Mathematical analysis 2
Chapter 4 : Sequences and Series of functions
Part 1: Sequences of functions

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Course outline

definition of sequence of functions

Let E be a non empty set of $\mathbb{R}$ and $\mathscr{F}(E,\mathbb{R})$ the set of all functions from E to $\mathbb{R}$ that is

$$\mathscr{F}(E,\mathbb{R}) = \{f \,/\, f : E \to \mathbb{R}\}$$

Definition

- *We call sequence of function all mapping from* $\mathbb{N}$ *to* $\mathscr{F}(E,\mathbb{R})$*:*

$$\begin{array}{rcl} f_n : \mathbb{N} & \longrightarrow & \mathscr{F}(E,\mathbb{R}) \\ n & \longmapsto & f_n \end{array}$$

- *A sequence of functions id denoted by* $\left(f_n\right)_n$.

Remark

It is important that all the functions f_n *are defined on the same set* E. *For example, one cannot consider the functions* $f_n : x \longmapsto \sqrt{x-n}$ *as a sequence of functions because each* f_n *is defined on* $[n,+\infty[$ *and* $E = \bigcap_{n\in\mathbb{N}} [n,+\infty[= \emptyset$

Generalities

Example.

- $f_n(x) = x^n$ *on* $]0,1[$.
- $f_n(x) = \cos(nx)$ *on* $\mathbb{R}$.
- $f_n(x) = \begin{cases} 2n^2x & \text{if} \quad 0 \le x \le 1/(2n) \\ 2n^2(1/n - x) & \text{if} \quad 1/(2n) < x < 1/n \\ 0 & \text{if} \quad 1/n \le x \le 1 \end{cases}$
- $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ *on* $\mathbb{R}$.

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Pointwise convergence

Definition

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on E.

1. *We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise at $x_0 \in E$ if the numerical sequence $(f_n(x_0))_{n\in\mathbb{N}}$ is convergent.*
2. *We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a function f on $I \subseteq E$ if for every fixed $x \in I$, the numerical sequence $(f_n(x))_n$ converges to $f(x)$, i.e.,*
$$\forall x \in I, \lim_{n\to+\infty} f_n(x) = f(x),$$
which means,
$$\forall x \in I, \forall \epsilon > 0, \exists N_{x,\epsilon}, \forall n \geq N_{x,\epsilon} \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

Remark

The integer N in this definition depends on both the choice of ϵ and the point $x \in I$.

Pointwise convergence

Example 1: Determine the pointwise limit of the sequence of functions (f_n) defined by

$$\forall n \in \mathbb{N}, f_n(x) = \frac{x}{1+nx}, E = [0,1].$$

We have:

- Case 1: If $x = 0$, $\lim_{n \to +\infty} f_n(0) = \lim_{n \to +\infty} 0 = 0$.
- Case 2: If $x \in (0,1]$, $\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \frac{x}{1+nx} = 0$.

Thus, $f_n \xrightarrow{\text{Simp}} f$ on E with $f(x) = 0$.

Pointwise convergence

Example 5.2. Suppose that $f_n : (0,1) \to \mathbb{R}$ is defined by $f_n(x) = \frac{n}{nx+1}$. Then, since $x \neq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x + \frac{1}{n}} = \frac{1}{x}.$$

So $f_n \to f$ pointwise, where $f : (0,1) \to \mathbb{R}$ is given by $f(x) = \frac{1}{x}$. We have $|f_n(x)| < n$ for all $x \in (0,1)$, so each f_n is bounded on $(0,1)$. However, their pointwise limit f is not bounded. Therefore, pointwise convergence does not, in general, preserve boundedness.

Pointwise convergence

Example 2: Study the pointwise convergence of the sequence of functions (f_n) where

$$\forall n \in \mathbb{N}, f_n(x) = \frac{nx}{1+nx}, E = [0,1].$$

Evaluating:

- Case 1: If $x = 0$, $\lim_{n\to+\infty} f_n(0) = \lim_{n\to+\infty} 0 = 0$.
- Case 2: If $x \in (0,1]$, $\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{nx}{1+nx} = \lim_{n\to+\infty} \frac{nx}{nx} = 1$.

Consequently, $f_n \xrightarrow{\text{Pointwise}} f$ on E with $f(x) = \begin{cases} 1 & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$.

Note that all functions f_n are continuous on E, but the limit function f is not.

