**FAST National University of Computer and Emerging Sciences, Karachi.**

Discrete Structures (CS 211)

MID I Examinations, Fall 2016

Time Allowed: 60 Minutes — Total Points: 38

Instructor: Nouman M Durrani and M Hussain Mughal — Dated: 24 September, 2016

Note: Answer all questions Q(x) and its subsequent parts P(x) in the given order O(x), otherwise marks will be deducted D(x). Q(y)'s marks deducted:

$$\forall x\, \exists y\, ((Q(x) \wedge P(x) \wedge O(x) \rightarrow D(x)) \wedge D(y))$$

**Propositional Logic and Equivalences** [10 points]

Q No. 1 (i) Let $p$, $q$, and $r$ be the propositions

$p$ :You get an A on the final exam.

$q$ :You do every exercise in this book.

$r$ :You get an A in this class.

Write these propositions $p$, $q$, and $r$ using logical connectives.

You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final. **$r \leftrightarrow (q \vee p)$**

[1 point]

(ii) An algorithm contains the following line: [1 point]

If not($x > 5$ and $x \leq 10$) then ...

How could this be written more simply?

Apply the first de Morgan's law: $\neg[(x > 5) \wedge (x \leq 10)]$ is equivalent to $\neg(x > 5) \vee \neg(x \leq 10)$, which in turn is equivalent to $(x \leq 5) \vee (x > 10)$. The line of the algorithm can therefore be written:

If $x \leq 5$ or $x > 10$ then ...

(iii) Write each of these statements in the form "if $p$, then $q$" or "$p$ if and only if $q$" in English. [2 points]

(a) If you keep your textbook, it will be a useful reference in your future courses.

(b) If you read the newspaper every day, you will be informed, and conversely.

(iv) Construct a truth table for the following compound proposition. [2 points]

$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

(v) Use the laws of logic to simplify the expression: [2 points]

$p \vee \neg(\neg p \rightarrow q)$

| | |
|---|---|
| $p \lor \neg(\neg p \to q) \equiv p \lor \neg(\neg\neg p \lor q)$ | implication law (with $\neg p$ in place of $p$) |
| $\equiv p \lor \neg(p \lor q)$ | double negation law |
| $\equiv p \lor (\neg p \land \neg q)$ | second de Morgan's law |
| $\equiv (p \lor \neg p) \land (p \lor \neg q)$ | second distributive law (with $\neg p$ and $\neg q$ in place of $q$ and $r$ respectively) |
| $\equiv \mathrm{T} \land (p \lor \neg q)$ | second inverse law |
| $\equiv (p \lor \neg q) \land \mathrm{T}$ | first commutative law (with T and $(p \lor \neg q)$ in place of $p$ and $q$ respectively) |
| $\equiv p \lor \neg q$ | first identity law (with $(p \lor \neg q)$ in place of $p$) |

*(vi)* Use the laws of logic to show that $[(p \to q) \land \neg q] \to \neg p$ is a tautology. [2 points]

| | |
|---|---|
| $[(p \to q) \land \neg q] \to \neg p \equiv \neg[(\neg p \lor q) \land \neg q] \lor \neg p$ | implication law (twice) |
| $\equiv \neg[\neg q \land (\neg p \lor q)] \lor \neg p$ | first commutative law |
| $\equiv \neg[(\neg q \land \neg p) \lor (\neg q \land q)] \lor \neg p$ | first distributive law |
| $\equiv \neg[(\neg q \land \neg p) \lor (q \land \neg q)] \lor \neg p$ | first commutative law |
| $\equiv \neg[(\neg q \land \neg p) \lor \mathrm{F}] \lor \neg p$ | first inverse law |
| $\equiv \neg(\neg q \land \neg p) \lor \neg p$ | second identity law |
| $\equiv (\neg\neg q \lor \neg\neg p) \lor \neg p$ | first de Morgan's law |
| $\equiv (q \lor p) \lor \neg p$ | double negation law (twice) |
| $\equiv q \lor (p \lor \neg p)$ | second associative law |
| $\equiv q \lor \mathrm{T}$ | second inverse law |
| $\equiv \mathrm{T}$ | second annihilation law |

**Predicate, Quantifiers and Rules of Inference** [12 points]

Q. No. 2 (i) What are the truth values of $\forall x P(x)$ *and* $\exists x P(x)$, where $P(x)$ is the statement "$x^2 < 10$". The domain consists of the positive integers not exceeding 4? [2 points]

- $\forall x\, P(x)$ ?

$$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \equiv \mathbf{F}$$

- $\exists x\, P(x)$ ?

$$\exists x\, P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \equiv \mathbf{T}$$

(ii) Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

[2 points]

as a logical expression

Let

- F(x): x is female
- P(x): x is a parent
- M(x,y): x is y's mother

At least one y

$$(F(x) \wedge P(x)) \rightarrow M(x, y)$$

All x

The domain is the set of all people

$$\forall x\, ( (F(x) \wedge P(x)) \rightarrow \exists y\, M(x, y) ), \text{ or}$$

$$\forall x\, \exists y\, ( (F(x) \wedge P(x)) \rightarrow M(x, y) )$$

(iii) Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

[2 points]

