

# Mathematical analysis 2

## Chapter 3 : Numerical series

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# Course outline

## 1 Generalities

- Convergence of a series
- Divergence Test
- Propriétés et opérations sur les séries

## 2 Positive term series

- Convergence criteria for Positive terms series

## 3 Arbitrary term series

## 4 Alternating series

# Generalities

## Definition

- Let  $(u_n)$  be a sequence of real numbers. The expression

$$u_0 + u_1 + \cdots + u_n + \cdots$$

is called **numerical series of general term  $u_n$** .

- A series of general term  $u_n$  is denoted by  $\sum_{n=0}^{+\infty} u_n$ ,  $\sum_{n \geq 0} u_n$  or simply  $\sum u_n$ .

## Definition

- The sum of the  $n$  first terms of the series is denoted by  $S_n$  and is called **partial sum**

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^n u_k.$$

- The sequence  $(S_n)$  is called **sequence of partial sum**.

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## Convergence of a series

### Definition

- A series of general term  $u_n$  is said to be **convergent** to  $S$  if the sequence of partial sum  $(S_n)$  is convergent. In this case we have

$$S = \lim_{n \rightarrow \infty} S_n = \sum_{n=0}^{+\infty} u_n.$$

- $S$  is called **the sum of the series** and we have

$$\sum u_n \text{ converges to } S \iff \lim_{n \rightarrow \infty} S_n = S$$

- A series that is not convergent is called **divergent**.

### Remark

*The nature of a series is by definition its convergence or divergence.*

## Geometric series

### Example.

Let  $(u_n)$  be a geometric series with the first term  $u_0 = a \neq 0$  and common ratio  $q$ . The general term is given by

$$u_n = aq^n \quad (a \neq 0).$$

The partial sum is given by

$$S_n = \begin{cases} a \left( \frac{1 - q^{n+1}}{1 - q} \right), & q \neq 1 \\ a(n+1), & q = 1 \end{cases}$$

**Question.** When does a geometric series  $\sum_{n=0}^{+\infty} aq^n$  converge?

## Geometric series

We have

$$S = \lim_{n \rightarrow \infty} S_n = \begin{cases} \frac{a}{1-q}, & \text{if } |q| < 1 \\ \text{The limit doesn't exist} & \text{if } q \leq -1 \\ \infty & \text{if } q \geq 1. \end{cases}$$

Consequently, the **geometric series**

- **Converges** if  $|q| < 1$ .
- **Diverges** if  $|q| \geq 1$ .

# Telescopic series

Example.

Let  $\sum_{n=1}^{+\infty} u_n$  be the series defined by the general term

$$u_n = \frac{1}{n(n+1)}, \quad n \geq 1.$$

## Telescopic series

Example.

Let  $\sum_{n=1}^{+\infty} u_n$  be the series defined by the general term

$$u_n = \frac{1}{n(n+1)}, \quad n \geq 1.$$

By decomposition to simple elements we can write the general term as follows

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

$$\begin{aligned} \text{Hence } S_n &= u_1 + u_2 + \cdots + u_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

# Telescopic series

Then

$$S_n = 1 - \frac{1}{n+1}.$$

And we have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefor the series  $\sum_{n \geq 1} \frac{1}{n(n+1)}$  converges to 1.

## Telescopic series

Example.

Let  $\sum_{n=1}^{+\infty} u_n$  be the series defined by the general term

$$u_n = \ln\left(1 + \frac{1}{n}\right), \quad n \geq 1.$$

## Telescopic series

Example.

Let  $\sum_{n=1}^{+\infty} u_n$  be the series defined by the general term

$$u_n = \ln\left(1 + \frac{1}{n}\right), \quad n \geq 1.$$

We have

$$\forall n \geq 1 : \quad \ln\left(1 + \frac{1}{n}\right) = \ln(n+1) - \ln(n).$$

Then

$$S_n = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n+1) - \ln(n)) = \ln(n+1) - \ln(1) = \ln(n+1)$$

The partial sum sequence is divergent then  $\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$  diverges.