Pointwise convergence

Example 3: Consider the sequence of functions

$$\forall n \geq 1, f_n(x) = \left(x^2 + \frac{1}{n^2}\right)^{\frac{1}{2}}, \quad E = \mathbb{R}$$

For $\forall x \in \mathbb{R}$, $\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \left(x^2 + \frac{1}{n^2}\right)^{\frac{1}{2}} = |x|$.
Hence, $f_n \xrightarrow{\text{Pointwise}} f$ on E where $f(x) = |x|$.
It's evident that all functions f_n are differentiable on E, but the limit function f is not.

Course outline

Uniform convergence

Definition

Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ uniformly on A if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.

Since $\left(\forall x \in I, |f_n(x) - f(x)| < \epsilon\right) \iff \sup_{x \in I} |f_n(x) - f(x)| < \epsilon$, the previous definition is thus equivalent to: $\forall \epsilon > 0, \exists N, \forall n \geq N \Rightarrow \sup_{x \in I} |f_n(x) - f(x)| < \epsilon$.

In other words: f is a uniform limit of the sequence $(f_n)_n$ on I if the sequence of terms $\sup_{x \in I} |f_n(x) - f(x)|$ tends to 0 as n tends to infinity. Therefore, we have the following proposition:

Proposition (Necessary and Sufficient Condition)

Let $(f_n)_n$ be a sequence of functions defined on E. The sequence $(f_n)_n$ converges uniformly to a function f on $I \subseteq E$ if and only if $\lim_{n \to +\infty} ||f_n - f|| = \lim_{n \to +\infty} \sup_{x \in I} |f_n(x) - f(x)| = 0$.

Uniform convergence

Example.

Study the pointwise and uniform convergence of the sequence of functions $(f_n)_n$ where $\forall n \geq 0, f_n(x) = \frac{x}{(1+x^2)^n}$, $E = \mathbb{R}^+$.

Pointwise convergence:

- If $x = 0$, $\lim\limits_{n \to +\infty} f_n(0) = \lim\limits_{n \to +\infty} 0 = 0$.
- If $x \in \mathbb{R}^+$,
$$\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \frac{x}{(1+x^2)^n} = \lim_{n \to +\infty} x e^{-n \log(1+x^2)} = 0.$$

Thus, $f_n \xrightarrow{\text{Pointwise}} f$ on E with $f(x) = 0$.

Uniform convergence: Verify if $\lim\limits_{n \to +\infty} ||f_n - f|| = 0$:

Calculate $\sup\limits_{x \in \mathbb{R}^+} |f_n(x) - 0| = \sup\limits_{x \in \mathbb{R}^+} \left| \frac{x}{(1+x^2)^n} \right| = \sup\limits_{x \in \mathbb{R}^+} \frac{x}{(1+x^2)^n}$. We just need to study the variation of f_n:

For $\forall x \in \mathbb{R}^+, f_n'(x) = \frac{1-(2n-1)x^2}{(1+x^2)^{n+1}}$. f_n' vanishes at $x = \frac{1}{\sqrt{2n-1}}$. The table of variation of f_n is as follows:

Uniform convergence

x	0	$\frac{1}{\sqrt{2n-1}}$	$+\infty$
$f_n(x)$	0 $\nearrow$	$f_n\left(\frac{1}{\sqrt{2n-1}}\right)$	$\searrow$ 0

Then we deduce that: $\sup\limits_{x\in\mathbb{R}^+} |f_n(x)-0| = f_n\left(\frac{1}{\sqrt{2n-1}}\right) = \frac{1}{\sqrt{2n-1}}\left(1-\frac{1}{2n}\right)^n.$

But, $\lim\limits_{n\to+\infty}\left(1-\frac{1}{2n}\right)^n = \lim\limits_{n\to+\infty} e^{n\log\left(1-\frac{1}{2n}\right)} = e^{-\frac{1}{2}}.$

Thus, $\lim\limits_{n\to+\infty}\frac{1}{\sqrt{2n-1}}\cdot\left(1-\frac{1}{2n}\right)^n = 0.$

We conclude that: $f_n \xrightarrow{\text{Unif}} f$ on $\mathbb{R}^+$.

