The National School of Artificial Intelligence ENSIA		**Probability Semester 3 2024/2025**

Worksheet n°1

Exercise 1. A die is rigged in the following way: when it is thrown, the probability P that a number appears is proportional to that number. Determine P(1), ..., P(6).

Exercise 2. A queue is formed by randomly assigning sequence numbers to n people (n ≥ 2). Two friends A and B are in this queue.

1. What is the probability that the two friends are located one behind the other?
2. What is the probability that the two friends are r seats apart (i.e. separated by r - 1 people)?

Exercise 3. A letter is stored with probability p in a chest of drawers. This chest has seven drawers. Six drawers have been looked at without success.
Calculate the probability that the letter is in the seventh drawer.

Exercise 4. A runner is randomly selected from a group of athletes to undergo a doping control.
We call T the event: "The test is positive". According to statistics, we admit that P(T)=0,05.
We call D the event: "The runner is doped".
The anti-doping control is not 100% reliable, we know that:

If a runner is doped, the test is positive in 97% of the cases.
If a runner is not doped, the test is positive in 1% of the cases.

1. We note p the probability of D. Determine the p value.
2. A runner has a positive test. What is the probability that he is not doped?

Exercise 5. We consider a succession of bags that we call $S_1, S_2, \dots, S_n, \dots$
At the beginning the bag S_1 contains 2 black chips and 1 white chip; all the other bags contain 1 black chip and 1 white chip each. We randomly draw a chip from bag S_1 and place it in bag S_2. Then, we randomly draw a chip from bag S_2, which we place in bag S_3, and so on.
Let us note B_k the event: "the chip drawn from the bag S_k is white", and $p_k = P(B_k)$ its probability.

a. Show that $P(B_2/B_1) = \frac{2}{3}$ and $P(B_2/\bar{B}_1) = \frac{1}{3}$.

b. Show that for any integer $n \geq 1$: $p_{n+1} = \frac{1}{3}(p_n + 1)$.

c. For any $n \in \mathbb{N}^*$, we pose $q_n = p_n - \frac{1}{2}$. Show that the sequence $(p_n)_{n\in\mathbb{N}^*}$ is convergent and determine its limit.

Exercise 6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space where $\Omega = \{\omega_1, \omega_2, \dots, \omega_6\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and the probability is given by

$$\mathbb{P}(\{\omega_i\}) = \frac{i}{21}, i = 1,2,\dots,6.$$

Let the events

$$A = \{\omega_1, \omega_3\}; B = \{\omega_2, \omega_4\} \; and \; C = \{\omega_6\}.$$

Let us consider the application $X(.)$ from Ω to $\mathbb{R}$, given by

$$X(\omega) = 3\mathbb{I}_A - 4\mathbb{I}_B + \mathbb{I}_C,$$

Where $\mathbb{I}_A$ is the indicator function of A.

1. Show that X is a random variable on $(\Omega, \mathcal{A})$.
2. Determine the distribution of X.
3. Determine the cumulative distribution function of X.
4. Calculate the probabilities

$$\mathbb{P}(X > -4); \mathbb{P}(-4 \leq X < 3).$$

Exercise 7. Let X be the discrete random variable of probability distribution

X=k	-1	1	3
P(X=k)	1/3	1/4	α

1. Determine α.
2. Plot the cumulative distribution function of X.
3. Calculate the expectation of X, the variance and P(X<2,5 / X≥0,5).

Exercise 8. An urn contains 3 white balls and 2 black balls. The number of balls drawn is a random variable Y which depends on the number displayed by a die.

$$Y = \begin{cases} 1 \text{ } ball \text{ } is \text{ } drawn \text{ } if \text{ } the \text{ } die \text{ } display \text{ } 1 \text{ } or \text{ } 2 \text{ } or \text{ } 3 \\ 2 \text{ } balls \text{ } are \text{ } drawn \text{ } if \text{ } the \text{ } die \text{ } display \text{ } 4 \text{ } or \text{ } 5 \\ 3 \text{ } balls \text{ } are \text{ } drawn \text{ } if \text{ } the \text{ } die \text{ } display \text{ } 6 \end{cases}$$

1. Give the probability distribution of Y then calculate $E[Y]$ and Var(Y).
2. What is the probability of drawing at least one black ball? (The draw being without put-back).

Exercise 9. For $\theta \in]0,1[$ and $n \geq 2$, we define the sequence p_k by

$$p_k = \begin{cases} C_n(1-\theta)\min\{k, n-k\} \text{ } if \text{ } k = 1, \dots, n-1 \\ \theta \qquad\qquad\qquad if \text{ } k = n \end{cases}$$

Where C_n is a positive constant. Determine C_n such that p_k is a probability on $\{1, \dots, n\}$.

Exercise 10. Let X be a random variable with values in $\mathbb{N}$ and having expectation $\mathbb{E}[X] = m$ where $1 \leq m \leq 4$. We suppose that the distribution of X verify

$$\forall n \in \mathbb{N}, 6\mathbb{P}(x = n+2) - 7\mathbb{P}(x = n+1) + 2\mathbb{P}(x = n) = 0.$$

1. Determine $p_n = \mathbb{P}(x = n)$. Deduce the value of p_0 and p_1 (We suppose that $p_n = p^n, 0 < p < 1$).
2. Calculate $Var(X)$ when
   A. $m = 1$.
   B. $m = 2$.

