Set Theory

- Sets: Definition, Universal Set, Complement, Cardinality
- Subset and Power Set
- Sets Operations
- Set Equality
- Characteristic Vectors: Sets as Bit-Vectors
- Multisets

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Set

A set is an unordered collection of objects

- A : $\{1, 2, a, b, Fred, LUMS\}$
- $\mathbb{N}$: $\{0, 1, 2, 3 \ldots\}$
- B : the set of all professors in LUMS
- B : $\{x \,|\, x \text{ is a professor in LUMS }\}$

Order of elements is not significant

- $\{1, 2, 3\}$ is the same as $\{2, 3, 1\}$

Repetition does not count

- $\{1, 2, 2, 2, 3\}$ is the same as $\{1, 2, 3\}$

Different from arrays in C++/Java (order, same type)

Sets: Notation

Notation:

- Names of sets usually denoted by upper case letters
- Objects in a set are called it's elements/members
- $x \in A$: x is an element of A
- $A \ni x$: A contains x

Sets: Description

- Sets can be described by listing it's elements
  - TAs $:= \{20102222, 20103333, 20104444, 20105555\}$
  - $\mathbb{N} := \{0, 1, 2, 3 \ldots\}$
- Sets can be described by an English phrase
  - Set of students in CS-210
  - Set of all professors in LUMS
- Sets can be described by providing a membership predicate
  - Objects for which the predicate is true will be members of the set
  - $B := \{x \mid x \text{ is a professor in LUMS}\}$
  - $C := \{x \in \mathbb{N} \mid x > 10\}$

Sets: Description

- Sets could have many different equivalent descriptions
- Follows from logical equivalence of membership predicates

ICP 4-1 List 5 elements of each of the following sets

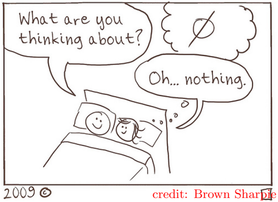
1. $A = \{x \mid x \text{ is even and a perfect square}\}$
2. $B = \{x \mid x + 1 \text{ is a multiple of } 4\}$
3. $C = \{x \text{ rational number} \mid x^2 < 2\}$
4. $X = \{x^2 \mid x \text{ is even}\}$
5. $Y = \{4x - 1 \mid x \in \mathbb{Z}\}$
6. $Z = \{\sqrt{x} \mid x < 2\}$

Standard Numerical Sets

- $\mathbb{N} := \{0, 1, 2, 3 \ldots\}$ ▷ Natural numbers
- $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ Integers
- $\mathbb{Z}^+ := \{\ldots, 1, 2, 3, \ldots\}$ ▷ Positive integers
- $\mathbb{Q} := \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ Rational numbers
- $\mathbb{R} :=$ the set of **Real numbers**
- **int** $:=$ set of integers that can be expressed in 32 bits
- **double** $:=$ the set of real numbers that can be expressed in 64 bits

Some other sets

- Empty Set
  - $A = \emptyset = \{\ \}$
- Family/Collection of sets: Set of sets
  - $A = \Big\{\{x, y\}, \{y, z\}, \{z\}\Big\}$
- Set containing sets
  - $A = \Big\{\{p, q\}, x, \{x\}, \{y\}\Big\}$



credit: Brown Sharpie

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

$\boxed{\textbf{ICP 4-2}}$ $A = \{3, 1, -3, 9\}$ $\qquad$ $B = \{-3, 1, 3, 9\}$

Is $A = B$? why?

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

ICP 4-3 $A = \{p, q, r, s\}$ $\qquad B = \{r, q, s\}$

Is $A = B$? why?

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

ICP 4-3 $A = \{p, q, r\}$ $\qquad B = \{p, r, q, p\}$

Is $A = B$? why?

Set Complement

The complement of a set A contains those elements that are not in A

- $\mathbb{Q} = \left\{ {}^{p}/_{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}$ (**rationals**)
- **Irrational numbers**: non-rational number
- $\left\{ \sqrt{2},\ \pi,\ \ e,\ \ \text{cat, dog, calculus} \ldots \right\}$
- Universal Set ▷ recall Universe of Discourse

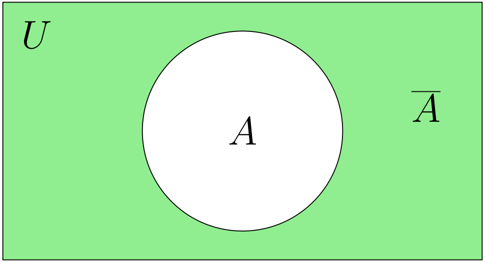
The complement of a set A contains those elements (of the universal set) that are not in A

Set Complement

The complement of a set A consists of all elements of U that are not in A

Denoted by $\overline{A}$

$$\overline{A} := \{x \in U \mid x \notin A\}$$



$\overline{A}$

Puzzle

In a town, there is a male barber who shaves all the men and only those men who do not shave themselves

Does the barber shave himself?

- Let $M = \{\text{men who shave themselves}\}$
- M is the set of people that the barber b does not shave
- If the barber b shaves himself, then $b \in M$
- Then barber must not shave himself, i.e. $b \notin M$
- But barber shaves all those not in M, so $b \in M$
- $b \in M$ is neither true nor false

Russell's Paradox

Let S be the set that contains all sets, which do not contain themselves

Can this set exist?

