Continuing with the premise $r \to \neg s$, we find that because s has the truth value 1, the truth value of r must be 0. Hence r is false. But with $\neg p$ false and the premise $\neg p \veebar r$ true, we also have r true. Therefore we find that $p \Rightarrow (\neg r \wedge r)$.

We have failed in our attempt to find a counterexample to the validity of the given argument. However, this failure has shown us that the given argument is valid — and the validity follows by using the method of Proof by Contradiction.

This introduction to the rules of inference has been far from exhaustive. Several of the books cited among the references listed near the end of this chapter offer additional material for the reader who wishes to pursue this topic further. In Section 2.5 we shall apply the ideas developed in this section to statements of a more mathematical nature. For we shall want to learn how to develop a proof for a theorem. And then in Chapter 4 another very important proof technique called *mathematical induction* will be added to our arsenal of weapons for proving mathematical theorems. First, however, the reader should carefully complete the exercises for this section.

EXERCISES 2.3

1. The following are three valid arguments. Establish the validity of each by means of a truth table. In each case, determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

a) $[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$

b) $[[(p \wedge q) \to r] \wedge \neg q \wedge (p \to \neg r)] \to (\neg p \vee \neg q)$

c) $[[p \vee (q \vee r)] \wedge \neg q] \to (p \vee r)$

2. Use truth tables to verify that each of the following is a logical implication.

a) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

b) $[(p \to q) \wedge \neg q] \to \neg p$

c) $[(p \vee q) \wedge \neg p] \to q$

d) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

3. Verify that each of the following is a logical implication by showing that it is impossible for the conclusion to have the truth value 0 while the hypothesis has the truth value 1.

a) $(p \wedge q) \to p$

b) $p \to (p \vee q)$

c) $[(p \vee q) \wedge \neg p] \to q$

d) $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$

e) $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$

4. For each of the following pairs of statements, use Modus Ponens or Modus Tollens to fill in the blank line so that a valid argument is presented.

a) If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
Janice had trouble starting her car.
$\therefore$ ______________________

b) If Brady solved the first problem correctly, then the answer he obtained is 137.
Brady's answer to the first problem is not 137.
$\therefore$ ______________________

c) If this is a **repeat-until** loop, then the body of this loop is executed at least once.
____
$\therefore$ The body of the loop is executed at least once.

d) If Tim plays basketball in the afternoon, then he will not watch television in the evening.
____
$\therefore$ Tim didn't play basketball in the afternoon.

5. Consider each of the following arguments. If the argument is valid, identify the rule of inference that establishes its validity. If not, indicate whether the error is due to an attempt to argue by the converse or by the inverse.

a) Andrea can program in C++, and she can program in Java.
Therefore Andrea can program in C++.

b) A sufficient condition for Bubbles to win the golf tournament is that her opponent Meg not sink a birdie on the last hole.
Bubbles won the golf tournament.
Therefore Bubbles' opponent Meg did not sink a birdie on the last hole.

c) If Ron's computer program is correct, then he'll be able to complete his computer science assignment in at most two hours.
It takes Ron over two hours to complete his computer science assignment.
Therefore Ron's computer program is not correct.

d) Eileen's car keys are in her purse, or they are on the kitchen table.

Eileen's car keys are not on the kitchen table.
Therefore Eileen's car keys are in her purse.

e) If interest rates fall, then the stock market will rise.
Interest rates are not falling.
Therefore the stock market will not rise.

6. For primitive statements p, q, and r, let P denote the statement

$$[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)],$$

while P_1 denotes the statement

$$[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)].$$

a) Use the rules of inference to show that

$$q \wedge r \Rightarrow q \vee r.$$

b) Is it true that $P \Rightarrow P_1$?

7. Give the reason(s) for each step needed to show that the following argument is valid.

$$[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)] \rightarrow (s \vee t)$$

Steps	Reasons
1) p	
2) $p \rightarrow q$	
3) q	
4) $r \rightarrow \neg q$	
5) $q \rightarrow \neg r$	
6) $\neg r$	
7) $s \vee r$	
8) s	
9) $\therefore s \vee t$	

8. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \rightarrow \neg t} \\ \therefore p \end{array}$$