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- **Divergence Test**
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## Divergence Test

### Proposition

If  $\lim_{n \rightarrow \infty} u_n \neq 0$  or  $\lim_{n \rightarrow \infty} u_n$  doesn't exist, then the series  $\sum u_n$  diverges.

### Example.

The series  $\sum_{n \geq 0} \frac{n}{n+1}$  is divergent since  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$

⚠ The divergence test provides a way of proving that a series diverges but there exist divergent series with the general term going to zero.

### Example.

**Harmonic series**  $\sum_{n=1}^{+\infty} \frac{1}{n}$  is divergent but  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

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## Proposition

*If the series  $\sum u_n$  and  $\sum v_n$  differ only for a finite number of terms, then the two series are of the same nature.*

## Remark

*The nature of a series remains unchanged when adding or subtracting a finite number of terms.*

## Proposition (Operations on series)

Let  $\sum u_n$  and  $\sum v_n$  be two series convergent respectively to  $S$  and  $L$  then

- ① The series  $\sum (u_n + v_n)$  is convergent to  $S + L$  and we have

$$\sum_{n=0}^{+\infty} (u_n + v_n) = \sum_{n=0}^{+\infty} u_n + \sum_{n=0}^{+\infty} v_n = S + L$$

- ② For all  $\alpha \in \mathbb{R}$  the series  $\sum (\alpha u_n)$  converges to  $(\alpha S)$  and we have

$$\sum_{n=0}^{+\infty} (\alpha u_n) = \alpha \sum_{n=0}^{+\infty} u_n = \alpha S.$$



## Important remark

### Remark

*In the cases:*

- ① If  $\sum u_n$  is convergent and  $\sum v_n$  is divergent then the series  $\sum(u_n + v_n)$  is divergent.
- ② If  $\sum u_n$  and  $\sum v_n$  diverge, their sum  $\sum(u_n + v_n)$  is not necessary divergent.

## Example.

*Study the nature of the series*

$$\sum_{n=1}^{+\infty} \left( \frac{3}{2^n} + \frac{2}{n(n+1)} \right)$$

## Example.

*Study the nature of the series*

$$\sum_{n=1}^{+\infty} \left( \frac{3}{2^n} + \frac{2}{n(n+1)} \right)$$

**Solution.** We have

- $\sum_{n=1}^{+\infty} \frac{1}{2^n}$  is a geometric series with common ratio  $q = \frac{1}{2} \in ]-1, 1[$  then it converges to  $\frac{1/2}{1 - 1/2} = 1$ .
- The series  $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$  converges to 1.

Therefore

$$\sum_{n=1}^{+\infty} \left( \frac{3}{2^n} + \frac{2}{n(n+1)} \right) = \sum_{n=1}^{+\infty} \frac{3}{2^n} + \sum_{n=1}^{+\infty} \frac{2}{n(n+1)} = 3 \sum_{n=1}^{+\infty} \frac{1}{2^n} + 2 \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 3 + 2 = 5.$$

Then the series  $\sum_{n=1}^{+\infty} \left( \frac{3}{2^n} + \frac{2}{n(n+1)} \right)$  is convergent to 5.

### Example.

The series  $\sum \frac{1}{n(n+1)}$  is **convergent**, even

$$\sum \frac{1}{n(n+1)} = \sum \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

Where  $\sum \frac{1}{n}$  **diverges** and  $\sum \frac{1}{n+1}$  **diverges** also.

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## Positive term series

### Definition

A series  $\sum u_n$  is said to be a positive term series if  $u_n \geq 0 \forall n \geq n_0$ ;  
 $n_0 \in \mathbb{N}$

### Example.

The series  $\sum_{n=1}^{+\infty} \frac{n+2}{n^2}$  is a positive terms since:  $\forall n \geq 1, \frac{n+2}{n^2} \geq 0$

### Remark

If a series  $\sum u_n$  is a positive term series then the sequence of partial sum  $(S_n)_n$  is increasing.

## Positive term series

### Proposition

Let  $\sum u_n$  be a positive term series

$\sum u_n$  converges  $\iff (S_n)$  is upper bounded.