- Let say S exists. Then either $S \in S$ or $S \notin S$
- If $S \in S$, since all elements of S do not contain themselves, thus S does not contain S, i.e. $S \notin S$
- If $S \notin S$, since S contains all those sets that do not contain themselves, so S should contain S, i.e. $S \in S$
- In both cases the assumption is contradicted

Cardinality of finite sets

For a finite set A it's cardinality is the number of distinct elements in A

Denoted by $|A|$

$A = \{\text{Even integers among the first 100 positive integers}\}$

ICP 4-4 $|A| = ?$

$B = \{\text{Positive factors of 16}\}$

ICP 4-5 $|B| = ?$

Sets Summary

- A set is an ordered collection of objects
- Order and repetition of objects do not matter
- Sets can be described in various ways
- Empty set is a well-defined set with zero objects
- Two sets are equal if and only if they have the same elements
- $\overline{A}$ is the collection of all objects in universal set that are not in A
- Cardinality of A is the number of distinct elements in A

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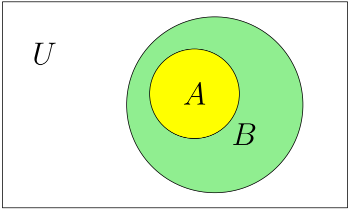
Subset

A is a subset of B if and only if every element of A is an element of B

Denoted by $A \subseteq B$

$$A \subseteq B \quad \text{means} \quad \forall x \, (x \in A \rightarrow x \in B)$$



$A \subseteq B$

When $A \subseteq B$, B is superset of A

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-6 $\mathbb{N} \subseteq \mathbb{Z}$

a) True

b) False

ICP 4-7 $\mathbb{Z} \subseteq \mathbb{Z}^+$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-8 $\mathbb{Z}^+ \subseteq \mathbb{N}$

a) True

b) False

ICP 4-9 $\mathbb{Q} \subseteq \mathbb{N}$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \dots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-10 $\mathbb{Z} \subseteq \mathbb{Q}$

a) True

b) False

ICP 4-11 $\mathbb{Z}^+ \subseteq \mathbb{Q}$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which one of the following is True/False?

ICP 4-12 $\mathbb{Z} \subseteq \mathbb{N}$

a) True

b) False

ICP 4-13 $\mathbb{Z}^+ \subseteq \mathbb{Z}$

a) True

b) False

Empty set is subset of every set

A is a subset of B if and only if every element of A is an element of B

$$A \subseteq B \quad \text{means} \quad \forall x \; (x \in A \rightarrow x \in B)$$

$$\forall A \quad \emptyset \subseteq A$$

Need to show that the following is true

$$\forall x \; (x \in \emptyset \rightarrow x \in A)$$

$$x \in \emptyset \text{ is always false } \; (\text{ for every } x)$$

thus

$$(x \in \emptyset \rightarrow x \in A) \text{ is always true}$$

Every set is subset of itself

A is a subset of B if and only if every element of A is an element of B

$$A \subseteq B \quad \text{means} \quad \forall x \, \big(x \in A \rightarrow x \in B\big)$$

$$\forall A \quad A \subseteq A$$

Need to show that the following is true

$$\forall x \, \big(x \in A \rightarrow x \in A\big)$$

$$\big(x \in A \rightarrow x \in A\big) \text{ is always true } \; (\text{ for every } x)$$

Proper Subset

A set A is called a proper subset of B if $A \subseteq B$ but $A \neq B$

Denoted by $A \subset B$ or $A \subsetneq B$

$$A \subset B \quad \text{means} \quad \forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$$

Set Equality $A = B$

Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$

- $A = B$ means $\forall x \, (x \in A \leftrightarrow x \in B)$ ▷ earlier definintion
- $A = B$ means $A \subseteq B$ AND $B \subseteq A$

$$A = B \text{ means } \underbrace{\forall x \, (x \in A \rightarrow x \in B)}_{A \subseteq B} \text{ AND } \underbrace{\forall x \, (x \in B \rightarrow x \in A)}_{B \subseteq A}$$

- Combining the two we get our old definition of $A = B$

Power set

The power set of a set A is the set of all subsets of A

Denoted by $\mathcal{P}(A)$

$A = \{p, q, r\}$

$$\mathcal{P}(A) = \Big\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\Big\}$$

- $\emptyset \notin A$ but $\emptyset \subseteq A$, thus $\emptyset \in \mathcal{P}(A)$
- $A \notin A$ but $A \subseteq A$, thus $A \in \mathcal{P}(A)$

Power set

Power set of the empty set, $\mathcal{P}(\emptyset)$

$$\mathcal{P}(\emptyset) \;=\; \Big\{\emptyset\Big\} \;=\; \Big\{\;\{\}\;\Big\}$$

Power set of the set containing $\emptyset$, $\mathcal{P}\Big(\{\emptyset\}\Big)$

$$\mathcal{P}\Big(\{\emptyset\}\Big) \;=\; \Big\{\emptyset, \{\emptyset\}\Big\}$$

Cardinality of Power set

If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

- $A = \{p, q, r\}$
- $\mathcal{P}(A) = \Big\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\Big\}$
- $|\mathcal{P}(A)| = 2^3 = 8$

ICP 4-14 Let $B = \{p\}$. What is $|\mathcal{P}(B)|$?

ICP 4-15 Let $C = \{\} = \emptyset$. What is $|\mathcal{P}(C)|$?

ICP 4-16 Let $D = \Big\{\{\}\Big\} = \{\emptyset\}$. What is $|\mathcal{P}(D)|$?