Hypothesis:

$\neg p \wedge q$

$r \rightarrow p$

$\neg r \rightarrow s$

$s \rightarrow t$

Conclusion:

$t$

| Step | | Reason |
|---|---|---|
| 1. | $\neg p \wedge q$ | Premise |
| 2. | $\neg p$ | Simplification using (1) |
| 3. | $r \rightarrow p$ | Premise |
| 4. | $\neg r$ | Modus tollens using (2) and (3) |
| 5. | $\neg r \rightarrow s$ | Premise |
| 6. | $s$ | Modus ponens using (4) and (5) |
| 7. | $s \rightarrow t$ | Premise |
| 8. | $t$ | Modus ponens using (6) and (7) |

(iv) For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

"I am either clever or lucky." "I am not lucky." "If I am lucky, then I will solve all the questions in this paper." [2 points]

$C \vee L$
$\neg L$
$L \to T$

**C  Disjunctive syllogism**

**$C \wedge L \to T$**

(v) For each of these arguments, explain which rules of inference are used for each step.

"Marry, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job." [2 points]

$C(x)$ = "$x$ is in this class"
$J(x)$ = "$x$ knows how to write programs in JAVA"
$H(x)$ = "$x$ can get a high paying job"

| | |
|---|---|
| Premise 1 | $C(\text{Doug})$ |
| Premise 2 | $J(\text{Doug})$ |
| Premise 3 | $\forall x(J(x) \to H(x))$ |
| Conclude | $\exists x(C(x) \wedge H(x))$ |

| **Step** | | **Reason** |
|---|---|---|
| 1 | $\forall x(J(x) \to H(x))$ | Premise 3 |
| 2 | $J(\text{Doug}) \to H(\text{Doug})$ | Universal Instantiation from (1) |
| 3 | $J(\text{Doug})$ | Premise 2 |
| 4 | $H(\text{Doug})$ | Modus Ponens from (2) and (3) |
| 5 | $C(\text{Doug})$ | Premise 1 |
| 6 | $C(\text{Doug}) \wedge H(\text{Doug})$ | Conjunction from (4) and (5) |
| $\therefore$ | $\exists x(C(x) \wedge H(x))$ | Existential generalization from (6) |

(vi) Determine whether the following argument is *valid or not:*

"The file is either a binary file or a text file. If it is a binary file, then my program won't accept it. My program will accept the file. Therefore, the file is a text file." [2 points]

Let $p$ denote the proposition 'The file is a binary file', let $q$ denote 'The file is a text file', and let $r$ denote 'My program will accept the file'. Then:

$$P_1 \equiv p \vee q$$
$$P_2 \equiv p \to \neg r$$
$$P_3 \equiv r$$
$$Q \equiv q$$

The argument now takes the form:

$$[(p \vee q) \wedge (p \to \neg r) \wedge r] \to q$$

We can find out whether this expression is a tautology either by constructing a truth table or by trying to simplify it using the laws of logic. Since a truth table would require eight rows and a large number of columns, and would be fairly tedious to construct, we try the latter approach.

$$\begin{aligned}
[(p \vee q) \wedge (p \to \neg r) \wedge r] \to q &\equiv \neg[(p \vee q) \wedge (\neg p \vee \neg r) \wedge r] \vee q \\
&\equiv \neg[(p \vee q) \wedge r \wedge (\neg p \vee \neg r)] \vee q \\
&\equiv \neg\{(p \vee q) \wedge [(r \wedge \neg p) \vee (r \wedge \neg r)]\} \vee q \\
&\equiv \neg\{(p \vee q) \wedge [(r \wedge \neg p) \vee \mathrm{F}]\} \vee q \\
&\equiv \neg[(p \vee q) \wedge r \wedge \neg p] \vee q \\
&\equiv \neg[\neg p \wedge (p \vee q) \wedge r] \vee q \\
&\equiv \neg\{[(\neg p \wedge p) \vee (\neg p \wedge q)] \wedge r\} \vee q
\end{aligned}$$

Finally we get =T

Rules of inference can also be used.

**Sets, Functions and Relations** [16 points]

Q. No. 3 (i) Use set builder notation and logical equivalences to establish the first *De Morgan law*:

$\overline{A \cap B} = \overline{A} \cup \overline{B}$. [2 points]

$$\overline{A \cap B}$$

$$= \{ x \mid x \notin (A \cap B) \}$$

$$= \{ x \mid \neg((x \in A) \wedge (x \in B)) \}$$

$$= \{ x \mid \neg(x \in A) \vee \neg(x \in B)) \}$$

$$= \{ x \mid (x \notin A) \vee (x \notin B) \}$$

$$= \{ x \mid \overline{(x \in A)} \vee \overline{(x \in B)} \}$$

$$= \overline{A} \cup \overline{B}$$

(ii) (a) For each of the following functions, determine whether the function is *onto* and whether it is *one-to-one*: $f : \mathrm{R} \to \mathrm{R}$, $f(x) = 2x + 1$ [2 points]

If $y \in \mathrm{R}$, then we can set $y = 2x + 1$ and solve for $x$, obtaining $x = (y - 1)/2$. Therefore every element $y$ of the codomain is the image of the element $(y - 1)/2$ of the domain. Thus $f$ is onto.

