University Calculus Reference Guide

1 Sequences and Limits

1.1 Sequences

A sequence is an ordered list of numbers $\{a_n\}_{n=1}^{\infty}$.

Definition 1. *A sequence $\{a_n\}$ converges to a limit L if for every $\epsilon > 0$, there exists an integer N such that for all $n > N$, $|a_n - L| < \epsilon$. Notation: $\lim_{n\to\infty} a_n = L$.*

1.2 Limits of Functions

Definition 2. *Let $f(x)$ be defined on an open interval containing c (except possibly at c). We say $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.*

1.2.1 Limit Laws

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$:

- **Sum:** $\lim_{x\to c}[f(x) + g(x)] = L + M$
- **Product:** $\lim_{x\to c}[f(x)g(x)] = LM$
- **Quotient:** $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ (provided $M \neq 0$)

1.2.2 The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ near c and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.

1.2.3 Special Trigonometric Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

1.3 Continuity

A function f is continuous at c if: 1. $f(c)$ is defined. 2. $\lim_{x\to c} f(x)$ exists. 3. $\lim_{x\to c} f(x) = f(c)$.

Theorem 1 (Intermediate Value Theorem)**.** *If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ such that $f(c) = k$.*

2 Differentiation

2.1 Definition of the Derivative

The derivative of f at x is defined as:

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

2.2 Differentiation Rules

Let u and v be differentiable functions of x.

- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Product Rule:** $\frac{d}{dx}(uv) = u'v + uv'$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

2.3 Common Derivatives

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

2.4 Theorems

Theorem 2 (Mean Value Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b), there exists $c \in (a,b)$ such that:*

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem 3 (L'Hôpital's Rule)**.** *If $\lim_{x\to c} \frac{f(x)}{g(x)}$ yields $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

3 Integration

3.1 Definite and Indefinite Integrals

- **Indefinite:** $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$.
- **Definite:** $\int_a^b f(x)\,dx$ represents the net signed area under the curve.

3.2 Fundamental Theorem of Calculus

Part 1: If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
Part 2: If F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

3.3 Integration Techniques

3.3.1 u-Substitution (Reverse Chain Rule)

Let $u = g(x)$, then $du = g'(x)dx$.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

3.3.2 Integration by Parts (Reverse Product Rule)

$$\int u\,dv = uv - \int v\,du$$

3.4 Common Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \quad (-1 < x < 1)$$