Subsets: Summary

- A is a subset of B if and only if every element of A is an element of B
- $A \subseteq B$, A is subset of B, B is superset of A
- Empty set is a subset of every set
- Every set is a subset of itself
- Power Set of A is the set of all subsets of A
- Cardinality of power set of A with $|A| = n$ is 2^n

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Set Operations

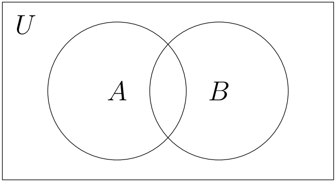
- Set operations (typically) take two sets and return another set
  - Set complement is a unary operation (takes one set)
  - Universal set is also involved in the background
- Set Algebra is built upon these set operations

Set Operations: Union

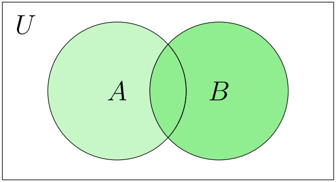
The union of two sets A and B is the set containing all elements that are in A or B or both

$$A \cup B \; = \; \Big\{ x \big| x \in A \vee x \in B \Big\}$$

$$\Big\{1, 3, 5\Big\} \cup \Big\{1, 5, 6, 7\Big\} \; = \; \Big\{1, 3, 5, 6, 7\Big\}$$



Two sets A and B



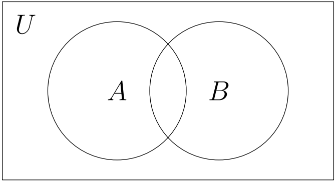
$A \cup B$

Set Operations: Intersection

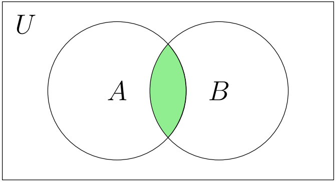
The intersection of two sets A and B is the set containing all elements that are both in A and B

$$A \cap B \;=\; \{x | x \in A \wedge x \in B\}$$

$$\{1,3,5\} \cap \{1,5,6,7\} \;=\; \{1,5\}$$



Two sets A and B



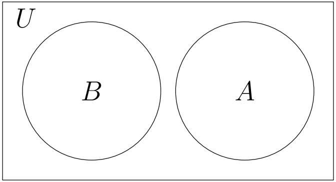
$A \cap B$

Disjoint Sets

Two sets A and B are disjoint if $A \cap B = \emptyset$

No elements in common. Logical expression!

$\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$



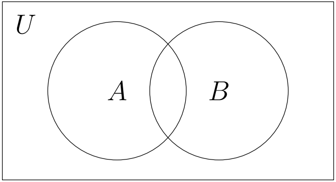
Disjoint sets A and B

Set Operations: Set Difference

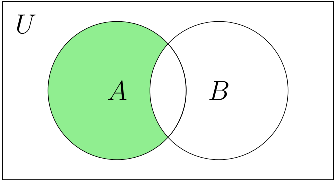
The difference of two sets A and B is the set containing those elements that are in A but not in B

$$A \setminus B = \{x | x \in A \wedge x \notin B\}$$

$\{1, 3, 5\} \setminus \{1, 5, 6, 7\} = \{3\}$



Two sets A and B



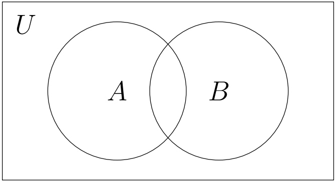
$A \setminus B$

Set Operations: Symmetric Difference

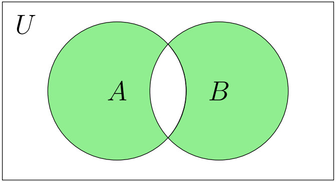
The symmetric difference of two sets A and B is the set containing those elements that are in exactly one of the two sets

$$A \oplus B \; = \; \{x | x \in A \oplus x \in B\}$$

$$\{1, 3, 5\} \oplus \{1, 5, 6, 7\} \; = \; \{3, 6, 7\}$$



Two sets A and B

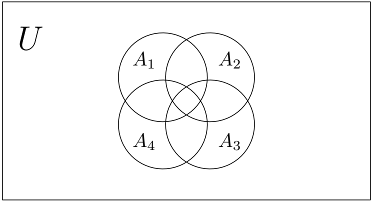


$A \oplus B$

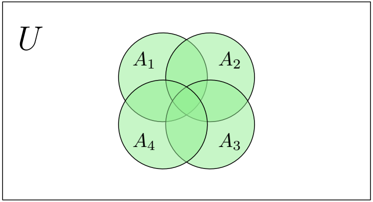
Generalized Union

The union of a collection of sets is the set containing those elements that are members of at least one set in the collection

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x \mid \exists\, i\ x \in A_i\}$$



n sets $A_1, A_2, \ldots, A_n$



$A_1 \cup A_2 \cup A_3 \cup A_4$

Generalized Union

The union of a collection of sets is the set containing those elements that are members of at least one set in the collection

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x | \exists\, i\; x \in A_i\}$$

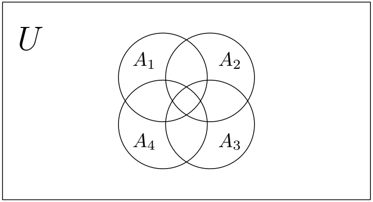
- Let $A_i = \{i, i+1, i+2, \ldots\}$
  - $A_1 = \{1, 2, 3, \ldots\}$
  - $A_2 = \{2, 3, 4, \ldots\}$
  - **ICP 4-17** $A_3 = ?$
  - **ICP 4-18** $A_4 = ?$