Uniform convergence

The sequence $(v_n)_n$ where $v_n = \sup_{x \in I} |f_n(x) - f(x)|$ being a sequence with positive terms, a sufficient condition for it to converge to 0 is that the sequence $(v_n)_n$ be bounded by a sequence that converges to 0.
Consequently, we have the following result:

Proposition (Sufficient condition for uniform convergence)

For a sequence of functions $(f_n)_n$ converging uniformly on I to a function f, it suffices that there exists a numerical sequence $(u_n)_n$ such that:

$$\forall x \in I, \forall n \geq n_0, |f_n(x) - f(x)| \leq u_n \quad \text{and} \quad \lim_{n \to +\infty} u_n = 0.$$

Uniform convergence

Example.

Study the pointwise and uniform convergence of the sequence of functions $(f_n)_n$ on $[0,1]$ defined as $f_n(x) = \frac{ne^{-x}+x^2}{n+x}$.

For all $x \in [0,1]$, $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \frac{ne^{-x}+x^2}{n+x} = e^{-x}$. Hence, $f_n \xrightarrow{\text{Simp}} f$ on $[0,1]$ with $f(x) = e^{-x}$.

For uniform convergence: For all $x \in I$ and $n \geq 1$,

$$|f_n(x) - f(x)| = \left|\frac{ne^{-x}+x^2}{n+x} - e^{-x}\right| = |x| \left|\frac{x - e^{-x}}{n+x}\right| \leq \frac{2}{n},$$

and $\lim\limits_{n\to+\infty} \frac{2}{n} = 0$. Hence, $\lim\limits_{n\to+\infty} ||f_n - f|| = 0$, and thus $f_n \xrightarrow{\text{Unif}} f$ on $[0,1]$.

Uniform convergence

Proposition (Necessary condition 1)

Let $(f_n)_n$ *be a sequence of functions defined on* E. *Then,* $f_n \xrightarrow{Unif} f$ *on* $I \subseteq E$ *implies* $f_n \xrightarrow{Pointwise} f$ *on* $I \subseteq E$.

Proposition (Necessary condition 2)

Let $(f_n)_n$ *be a sequence of functions defined on* E. *If* $f_n \xrightarrow{Unif} f$ *on* $I \subseteq E$, *then for any sequence* $(x_n)_n$ *of points in* I *converging to* $x \in I$, $\lim\limits_{n \to +\infty} |f_n(x_n) - f(x_n)| = 0$.

Uniform convergence

Remark

By contraposition, from the previous proposition, if there exists a sequence $(x_n)_n \in I$ converging to $x \in I$ such that $\lim\limits_{n\to+\infty} f_n(x_n) - f(x_n) \neq 0$, then $f_n \nrightarrow$ Uniff on I.

Example.

Study the pointwise and uniform convergence of the sequence of functions $(f_n)_n$ on $E = [0,1]$ defined as $f_n(x) = \frac{nx}{1+n^3x^3}$.

Pointwise convergence:

- If $x = 0$, $f_n(0) = 0$, so $\lim\limits_{n\to+\infty} f_n(0) = 0$.
- For all $x \in (0,1)$, $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \dfrac{nx}{1+n^3x^3} = \lim\limits_{n\to+\infty} \dfrac{1}{n^2x^2} = 0$.

Thus, $f_n \xrightarrow{\text{Pointwise}} f$ on $[0,1]$ with $f(x) = 0$.

Uniform convergence: For all $n \in \mathbb{N}^*$, $x_n = \frac{1}{n} \in [0,1]$ and $\lim\limits_{n\to+\infty} x_n = 0 \in [0,1]$. As $\lim\limits_{n\to+\infty} \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} \neq 0$, hence $\sup\limits_{x\in E} |f_n(x) - f(x)| \geq \frac{1}{2} \nrightarrow 0$ as $n \to +\infty$. Therefore, $f_n \nrightarrow f$ Unif on $[0,1]$.

Course outline

Course outline

Boundedness

Theorem

Suppose that $f_n : E \to \mathbb{R}$ is bounded on E for every $n \in \mathbb{N}$, and $f_n \to f$ uniformly on E. Then $f : E \to \mathbb{R}$ is bounded on E.

Example.

The sequence of functions $f_n : (0,1) \to \mathbb{R}$ defined by $f_n(x) = \dfrac{n}{nx+1}$, cannot converge uniformly on $(0,1)$ since each f_n is bounded on $(0,1)$, but their pointwise limit $f(x) = \dfrac{1}{x}$ is not.

