \leq 20$

$x_1, x_2, x_3 \geq 0$

		x_1	x_2	x_3	S_1	S_2	S_3	
		20	10	1	0	0	0	
S_1	0	3	-3	5	1	0	0	50
S_2	0	1*	0	1	0	1	0	10
S_3	0	1	-1	4	0	0	1	20
		-20	-10	-1	0	0	0	

E.V

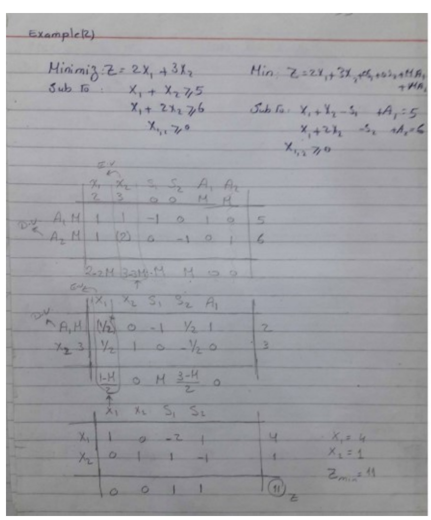
	x_1	x_2	x_3	S_1	S_2	S_3	
S_1	0	-3		1		0	20
x_1	1	0	1	0	1	0	10
S_3	0	-1		0		1	10
	0	-10	19	0	20	0	

no positive ratio

This means that there is no leaving Variable and that x_2 Can be increased indefinitely without Violating any of the Constraints

can infinite increase in x_2 leads To an infinite increase in z, thus, the problem has no bounded solution.

Example including Artificial Variable



Example(2)

Minimiz: $Z = 2X_1 + 3X_2$

Sub To: $X_1 + X_2 \geq 5$

$X_1 + 2X_2 \geq 6$

$X_{1,2} \geq 0$

Min. $Z = 2X_1 + 3X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Sub To: $X_1 + X_2 - S_1 + A_1 = 5$

$X_1 + 2X_2 - S_2 + A_2 = 6$

$X_{1,2} \geq 0$

E.V

		X_1	X_2	S_1	S_2	A_1	A_2	
		2	3	0	0	M	M	
A_1	M	1	1	-1	0	1	0	5
D.V A_2	M	1	(2)	0	-1	0	1	6
		2-2M	3-3M	-M	M	0	0	

E.V

		X_1	X_2	S_1	S_2	A_1	
D.V A_1	M	(1/2)*	0	-1	1/2	1	2
X_2	3	1/2	1	0	-1/2	0	3
		$\frac{1-M}{2}$	0	M	$\frac{3-M}{2}$	0	

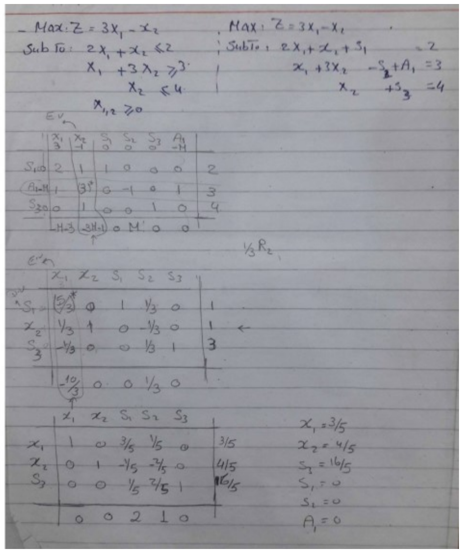
	X_1	X_2	S_1	S_2	
X_1	1	0	-2	1	4
X_2	0	1	1	-1	1
	0	0	1	1	11 Z

$X_1 = 4$

$X_2 = 1$

$Z_{min} = 11$

Example including Artificial Variable



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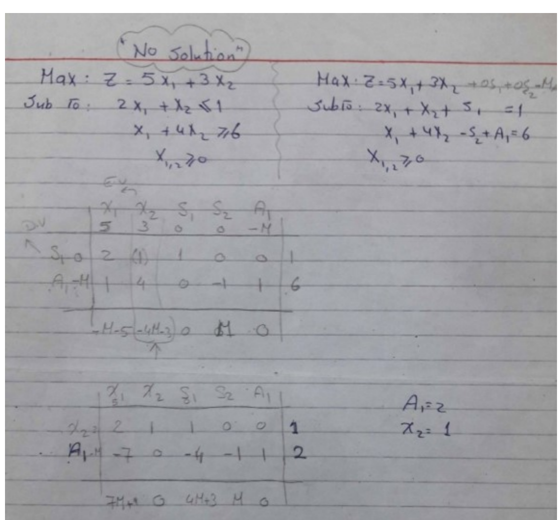
No Solution case



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