ICP 4-19 $\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \ldots\}$

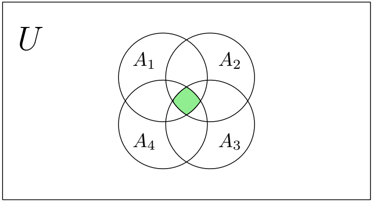
Generalized Intersection

The intersection of a collection of sets is the set containing those elements that are members of all the sets in the collection

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x | \; \forall \; i \; x \in A_i\}$$



n sets $A_1, A_2, \ldots, A_n$



$A_1 \cap A_2 \cap A_3 \cap A_4$

Generalized Intersection

The intersection of a collection of sets is the set containing those elements that are members of all the sets in the collection

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x \mid \forall\ i\ x \in A_i\}$$

- Let $A_i = \{i, i+1, i+2, \ldots\}$
  - $A_1 = \{1, 2, 3, \ldots\}$
  - $A_2 = \{2, 3, 4, \ldots\}$
  - **ICP 4-20** $A_5 = ?$
  - **ICP 4-21** $A_6 = ?$

ICP 4-22 $$\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \ldots\}$$

Set Operations

- Set Operation (Binary)
  - Union
  - Intersection
  - Difference
  - Symmetric Difference
- Generalized Union
- Generalized Intersection

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Set Equality using Identities

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

- To prove two sets A and B to be equal
- Start with one set (say A) and replace it with an equal set
- These established equalities between sets are called "set identities"
- Continue doing this until we get the set B

Set Identities

Identity	Name
$A \cup \emptyset = A$ $A \cap U = A$	Identity Laws
$A \cap \emptyset = \emptyset$ $A \cup U = U$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws

Set Identities

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws

Set Identities

Identity	Name
$\overline{(\overline{A})} = A$	Double Complement Law
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement Laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws

Set Identities: Demonstration

$U = \{1, 2, 3, 4, 5, 6\} \qquad A = \{2, 3, 5\} \qquad B = \{2, 3, 4\}$

- $\overline{B} = \{1, 5, 6\}$
- $\overline{A} = \{1, 4, 6\}$
- $\overline{(\overline{A})} = \{2, 3, 5\}$
- $A \cup \overline{A} = \{1, 2, 3, 4, 5, 6\}$
- $A \cap \overline{A} = \{\}$

Set Identities: Demonstration

$U = \{1,2,3,4,5,6\} \qquad A = \{2,3,5\} \qquad B = \{2,3,4\}$

- $\overline{B} = \{1,5,6\}$
- $\overline{A} = \{1,4,6\}$
- $\overline{(\overline{A})} = \{2,3,5\}$
- $A \cup \overline{A} = \{1,2,3,4,5,6\}$
- $A \cap \overline{A} = \{\}$
- $A \cap B = \{2,3\}$
- $A \cup B = \{2,3,4,5\}$
- $\overline{A \cap B} = \{1,4,5,6\}$
- $\overline{A} \cup \overline{B} = \{1,4,5,6\}$
- $\overline{A \cup B} = \{1,6\}$
- $\overline{A} \cap \overline{B} = \{1,6\}$

Set Identities: Demonstration

$U = \{1,2,3,4,5,6\} \qquad A = \{2,3,5\} \qquad B = \{2,3,4\}$

- $\overline{B} = \{1,5,6\}$
- $\overline{A} = \{1,4,6\}$
- $\overline{(\overline{A})} = \{2,3,5\}$
- $A \cup \overline{A} = \{1,2,3,4,5,6\}$
- $A \cap \overline{A} = \{\}$

- $A \cap B = \{2,3\}$
- $A \cup B = \{2,3,4,5\}$
- $\overline{A \cap B} = \{1,4,5,6\}$
- $\overline{A} \cup \overline{B} = \{1,4,5,6\}$
- $\overline{A \cup B} = \{1,6\}$
- $\overline{A} \cap \overline{B} = \{1,6\}$

- **ICP 4-23** $B \cap \overline{B} = ?$
- **ICP 4-24** $B \cup \overline{B} = ?$
- **ICP 4-25** $\overline{(\overline{B})} = ?$

Set Equalities using Identities

Show using set identities that

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$$

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$$LHS = \overline{A \cup B \cup C}$$

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$$\begin{aligned} LHS &= \overline{A \cup B \cup C} \\ &= \overline{A} \cap \overline{(B \cup C)} \qquad \text{DeMorgan's Law} \end{aligned}$$

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$$\begin{aligned} LHS &= \overline{A \cup B \cup C} \\ &= \overline{A} \cap \overline{(B \cup C)} && \text{DeMorgan's Law} \\ &= \overline{A} \cap (\overline{B} \cap \overline{C}) && \text{DeMorgan's Law} \end{aligned}$$

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Show using set identities that

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$$\begin{aligned} LHS &= \overline{A \cup B \cup C} && \\ &= \overline{A} \cap \overline{(B \cup C)} && \text{DeMorgan's Law} \\ &= \overline{A} \cap (\overline{B} \cap \overline{C}) && \text{DeMorgan's Law} \\ &= \overline{A} \cap \overline{B} \cap \overline{C} && \text{Associative Law} \end{aligned}$$

Set Equalities using Identities

Show using set identities that

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$$

$$\begin{aligned}
LHS &= \overline{A \cup B \cup C} & \\
&= \overline{A} \cap \overline{(B \cup C)} & \text{DeMorgan's Law} \\
&= \overline{A} \cap (\overline{B} \cap \overline{C}) & \text{DeMorgan's Law} \\
&= \overline{A} \cap \overline{B} \cap \overline{C} & \text{Associative Law} \\
&= RHS &
\end{aligned}$$

