Academic year (2024/2025)
Mathematical analysis 2

Work sheet N 4
Sequences and series of functions

Exercise 1. Let (f_n) be the sequence of functions defined on $\mathbb{R}$ by $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

1. Study the pointwise convergence.
2. Show that the sequence converges uniformly on $[a, \infty)$ if $a > 0$.
3. Show that the sequence does not converge uniformly on $[0, \infty)$.

Exercise 2. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ where $f_n : [-1; 1] \to \mathbb{R}$ is defined by:

$$f_n(x) = \sin(nx) \cdot e^{-n \cdot x^2} + \sqrt{1-x^2}; \quad n \in \mathbb{N}$$

1. Show that the sequence of functions (f_n) converges on $[-1; 1]$ to a function f, which we will determine.
2. Show that the sequence (f_n) converges uniformly to f on any $[\alpha; 1]$, where $0 < \alpha < 1$.
3. Show that the sequence (f_n) does not converge uniformly to f on $[0; 1]$.

Exercise 3. Study the pointwise and uniform convergence of the sequence of functions $(f_n)_{n\geq 1}$ in each of the following cases (provide the domains where there is uniform convergence):

$$1)\ f_n(x) = \frac{ne^{-x} + x^2}{n+x} \text{ on } [0; 1]. \quad 2) f_n(x) = \frac{\ln(1+nx)}{1+nx} \text{ on } [0; +1[.$$

Exercise 4. Study the pointwise, uniform, and normal convergence of the series of functions $\sum f_n(x)$ in the following cases:

$$1)\ f_n(x) = \frac{x}{n(1+nx^2)} \text{ on } \mathbb{R}^+ \text{ for } n \geq 1. \qquad 2)\ f_n(x) = \frac{e^{-nx}}{1+n^2} \text{ for } n \geq 0$$

Exercise 5. Consider the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(1+e^x)}}$.

1. Find the domain D of convergence of the series.
2. Let $F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(1+e^x)}}$ for $x \in D$. Study the continuity and then the differentiability of F on D.