Steps	Reasons
1) $\neg s \wedge \neg u$	
2) $\neg u$	
3) $\neg u \rightarrow \neg t$	
4) $\neg t$	
5) $\neg s$	
6) $\neg s \wedge \neg t$	
7) $r \rightarrow (s \vee t)$	
8) $\neg(s \vee t) \rightarrow \neg r$	
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	
10) $\neg r$	
11) $(\neg p \vee q) \rightarrow r$	
12) $\neg r \rightarrow \neg(\neg p \vee q)$	
13) $\neg r \rightarrow (p \wedge \neg q)$	
14) $p \wedge \neg q$	
15) $\therefore p$	

9. a) Give the reasons for the steps given to validate the argument

$$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)] \rightarrow (\neg q \rightarrow s).$$

Steps	Reasons
1) $\neg(\neg q \rightarrow s)$	
2) $\neg q \wedge \neg s$	
3) $\neg s$	
4) $\neg r \vee s$	
5) $\neg r$	
6) $p \rightarrow q$	
7) $\neg q$	
8) $\neg p$	
9) $p \vee r$	
10) r	
11) $\neg r \wedge r$	
12) $\therefore \neg q \rightarrow s$	

b) Give a direct proof for the result in part (a).

c) Give a direct proof for the result in Example 2.32.

10. Establish the validity of the following arguments.

a) $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$

b) $[p \wedge (p \rightarrow q) \wedge (\neg q \vee r)] \rightarrow r$

c) $\begin{array}{l} p \rightarrow q \\ \neg q \\ \underline{\neg r} \\ \therefore \neg(p \vee r) \end{array}$

d) $\begin{array}{l} p \rightarrow q \\ r \rightarrow \neg q \\ \underline{r} \\ \therefore \neg p \end{array}$

e) $\begin{array}{l} p \rightarrow (q \rightarrow r) \\ \neg q \rightarrow \neg p \\ \underline{p} \\ \therefore r \end{array}$

f) $\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ \underline{\neg s} \\ \therefore t \end{array}$

g) $\begin{array}{l} p \rightarrow (q \rightarrow r) \\ p \vee s \\ t \rightarrow q \\ \underline{\neg s} \\ \therefore \neg r \rightarrow \neg t \end{array}$

h) $\begin{array}{l} p \vee q \\ \neg p \vee r \\ \underline{\neg r} \\ \therefore q \end{array}$

11. Show that each of the following arguments is invalid by providing a counterexample—that is, an assignment of truth values for the given primitive statements p, q, r, and s such that all premises are true (have the truth value 1) while the conclusion is false (has the truth value 0).

a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

b) $[[(p \wedge q) \rightarrow r] \wedge (\neg q \vee r)] \rightarrow p$

c) $\begin{array}{l} p \leftrightarrow q \\ q \rightarrow r \\ r \vee \neg s \\ \underline{\neg s \rightarrow q} \\ \therefore s \end{array}$

d) $\begin{array}{l} p \\ p \rightarrow r \\ p \rightarrow (q \vee \neg r) \\ \underline{\neg q \vee \neg s} \\ \therefore s \end{array}$

12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. a) Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

b) The tautology in part (a) provides the rule of inference known as *resolution*, where the conclusion $(q \vee r)$ is called the *resolvent*. This rule was proposed in 1965 by J. A. Robinson and is the basis of many computer programs designed to automate a reasoning system.

In applying resolution each premise (in the hypothesis) and the conclusion are written as *clauses*. A clause is a primitive statement or its negation, or it is the disjunction of terms each of which is a primitive statement or the negation of such a statement. Hence the given rule has the clauses $(p \vee q)$ and $(\neg p \vee r)$ as premises and the clause $(q \vee r)$ as its conclusion (or, resolvent). Should we have the premise $\neg(p \wedge q)$, we replace this by the logically equivalent clause $\neg p \vee \neg q$, by the first of DeMorgan's Laws. The premise $\neg(p \vee q)$ can be replaced by the two clauses $\neg p$, $\neg q$. This is due to the second DeMorgan Law and the Rule of Conjunctive Simplification. For the premise $p \vee (q \wedge r)$, we apply the Distributive Law of $\vee$ over $\wedge$ and the Rule of Conjunctive Simplification to arrive at either of the two clauses $p \vee q$, $p \vee r$. Finally, the premise $p \rightarrow q$ becomes the clause $\neg p \vee q$.

Establish the validity of the following arguments, using resolution (along with the rules of inference and the laws of logic).