Set Equalities using Membership Table

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x\ (x \in A \leftrightarrow x \in B)$$

- To prove two sets A and B to be equal
- We directly prove the above definition of equality
- For *every element* $x \in U$, we prove that it is either both in A and B or none of them
- When A and B are defined in terms of sets operations on other sets, *every element* $x \in U$ means all types of elements

Set Equalities using Membership Table

Prove using membership table that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For very element $x \in U$ there are exactly 8 possibilities based on it's membership (denoted by $0/1$) in some combination of A, B, and C

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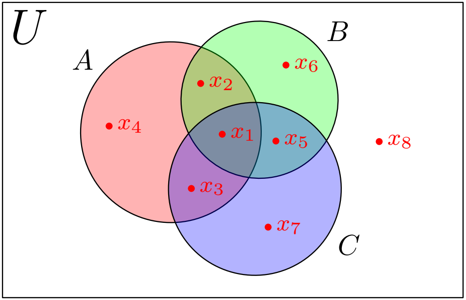

element type	A	B	C
x_1	1	1	1
x_2	1	1	0
x_3	1	0	1
x_4	1	0	0
x_5	0	1	1
x_6	0	1	0
x_7	0	0	1
x_8	0	0	0

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A	B	C					
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
0	0	0					

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A	B	C	$B \cup C$				
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
0	0	0					

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A	B	C	$B \cup C$				
1	1	1	1				
1	1	0	1				
1	0	1	1				
1	0	0	0				
0	1	1	1				
0	1	0	1				
0	0	1	1				
0	0	0	0				

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- For very element $x \in U$ there are exactly 8 possibilities based on it's membership (denoted by 0/1) in some combination of A, B, and C

A	B	C	$B \cup C$	$A \cap (B \cup C)$			
1	1	1	1				
1	1	0	1				
1	0	1	1				
1	0	0	0				
0	1	1	1				
0	1	0	1				
0	0	1	1				
0	0	0	0				

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A	B	C	$B \cup C$	$A \cap (B \cup C)$			
1	1	1	1	1			
1	1	0	1	1			
1	0	1	1	1			
1	0	0	0	0			
0	1	1	1	0			
0	1	0	1	0			
0	0	1	1	0			
0	0	0	0	0			

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A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$		
1	1	1	1	1			
1	1	0	1	1			
1	0	1	1	1			
1	0	0	0	0			
0	1	1	1	0			
0	1	0	1	0			
0	0	1	1	0			
0	0	0	0	0			

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A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$		
1	1	1	1	1	1		
1	1	0	1	1	1		
1	0	1	1	1	0		
1	0	0	0	0	0		
0	1	1	1	0	0		
0	1	0	1	0	0		
0	0	1	1	0	0		
0	0	0	0	0	0		

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- For very element $x \in U$ there are exactly 8 possibilities based on it's membership (denoted by 0/1) in some combination of A, B, and C

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	
1	1	1	1	1	1		
1	1	0	1	1	1		
1	0	1	1	1	0		
1	0	0	0	0	0		
0	1	1	1	0	0		
0	1	0	1	0	0		
0	0	1	1	0	0		
0	0	0	0	0	0		

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- For very element $x \in U$ there are exactly 8 possibilities based on it's membership (denoted by 0/1) in some combination of A, B, and C

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	
1	1	1	1	1	1	1	
1	1	0	1	1	1	0	
1	0	1	1	1	0	1	
1	0	0	0	0	0	0	
0	1	1	1	0	0	0	
0	1	0	1	0	0	0	
0	0	1	1	0	0	0	
0	0	0	0	0	0	0	

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A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	
1	1	0	1	1	1	0	
1	0	1	1	1	0	1	
1	0	0	0	0	0	0	
0	1	1	1	0	0	0	
0	1	0	1	0	0	0	
0	0	1	1	0	0	0	
0	0	0	0	0	0	0	

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A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

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- For very element $x \in U$ there are exactly 8 possibilities based on it's membership (denoted by 0/1) in some combination of A, B, and C

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

For each type of element in U the entries in red columns are the same

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C					
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
0	0	0					

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C	$A \setminus C$				
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
0	0	0					

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C	$A \setminus C$				
1	1	1	0				
1	1	0	1				
1	0	1	0				
1	0	0	1				
0	1	1	0				
0	1	0	0				
0	0	1	0				
0	0	0	0				

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C	$A \setminus C$	$B \setminus C$			
1	1	1	0				
1	1	0	1				
1	0	1	0				
1	0	0	1				
0	1	1	0				
0	1	0	0				
0	0	1	0				
0	0	0	0				

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A	B	C	$A \setminus C$	$B \setminus C$			
1	1	1	0	0			
1	1	0	1	1			
1	0	1	0	0			
1	0	0	1	0			
0	1	1	0	0			
0	1	0	0	1			
0	0	1	0	0			
0	0	0	0	0			

Set Equalities using Membership Table

Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$		
1	1	1	0	0			
1	1	0	1	1			
1	0	1	0	0			
1	0	0	1	0			
0	1	1	0	0			
0	1	0	0	1			
0	0	1	0	0			
0	0	0	0	0			

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A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$		
1	1	1	0	0	0		
1	1	0	1	1	1		
1	0	1	0	0	0		
1	0	0	1	0	1		
0	1	1	0	0	0		
0	1	0	0	1	1		
0	0	1	0	0	0		
0	0	0	0	0	0		

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A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$	$A \cup B$	
1	1	1	0	0	0		
1	1	0	1	1	1		
1	0	1	0	0	0		
1	0	0	1	0	1		
0	1	1	0	0	0		
0	1	0	0	1	1		
0	0	1	0	0	0		
0	0	0	0	0	0		

