The National School of Artificial Intelligence ENSIA

Probability
Semester 3
2024/2025

Worksheet n°2

Exercise 1. Let X be a r.v. Show that if $\mathbb{E}[|X|] = 0$, then $X = 0\ a.s.$

Exercise 2. A r.v. X, with unknown distribution, have an expectation $\mu = 10$ and a standard deviation $\sigma = 5$. Show that for $n \geq 50$, the probability of the event $\{10 - n < X < 10 + n\}$ is at least equal to 0,99.

Exercise 3. For all non-zero integer n, we consider the function f_n defined by

$$f_n(x) = n^2 x \boldsymbol{e}^{\left(-\boldsymbol{n}^2\boldsymbol{x}^2/2\right)} \mathbb{I}_{\mathbb{R}_+}(x)$$

1. Show that f_n is a density of a random variable.
2. Let (X_n) be a sequence of random variables such that, for all $n \geq 1, X_n$ converges in probability to a random variable X that we will specify.

Exercise 4. For all $n \in \mathbb{N}$, we define :

$$f_n(x) = (1 - \cos(2\pi n x))\mathbb{I}_{[\mathbf{0},\mathbf{1}]}(x).$$

1. Verify that f_n is a density.
2. For all $n \in \mathbb{N}$, calculate the cumulative distribution function F_n associated to f_n.
3. Show that for all $x \in \mathbb{R}, F_n(x)$ converges to the cumulative distribution function of the uniforme law on $[0,1]$.
4. For $x \in \mathbb{R}$, does $f_n(x)$ converge when n goes to infinity ? Conclude.

Exercise 5. Let (U_n) be a sequence of independent random variables all following the uniform distribution on $[0,1]$. Let $M_n = \max(U_1, \dots, U_n)$ and $X_n = n(1 - M_n)$.

1. What is the cumulative distribution function of X_n ?
2. Study the convergence in law of the sequence (X_n).

Exercise 6. A factory manufactures parts of which an unknown proportion p is defective, and one wishes to find an approximate value of p. A sample of n parts is taken. It is assumed that the sample is taken from a very large population, and therefore that it can be compared to a series of n independent draws with replacement. We note X_n the random variable equal to the number of defective parts and we wish to quantify the fact that $\frac{X_n}{n}$ approaches p.

1. What is the distribution of X_n? Its mean? Its variance?
2. Show that, for all $\varepsilon > 0, \mathbb{P}\left(\left|\frac{X_n}{n} - p\right| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2}$.
3. Deduce a condition on n for $\frac{X_n}{n}$ to be an approximate value of p to the nearest 10^{-2} with probability greater than or equal to 95%.
4. Answer the previous question using this time an approximation of X_n by the central limit theorem. What do you think about it?

Exercise 7. Quite often, the number of reservations for an air route is higher than the number of passengers who actually show up on the day of the flight. This is due to unforeseeable impediments of some passengers and to a systematic policy of some of them who book seats on several flights in order to choose at the last moment the one that suits them best (because of the competition, and depending on the tariffs chosen, the airlines do not penalize the customers who withdraw and only charge those who actually board). To compensate for this, an airline operating a 300-seat aircraft decides to overbook by taking a number $n > 300$ reservations for each flight. If more than 300 passengers show up for boarding, the first 300 to arrive take their flight and the others are compensated financially.

1. We consider that the passengers are mutually independent and that the probability of withdrawal of each of them is 10%. Let us note n the number of reservations taken by the airline for a given flight and S_n the (random) number of passengers presenting themselves for boarding for this flight. Give the distribution of S_n, $\mathbb{E}[S_n]$ and $Var\ (S_n)$.
2. The company's sales manager would like to know the maximum value of n such that $\mathbb{P}(S_n \leq 300) \geq 0{,}99$.

Exercise 8. (Mellin transform). Let X be a positive random variable. Its Mellin transform is the function

$$T_X(t) = \mathbb{E}(X^t)$$

for all values of t for which the expectation of X^t exists.