### Example.

Let's consider the positive term series  $\sum \frac{1}{n(n+1)}$ .

## Positive term series

### Proposition

Let  $\sum u_n$  be a positive term series

$\sum u_n$  converges  $\iff (S_n)$  is upper bounded.

### Example.

Let's consider the positive term series  $\sum \frac{1}{n(n+1)}$ . We have

$$S_N = \sum_{n=1}^N \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.$$

For all  $n \geq 1$ :  $S_n \leq 1$  then  $(S_n)$  is upper bounded, therefore  $\sum \frac{1}{n(n+1)}$  is convergent.

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## Comparison test

### Theorem

Let  $\sum u_n$  et  $\sum v_n$  be two positive term series such that for all  $n \geq n_0$ ,  $n_0 \in \mathbb{N}$ , we have

$$0 \leq u_n \leq v_n$$

Then

- ① If  $\sum v_n$  converges  $\implies \sum u_n$  converges.
- ② If  $\sum u_n$  diverges  $\implies \sum v_n$  diverges.

### Example.

Study the nature of the series  $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$  using the comparison test.

## Comparison test

In one hand  $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$  is a positive term series since

$$\frac{|\cos n|}{5^n} \geq 0 \quad \forall n \in \mathbb{N}$$

In the other hand we have

$$\forall n \in \mathbb{N}, \quad |\cos n| \leq 1 \implies \frac{|\cos n|}{5^n} \leq \frac{1}{5^n}.$$

In this case we choose  $v_n = \frac{1}{5^n}$ , which is a convergent geometric series

(since  $q = 1/5 \in ]-1, 1[$ ), Consequently the series  $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$  is convergent.

## Comparison test

Example.

*Study the nature of the series of general term*

$$u_n = \frac{3 + \sin(\ln n)}{n}$$

## Comparison test

### Example.

Study the nature of the series of general term  $u_n = \frac{3 + \sin(\ln n)}{n}$

We have for all  $n \geq 1$ :

$$-1 \leq \sin(\ln n) \leq 1$$

$$2 \leq 3 + \sin(\ln n) \leq 4$$

$$\frac{2}{n} \leq \frac{3 + \sin(\ln n)}{n} \leq \frac{4}{n}$$

We can see that  $\sum_{n=1}^{+\infty} \frac{3 + \sin(\ln n)}{n}$  is a positive term series and

$$\sum_{n=1}^{+\infty} \frac{2}{n} \leq \sum_{n=1}^{+\infty} \frac{3 + \sin(\ln n)}{n}$$

Since  $\sum_{n=1}^{+\infty} \frac{2}{n}$  is divergent then  $\sum_{n=1}^{+\infty} \frac{3 + \sin(\ln n)}{n}$  is divergent.

## Limit comparison test

### Theorem

Let  $\sum u_n$  and  $\sum v_n$  two positive term series. If  $u_n \sim v_n$  or  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \ell$ ,  $\ell \neq 0$ ,  $\ell \neq +\infty$  then the two series have the same nature.

### Example.

Study the nature of the series  $\sum_{n \geq 0} \frac{n^3 + 1}{n^5 + 2n^3 + 2}$

## Limit comparison test

### Theorem

Let  $\sum u_n$  and  $\sum v_n$  two positive term series. If  $u_n \sim v_n$  or  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \ell$ ,  $\ell \neq 0$ ,  $\ell \neq +\infty$  then the two series have the same nature.

### Example.

Study the nature of the series  $\sum_{n \geq 0} \frac{n^3 + 1}{n^5 + 2n^3 + 2}$

We have for all  $n \in \mathbb{N}$ ,  $u_n > 0$  and

$$\frac{n^3 + 1}{n^5 + 2n^3 + 2} \sim \frac{n^3}{n^5} = \frac{1}{n^2}$$

Since  $\sum_{n=1}^{+\infty} \frac{1}{n^2}$  is convergent ( Riemann Series) then  $\sum_{n \geq 0} \frac{n^3 + 1}{n^5 + 2n^3 + 2}$  is convergent.