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A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$	$A \cup B$	
1	1	1	0	0	0	1	
1	1	0	1	1	1	1	
1	0	1	0	0	0	1	
1	0	0	1	0	1	1	
0	1	1	0	0	0	1	
0	1	0	0	1	1	1	
0	0	1	0	0	0	0	
0	0	0	0	0	0	0	

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A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$	$A \cup B$	$(A \cup B) \setminus C$
1	1	1	0	0	0	1	
1	1	0	1	1	1	1	
1	0	1	0	0	0	1	
1	0	0	1	0	1	1	
0	1	1	0	0	0	1	
0	1	0	0	1	1	1	
0	0	1	0	0	0	0	
0	0	0	0	0	0	0	

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A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$	$A \cup B$	$(A \cup B) \setminus C$
1	1	1	0	0	0	1	0
1	1	0	1	1	1	1	1
1	0	1	0	0	0	1	0
1	0	0	1	0	1	1	1
0	1	1	0	0	0	1	0
0	1	0	0	1	1	1	1
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

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Prove using membership table that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$

A	B	C	$A \setminus C$	$B \setminus C$	$(A \setminus C) \cup (B \setminus C)$	$A \cup B$	$(A \cup B) \setminus C$
1	1	1	0	0	0	1	0
1	1	0	1	1	1	1	1
1	0	1	0	0	0	1	0
1	0	0	1	0	1	1	1
0	1	1	0	0	0	1	0
0	1	0	0	1	1	1	1
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

Set Equalities using Logical Equivalence

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

Set Equalities using Logical Equivalence

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

- To prove two sets R and S to be equal
- Prove that the membership predicate of R is logically equivalent to the membership predicate of S
- Recall the membership predicate decides whether or not x is in a set
- When the two membership predicates are logically equivalent, for any x they will either both be True or both be False
- We get that $\forall x \, (x \in R \leftrightarrow x \in S)$ is true

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$ $\qquad \triangleright$ LHS

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$ ▷ LHS

$\equiv x \in A \wedge x \in (B \cup C)$ ▷ Intersection

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$ ▷ LHS

$\equiv x \in A \wedge x \in (B \cup C)$ ▷ Intersection

$\equiv x \in A \wedge (x \in B \vee x \in C)$ ▷ Union

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$ ▷ LHS

$\equiv x \in A \wedge x \in (B \cup C)$ ▷ Intersection

$\equiv x \in A \wedge (x \in B \vee x \in C)$ ▷ Union

$\equiv (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ ▷ Distributive Law

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$
\begin{aligned}
& x \in A \cap (B \cup C) && \triangleright \text{ LHS} \\
\equiv\ & x \in A \land x \in (B \cup C) && \triangleright \text{ Intersection} \\
\equiv\ & x \in A \land (x \in B \lor x \in C) && \triangleright \text{ Union} \\
\equiv\ & (x \in A \land x \in B) \lor (x \in A \land x \in C) && \triangleright \text{ Distributive Law} \\
\equiv\ & x \in (A \cap B) \lor x \in (A \cap C) && \triangleright \text{ Intersection}
\end{aligned}
$$

Set Equalities using Logical Equivalence

Prove using logical equivalences that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$
\begin{aligned}
& x \in A \cap (B \cup C) && \triangleright \text{ LHS} \\
& \equiv x \in A \land x \in (B \cup C) && \triangleright \text{ Intersection} \\
& \equiv x \in A \land (x \in B \lor x \in C) && \triangleright \text{ Union} \\
& \equiv (x \in A \land x \in B) \lor (x \in A \land x \in C) && \triangleright \text{ Distributive Law} \\
& \equiv x \in (A \cap B) \lor x \in (A \cap C) && \triangleright \text{ Intersection} \\
& \equiv x \in (A \cap B) \cup (A \cap C) && \triangleright \text{ Union}
\end{aligned}
$$

Set Equality using Subset Relations

Two sets R and S are equal if $R \subseteq S$ and $S \subseteq R$

Set Equality using Subset Relations

Two sets R and S are equal if $R \subseteq S$ and $S \subseteq R$

- To prove $R = S$
- Prove $R \subseteq S$ and
- Prove $S \subseteq R$
- By the above definition we get that $R = S$

Set Equality using Subset Relations

Proving using subset relations that $\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A})$

Set Equality using Subset Relations

Proving using subset relations that $\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A})$

$$(X \subseteq Y) \wedge (Y \subseteq X) \equiv X = Y$$

Set Equality using Subset Relations

Proving using subset relations that $\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A})$

$$(X \subseteq Y) \wedge (Y \subseteq X) \equiv X = Y$$

First show that

$\boxed{1}$ $\overline{(A \cup C) \cap B} \subseteq \overline{B} \cup (\overline{C} \cap \overline{A})$

Set Equality using Subset Relations

Proving using subset relations that $\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A})$

$$(X \subseteq Y) \wedge (Y \subseteq X) \equiv X = Y$$

First show that

$\boxed{\mathbf{1}}$ $\overline{(A \cup C) \cap B} \subseteq \overline{B} \cup (\overline{C} \cap \overline{A})$

Next show that

$\boxed{\mathbf{2}}$ $\overline{B} \cup (\overline{C} \cap \overline{A}) \subseteq \overline{(A \cup C) \cap B}$