Exercise 11. The cumulative distribution function of a random variable X is given by

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 2 \\ (x-2)^2 & \text{if} \quad 2 \leq x < 3. \\ 1 & \text{if} \quad x \geq 3 \end{cases}$$

1. Calculate $P\left(1 \leq X \leq \frac{5}{2}\right); P\left(X \geq \frac{5}{2} \middle/ X \leq \frac{7}{2}\right)$.
2. Find the density function of the random variable X.

Exercise 12.

1. Is there a constant C such that the function f defined by
$$f(x) = C(x^2 - 4x)1_{[-1,1]}(x)$$
is the density of a random variable X.
2. Consider X the random variable of density f: $f(x) = C(x^2 - 4x)1_{[0,2]}(x)$.
   a. Calculate C and determine the cumulative distribution function of X.
   b. Calculate $E[X^k], k \geq 1$, deduce $E[X]$ and Var(X).
   c. Calculate $P\left(X \geq \frac{1}{2} / X < 1\right)$.
   d. Determine the density of the random variable $Y = \sqrt{X}$.

Exercise 13. Let be the random variable of density $f(x) = \frac{\alpha}{1+x^2}, x \in \boldsymbol{R}$

1. Determine α, F and $E[X]$.
2. Determine the density of the random variable $Y = |X|$.

Exercise 14. A die is rolled 10 times. Let X be the random variable representing the number of marks multiple of 3 obtained. What is the probability distribution of X? Determine E[X] and Var(X).

Exercise 15. A player tosses a coin n times. The probability of getting tails on a toss is p. The player wins if he gets tails exactly once. For what values of p is he most likely to win?

Exercise 16. Two players of equal strength are playing against each other. What is the probability of one of the players winning 5 out of 8 games? At least 5 games out of 8?

Exercise 17. A doctor knows that one tenth of his patients have a disease A. In one morning he makes 20 visits.

1. What is the probability distribution of the variable N expressing the number of patients suffering from A.
2. Calculate P(N≥2).
3. Determine E[N] and Var(N).

Exercise 18. A die is rolled until a "6" is obtained and the experiment is stopped.

1. Give the probability distribution of the random variable X corresponding to the number of throws until a "6" appears.
2. Calculate the mathematical expectation of X and its variance.

Exercise 19. An urn contains white balls in proportion p=0.4 and black balls in proportion q=1-p. We draw a ball, note its color and put it back in the urn. We note by X the random variable equal to the number of draws at the end of which appears the first time a white ball. Determine the distribution of X. Calculate E[X] and Var(X).

Exercise 20. The proportion of defective tubes produced by a company is 2%.

a. What is the distribution of the number of defective tubes in a sample of 200 tubes? Determine its expectation and variance.
b. Are the conditions required to approximate this distribution by a Poisson distribution satisfied?
   Calculate the probabilities of obtaining a number of defective tubes:
- Zero
- equal to 5
- less than or equal to 6
- greater than or equal to 10

Exercise 21. A switchboard receives on average 2 calls per minute. The calls are randomly distributed in time.

1. What is the probability law governing the number of calls received in 3 minutes?

2. What is the probability that there are no calls in 3 minutes?
3. What is the probability that the number of calls in 2 minutes is greater than or equal to 5?

Exercise 22. Assume that the lifetime of a hard disk is distributed according to an exponential law. The manufacturer wants to guarantee that the hard drive has a probability of less than 0.001 of failing over one year. What is the minimum average life of the hard drive?

Exercise 23. Let X be a centered reduced normal random variable.

1. Determine the following probabilities:
    - $\mathbb{P}(X \geq 0)$ *and* $\mathbb{P}(X > 0)$;
    - $\mathbb{P}(X \leq 1), \mathbb{P}(X \geq 1), \mathbb{P}(X \leq -1), \mathbb{P}(X \geq -1), \mathbb{P}(X > -2.33)$ *and* $\mathbb{P}(X < 5.3)$;
    - $\mathbb{P}(-1 \leq X \leq 1), \mathbb{P}(-3 \leq X \leq -2)$ *and* $\mathbb{P}(|X| \geq 2.2)$.
2. Find x such that $\mathbb{P}(X \leq x) = 0.975, \mathbb{P}(X \geq x) = 0.90, \mathbb{P}(X \leq x) = 0.95$ *and* $\mathbb{P}(|X| < x) = 0.90$.

Exercise 24. Machines make sheet metal plates for stacking.

1. Let X be the Random Variable: "thickness of the plate in mm"; it is assumed that X follows the normal distribution of parameters μ=0.3 and σ=0.1.
   Calculate the probability that X is less than 0.36 mm and the probability that X is between 0.25 and 0.35?
2. The use of these plates consists in stacking n of them, numbered from 1 to N by taking them at random: let X_i be the random variable: "thickness of plate number i in mm" and Z the random variable: "thickness of n plates in mm".
   For n=20, what is the law of Z, its expectation and its variance?