Elementary Differential Geometry

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1 Calculus on Euclidean Space

Definition 1.1. The *Euclidean 3-space*, denoted $\mathbb{R}^3$, is the set of ordered triples of the form $p = (p_1, p_2, p_3)$, where $p_i \in \mathbb{R}$. An element of $\mathbb{R}^3$ is called a *point*.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. Define the addition to be $p + q = (p_i + q_i)$ and define the scalar multiplication to be $ap = (ap_i)$. The additive identity $0 = (0, 0, 0)$ is called the *origin* of $\mathbb{R}^3$. It is trivial that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Definition 1.2. Let x, y, and z be real-valued functions on $\mathbb{R}^3$ such that for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $x(p) = p_1$. $y(p) = p_2$, and $z(p) = p_3$. We call x, y, and z the *natural coordinate functions* of $\mathbb{R}^3$.

Let x, y, and z be the natural coordinate functions, rewrite $x = x_1$, $y = x_2$, and $z = x_3$. Then we have $p = (p_i) = (x_i(p))$.

Definition 1.3. A real-valued function f on $\mathbb{R}^3$ is *differentiable* if all partial derivatives exist and continuous.

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we define the norm to be $\|x\| = \sqrt{\sum {x_i}^2}$.

Definition 1.4. A subset $O \subset \mathbb{R}^3$ is *open* if for all $p \in O$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$.

Let $f : O \to \mathbb{R}$ be a function defined on an open set. The differetiability of f at p can be determined entirely from values of f on O. This means that differetiation is a local operation. We will discuss this later.

Definition 1.5. A *tangent vector* v_p is an ordered pair $v_p = (v, p)$, where $v, p \in \mathbb{R}^3$. Here v is called the *vector part* and p is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

Definition 1.6. Let $p \in \mathbb{R}^3$. The *tangent space* at p, denoted $T_p(\mathbb{R}^3)$, is the set of all tangent vectors that have p as point of application.

Fix a tangent space $T_p(\mathbb{R}^3)$ and let $T_p(\mathbb{R}^3)$ adapt the operations from $\mathbb{R}^3 \times \mathbb{R}^3$. We have a natural linear map $f : T_p(\mathbb{R}^3) \to \mathbb{R}^3$ defined by $v_p \to v$ and it is trivially an isomorphism.

Definition 1.7. A *vector field* V on $\mathbb{R}^3$ is a function $V : \mathbb{R}^3 \to \mathbb{R}^3$ such that for all $p \in \mathbb{R}^3$, $V(p) \subset T_p(\mathbb{R}^3)$.

Let V and W be vector field. Let f be a real-valued function. For all $p \in \mathbb{R}^3$, define $V + W$ by $(V + W)(p) = V(p) + W(p)$ and $(fV)(p) = f(p)V(p)$.

Definition 1.8. Let U_1, U_2, and U_3 be vector fields on $\mathbb{R}^3$ such that $U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, and $U_3(p) = (0, 0, 1)_p$ for all $p \in \mathbb{R}^3$. We call (U_1, U_2, U_3) the *natural frame field* on $\mathbb{R}^3$.

Proposition. Let V be a vector field on $\mathbb{R}^3$. There are three uniquely determined real-valued functions v_1, v_2, and v_3 on $\mathbb{R}^3$ such that $V = v_1U_1 + v_2U_2 + v_3U_3$.

Proof. For all $p \in \mathbb{R}^3$, $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1,0,0)_p + v_2(p)(0,1,0)_p + v_3(p)(0,0,1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3U_3(p)$, hence $V = \sum v_iU_i$. □

The functions v_1, v_2, and v_3 are called the *Euclidean coordinate functions* on V.

Definition 1.9. A vector field V is *differentiable* if its Euclidean coordinate functions are differentiable.

Definition 1.10. Let f be a differentiable real-valued function on $\mathbb{R}^3$ and let v_p be a tangent vector on $\mathbb{R}^3$. The *directional derivative* of f with respect to v_p, denoted $v_p[f]$, is defined to be $(\mathrm{d}/\mathrm{d}t)f(p+tv)$ at $t = 0$.

Remark. We will not write the restriction every time for convenience.

Proposition. Let $v_p = (v_1, v_2, v_3)_p$ be a tangent vector, then $v_p[f] = \sum v_i(\partial f/\partial x_i)(p)$.