Set Equality using Subset Relations

We need to prove that

$\boxed{\mathbf{1}}$ $\overline{(A \cup C) \cap B} \subseteq \overline{B} \cup (\overline{C} \cap \overline{A})$ $\boxed{\mathbf{2}}$ $\overline{B} \cup (\overline{C} \cap \overline{A}) \subseteq \overline{(A \cup C) \cap B}$

Set Equality using Subset Relations

We need to prove that

$\boxed{1}$ $\overline{(A \cup C) \cap B} \subseteq \overline{B} \cup (\overline{C} \cap \overline{A})$ $\boxed{2}$ $\overline{B} \cup (\overline{C} \cap \overline{A}) \subseteq \overline{(A \cup C) \cap B}$

ICP-4-26 Prove $\boxed{1}$: if $x \in \overline{(A \cup C) \cap B}$, then $x \in \overline{B} \cup (\overline{C} \cap \overline{A})$

Set Equality using Subset Relations

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$$x \in \overline{(A \cup C) \cap B}$$
$$x \notin (A \cup C) \cap B \begin{cases} x \notin B & x \in \overline{B} \\ \\ x \notin (A \cup C) \end{cases}$$

Set Equality using Subset Relations

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$$
\begin{array}{l}
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x \notin (A \cup C) \cap B
\end{array}
\begin{cases}
x \notin B & \begin{array}{l} x \in \overline{B} \\ x \in \overline{B} \cup (\overline{C} \cap \overline{A}) \end{array} \\
 & \\
x \notin (A \cup C) & \begin{array}{l} x \notin A \wedge x \notin C \\ x \in \overline{A} \cap \overline{C} \\ x \in \overline{B} \cup (\overline{C} \cap \overline{A}) \end{array}
\end{cases}
$$

Set Equality using Subset Relations

We need to prove that

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Set Equality

- Equality of two sets can be proved using
  - Algebraic Rules (Set Identities)
  - Set Membership Tables
  - Logical Equivalence of membership predicates
  - By proving bidirectional subset relationships

Set Theory

- Sets: Definition, Universal Set, Complement, Cardinality
- Subset and Power Set
- Sets Operations
- Set Equality
- Characteristic Vectors: Sets as Bit-Vectors
- Multisets

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Sets as bit-strings (bit vectors)

- Sets stored in an unordered fashion in memory
- Union/Intersection etc. are computationally expensive
- When $|U|$ is small compared to computer memory, then we can do set operations efficiently
- Impose any fixed ordering on elements of U
- $U = \{DM, Cal, Chem, Bio, Phy, Pro\}$ (in order)
- Sets (subsets of U) are represented by bit-string of length 6
- Each bit signifies whether the corresponding element is in the set
- Called bit-vector representation of sets or characteristic vector of a set

Sets as bit-strings (bit vectors)

DM	Calc	Chem	Bio	Phy	Prog

The set $\{Calc, Chem, Phy\}$ is

0	1	1	0	1	0

The set $\{Prog, DM, Calc, Phy\}$ is

1	1	0	0	1	1

ICP 4-28 What is the characteristic vector of the set

$$\{Chem, DM\}?$$

ICP 4-29 What is the characteristic vector of the set

$$\{Calc, DM, Chem, Phy, Prog, Bio\}?$$

Sets as bit-strings (bit vectors)

DM	Calc	Chem	Bio	Phy	Prog

The set

1	0	0	0	0	1

is $\{DM, Prog\}$

The set

0	0	0	0	0	0

is the empty set

ICP 4-30 What is the set corresponding to the characteristic vector

1	1	1	1	1	1

Sets operations using bit-strings

$$A \cup B \;=\; \{x | x \in A \vee x \in B\}$$

$A = \{Calc, Chem, Phy\}$	0	1	1	0	1	0
$B = \{Prog, DM, Calc, Phy\}$	1	1	0	0	1	1
$A \cup B$	$A \vee B$					
$\{DM, Calc, Chem, Prog, Phy\}$	1	1	1	0	1	1

Sets operations using bit-strings

$$A \cap B \;=\; \{x | x \in A \wedge x \in B\}$$

$A = \{Calc, Chem, Phy\}$	0	1	1	0	1	0
$B = \{Prog, DM, Calc, Phy\}$	1	1	0	0	1	1
$A \cap B$	$A \wedge B$					
$\{Calc, Phy\}$	0	1	0	0	1	0

Sets operations using bit-strings

$$A \oplus B \; = \; \{x | x \in A \oplus x \in B\}$$

$A = \{Calc, Chem, Phy\}$	0	1	1	0	1	0
$B = \{Prog, DM, Calc, Phy\}$	1	1	0	0	1	1
$A \oplus B$	$A \oplus B$					
$\{DM, Chem, Prog\}$	1	0	1	0	0	1

Sets as bit-vectors: Summary

- Sets can be represented as bit vectors, when universal set is *'small'*
- Also called characteristic vectors of sets
- Order of U is critical
- Sets operations can be performed using bit-wise operators of programming language
- More suitable for computer implementations
- Only feasible when U is small

Set Theory

- Sets: Definition, Universal Set, Complement, Cardinality
- Subset and Power Set
- Sets Operations
- Set Equality
- Characteristic Vectors: Sets as Bit-Vectors
- Multisets

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Multiset

A multiset is an unordered collection of objects where repetition of the elements matters

- $A \ : \ \{1, 2, 2, 2, 3, 3\}$
- $B \ : \ \{CS100, CS100, CS100, CS210, CS210\}$
- $C \ :$ the multiset of last names of the all professors in LUMS

Order of elements is not significant

- $\{1, 2, 2, 3\}$ is the same as $\{2, 3, 1, 2\}$

Repetition counts!