We show that $f$ is one-to-one by proving the contrapositive of the definition. Suppose $f(x_1) = f(x_2)$. Then:

$$2x_1 + 1 = 2x_2 + 1$$

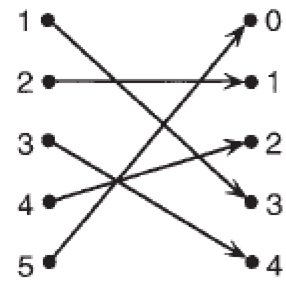
Hence $$2x_1 = 2x_2$$

so $$x_1 = x_2$$

Therefore $f$ is one-to-one.

(b) A *function f:* $\{1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4\}$ is defined by the rule: [2 points]

$f(n)$ is the remainder after $3n$ is divided by 5. Also draw an arrow diagram for this *function*.




One-to-one and onto.

(iii) Let $f$ be the function from $Z \rightarrow Z$ with $f(x) = x^2$ is not invertible. Why? Change the domain and codomain so that the function f(x) becomes *invertible*. [2 points]

Because $f(-2) = f(2) = 4$, $f$ is not one-to-one.

If an inverse function were defined, it would have to assign two elements to 4

Hence, $f$ is not invertible

Show that if we restrict the function $f(x) = x^2$ to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then $f$ is invertible

**One-to-One Function Proof**

- If $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$
- This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$
- Because both x and y are nonnegative, we must have $x = y$. It is **one-to-one**

**Onto Function Proof**

- The codomain is the set of all nonnegative real numbers, so each nonnegative real number has a square root. It is onto

Therefore, $f$ **is invertible**

(iv) Let $f: R \rightarrow R$, $f(x) = x^2$ and $g: R \rightarrow R$, $g(x) = 3x - 1$. Find $f \circ g$ and $g \circ f$. [2 points]

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = 3x - 1$. Find $f \circ g$ and $g \circ f$.

Note firstly that the composite function $f \circ g$ exists because the codomain of $g$ equals the domain of $f$. Similarly, $g \circ f$ exists because the codomain of $f$ equals the domain of $g$.

The function $f \circ g$ is found as follows:

$$\begin{aligned} f \circ g : \mathbb{R} \to \mathbb{R}, (f \circ g)(x) &= f(g(x)) \\ &= f(3x - 1) \\ &= (3x-1)^2 \end{aligned}$$

where the last line is obtained by substituting $3x - 1$ in place of $x$ in the formula for $f(x)$.

The function $g \circ f$ is obtained in a similar manner:

$$\begin{aligned} g \circ f : \mathbb{R} \to \mathbb{R}, (g \circ f)(x) &= g(f(x)) \\ &= g(x^2) \\ &= 3x^2 - 1 \end{aligned}$$

(v) Prove that if $x$ is a real number, then $[\ 2x\ ] = [\ x\ ] + [\ x + 1/2\ ]$. Assume that the *fractional part* $\varepsilon \geq 1/2$. [2 points]

# Example 1

$\boxed{x = n + \varepsilon}$

- **Second case: $1/2 \leq \varepsilon < 1$**

$\boxed{\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor}$

$$\begin{aligned} \text{LHS} &= \lfloor 2x \rfloor \\ &= \lfloor 2n + 2\varepsilon \rfloor \\ &= \lfloor 2n + 1 + 2\varepsilon - 1 \rfloor \\ &= 2n+1 \quad \text{because } (0 \leq 2\varepsilon - 1 < 1) \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \lfloor x \rfloor + \lfloor x + 1/2 \rfloor \\ &= \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor \\ &= \lfloor n + \varepsilon \rfloor + \lfloor n + 1 + \varepsilon - 1/2 \rfloor \\ &= n + n+1 \quad \text{because } (1/2 \leq \varepsilon < 1) \\ & \qquad\qquad\qquad (0 \leq \varepsilon - 1/2 < 1/2) \\ &= 2n+1 \end{aligned}$$

LHS = RHS

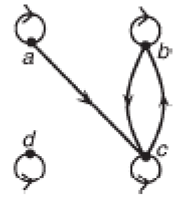
(vi) Let $R$ be the *relation* on {a, b, c, d} defined by the following *matrix*: [1+ 3 points]

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} T & F & T & F \\ F & T & T & F \\ F & T & T & F \\ F & F & F & T \end{bmatrix} \end{array}$$

(a) Draw the graphical representation of *R*.

(b) State, giving reasons, whether *R* is reflexive, symmetric or transitive.

(a)




(b) Reflexive: the matrix has T along its principal diagonal. Not symmetric: a *R* c but c is not related to a. Not transitive: a *R* c and c *R* b but a is not related to b.