Course outline

1. Generalities
2. Pointwise convergence and uniform convergence of sequences of functions
   - Pointwise convergence
   - Uniform convergence
3. Properties of uniform convergence
   - Boundedness
   - Uniform convergence and continuity
   - Uniform convergence and differentiation
   - Uniform convergence and integration

Uniform convergence and continuity

Theorem

Let (f_n) be a sequence of functions defined on $I \subseteq \mathbb{R}$. If:

1. *All functions f_n are continuous at $a \in I$.*
2. *The sequence of functions (f_n) converges uniformly on I to a function f.*

Then f is continuous at a, and $\lim_{x \to a} \lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \lim_{x \to a} f_n(x)$.

Corollary

Let (f_n) be a sequence of functions defined on $I \subseteq \mathbb{R}$. If:

1. *All functions f_n are continuous on I.*
2. *The sequence of functions (f_n) converges uniformly on I to a function f.*

Then f is continuous on I.

Uniform convergence and continuity

Remark

By contraposition, from the previous Corollary, we deduce: If all f_n are continuous on I and the limit f_n is not continuous on I, then $f_n \nrightarrow f$ Unif on I.

Example.

Consider the sequence of functions $(f_n)_n$ defined by $f_n(x) = \frac{1}{1+nx}$ on $E = \mathbb{R}^+$.

Pointwise convergence: *We have $f_n \xrightarrow{Pointwise} f$ on $\mathbb{R}^+$ with*

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^+_* \\ 1 & \text{if } x = 0 \end{cases}.$$

Uniform convergence: *All f_n are continuous on $\mathbb{R}^+$, however, f is not continuous on $\mathbb{R}^+$. Hence, (f_n) is not uniformly convergent to f on $\mathbb{R}^+$.*

Course outline

Uniform convergence and differentiation

Theorem

Let $(f_n)_n$ *be a sequence of functions defined on* $E \subseteq \mathbb{R}$. *If:*

1. *All* f_n *are of class* C^1 *on* E.
2. $\exists x_0 \in I$ *such that the numerical sequence* $(f_n(x_0))_n$ *is convergent.*
3. *The sequence of derivative functions* $(f_n')_n$ *converges uniformly on* E *to a function* g.

Then:

1. *The sequence of functions* $(f_n)_n$ *converges uniformly on* E *to* f.
2. f *is of class* C^1 *on* E, *and we have* $(\lim_{n\to+\infty} f_n(x))' = \lim_{n\to+\infty} f_n'(x)$,

Uniform convergence and differentiation

Example. Consider the sequence of functions defined by:

$$f_n(x) = \frac{\sin(nx)}{n^2}, \quad x \in [0, 2\pi].$$

1. f_n **is** C^1 **on** $[0, 2\pi]$
2. **Simple convergence at a point** $x_0 = 0$**:** At $x_0 = 0$, we have:

$$f_n(0) = \frac{\sin(0)}{n^2} = 0, \quad \forall n.$$

The sequence trivially converges at this point.
3. **Uniform convergence of the derivatives** $f_n'(x)$**:** we have

$$f_n'(x) = \frac{\cos(nx)}{n}$$

Observe that:

$$|f_n'(x)| = \frac{|\cos(nx)|}{n}.$$

Uniform convergence and differentiation

Since $|\cos(nx)| \leq 1$ for all $x \in [0, 2\pi]$, it follows that:

$$\sup_{x\in[0,2\pi]} |f_n'(x)| = \frac{1}{n}.$$

This uniformly tends to 0 as $n \to \infty$. Therefore, $f_n'(x)$ uniformly converges to $g(x) = 0$ on $[0, 2\pi]$.
Then
The sequence $f_n(x)$ converges uniformly on $[0, 2\pi]$. Let's compute the pointwise limit:

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{n^2} = 0, \quad \forall x \in [0, 2\pi].$$

Thus, $f_n(x) \to 0$ uniformly on $[0, 2\pi]$.

Course outline

Uniform convergence and integration

Theorem

Let $(f_n)_n$ be a sequence of functions defined on $I = [a,b]$. If:

1. *All f_n are integrable on $[a,b]$.*
2. *The sequence $(f_n)_n$ is uniformly convergent on $[a,b]$ towards f.*

Then, the function f is integrable on $[a,b]$, and we have

$$\lim_{n\to+\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to+\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$