Chapter 3. Random Vectors

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Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilized space and $V = (X_1, \cdots, X_n)$. An application of Ω in $\mathbb{R}^n$, which to any $\omega \in \Omega$ maps a sequence $V(\omega) = (X_1(\omega), \cdots, X_n(\omega))$ is a random vector, if for all $i = 1, \cdots, n$, the application X_i is a random variable.

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$\{V(\omega) \leq x\} = \{X_1(\omega) \leq x_1, \cdots, X_n(\omega) \leq x_n\} \Longleftrightarrow \{\omega \in V^{-1}(x)\}$

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$\{V(\omega) \leq x\} = \{X_1(\omega) \leq x_1, \cdots, X_n(\omega) \leq x_n\} \Longleftrightarrow \{\omega \in V^{-1}(x)\}$
$V^{-1}(x)$ being the inverse image of the $\mathbb{R}^n$: $(]-\infty, x_1], \cdots,]-\infty, x_n])$.

Cumulative distribution function

The r.v. V is characterized by its cumulative distribution function F defined by $F(x) = \mathbb{P}(V(\omega) \leq x)$.

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Discrete case

The distribution of the random vector $V = \left(X_1, \cdots, X_n\right)$ is defined by

$$\mathbb{P}_X\left(k_1, \cdots, k_n\right) = \mathbb{P}\left(X_1 = k_1, \cdots, X_n = k_n\right),$$

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$$\mathbb{P}_X(k_1, \cdots, k_n) = \mathbb{P}(X_1 = k_1, \cdots, X_n = k_n),$$

and for all $A_1 \in \mathcal{F}_1, \cdots, A_n \in \mathcal{F}_n$

$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \sum_{k_1 \in A_1, \cdots, k_n \in A_n} \mathbb{P}_X(k_1, \cdots, k_n).$$

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The cumulative distribution function of the random vector V is defined by

$$F_V(x_1, \cdots, x_n) = \sum_{k_1 \leq x_1} \cdots \sum_{k_n \leq x_n} \mathbb{P}_X(k_1, \cdots, k_n).$$

Cumulative distribution function

Continuous case

F_X is defined by

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Hence the density of the random vector V is defined by

$$f_V\left(x_1, \cdots, x_n\right) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_V\left(x_1, \cdots, x_n\right).$$

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Then, for all $A_1 \in \mathcal{F}_1, \cdots, A_n \in \mathcal{F}_n$

$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \int_{A_1} \cdots \int_{A_n} f_V(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.$$

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Definition

The random variables $X_1, \cdots, X_n$ are said to be mutually independent if

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$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \int_{A_1} \cdots \int_{A_n} f_V(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.$$

Definition

The random variables $X_1, \cdots, X_n$ are said to be mutually independent if

$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in A_i).$$

Joint distribution. Marginal distribution - Discrete case

From now on, we'll consider the case n $= 2$, i.e. we'll be interested in the random pair (X, Y).

Let p_{ij} be the probability associated with the point (x_i, y_j)

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The distribution of X and the distribution of Y are called marginal distributions. We have
$\mathbb{P}(X = x_i) = \sum_j \mathbb{P}(X = x_i, Y = y_j) = \sum_j p_{ij} = p_{i\bullet}$ and
$\mathbb{P}(Y = y_i) = p_{\bullet j}$.
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Similarly, the cumulative distribution function of the marginal Y is

$$F_Y(y) = \mathbb{P}(Y \leq y) = F_V(+\infty, y).$$

Joint distribution. Marginal distribution - Discrete case

Conditional distribution

Let the event $\{X = x_i\}$ be such that $\mathbb{P}(X = x_i) \neq 0$ i.e. $p_{i\cdot} \neq 0$.

Joint distribution. Marginal distribution - Discrete case

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$$p_j^i = \mathbb{P}\left(Y = y_j \middle| X = x_i \right) = \frac{\mathbb{P}\left(X = x_i, Y = y_j\right)}{\mathbb{P}\left(X = x_i\right)} = \frac{p_{ij}}{p_{i.}},$$

in the same way

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Remarque

We can then determine the cumulative distribution function, expectation and variance of $Y|_{X=x_i}$ *and* $X|_{Y=y_j}$.

Joint distribution. Marginal distribution - Discrete case

Distribution of the sum of two random variables

The probability $\mathbb{P}\left(Z=k\right)$ of the sum $Z=X+Y$ of two random variables X and Y is the sum of the probabilities $\mathbb{P}\left(X=x_i, Y=y_j\right)$ extended to all pairs $\left(x_i, y_j\right)$ linked by the relation $k=x_i+y_j$.

$$\mathbb{P}\left(Z=k\right)=\sum_{i+j=k}\mathbb{P}\left(X=x_i, Y=y_j\right).$$

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If the random variables X and Y are independent, we have

Joint distribution. Marginal distribution - Discrete case

Distribution of the sum of two random variables

The probability $\mathbb{P}(Z = k)$ of the sum $Z = X + Y$ of two random variables X and Y is the sum of the probabilities $\mathbb{P}(X = x_i, Y = y_j)$ extended to all pairs (x_i, y_j) linked by the relation $k = x_i + y_j$.

$$\mathbb{P}(Z = k) = \sum_{i+j=k} \mathbb{P}(X = x_i, Y = y_j).$$

If the random variables X and Y are independent, we have

$$\mathbb{P}(Z = k) = \sum_{i+j=k} \mathbb{P}(X = x_i)\, \mathbb{P}(Y = y_j).$$

Joint distribution. Marginal distribution - Discrete case

Sum of of two binomial random variables

Example

Let X_1 and X_2 be two independent random variables such that $X_1 \rightsquigarrow \mathcal{B}(n_1, p)$ and $X_2 \rightsquigarrow \mathcal{B}(n_2, p)$.

