

Algebraic Geometry

Hassium

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1 Varieties

Definition 1.1. Let K be an algebraically closed field. The *affine n -space* \mathbb{A}_K^n is a set $\{(a_1, \dots, a_n) \mid a_i \in K\}$. An element $P \in \mathbb{A}_K^n$ is called a *point*, if $P = (a_1, \dots, a_n)$, each a_i is called a *coordinate* of P .

Remark. We will write \mathbb{A}^n for \mathbb{A}_K^n .

Let $A = K[x_1, \dots, x_n]$ be a polynomial ring, then A can be expressed as a function such that for all $f \in A$ and $P = (a_1, \dots, a_n) \in \mathbb{A}^n$, $f(P) = f(a_1, \dots, a_n)$, which substitutes x_i by a_i .

Definition 1.2. Let K be an algebraically closed field and $A = K[x_1, \dots, x_n]$. Let $T \subset A$, the *zero set* of T is the set $Z(T) = \{P \in \mathbb{A}^n \mid f \in T \text{ and } f(P) = 0\}$.

Let $T \subset A$ and let J be the ideal generated by T . For all $P \in Z(J)$ and $f \in T \subset J$, $f(P) = 0$, so $Z(J) \subset Z(T)$. For all $f = \sum_1^k a_i t_i$, where $a_i \in A$ and $t_i \in T$, and $P \in Z(T)$, we have $t_i(P) = 0$, so $f(P) = 0$, hence $Z(T) = Z(J)$. Since K is a field, K is a PID, so K is noetherian. By Hilbert basis theorem, A is noetherian, then all ideal are finitely generated. Let an ideal $J = (f_1, \dots, f_r)$, since $Z(T) = Z(J)$, $Z(T)$ is the set of common zeros of those polynomials.

Definition 1.3. A subset Y of \mathbb{A}^n is an *algebraic set* if there exists a subset $T \subset A$ such that $Y = Z(T)$.

Proposition. The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

Proof. Let $\{Y_i\}_{i \in I}$ be an arbitrary family of algebraic sets with $Y_i = Z(T_i)$. (i) Consider $Y_1 \cup Y_2$. For all $P \in Y_1$, $f \in T_1$, and $g \in T_2$, then $fg(P) = f(P)g(P) = 0$, so $P \in Z(T_1 T_2)$. For all $P' \in Z(T_1 T_2)$, $fg(P') = f(P')g(P')$, since A is an integral domain, either $f(P')$ or $g(P')$ is 0, then $P' \in Z(T_1) \cup Z(T_2) = Y_1 \cup Y_2$. (ii) Consider $\bigcap Y_i = \bigcap Z(T_i)$. For all $P \in Z(T_i)$ and $f_i \in T_i$, $f_i(P) = 0$, then $P \in Z(\bigcup T_i)$. For all $P' \in Z(\bigcup T_i)$, $f_i(P) = 0$, so $P \in Z(T_i)$ for all $i \in I$, which implies $P \in \bigcap Z(T_i) = \bigcap Y_i$. (iii) Let $T = (1)$, then $Z(T) = \emptyset$. Let $T = \{0\}$, then $Z(T) = \mathbb{A}^n$. \square

Definition 1.4. The open subsets of the *Zariski topology* on \mathbb{A}^n is the complements of the algebraic sets.

By the previous proposition, it is trivial that this defines a topology.

Example. The open sets of the Zariski topology on \mathbb{A}^1 are the empty set and the complements of finite subsets. This topology is not Hausdorff.

Proof. The space $\mathbb{A}^1 = K[x]$. Since K is a field, $K[x]$ is a PID. Since K is algebraically closed, any polynomial can be factorized as $f(x) = c(x_1 - a_1) \cdots (x_n - a_n)$, where $c, a_i \in K$, then $Z(f) = \{a_1, \dots, a_n\}$. Hence all closed subsets are either finite or \mathbb{A}^1 , which is $\mathbb{A}^1 \setminus \emptyset$. Suppose the space is Hausdorff. For all $x, y \in \mathbb{A}^1$, let a desired $U_x = \mathbb{A}^1 / \{a_1, \dots, a_n\} \neq \emptyset$, then $U_y = \{a_1, \dots, a_n\}$, which is finite, yet contradiction. \square

Definition 1.5. A nonempty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y , the subspace topology.

Remark. The empty set is not considered to be irreducible.

Example. The space \mathbb{A}^1 is irreducible.

Proof. All proper closed sets of \mathbb{A}^1 is finite. Since \mathbb{A}^1 is infinite, \mathbb{A}^1 cannot be the union of two proper closed subsets, yet contradiction. \square

Example. Any nonempty open subset of an irreducible space is irreducible and dense.

Proof. Let X be an irreducible space and let Y be a nonempty open irreducible subset of X . (irreducible) Suppose Y is reducible, then there exists open subsets $Y_1, Y_2 \subset Y$ such that $Y = (Y/Y_1) \cup (Y/Y_2) = (Y/X_1) \cup (Y/X_2)$, where X_i is open in X . Let $x \in X$. If $x \notin Y$, then $x \in X_1$ \square

Example. If Y is an irreducible subset of X , then its closure \overline{Y} in X is also irreducible.

Proof. \square

Definition 1.6. An *affine algebraic variety* is an irreducible closed subset of \mathbb{A}^n . An open subset of an affine variety is called a *quasi-affine variety*.

Remark.

Theorem 1.1 (Hilbert's Nullstellensatz). Let K be an algebraically closed field, let J be an ideal in the polynomial ring $A = K[x_1, \dots, x_n]$, and let $f \in A$ vanishes at all points of $Z(J)$. Then $f^r \in J$ for some integer $r > 0$.

Proof. Consider the *Rabinowitsch trick*. \square

Proposition. If $T_1 \subset T_2$ are subsets of A , then $Z(T_2) \subset Z(T_1)$. If $Y_1 \subset Y_2$ are subsets of \mathbb{A}^1 , then $I(Y_2) \subset I(Y_1)$. For any subsets $Y_1, Y_2 \subset \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

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Exercises and Proofs

Exercise 1.1.1. Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over K . Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over K . Let f be any irreducible quadratic polynomial in $K[x, y]$, and let W be the conic defined by f . Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof. (i)

□

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