(i)
$$\begin{array}{l} p \vee (q \wedge r) \\ \underline{p \rightarrow s} \\ \therefore r \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \\ \underline{p \leftrightarrow q} \\ \therefore q \end{array}$$

(iii)
$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore q \vee s \end{array}$$

(iv)
$$\begin{array}{l} \neg p \vee q \vee r \\ \neg q \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(v)
$$\begin{array}{l} \neg p \vee s \\ \neg t \vee (s \wedge r) \\ \neg q \vee r \\ \underline{p \vee q \vee t} \\ \therefore r \vee s \end{array}$$

c) Write the following argument in symbolic form, then use resolution (along with the rules of inference and the laws of logic) to establish its validity.

Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

2.4 The Use of Quantifiers

In Section 2.1, we mentioned how sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1, or 3, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

We refer to the sentence "The number $x + 2$ is an even integer" as an *open statement*, which we formally define as follows.

Definition 2.5 A declarative sentence is an *open statement* if

1) it contains one or more variables, and

Section 2.3

1. (a)

p	q	r	$p \rightarrow q$	$(p \vee q)$	$(p \vee q) \rightarrow r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The validity of the argument follows from the results in the last row. (The first seven rows may be ignored.)

(b)

p	q	r	$(p \wedge q) \rightarrow r$	$\neg q$	$p \rightarrow \neg r$	$\neg p \vee \neg q$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	1	1
0	1	1	1	0	1	1
1	0	0	1	1	1	1
1	0	1	1	1	0	1
1	1	0	0	0	1	0
1	1	1	1	0	0	0

The validity of the argument follows from the results in rows 1, 2, and 5 of the table. The results in the other five rows may be ignored.

(c)

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$[p \vee (q \vee r)] \wedge \neg q$	$p \vee r$
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	0	1
1	1	1	1	1	0	1

Consider the last two columns of this truth table. Here we find that whenever the truth value of $[p \vee (q \vee r)] \wedge \neg q$ is 1 then the truth value of $p \vee r$ is also 1. Consequently,

$$[[p \vee (q \vee r)] \wedge \neg q] \Rightarrow p \vee r.$$

(The rows of the table that are crucial for assessing the validity of the argument are rows 2, 5, and 6. Rows 1, 3, 4, 7, and 8 may be ignored.)

2. (a)

p	q	r	$p \to q$	$q \to r$	$p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	1	0	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

(b)

p	q	$p \to q$	$(p \to q) \land \neg q$	$[(p \to q) \land \neg q] \to \neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	0	1
1	1	1	0	1

(c)

p	q	$\neg p$	$p \lor q$	$(p \lor q) \land \neg p$	$[(p \lor q) \land \neg p] \to q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

(d)

p	q	r	$p \to r$	$q \to r$	$\overbrace{(p \lor q) \to r}^{s}$	$[(p \to r) \land (q \to r)] \to s$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	1
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	1	1	1	1	1

3. (a) If p has the truth value 0, then so does $p \land q$.
(b) When $p \lor q$ has the truth value 0, then the truth value of p (and that of q) is 0.
(c) If q has truth value 0, then the truth value of $[(p \lor q) \land \neg p]$ is 0, regardless of the truth value of p.
(d) The statement $q \lor s$ has truth value 0 only when each of q, s has truth value 0. Then

$(p \rightarrow q)$ has truth value 1 when p has truth value 0; $(r \rightarrow s)$ has truth value 1 when r has truth value 0. But then $(p \vee r)$ must have truth value 0, not 1.
(e) For $(\neg p \vee \neg r)$ the truth value is 0 when both p, r have truth value 1. This then forces q, s to have truth value 1, in order for $(p \rightarrow q)$, $(r \rightarrow s)$ to have truth value 1. However, this results in truth value 0 for $(\neg q \vee \neg s)$.

4. (a) Janice's daughter Angela will check Janice's spark plugs. (Modus Ponens)
(b) Brady did not solve the first problem correctly. (Modus Tollens)
(c) This is a **repeat-until** loop. (Modus Ponens)
(d) Tim watched television in the evening. (Modus Tollens)

5. (a) Rule of Conjunctive Simplification
(b) Invalid – attempt to argue by the converse
(c) Modus Tollens
(d) Rule of Disjunctive Syllogism
(e) Invalid – attempt to argue by the inverse

6. (a)

	Steps	Reasons
(1)	$q \wedge r$	Premise
(2)	q	Step (1) and the Rule of Conjunctive Simplification
(3)	$\therefore q \vee r$	Step (2) and the Rule of Disjunctive Amplification

Consequently, $(q \wedge r) \rightarrow (q \vee r)$ is a tautology, or $q \wedge r \Rightarrow q \vee r$.