## Limit comparison test

Example.

*Study the nature of the series defined by*  $\sum_{n=0}^{+\infty} \ln \left( 1 + \frac{1}{3^n} \right)$

## Limit comparison test

Example.

Study the nature of the series defined by  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$

We know that  $\ln\left(1 + \frac{1}{x}\right) \sim \frac{1}{x}$  then  $\ln\left(1 + \frac{1}{3^n}\right) \sim \frac{1}{3^n}$

Since  $\sum_{n=0}^{+\infty} \frac{1}{3^n}$  is a convergent geometric series then  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$  is convergent.

## Limit comparison test

Example.

Study the nature of the series defined by  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$

We know that  $\ln\left(1 + \frac{1}{x}\right) \underset{\infty}{\sim} \frac{1}{x}$  then  $\ln\left(1 + \frac{1}{3^n}\right) \underset{\infty}{\sim} \frac{1}{3^n}$

Since  $\sum_{n=0}^{+\infty} \frac{1}{3^n}$  is a convergent geometric series then  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$  is convergent.

Example.

The series  $\sum_{n=1}^{+\infty} \left| \sin\left(\frac{1}{n}\right) \right|$

## Limit comparison test

### Example.

Study the nature of the series defined by  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$

We know that  $\ln\left(1 + \frac{1}{x}\right) \underset{\infty}{\sim} \frac{1}{x}$  then  $\ln\left(1 + \frac{1}{3^n}\right) \underset{\infty}{\sim} \frac{1}{3^n}$

Since  $\sum_{n=0}^{+\infty} \frac{1}{3^n}$  is a convergent geometric series then  $\sum_{n=0}^{+\infty} \ln\left(1 + \frac{1}{3^n}\right)$  is convergent.

### Example.

The series  $\sum_{n=1}^{+\infty} \left| \sin\left(\frac{1}{n}\right) \right|$  is divergent since  $\left| \sin\left(\frac{1}{n}\right) \right| \underset{\infty}{\sim} \frac{1}{n}$  since  $(\sin x \underset{0}{\sim} x)$ , and the series  $\sum_{n=1}^{+\infty} \frac{1}{n}$  is divergent.

## Integral test

### Theorem

Let  $f : [a, +\infty[ \rightarrow \mathbb{R}^+$  be a **continuous positive decreasing** mapping.  
 We set  $u_n = f(n)$  for all  $n \in \mathbb{N}^*$ , ( $n \geq a$ ) Then

$$\begin{aligned} \sum u_n \text{ converges} &\iff \int_a^{+\infty} f(x) dx \text{ exist} \\ &\iff \lim_{t \rightarrow +\infty} \int_a^t f(x) dx = \ell. \quad (\ell \text{ finite}). \end{aligned}$$

## Integral test

### Example. (Harmonic series)

*Study the nature of the series*  $\sum_{n=1}^{+\infty} \frac{1}{n}.$

## Integral test

### Example. (Harmonic series)

Study the nature of the series  $\sum_{n=1}^{+\infty} \frac{1}{n}$ .

We set  $f(n) = \frac{1}{n}$  we consider the mapping  $f: [1, +\infty[ \rightarrow \mathbb{R}^+ / x \mapsto f(x) = \frac{1}{x}$ .  
The mapping  $f$  is continuous, positive and decreasing on  $[1, +\infty[$

$$\int_1^t f(x) dx = \int_1^t \frac{1}{x} = \ln x \Big|_1^t = \ln t - \ln 1 = \ln t.$$

and

$$\lim_{t \rightarrow +\infty} \int_1^t f(x) dx = \lim_{t \rightarrow +\infty} \ln t = +\infty.$$

Then the series  $\sum_{n=1}^{+\infty} \frac{1}{n}$  is divergent.

## Integral test

Example.

*Study the nature of the series*  $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$

# Integral test

Example.

