Algebraic Geometry

Hassium

1 Varieties

Definition 1.1. Let K be an algebraically closed field. The *affine n-space* $\mathbb{A}^n_K$ is a set $\{(a_1, \ldots, a_n) \mid a_i \in K\}$. An element $P \in \mathbb{A}^n_K$ is called a *point*, if $P = (a_1, \ldots, a_n)$, each a_i is called a *coordinate* of P.

Remark. We will write $\mathbb{A}^n$ for $\mathbb{A}^n_K$.

Let $A = K[x_1, \ldots, x_n]$ be a polynomial ring, then A can be expressed as a function such that for all $f \in A$ and $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$, $f(P) = f(a_1, \ldots, a_n)$, which substitues x_i by a_i.

Definition 1.2. Let K be an algebraically closed field and $A = K[x_1, \ldots, x_n]$. Let $T \subset A$, the *zero set* of T is the set $Z(T) = \{P \in \mathbb{A}^n \mid f \in T \text{ and } f(P) = 0\}$.

Let $T \subset A$ and let J be the ideal generated by T. For all $P \in Z(J)$ and $f \in T \subset J$, $f(P) = 0$, so $Z(J) \subset Z(T)$. For all $f = \sum_1^k a_i t_i$, where $a_i \in A$ and $t_i \in T$, and $P \in Z(T)$, we have $t_i(P) = 0$, so $f(P) = 0$, hence $Z(T) = Z(J)$. Since K is a field, K is a PID, so K is noetherian. By Hilbert basis theorem, A is noetherian, then all idea are finitely generated. Let an ideal $J = (f_1, \ldots, f_r)$, since $Z(T) = Z(J)$, $Z(T)$ is the set of common zeros of those polynomials.

Definition 1.3. A subset Y of $\mathbb{A}^n$ is an *algebraic set* if there exists a subset $T \subset A$ such that $Y = Z(T)$.

Proposition. The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

Proof. Let $\{Y_i\}_{i \in I}$ be an arbitrary family of algebraic sets with $Y_i = Z(T_i)$. (i) Consider $Y_1 \cup Y_2$. For all $P \in Y_1$, $f \in T_1$, and $g \in T_2$, then $fg(P) = f(P)g(P) = 0$, so $P \in Z(T_1T_2)$. For all $P' \in Z(T_1T_2)$, $fg(P') = f(P')g(P')$, since A is an integral domain, either $f(P')$ or $g(P')$ is 0, then $P' \in Z(T_1) \cup Z(T_2) = Y_1 \cup Y_2$. (ii) Consider $\bigcap Y_i = \bigcap Z(T_i)$. For all $P \in Z(T_i)$ and $f_i \in T_i$, $f_i(P) = 0$, then $P \in Z(\bigcup T_i)$. For all $P' \in Z(\bigcup T_i)$, $f_i(P) = 0$, so $P \in Z(T_i)$ for all $i \in I$, which implies $P \in \bigcap Z(T_i) = \bigcap Y_i$. (iii) Let $T = (1)$, then $Z(T) = \varnothing$. Let $T = \{0\}$, then $Z(T) = \mathbb{A}^n$. $\square$

Definition 1.4. The open subsets of the *Zariski topology* on $\mathbb{A}^n$ is the complements of the algebraic sets.

By the previous proposition, it is trivial that this defines a topology.

Example. The open sets of the Zariski topology on $\mathbb{A}^1$ are the empty set and the complements of finite subsets. This topology is not Hausdorff.

Proof. The space $\mathbb{A}^1 = K[x]$. Since K is a field, $K[x]$ is a PID. Since K is algebraically closed, any polynomial can be factorized as $f(x) = c(x_1 - a_1) \cdots (x_n - a_n)$, where $c, a_i \in K$, then $Z(f) = \{a_1, \ldots, a_n\}$. Hence all closed subsets are either finite or $\mathbb{A}^1$, which is $\mathbb{A}^1 \setminus \varnothing$. Suppose the space is Hausdorff. For all $x, y \in \mathbb{A}^1$, let a desired $U_x = \mathbb{A}^1 / \{a_1, \ldots, a_n\} \neq \varnothing$, then $U_y = \{a_1, \ldots, a_n\}$, which is finite, yet contradiction. $\square$

Definition 1.5. A nonempty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y, the subspace topology.

Remark. The empty set is not considered to be irreducible.

Example. The space $\mathbb{A}^1$ is irreducible.

Proof. All proper closed sets of $\mathbb{A}^1$ is finite. Since $\mathbb{A}^1$ is infinite, $\mathbb{A}^1$ cannot be the union of two proper closed subsets, yet contradiction. □

Example. Any nonempty open subset of an irreducible space is irreducible and dense.

Proof. Let X be an irreducible space with $S \subset X$ and $S \neq \varnothing$. Consider □

Example. If Y is an irreducible subset of X, then its closure $\overline{Y}$ in X is also irreducible.

Proof. □

Definition 1.6. An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is called a *quasi-affine variety*.

Theorem 1.1 (Hilbert's Nullstellensatz)**.** Let K be an algebraically closed field, let J be an ideal in the polynomial ring $A = K[x_1, \dots, x_n]$, and let $f \in A$ vanishes at all points of $Z(J)$. Then $f^r \in J$ for some integer $r > 0$.

Proof. Consider the *Rabinowitsch trick*. □

Proposition. If $T_1 \subset T_2$ are subsets of A, then $Z(T_2) \subset Z(T_1)$. If $Y_1 \subset Y_2$ are subsets of $\mathbb{A}^1$, then $I(Y_2) \subset I(Y_2)$. For any subsets $Y_1, Y_2 \subset \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

2 Schemes

3 Cohomology

4 Curves

5 Surfaces

Exercises and Proofs

Exercise 1.1.1. Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over K. Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over K. Let f be any irreducible quadratic polynomial in $K[x, y]$, and let W be the conic defined by f Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof. (i) $\square$

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