1. Show that $T_X(t) = \varphi_{lnX}\left({}^{t}/_{i}\right)$ when the two sides are well defined.
2. Show that if X and Y are independent and positive, we have $T_{XY}(t) = T_X(t)T_Y(t)$.
3. Show that $T_{bX^a}(t) = b^t T_X(at)$ for $b > 0$ and at in the definition domain of $T_X(t)$.
4. Find the Mellin transform of a log-normale random variable X of parameters (m, σ). Use the fact that $T_X(k) = \mathbb{E}(X^k)$ to calculate the k^{th} moment of X for $k = 1,2, \ldots$

Exercise 9. The following table represents the joint probability law of a couple (X, Y).

Y \ X	-1	1	2
-1	0,1	0,2	0,1
2	p	0,15	0,25

1. a) Determine the constant p.
   b) Determine the distribution of X and of Y. Calculate $\mathbb{E}[X]$ and Var(X).
2. a) Determine the distribution of $X|_{Y=2}$.
   b) Are the variables X and Y independent?
   c) Calculate $\mathbb{E}[X|_{Y=2}]$ and Var($X|_{Y=2}$).
3. Calculate $\mathbb{P}(Y < 1{,}5 | X \geq 0{,}5)$.
4. Determine the distribution of S=X+Y. Deduce $\mathbb{E}[S]$.
5. Calculate $Cov(X, Y)$, what can we conclude from this?

Exercise 10. Let (X, Y) be a couple of r.v. with the joint distribution

$$p_{ij} = \mathbb{P}(X = i, Y = j) = \frac{\lambda^i}{i!}\frac{e^{-(1+\lambda)}}{(j-i)!}, \lambda > 0, (i, j) \in \mathbb{N}^2\ with\ i \leq j.$$

1. a) Determine the marginal of X and Y.
   b) Are the variables X and Y independent? Deduce $\mathbb{E}[X]$ and Var(X).

2. a) Determine the conditional distribution of $Y|_{X=i}$ and $X|_{Y=j}$.
   b) Calculate $\mathbb{E}[Y|_{X=i}]$ deduce $\mathbb{E}[Y|X]$. Calculate $\mathbb{E}\big[\mathbb{E}[Y|X]\big]$.
3. Same questions b et c for $\mathbb{E}\big[X|_{Y=j}\big]$ and $\mathbb{E}\big[\mathbb{E}[X|Y]\big]$.
4. Determine the distribution of Z=Y-X. Deduce $\mathbb{E}[Z]$ and $Var(Z)$.

Exercise 11. Let be the couple (X, Y) of joint density $f_{(X,Y)}(x, y) = ky\mathbb{I}_D(x, y)$
Where D being the interior of the triangle of vertices $(0,0), (0,1), (1,0)$.

1. a) Determine k.
   b) Determine the marginal distribution of X and of Y. Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
2. a) Determine the density of $Y|_{X=x}$. Are the variables X and Y independent?
   b) Calculate $\mathbb{E}[Y|_{X=x}]$ deduce $\mathbb{E}[Y|X]$ and $\mathbb{E}\big[\mathbb{E}[Y|X]\big]$.
3. Calculate $\mathbb{P}(Y > X)$.
4. a) Determine the joint distribution of the couple (Z, T) where $Z = X + Y$ and $T = X$.
   b) Deduce the density of Z.
5. Calculate $Cov(X, Y)$. What can we conclude from this?

Exercise 12. Let be the couple (X, Y) of joint density $f_{(X,Y)}(x, y) = ky|x|\mathbb{I}_D(x, y)$
Where $D = \left\{(x, y) \in \frac{\mathbb{R}^2}{y} \geq 0, x^2 + y^2 < 1\right\}$.

1. a) Determine $k, f_X(x)$ and $f_Y(y)$ then calculate $\mathbb{E}[X]$ et $\mathbb{E}[Y]$.
   c) Are the variables X and Y independent?
2. a) Determine $f_{X=x|}(y)$. Calculate $\mathbb{E}[Y|_{X=x}]$.
   c) Determine the density of $Z = \mathbb{E}\left[\frac{3}{2}Y \middle| X\right]$.
3. Calculate $\mathbb{P}\left(X \leq Y < \frac{1}{2}\right)$.
4. Calculate $Cov(X, Y)$. What can we conclude from this?
5. a) Determine the joint distribution of the couple (Z, T) where $Z = X^2$ and $T = X^2 + Y^2$.
   b) Deduce the density of T.