Study the nature of the series  $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$

We set  $f(n) = \frac{1}{n(n+1)}$  and consider the mapping  
 $f : [1, +\infty[ \rightarrow \mathbb{R}^+ / x \mapsto f(x) = \frac{1}{x(x+1)}.$

The mapping  $f$  is continuous, positive and decreasing on  $[1, +\infty[$   

$$\begin{aligned} \int_1^t f(x) dx &= \int_1^t \frac{1}{x(x+1)} = \int_1^t \frac{1}{x} dx - \int_1^t \frac{1}{x+1} dx \quad \left( \text{since } \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \right) \\ &= \ln x \Big|_1^t - \ln(x+1) \Big|_1^t = \ln t - \ln 1 - \ln(t+1) + \ln 2 \\ &= \ln \left( \frac{t}{t+1} \right) + \ln 2. \end{aligned}$$

Therefor  $\lim_{t \rightarrow +\infty} \int_1^t f(x) dx = \lim_{t \rightarrow +\infty} \ln \left( \frac{t}{t+1} \right) + \ln 2 = \ln 2.$

Then the series  $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$  is convergent.

## Riemann series

### Definition

Let  $\alpha \in \mathbb{R}$ , we call Riemann series all series with general term

$$u_n = \frac{1}{n^\alpha}, \quad n \geq 1, \quad \alpha \in \mathbb{R}.$$

### Proposition

Riemann series  $\sum \frac{1}{n^\alpha}$ ,  $\alpha \in \mathbb{R}$  converges if and only if  $\alpha > 1$  and diverges if  $\alpha \leq 1$ .

## Riemann series

### Example.

*We have*

- *The series  $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$  is convergent*

*since*

# Riemann series

## Example.

We have

- The series  $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$  is convergent

since  $\sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{(n^3)^{1/2}} = \sum \frac{1}{n^{3/2}}$ , it is a Riemann series with

$\alpha = \frac{3}{2} > 1$  then  $\sum \frac{1}{\sqrt{n^3}}$  converges.

- The series  $\sum \sqrt{n}$  is divergent since

## Riemann series

### Example.

We have

- The series  $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$  is convergent

since  $\sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{(n^3)^{1/2}} = \sum \frac{1}{n^{3/2}}$ , it is a Riemann series with  $\alpha = \frac{3}{2} > 1$  then  $\sum \frac{1}{\sqrt{n^3}}$  converges.

- The series  $\sum \sqrt{n}$  is divergent since  $\sum \sqrt{n} = \sum n^{1/2} = \sum \frac{1}{n^{-1/2}}$ , it is a Riemann series with  $\alpha = \frac{-1}{2} \leq 1$  then  $\sum \sqrt{n}$  diverges.

# Riemann series

## Proposition

Let  $\sum u_n$  a positive term series

- ① If there exist  $\alpha > 1$  such that the sequence  $(n^\alpha u_n)$  is upper bounded by a constant  $M > 0$  then  $\sum u_n$  converges.
- ② If there exist  $\alpha \leq 1$  such that the sequence  $(n^\alpha u_n)$  is lower bounded by a constant  $m > 0$  then  $\sum u_n$  diverges.

## Corollaire

Let  $\sum u_n$  be a positive term series. We suppose that there exist  $\alpha \in \mathbb{R}$  such that

- ① If  $\lim_{n \rightarrow \infty} n^\alpha u_n = \ell$ , ( $\ell \neq 0$  et  $\ell \neq +\infty$ ) the series  $\sum u_n$  and  $\sum \frac{1}{n^\alpha}$  are of the same nature.
- ② If  $\lim_{n \rightarrow \infty} n^\alpha u_n = 0$  and  $\sum \frac{1}{n^\alpha}$  converges then  $\sum u_n$  converges.
- ③ If  $\lim_{n \rightarrow \infty} n^\alpha u_n = \infty$  and  $\sum \frac{1}{n^\alpha}$  diverges then  $\sum u_n$  diverges.

## Riemann series

Example.