(b) Consider the truth value assignments $p: 0$, $q: 1$, and $r: 0$. For these assignments $[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$ has truth value 1, while $[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)]$ has truth value 0. Therefore, $P \rightarrow P_1$ is *not* a tautology, or $P \not\Rightarrow P_1$.

7.

(1) & (2)	Premise
(3)	Steps (1), (2) and the Rule of Detachment
(4)	Premise
(5)	Step (4) and $(r \rightarrow \neg q) \Longleftrightarrow (\neg\neg q \rightarrow \neg r) \Longleftrightarrow (q \rightarrow \neg r)$
(6)	Steps (3), (5) and the Rule of Detachment
(7)	Premise
(8)	Steps (6), (7) and the Rule of Disjunctive Syllogism
(9)	Step (8) and the Rule of Disjunctive Amplification

8.

(1)	Premise
(2)	Step (1) and the Rule of Conjunctive Simplification
(3)	Premise
(4)	Steps (2), (3) and the Rule of Detachment

(5) Step (1) and the Rule of Conjunctive Simplification
(6) Steps (4), (5) and the Rule of Conjunction
(7) Premise
(8) Step (7) and $[r \rightarrow (s \vee t)] \Longleftrightarrow [\neg(s \vee t) \rightarrow \neg r]$
(9) Step (8) and DeMorgan's Laws
(10) Steps (6), (9) and the Rule of Detachment
(11) Premise
(12) Step (11) and $[(\neg p \vee q) \rightarrow r] \Longleftrightarrow [\neg r \rightarrow \neg(\neg p \vee q)]$
(13) Step (12) and DeMorgan's Laws and the Law of Double Negation
(14) Steps (10), (13) and the Rule of Detachment
(15) Step (14) and the Rule of Conjunctive Simplification

9. (a)

(1) Premise (The Negation of the Conclusion)
(2) Step (1) and $\neg(\neg q \rightarrow s) \Longleftrightarrow \neg(\neg\neg q \vee s) \Longleftrightarrow \neg(q \vee s) \Longleftrightarrow \neg q \wedge \neg s$
(3) Step (2) and the Rule of Conjunctive Simplification
(4) Premise
(5) Steps (3), (4) and the Rule of Disjunctive Syllogism
(6) Premise
(7) Step (2) and the Rule of Conjunctive Simplification
(8) Steps (6), (7) and Modus Tollens
(9) Premise
(10) Steps (8), (9) and the Rule of Disjunctive Syllogism
(11) Steps (5), (10) and the Rule of Conjunction
(12) Step (11) and the Method of Proof by Contradiction

(b)

(1)	$p \rightarrow q$	Premise
(2)	$\neg q \rightarrow \neg p$	Step (1) and $(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p)$
(3)	$p \vee r$	Premise
(4)	$\neg p \rightarrow r$	Step (3) and $(p \vee r) \Longleftrightarrow (\neg p \rightarrow r)$
(5)	$\neg q \rightarrow r$	Steps (2), (4) and the Law of the Syllogism
(6)	$\neg r \vee s$	Premise
(7)	$r \rightarrow s$	Step (6) and $(\neg r \vee s) \Longleftrightarrow (r \rightarrow s)$
(8) $\therefore$	$\neg q \rightarrow s$	Steps (5), (7) and the Law of the Syllogism

(c)

(1)	$\neg p \leftrightarrow q$	Premise
(2)	$(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Longleftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$

(3)	$\neg p \rightarrow q$	Step (2) and the Rule of Conjunctive Simplification
(4)	$q \rightarrow r$	Premise
(5)	$\neg p \rightarrow r$	Steps (3), (4) and the Law of the Syllogism
(6)	$\neg r$	Premise
(7)	$\therefore p$	Steps (5), (6) and Modus Tollens.