Proof. Let $p = (p_1, p_2, p_3)$. Then $v_p[f] = (\mathrm{d}/\mathrm{d}t)f(p+tv)|_{t=0} = \sum(\partial f/\partial z)(p)\cdot(\mathrm{d}/\mathrm{d}t)(p_i+tv_i) = \sum(\partial f/\partial x_i)(p)v_i$. □

Example. Consider $f = x^2yz$ with $p = (1,1,0)$ and $v = (1,0,-3)$. By the definition, $p + tv = (1+t, 1, -3t)$, so $v_p[f] = (\mathrm{d}/\mathrm{d}t)(-3t^3 - 6t^2 - 3t) = -3$. Since $(\partial f/\partial x) = 2xyz$, $(\partial f/\partial y) = x^2z$, and $(\partial f/\partial z) = x^2y$, we have $(\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0$ and $(\partial f/\partial z)(p) = 1$, so $v_p[f] = -3$.

Proposition. Let f and g be functions on $\mathbb{R}^3$. Let v_p and w_p be tangent vectors. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.
2. $v_p[af + bg] = av_p[f] + bv_p[g]$.
3. $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$.

Proof. (i) We have $(av_p+bw_p)[f] = \sum(av_i+bw_i)(\partial f/\partial x_i)(p) = \sum av_i(\partial f/\partial x_i)+\sum bw_i(\partial f/\partial x_i)(p) = av_p[f]+bw_p[f]$. (ii) We have $v_p[af+bg] = \sum v_i(\partial(af+bg)/\partial x_i)(p) = \sum v_i(\partial(af)/\partial x_i)(p)+\sum v_i(\partial(bg)/\partial x_i)(p) = av_p[f]+bv_p[g]$. (iii) We have $v_p[fg] = \sum v_i(\partial(fg)/\partial x_i)(p) = \sum v_i(\partial f/\partial x_i)(p)g(p) + f(p)\sum v_i(\partial g/\partial x_i)(p) = v_p[f]g(p) + f(p)v_p[g]$. □

Let V be a vector field, we define $V[f]$ at $p \in \mathbb{R}^3$ to be $V(p)[f]$. By the convention, $U_i(p)[f] = (\partial f/\partial x_i)(p)$.

Proposition. Let V and W be vector fields. Let f, g, and h be real-valued functions. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(fV + gW)[h] = fV[h] + gW[h]$.
2. $V[af + bg] = aV[f] + bV[g]$.
3. $V[fg] = V[f]g + fV[g]$.

Proof. (i) For all $p \in \mathbb{R}^3$, $(fV + gW)(p)[h] = (f(p)V(p) + g(p)W(p))[h] = fV[h] + gW[h]$. (ii) For all $p \in \mathbb{R}^3$, $V(p)[af + bg] = aV(p)[f] + bV(p)[g]$. (iii) For all $p \in \mathbb{R}^3$, $V(p)[f]g(p) + f(p)V(p)[g] = V[f](p)g(p) + f(p)V[g](p) = (V[f]g + fV[g])(p)$. □

Example. Let $V = xU_1 - y^2U_3$ and let $f = x^2y + z^3$. Then $V[f] = xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3] = 2x^2y - 3y^2z^2$.

Let $I \subset \mathbb{R}$ be an open interval. Let $\alpha : I \to \mathbb{R}^3$ be a function. We can rewrite $\alpha(t)$ as $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where $\alpha_i : I \to \mathbb{R}$. We say α is *differentiable* if α_i are differentiable.

Definition 1.11. A *curve* in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an open interval.

Example. A curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = p + tq$, where $\alpha(0) = p$ and $q \neq 0$, is called a *straight line*.

Example. Here are some examples of curves.

1. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (a \cos t, a \sin t, bt)$.
2. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (1 + \cos t, \sin t, 2 \sin(t/2))$.
3. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$.
4. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$.

Definition 1.12. Let $\alpha : I \to \mathbb{R}^3$ be a curve with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For all $t \in I$, the *velocity vector* of α at t is the tangent vector $\alpha'(t) = ((\mathrm{d}\alpha_1/\mathrm{d}t)(t), (\mathrm{d}\alpha_2/\mathrm{d}t)(t), (\mathrm{d}\alpha_3/\mathrm{d}t)(t))_{\alpha(t)}$ at the point $\alpha(t) \in \mathbb{R}^3$. The curve α is said to be *regular* if $\alpha_i \neq 0$ for all i.

Consider the velocity vector $\alpha'(t)$, we can rewrite it by the natural frame fields, so $\alpha'(t) = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t)U_i(\alpha(t))$.

Definition 1.13. Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $h : J \to I$ be differentiable, where J is an open interval of $\mathbb{R}$. The *reparametrization* of α by h is the composition $\alpha \circ h : J \to \mathbb{R}^3$.

The composition of differentiable functions is differentiable, so any reparametrization is differentiable, which means it is a curve.