- $\{1, 2, 2, 2, 3\}$ is not the same as $\{1, 2, 3\}$

Multisets: Terminology

- Multiset is also termed as bag or mset
- Number of instances of each element in a multiset is called multiplicity

An infinite number of multisets exist which contain only elements a and b, but vary in the multiplicities of their elements

- $\{a, b\}$
- $\{a, a, b\}$
- $\{a, a, a, b, b, b\}$

All of these are different multisets

All of these represent the same set

Multisets: Multiplicity

Multisets can be represented as a set of ordered pairs $\big(x, m_A(x)\big)$

- x is an element in the multiset A
- $m_A(x)$ is the multiplicity of x in the multiset A

$$\{a, a, b, b, b\} \quad \longrightarrow \quad \big\{(a, 2), (b, 3)\big\}$$

$$\{1, 2, 3, 2, 1\} \quad \longrightarrow \quad \big\{(1, 2), (2, 2), (3, 1)\big\}$$

$$\{Khan, Ali, Khan, Ali, Ayesha\} \quad \longrightarrow \quad \big\{(Ayesha, 1), (Khan, 2), (Ali, 2)\big\}$$

Multisets: Support and Cardinality

The support of a multiset A in a universe U is the underlying set of A

$$support(A) \;=\; \{x \in U \,|\, m_A(x) > 0\}$$

$$A = \big\{(Ayesha, 2), (Khan, 2), (Ali, 1)\big\} \implies support(A) = \{Ayesha, Khan, Ali\}$$

The cardinality of a multiset A is the sum of multiplicities of its elements

$$A \;=\; \big\{(Ayesha, 2), (Khan, 2), (Ali, 1)\big\}$$

$$|A| \;=\; 2 + 2 + 1 = 5$$

$$|Support(A)| \;=\; 3$$

Multisets: Inclusion

A multiset A is included in the multiset B if $\forall x \in U,\ m_A(x) \leq m_B(x)$

denoted as $A \subseteq B$

$$A = \{(Ayesha, 2), (Khan, 2), (Ali, 1)\}$$

$$B = \{(Ayesha, 3), (Khan, 2), (Ali, 2), (Imdad, 1)\}$$

Is $A \subseteq B$?

Is $B \subseteq A$?

Multisets: Union

The union of a multiset A and a multiset B is a multiset C such that

$$\forall x \in U,\ m_C(x) \ =\ max\big(m_A(x), m_B(x)\big)$$

denoted as $C \ = \ A \cup B$

$$A \ = \ \big\{(Ayesha, 2), (Khan, 2), (Ali, 1)\big\}$$

$$B \ = \ \big\{(Ayesha, 3), (Ali, 2), (Imdad, 1)\big\}$$

$$A \cup B \ = \ \big\{(Ayesha, 3), (Ali, 2), (Imdad, 1), (Khan, 2)\big\}$$

Multisets: Intersection

The intersection of a multiset A and a multiset B is a multiset C such that

$$\forall x \in U,\ m_C(x) = min\big(m_A(x), m_B(x)\big)$$

denoted as $C = A \cap B$

$$A = \big\{(Ayesha, 2), (Khan, 2), (Ali, 1)\big\}$$

$$B = \big\{(Ayesha, 3), (Ali, 2), (Imdad, 1)\big\}$$

$$A \cap B = \big\{(Ayesha, 2), (Ali, 1)\big\}$$

Multisets: Sum

The sum of a multiset A and a multiset B is a multiset C such that

$$\forall x \in U,\ m_C(x) = m_A(x) + m_B(x)$$

denoted as $C = A \sqcup B$

$$A = \{(Ayesha, 2), (Khan, 2), (Ali, 1)\}$$
$$B = \{(Ayesha, 3), (Ali, 2), (Imdad, 1)\}$$

$$A \sqcup B = \{(Ayesha, 5), (Ali, 3), (Imdad, 1), (Khan, 2)\}$$

Sum of two multisets ($\sqcup$) is also known as disjoint union

Multisets: Difference

The difference of a multiset B from a multiset A is a multiset C such that

$$\forall x \in U,\ m_C(x) \ = \ max\big(m_A(x) - m_B(x), 0\big)$$

denoted as $C \ = \ A \setminus B$

$$A \ = \ \big\{(Ayesha, 5), (Khan, 2), (Ali, 1)\big\}$$

$$B \ = \ \big\{(Ayesha, 3), (Ali, 2), (Imdad, 1)\big\}$$

$$A \setminus B \ = \ \big\{(Ayesha, 2), (Khan, 2)\big\}$$

The Set-of-Words Vector Model for Text Representation

Set-of-Words: Documents represented by vectors $\in \{0, 1\}^{|\Sigma|}$

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth	...
ANTHONY	1	1	0	0	0	1	
BRUTUS	1	1	0	1	0	0	
CAESAR	1	1	0	1	1	1	
CALPURNIA	0	1	0	0	0	0	
CLEOPATRA	1	0	0	0	0	0	
MERCY	1	0	1	1	1	1	
WORSER	1	0	1	1	1	0	
...							

The Bag of words Vector Model for Text Representation

Bag-of-Words: Documents represented by term-frequency vectors $\in \mathbb{N}^{|\Sigma|}$

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth	...
ANTHONY	157	73	0	0	0	1	
BRUTUS	4	157	0	2	0	0	
CAESAR	232	227	0	2	1	0	
CALPURNIA	0	10	0	0	0	0	
CLEOPATRA	57	0	0	0	0	0	
MERCY	2	0	3	8	5	8	
WORSER	2	0	1	1	1	5	
...							