*Study the nature of the series*

$$\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$$

## Riemann series

### Example.

*Study the nature of the series*

$$\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$$

we know that

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \implies e^x - 1 \underset{0}{\sim} x$$

Consequently

$$e^{\frac{3}{n^2}} - 1 \underset{\infty}{\sim} \frac{3}{n^2}.$$

Then

$$\lim_{n \rightarrow +\infty} n^2 \left( e^{\frac{3}{n^2}} - 1 \right) = \lim_{n \rightarrow +\infty} n^2 \left( \frac{3}{n^2} \right) = 3$$

Therefore the series  $\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$  converges.

## Riemann series

Example.

*study the nature of the series*

$$\sum_{n \geq 2} \frac{1}{\ln n}$$

## Riemann series

Example.

study the nature of the series  $\sum_{n \geq 2} \frac{1}{\ln n}$

We have  $\lim_{n \rightarrow \infty} n \frac{1}{\ln n} = \infty$  and  $\sum \frac{1}{n}$  diverges, then  $\sum_{n \geq 2} \frac{1}{\ln n}$  diverges.

## Riemann series

Example.

*study the nature of the series*

$$\sum_{n=0}^{+\infty} e^{-n}$$

## Riemann series

Example.

study the nature of the series  $\sum_{n=0}^{+\infty} e^{-n}$

We have  $\lim_{n \rightarrow \infty} n^2 e^{-n} = 0$  and  $\sum \frac{1}{n^2}$  converges, then  $\sum_{n=0}^{+\infty} e^{-n}$  converges.

## D'Alembert ratio test

### Proposition

Let  $\sum u_n$  be a series of positive terms. We set

$$\lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = \ell$$

- 1 If  $\ell < 1 \implies \sum u_n$  converges.
- 2 If  $\ell > 1 \implies \sum u_n$  diverges.
- 3 If  $\ell = 1$  we can't say any thing of the nature of the series.

## D'Alembert ratio test

Example.

*Study the nature of the series*

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

## D'Alembert ratio test

Example.

*Study the nature of the series*

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

We have

$$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1}$$

and

$$\lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow +\infty} \frac{1}{n+1} = 0$$

Consequently  $\sum_{n=0}^{+\infty} \frac{1}{n!}$  converges.

## D'Alembert ratio test

Example.

*Study the nature of the series*

$$u_n = \frac{n^n}{n!}, \quad n \geq 0$$

## D'Alembert ratio test

Example.

*Study the nature of the series*

$$u_n = \frac{n^n}{n!}, \quad n \geq 0$$

$$\forall n \in \mathbb{N}: \frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = e^{n \ln\left(\frac{n+1}{n}\right)}$$

$$\lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow +\infty} e^{n \ln\left(\frac{n+1}{n}\right)} = \lim_{n \rightarrow +\infty} e^{n\left(\frac{1}{n}\right)} = e > 1. \quad \left(\text{car } \ln\left(\frac{n+1}{n}\right) \underset{\infty}{\sim} \frac{1}{n}\right)$$

Therefore the series  $\sum u_n = \frac{n^n}{n!}$  diverges.

## D'Alembert ratio test

Example.

*Study the nature of the series*  $\sum_{n \geq 1} \frac{2^{2n} e^{-2n}}{n}$

## D'Alembert ratio test

Example.

Study the nature of the series  $\sum_{n \geq 1} \frac{2^{2n} e^{-2n}}{n}$

We have

$$\begin{aligned} \forall n \in \mathbb{N}: \frac{u_{n+1}}{u_n} &= \frac{2^{2(n+1)} e^{-2(n+1)}}{n+1} \cdot \frac{n}{2^{2n} e^{-2n}} = 2^2 e^{-2} \frac{n}{n+1} \\ &= \left(\frac{2}{e}\right)^2 \frac{n}{n+1} \rightarrow \left(\frac{2}{e}\right)^2 < 1 \end{aligned}$$

Then the series  $\sum_{n \geq 1} \frac{2^{2n} e^{-2n}}{n}$  converges.

## Cauchy root test

### Proposition

Let  $\sum u_n$  be a positive term series. We set

$$\lim_{n \rightarrow +\infty} \sqrt[n]{u_n} = \ell$$

- ① If  $\ell < 1 \implies \sum u_n$  converges.
- ② If  $\ell > 1 \implies \sum u_n$  diverges.
- ③ If  $\ell = 1$  we can't say any thing on the nature of the series.

