Nested Quantified Expressions: Order is important

Order of quantifier is extremely important

$$\forall x \exists y\, P(x, y) \quad \textbf{is not the same as} \quad \exists y \forall x\, P(x, y)$$

$$Q(x, y, z) : x \text{ is an instructor of course } y \text{ in university } z$$

- $\forall z\ \forall y\ \exists x\ Q(x, y, z)$: In every uni. for every course there is an instructor
- $\forall z\ \exists x\ \forall y\ Q(x, y, z)$: In every uni. there is an instructor for all courses
- $\exists x\ \forall z\ \forall y\ Q(x, y, z)$: There is an instructor for every course in every uni.

- **However,** In some cases, the order of universal and existential quantifiers can be swapped without changing the meaning of a propositional function.
  This happens when the quantifiers are independent of each other.
  Here are a few examples

Example 1: Properties of Real Numbers

Let the domain be the set of real numbers $\mathbb{R}$.

Propositional function: $P(x, y) : x + y = y + x$

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y)$
   - "For every real number x, there exists a real number y such that $x + y = y + x$."
2. $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, P(x, y)$
   - "There exists a real number y such that for every real number x, $x + y = y + x$."

Since the commutative property of addition holds for all real numbers, both statements are true.

Therefore, the order of quantifiers does not matter here.

Example 2: Multiplicative Identity

Let the domain be $\mathbb{R}$.

Propositional function: $P(x, y) : x \cdot y = x$

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y)$
   - "For every real number x, there exists a real number y such that $x \cdot y = x$."
2. $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, P(x, y)$
   - "There exists a real number y such that for every real number x, $x \cdot y = x$."

Since $y = 1$ works as the multiplicative identity for all x, both statements are true, and the order of quantifiers can be switched without affecting the meaning.