Joint distribution. Marginal distribution - Discrete case

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Example

Let X_1 and X_2 be two independent random variables such that $X_1 \rightsquigarrow \mathcal{B}(n_1, p)$ and $X_2 \rightsquigarrow \mathcal{B}(n_2, p)$ .Find the distribution of the r.v. $Z = X_1 + X_2$

Joint distribution. Marginal distribution - Continuous case

Jointe distribution

A random variable $Z = (X, Y)$ is continuous if there exists an application $f(x, y)$ called density of the random pair (X, Y) verifying

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b. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dxdy = 1$

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The cumulative distribution function of the pair (X, Y) is defined by

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dudv$$

Joint distribution. Marginal distribution - Continuous case

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and we have

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}.$$

Joint distribution. Marginal distribution - Continuous case

Marginal distribution

The functions

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(u, v)\, dudv$$

Joint distribution. Marginal distribution - Continuous case

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and

$$F_Y(y) = \mathbb{P}(Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{+\infty} f(u, v)\, dudv$$

are called marginal cumulative ditribution functions of the random variables X and Y respectively.

Joint distribution. Marginal distribution - Continuous case

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The functions

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, v)\, dv \text{ et } f_Y(y) = \int_{-\infty}^{+\infty} f(u, y)\, du$$

are the marginal densities of X and Y respectively.

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$$f_{X|Y}\left(x|\, y\right) = \frac{f\left(x, y\right)}{f_Y\left(y\right)} \text{ if } f_Y\left(y\right) \neq 0.$$

Joint distribution. Marginal distribution - Continuous case

Conditional distribution

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$$f_{X|Y}(x|\, y) = \frac{f(x, y)}{f_Y(y)} \text{ if } f_Y(y) \neq 0.$$

The conditional density Y knowing $\{X = x\}$ is defined by

$$f_{Y|X}(y|\, x) = \frac{f(x, y)}{f_X(x)} \text{ if } f_X(x) \neq 0.$$

The random variables X and Y are independent if

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ \mathbb{P}\left(X \leq x, Y \leq y\right) = \mathbb{P}\left(X \leq x\right) \mathbb{P}\left(Y \leq y\right)$$

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or

$$F\left(x, y\right) = F_X\left(x\right)F_Y\left(y\right),$$

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or in the continuous case, deriving twice with respect to x and y

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Joint distribution. Marginal distribution - Continuous case

Independence of two random variables

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In this case

$$f_{X|Y}\left(x|\, y\right) = f_X\left(x\right) \text{ et } f_{Y|X}\left(y|\, x\right) = f_Y\left(y\right).$$

Joint distribution. Marginal distribution - Continuous case

Independence of two random variables

Example

A paire of random variables $Z = (X, Y)$ have the density

$$f(x, y) = \begin{cases} kxye^{-\left(x^2+y^2\right)} & \text{si } x > 0 \text{ et } y > 0 \\ 0 & \text{sinon} \end{cases}.$$

1. Find the coefficient k.
2. Find the marginal distributions of X and Y.
3. Calculate the conditional densities of X knowing $\{Y = y\}$ and of Y knowing $\{X = x\}$.

Joint distribution. Marginal distribution - Continuous case

Distribution of the sum of two random variables

Let $Z = X + Y$, we can determine the cumulative distribution function of the random variable Z. We have

Joint distribution. Marginal distribution - Continuous case

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\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) \\
&= \int\int_A f(x, y)\, dxdy,
\end{aligned}
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where $A = \left\{(x, y) \in \mathbb{R}^2 : x + y \leq z\right\}$

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where $A = \left\{(x, y) \in \mathbb{R}^2 : x + y \leq z\right\}$
If X and Y are independent, we have

$$
\begin{aligned}
F_Z(z) &= \int\int_A f(x, y)\, dxdy = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{z-x} f(x, y)\, dy\right] dx \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{z-x} f_Y(y)\, dy\right] f_X(x)\, dx \\
&= \int_{-\infty}^{+\infty} F_Y(z - x)\, f_X(x)\, dx.
\end{aligned}
$$

Joint distribution. Marginal distribution - Continuous case

Distribution of the sum of two random variables

In the same way, we can find

$$F_Z(z) = \int_{-\infty}^{+\infty} F_X(z-y)\, f_Y(y)\, dy.$$

Joint distribution. Marginal distribution - Continuous case

Distribution of the sum of two random variables

In the same way, we can find

$$F_Z\left(z\right) = \int_{-\infty}^{+\infty} F_X\left(z - y\right) f_Y\left(y\right) dy.$$

Deriving $F_Z\left(z\right)$ with respect to z we find

$$f_Z\left(z\right) = \int_{-\infty}^{+\infty} f_X\left(z - y\right) f_Y\left(y\right) dy \text{ et } f_Z\left(z\right) = \int_{-\infty}^{+\infty} f_Y\left(z - x\right) f_X\left(x\right) dx.$$

Joint distribution. Marginal distribution - Continuous case

Distribution of the sum of two random variables

In the same way, we can find

$$F_Z(z) = \int_{-\infty}^{+\infty} F_X(z-y)\, f_Y(y)\, dy.$$

Deriving $F_Z(z)$ with respect to z we find

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\, dy \text{ et } f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx.$$

f_Z is called the convolution product of f_X and f_Y noted $f_Z = f_X * f_Y$.

Joint distribution. Marginal distribution - Continuous case

Distribution of the sum of two random variables

Example

Let n random variables $X_i \rightsquigarrow \mathcal{N}(0,1)$ $(i = 1, \cdots, n)$. Determine the distribution of the random variable $Y = X_1^2 + \cdots + X_n^2$.