### Example.

Study the nature of the series

$$\sum_{n=1}^{+\infty} \left( \frac{n+1}{n} \right)^{-n^2}$$

## Cauchy root test

We have  $\forall n \in \mathbb{N}^* \ u_n \geq 0$  and

$$\sqrt[n]{u_n} = \left( \frac{n+1}{n} \right)^{-n} = \left( 1 + \frac{1}{n} \right)^{-n}$$

$$\lim_{n \rightarrow +\infty} \sqrt[n]{u_n} = \lim_{n \rightarrow +\infty} \left( \frac{n+1}{n} \right)^{-n} = \lim_{n \rightarrow +\infty} e^{-n \ln(1 + \frac{1}{n})} = \lim_{n \rightarrow +\infty} e^{-n(\frac{1}{n})} = e^{-1} < 1$$

Then the series  $\sum_{n=1}^{+\infty} \left( \frac{n+1}{n} \right)^{-n^2}$  converges.

## Cauchy root test

Example.

*Study the nature of the series*  $\sum_{n=0}^{+\infty} \left( \frac{n-1}{2n+3} \right)^n$

## Cauchy root test

### Example.

Study the nature of the series  $\sum_{n=0}^{+\infty} \left( \frac{n-1}{2n+3} \right)^n$

$\sum_{n=0}^{+\infty} \left( \frac{n-1}{2n+3} \right)^n$  is a positive term series and we have

$$\lim_{n \rightarrow +\infty} \sqrt[n]{u_n} = \lim_{n \rightarrow +\infty} \frac{n-1}{2n+3}$$

$$= \frac{1}{2} < 1$$

Then the series  $\sum_{n=0}^{+\infty} \left( \frac{n-1}{2n+3} \right)^n$  converges.

## Link between D'Alembert ratio test and Cauchy root test

### Proposition

Let  $\sum u_n$  be a positive term series then

$$\lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = l \implies \lim_{n \rightarrow +\infty} \sqrt[n]{u_n} = \ell$$

### Remark

*The converse is false.*

# Course outline

- 1 Generalities
  - Convergence of a series
  - Divergence Test
  - Propriétés et opérations sur les séries
- 2 Positive term series
  - Convergence criteria for Positive terms series
- 3 Arbitrary term series
- 4 Alternating series

## Arbitrary term series

### Definition

We call arbitrary term series all series  $\sum u_n$  which the general term can take positive or negative values.

### Example.

The series

$$\sum \frac{(-1)^n}{n^2}, \sum \sin\left(n\frac{\pi}{2}\right)$$

are arbitrary term series.

## Absolutely convergent series

### Definition

If  $\sum |u_n|$  converges we say that the series  $\sum u_n$  is absolutely convergent.

### Remark

All convergent positive term series is absolutely convergent.

### Theorem

Let  $\sum u_n$  an arbitrary term series

- If  $\sum u_n$  is absolutely convergent then  $\sum u_n$  is convergent. That is to say  $\sum |u_n|$  converges  $\implies \sum u_n$  converges.
- The converse is false.
- If  $\sum u_n$  diverges then  $\sum |u_n|$  diverges.

## Absolutely convergent series

Example.

*Study the nature of the series*

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$$

## Absolutely convergent series

Example.

Study the nature of the series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

We have

$$|u_n| = \left| \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$$

The series  $\sum \frac{1}{n^2}$  is convergent then the series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$  is absolutely convergent then it converges.

## Absolutely convergent series

Example.

*Study the nature of the series*  $\sum_{n=1}^{+\infty} \frac{\cos(n\pi)}{n^3}$

## Absolutely convergent series

Example.

Study the nature of the series  $\sum_{n=1}^{+\infty} \frac{\cos(n\pi)}{n^3}$

We have

$$|u_n| = \left| \frac{\cos(n\pi)}{n^3} \right| \leq \frac{1}{n^3}$$

The series  $\sum \frac{1}{n^3}$  is convergent then the series  $\sum \left| \frac{\cos(n\pi)}{n^3} \right|$  converges  
therefore the series  $\sum \frac{\cos(n\pi)}{n^3}$  converges.