10. (a)

(1)	$p \wedge \neg q$	Premise
(2)	p	Step (1) and the Rule of Conjunctive Simplification
(3)	r	Premise
(4)	$p \wedge r$	Steps (2), (3) and the Rule of Conjunction
(5)	$\therefore (p \wedge r) \vee q$	Step (4) and the Rule of Disjunctive Amplification

(b)

(1)	$p,\ p \rightarrow q$	Premises
(2)	q	Step (1) and the Rule of Detachment
(3)	$\neg q \vee r$	Premise
(4)	$q \rightarrow r$	Step (3) and $\neg q \vee r \Longleftrightarrow (q \rightarrow r)$
(5)	$\therefore r$	Steps (2), (4) and the Rule of Detachment

(c)

(1)	$p \rightarrow q,\ \neg q$	Premises
(2)	$\neg p$	Step (1) and Modus Tollens
(3)	$\neg r$	Premise
(4)	$\neg p \wedge \neg r$	Steps (2), (3) and the Rule of Conjunction
(5)	$\therefore \neg(p \vee r)$	Step (4) and DeMorgan's Laws

(d)

(1)	$r,\ r \rightarrow \neg q$	Premises
(2)	$\neg q$	Step (1) and the Rule of Detachment
(3)	$p \rightarrow q$	Premise
(4)	$\therefore \neg p$	Steps (2), (3) and Modus Tollens

(e)

(1)	p	Premise
(2)	$\neg q \rightarrow \neg p$	Premise
(3)	$p \rightarrow q$	Step (2) and $(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p)$
(4)	q	Steps (1), (3) and the Rule of Detachment
(5)	$p \wedge q$	Steps (1), (4) and the Rule of Conjunction
(6)	$p \rightarrow (q \rightarrow r)$	Premise
(7)	$(p \wedge q) \rightarrow r$	Step (6), and $[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \wedge q) \rightarrow r]$
(8)	$\therefore r$	Steps (5), (7) and the Rule of Detachment

(f)

(1)	$p \wedge q$	Premise
(2)	p	Step (1) and the Rule of Conjunctive Simplification
(3)	$p \rightarrow (r \wedge q)$	Premise
(4)	$r \wedge q$	Steps (2), (3) and the Rule of Detachment
(5)	r	Step (4) and the Rule of Conjunctive Simplification
(6)	$r \rightarrow (s \vee t)$	Premise
(7)	$s \vee t$	Steps (5), (6) and the Rule of Detachment
(8)	$\neg s$	Premise
(9)	$\therefore t$	Steps (7), (8) and the Rule of Disjunctive Syllogism

(g)

(1)	$\neg s,\ p \vee s$	Premises
(2)	p	Step (1) and the Rule of Disjunctive Syllogism
(3)	$p \rightarrow (q \rightarrow r)$	Premise
(4)	$q \rightarrow r$	Steps (2), (3) and the Rule of Detachment
(5)	$t \rightarrow q$	Premise
(6)	$t \rightarrow r$	Steps (4), (5) and the Law of the Syllogism
(7)	$\therefore \neg r \rightarrow \neg t$	Step (6) and $(t \rightarrow r) \Longleftrightarrow (\neg r \rightarrow \neg t)$

(h)

(1)	$\neg p \vee r$	Premise
(2)	$p \rightarrow r$	Step (1) and $(p \rightarrow r) \Leftrightarrow (\neg p \vee r)$
(3)	$\neg r$	Premise
(4)	$\neg p$	Steps (2), (3) and Modus Tollens
(5)	$p \vee q$	Premise
(6)	$\neg p \rightarrow q$	Step (5) and $(p \vee q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (\neg p \rightarrow q)$
(7)	$\therefore q$	Steps (4), (6) and Modus Ponens

11. (a) $p:1 \quad q:0 \quad r:1$
(b) $p:0 \quad q:0 \quad r:0$ or 1
$p:0 \quad q:1 \quad r:1$
(c) $p,q,r:1 \quad s:0$
(d) $p,q,r:1 \quad s:0$

12. a) p: Rochelle gets the supervisor's position.
q: Rochelle works hard.
r: Rochelle gets a raise.
s: Rochelle buys a new car.

$$\begin{array}{l} (p \wedge q) \rightarrow r \\ r \rightarrow s \\ \underline{\neg s \qquad\qquad} \\ \therefore \neg p \vee \neg q \end{array}$$

(1)	$\neg s$	Premise
(2)	$r \rightarrow s$	Premise
(3)	$\neg r$	Steps (1), (2) and Modus Tollens
(4)	$(p \wedge q) \rightarrow r$	Premise
(5)	$\neg(p \wedge q)$	Steps (3), (4) and Modus Tollens
(6)	$\therefore \neg p \vee \neg q$	Step (5) and $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$.

