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7.2 Orthogonal Diagonalization

Definition. If A and B are square matrices, then we say that B is orthogonally similar to A if there is an orthogonal matrix P such that $B = P^T A P$

NOTE: If B is orthogonally similar to A, then it is also true that A is orthogonally similar to B since we can express A as

$$A = PBP^T = Q^T B Q$$

where

$$Q = P^T$$

Conditions for orthogonal Diagonalizability

If A is an nxn matrix with real entries, then the following are equivalent.

a) A is orthogonally diagonalizable
b) A has an orthonormal set of n eigen vectors.
c) A is symmetric.

Properties of Symmetric Matrices

a. The eigen values of A are all $\mathbb{R}$.
b. Eigenvectors from different eigenspaces are orthogonal.

Orthogonally Diagonalizing an nxn symmetric Matrix

step #01: Find a basis for each eigenspace of A
step #02: Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.
step #03: Form the matrix P whose cols. are the vectors constructed in step 2. This matrix will orthogonally diagonalize A, and the eigenvalues ...

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Example #01 : Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix P that diagonalizes :

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution :

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{bmatrix}$$

$$\Rightarrow \quad (\lambda-2)^2(\lambda-8) = 0$$

$\lambda = 2$ and $\lambda = 8$

when $\lambda = 2$:
$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$$

$2R_1 + R_2$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \Rightarrow & \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

$x_2 = s$
$x_3 = t$
$x_1 = -s - t$

$$(x_1, x_2, x_3) = (-s-t, s, t)$$
$$= s(-1, 1, 0) + t(-1, 0, 1)$$

$u_1 = (-1, 1, 0) \qquad u_2 = (-1, 0, 1)$

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when $x = 8$:
$$\begin{bmatrix} -4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

after reducing, we get,

$u_3 = (1, 1, 1)$

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Example #02: Spectral Decomposition.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda+2 \end{bmatrix}$$

$$(\lambda-1)(\lambda+2) - \cancel{5}\,4 = 0$$
$$\lambda^2 + 2\lambda - \lambda - 2 - 4 = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$\lambda = -3, 2$$

when $\lambda = -3$:

$$\begin{bmatrix} -3-1 & -2 \\ -2 & -3+2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

$$q_1 = \left(\frac{1}{\sqrt5}, \frac{-2}{\sqrt5}\right) \qquad q_2 = \left(\frac{2}{\sqrt5}, \frac{1}{\sqrt5}\right)$$

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = (-3)\begin{bmatrix} \frac{1}{\sqrt5} \\ \frac{-2}{\sqrt5} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt5} & \frac{-2}{\sqrt5} \end{bmatrix} + (2)\begin{bmatrix} \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt5} , \frac{1}{\sqrt5} \end{bmatrix}$$

$$= (-3)\begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{4}{5} \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Now; let us see what this spectral decomposition tells us about the image of the vector $x = x_2 - x_1 = (1,1)$ under multiplication by A. Writing x in column form, it follows that

$$Ax = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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