## Conditionally convergent series

### Definition

*A convergent series but non absolutely convergent is called conditionally convergent series.*

### Example.

*The series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is conditionally convergent.*

## Conditionally convergent series

### Definition

*A convergent series but non absolutely convergent is called conditionally convergent series.*

### Example.

The series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is conditionally convergent.

$|u_n| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$  and the series  $\sum \frac{1}{n}$  is divergent then the series

$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is not absolutely convergent but  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is convergent then

$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is conditionally convergent.

## Abel test

### Theorem

Let  $\sum u_n$  an arbitrary term series such that  $u_n = a_n \cdot b_n$ , où  $a_n$  et  $b_n$  two sequences satisfying

① La suite  $b_n$  decreasing and positive.

②  $\lim_{n \rightarrow +\infty} b_n = 0$ .

③  $\exists M > 0$  such that  $\forall n \in \mathbb{N}: \left| \sum_{k=0}^n a_k \right| \leq M$ .

Then  $\sum u_n$  is convergent.

## Abel test

Example.

Show that the series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is convergent

## Abel test

Example.

Show that the series  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is convergent

Let  $a_n = (-1)^n$  and  $b_n = \frac{1}{n}$ , we have

- ①  $b_n$  is a positive decreasing sequence.
- ②  $\lim_{n \rightarrow +\infty} b_n = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$ .
- ③ We have  $\forall n \geq 1$

$$\sum_{k=0}^n a_k = \begin{cases} -1 \\ 0 \\ 1. \end{cases}$$

Then  $\left| \sum_{k=0}^n a_k \right| \leq 1$ , By Abel test  $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$  is convergent.

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## Alternating series & Leibnitz test

### Definition

A series  $\sum u_n$  is said to be alternating series if and only if for all  $n \geq n_0$ , ( $n_0 \in \mathbb{N}$ )  $u_n = (-1)^n v_n$  or  $u_n = (-1)^{n+1} v_n$  with  $v_n \geq 0$ . Then all series of the form  $\sum (-1)^n u_n$ ,  $u_n \geq 0$  is said to be alternating series.

### Example.

The series  $\sum \frac{(-1)^n}{n}$ ,  $\sum \frac{(-1)^n}{n^2 + 3}$ ,  $\sum \frac{\cos(n\pi)}{e^n}$  are alternating series.

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### Example.

The series  $\sum \frac{(-1)^n}{n}$ ,  $\sum \frac{(-1)^n}{n^2 + 3}$ ,  $\sum \frac{\cos(n\pi)}{e^n}$  are alternating series.

### Theorem

Let  $\sum (-1)^n u_n$  be an alternating series. If  $(u_n)$  a positive decreasing convergent sequence to 0 then  $\sum (-1)^n u_n$  is convergent.

## Alternating series & Leibnitz test

Example.

Study the nature of the series  $\sum_{n=1}^{+\infty} (-1)^n \ln(1 + \frac{1}{n})$

## Alternating series & Leibnitz test

Example.

Study the nature of the series  $\sum_{n=1}^{+\infty} (-1)^n \ln(1 + \frac{1}{n})$

The series  $\sum_{n=1}^{+\infty} (-1)^n \ln(1 + \frac{1}{n})$  is a convergent alternating series, since we have:

- ①  $u_n = \ln(1 + \frac{1}{n}) \geq 0$ .
- ②  $(u_n)$  is decreasing since

$$u_n = f(n), f'(n) = \frac{\frac{-1}{n^2}}{1 + \frac{1}{n}} = \frac{-1}{n^2} \cdot \frac{n}{n+1} = \frac{-1}{n(n+1)} < 0.$$

- ③  $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \ln(1 + \frac{1}{n}) = 0$

Then by Leibnitz test the series  $\sum_{n=1}^{+\infty} (-1)^n \ln(1 + \frac{1}{n})$  is convergent.