b) p: Dominic goes to the racetrack.
q: Helen gets mad.
r: Ralph plays cards all night.
s: Carmela gets mad.
t: Veronica is notified.

$$\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ (q \vee s) \rightarrow t \\ \underline{\neg t \qquad\qquad} \\ \therefore \neg p \wedge \neg r \end{array}$$

(1)	$\neg t$	Premise
(2)	$(q \vee s) \rightarrow t$	Premise
(3)	$\neg(q \vee s)$	Steps (1), (2) and Modus Tollens
(4)	$\neg q \wedge \neg s$	Step (3) and $\neg(q \vee s) \Longleftrightarrow \neg q \wedge \neg s$
(5)	$\neg q$	Step (4) and the Rule of Conjunctive Simplification
(6)	$p \rightarrow q$	Premise
(7)	$\neg p$	Steps (5), (6) and Modus Tollens
(8)	$\neg s$	Step (4) and the Rule of Conjunctive Simplification

(9) $r \rightarrow s$ Premise

(10) $\neg r$ Steps (8), (9) and Modus Tollens

(11) $\therefore \neg p \wedge \neg r$ Steps (7), (10) and the Rule of Conjunction

c) p: There is a chance of rain.
q: Lois' red head scarf is missing.
r: Lois does not mow her lawn.
s: The temperature is over 80^o F.

$$\begin{array}{l} (p \vee q) \rightarrow r \\ s \rightarrow \neg p \\ s \wedge \neg q \\ \hline \therefore \neg r \end{array}$$

The following truth value assignments provide a counterexample to the validity of this argument:
$p : 0;\ q : 0;\ r : 1;\ s : 1$

13. (a)

p	q	r	$p \vee q$	$\neg p \vee r$	t: $(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$t \rightarrow (q \vee r)$
0	0	0	0	1	0	0	1
0	0	1	0	1	0	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	1	1
1	1	1	1	1	1	1	1

From the last column of the truth table it follows that $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology.

Alternately we can try to see if there are truth values that can be assigned to p, q, and r so that $(q \vee r)$ has truth value 0 while $(p \vee q)$, $(\neg p \vee r)$ both have truth value 1.

For $(q \vee r)$ to have truth value 0, it follows that $q : 0$ and $r : 0$. Consequently, for $(p \vee q)$ to have truth value 1, we have $p : 1$ since $q : 0$. Likewise, with $r : 0$ it follows that $\neg p : 1$ if $(\neg p \vee) r$ has truth value 1. But we cannot have $p : 1$ and $\neg p : 1$. So whenever $(p \vee q)$, $(\neg p \vee r)$ have truth value 1, we have $(q \vee r)$ with truth value 1 and it follows that $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology.

Finally we can also argue as follows:

Steps	Reasons
1. $p \vee q$	1. Premise
2. $q \vee p$	2. Step (1) and the Commutative Law of $\vee$
3. $\neg(\neg q) \vee p$	3. Step (2) and the Law of Double Negation
4. $\neg q \rightarrow p$	4. Step (3), $\neg q \rightarrow p \Leftrightarrow \neg(\neg q) \vee p$
5. $\neg p \vee r$	5. Premise
6. $p \rightarrow r$	6. Step (5), $p \rightarrow r \Leftrightarrow \neg p \vee r$
7. $\neg q \rightarrow r$	7. Steps (4), (6), and the Law of the Syllogism
8. $\therefore q \vee r$	8. Step (7), $\neg q \rightarrow r \Leftrightarrow q \vee r$

(b)

(i) Steps	Reasons
1. $p \vee (q \vee r)$	1. Premise
2. $(p \vee q) \wedge (p \vee r)$	2. Step (1) and the Distribution Law of $\vee$ over $\wedge$
3. $p \vee r$	3. Step (2) and the Rule of Conjunctive Simplification
4. $p \rightarrow s$	4. Premise
5. $\neg p \vee s$	5. Step (4), $p \rightarrow s \Leftrightarrow \neg p \vee s$
6. $\therefore r \vee s$	6. Steps (3), (5), the Rule of Conjunction, and Resolution

(ii) Steps	Reasons
1. $p \leftrightarrow q$	1. Premise
2. $(p \rightarrow q) \wedge (q \rightarrow p)$	2. $(p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$
3. $p \rightarrow q$	3. Step (2) and the rule of Conjunctive Simplification
4. $\neg p \vee q$	4. Step (3), $p \rightarrow q \Leftrightarrow \neg p \vee q$
5. p	5. Premise
6. $p \vee q$	7. Step (5) and the Rule of Disjunctive Amplification
7. $[(p \vee q) \wedge (\neg p \vee q)]$	7. Steps (6), (4), and the Rule of Conjunction
8. $q \vee q$	8. Step (7) and Resolution
9. $\therefore q$	9. Step (8) and the Idempotent Law of $\vee$.

(iii) Steps	Reasons
1. $p \vee q$	1. Premise
2. $p \rightarrow r$	2. Premise
3. $\neg p \vee r$	3. Step (2), $p \rightarrow r \Leftrightarrow \neg p \vee r$
4. $[(p \vee q) \wedge (\neg p \vee r)]$	4. Steps (1), (3), and the Rule of Conjunction
5. $q \vee r$	5. Step (4) and Resolution
6. $r \rightarrow s$	6. Premise
7. $\neg r \vee s$	7. Step (6), $r \rightarrow s \Leftrightarrow \neg r \vee s$
8. $[(r \vee q) \wedge (\neg r \vee s)]$	8. Steps (5), (7), the Commutative Law of $\vee$, and the Rule of Conjunction
9. $\therefore q \vee s$	9. Step (8) and Resolution

(iv)

	Steps		Reasons
1.	$\neg p \vee q \vee r$	1.	Premise
2.	$q \vee (\neg p \vee r)$	2.	Step (1) and the Commutative and Associative Laws of $\vee$
3.	$\neg q$	3.	Premise
4.	$\neg q \vee (\neg p \vee r)$	4.	Step (3) and the Rule of Disjunctive Amplification
5.	$[[q \vee (\neg \vee r)] \wedge [\neg q \vee (\neg p \vee r)]]$	5.	Steps (2), (4), and the Rule of Conjunction
6.	$(\neg p \vee r)$	6.	Step (5), Resolution, and the Idempotent Law of $\wedge$
7.	$\neg r$	7.	Premise
8.	$\neg r \vee \neg p$	8.	Step (7) and the Rule of Disjunctive Amplification
9.	$[(r \vee \neg p) \wedge (\neg r \vee \neg p)]$	9.	Steps (6), (8), the Commutative Law of $\vee$, and the Rule of Conjunction
10.	$\therefore \neg p$	10.	Step (9), Resolution, and the Idempotent Law of $\vee$

(v) Steps	Reasons
1. $\neg p \vee s$	1. Premise
2. $p \vee q \vee t$	2. Premise
3. $p \vee (q \vee t)$	3. Step (2) and the Associative Law of $\vee$
4. $[[p \vee (q \vee t)] \wedge (\neg p \vee s)]$	4. Steps (3), (1), and the Rule of Conjunction
5. $(q \vee t) \vee s$	5. Step (4) and Resolution (and the First Substitution Rule)
6. $q \vee (t \vee s)$	6. Step (5) and the Associative Law of $\vee$
7. $\neg q \vee r$	7. Premise
8. $[[q \vee (t \vee s)]$	8. Steps (6), (7), and the Rule of Conjunction
9. $(t \vee s) \vee r$	9. Step (8) and Resolution (and the First Substitution Rule)
10. $t \vee (s \vee r)$	10. Step (9) and the Associative Law of $\vee$
11. $\neg t \vee (s \wedge r)$	11. Premise
12. $(\neg t \vee s) \wedge (\neg t \vee r)$	12. Step (11) and the Distributive Law of $\vee$ over $\wedge$
13. $\neg t \vee s$	13. Step (12) and the Rule of Conjunctive Simplification
14. $[[t \vee (s \vee r)] \wedge (\neg t \vee s)]$	14. Steps (10), (13), and the Rule of Conjunction
15. $(s \vee r) \vee s$	15. Step (14) and Resolution (and the First Substitution Rule)
16. $\therefore r \vee s$	16. Step (15) and the Commutative, Associative, and Idempotent Laws of $\vee$

(c) Consider the following assignments.

p: Jonathan has his driver's license.
q: Jonathan's new car is out of gas.
r: Jonathan likes to drive his new car.

Then the given argument can be written in symbolic form as

$$\begin{array}{l} \neg p \vee q \\ p \vee \neg r \\ \underline{\neg q \vee \neg r} \\ \therefore \neg r \end{array}$$