Proposition. Let β be the reparametrization of α by h, then $\beta'(s) = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$.

Proof. Rewrite $\beta(s) = \alpha(h(s))$, then we have $\beta'(s) = (\mathrm{d}(\alpha_i h_i)/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}\alpha_i/\mathrm{d}s)(h(s)) \cdot (\mathrm{d}h/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$. ☐

Proposition. Let α be a curve and let f be a differentiable function on $\mathbb{R}^3$, then $\alpha'(t)[f] = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$.

Proof. We have $\alpha'(t)[f] = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t) \cdot (\partial f/\partial x_i)(\alpha(t)) = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$ by the chain rule. ☐

Now we show a general idea of parametrizations. The proofs will be included in other sections when we have enough tools. Assume every result is correct for now.

Definition 1.14. A *1-form* φ on $\mathbb{R}^3$ is a function $\varphi : \coprod_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3) \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ and $v, w \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$, $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$.

Given a 1-form φ, for any point p, denote the restriction $\varphi|_{T_p(\mathbb{R}^3)} : T_p(\mathbb{R}^3) \to \mathbb{R}$ by φ_p, then φ_p is linear. Let φ and ψ be 1-forms. Define the addition and scalar multiplication by $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$ and $(f\varphi)(v_p) = f(p)\varphi(v_p)$. Given any 1-form φ and point p, φ_p is a linear functional in $T_p^*(\mathbb{R}^3)$, the dual space of $T_p(\mathbb{R}^3)$.

Definition 1.15. Let φ be a 1-form and let V be a vector field. For all $p \in \mathbb{R}^3$, define $\varphi(V)(p) = \varphi_p(V(p))$. We say φ is *differentiable* if for every differentiable vector field V, the function $\varphi(V)$ is differentiable.

Now let V and W be vector fields, we have $\varphi(fV + gW)(p) = \varphi((fV + gW)(p)) = \varphi(fV(p) + gW(p)) = (f\varphi(V) + g\varphi(W))(p)$. Similarly, $(f\varphi + g\psi)(V) = f\varphi(V) + g\psi(V)$.

Definition 1.16. If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable. The *differential* of f, denoted $\mathrm{d}f$, is the function $\mathrm{d}f(v_p) = v_p[f]$ for all tangent vectors v_p.

Let $v_p, w_p \in T_p(\mathbb{R}^3)$ and let $a, b \in \mathbb{R}$, then $\mathrm{d}f(av_p + bw_p) = (av_p + bw_p)[f] = av_p[f] + bw_p[f] = a\mathrm{d}f(v_p) + b\mathrm{d}f(w_p)$. Hence $\mathrm{d}f$ is a 1-form.

Example. Consider the natural coordinate functions x_i. We have $\mathrm{d}x_i(v_p) = v_p[x_i] = \sum v_i(\partial x_i/\partial x_j)(p) = v_i$.

Proposition. If φ is a 1-form on $\mathbb{R}^3$, then $\varphi = \sum f_i \mathrm{d}x_i$, where $f_i = \varphi(U_i)$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\varphi(v_p) = \varphi(\sum v_i U_i(p)) = \sum v_i \varphi(U_i(p)) = \sum v_i f_i(p) = \sum f_i(p)\mathrm{d}x_i(v_p) = (\sum f_i \mathrm{d}x_i)(v_p)$, hence $\varphi = \sum f_i \mathrm{d}x_i$. ☐

The functions f_1, f_2, and f_3 are called the *Euclidean coordinate functions* of the 1-form φ.

Proposition. Let f be a differentiable function on $\mathbb{R}^3$, then $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\mathrm{d}f(v_p) = v_p[f] = \sum v_i(\partial f/\partial x_i)(p) = \sum(\partial f/\partial x_i)(p)\mathrm{d}x_i(v_p) = (\sum(\partial f/\partial x_i)\mathrm{d}x_i)(v_p)$, hence $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$. □

Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(f+g) = \sum(\partial(f+g)/\partial x_i)\mathrm{d}x_i = \sum(\partial f/\partial x_i)\mathrm{d}x_i + \sum(\partial g/\partial x_i)\mathrm{d}x_i = \mathrm{d}f + \mathrm{d}g$. Now we denote the multiplication to be fg.

Proposition. Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$.

Proof. We have $\mathrm{d}(fg) = \sum(\partial(fg)/\partial x_i)\mathrm{d}x_i = \sum((\partial f/\partial x_i)g + (\partial g/\partial x_i)f)\mathrm{d}x_i = g\mathrm{d}f + f\mathrm{d}g$. □

Proposition. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable, then $\mathrm{d}(h(f)) = (\mathrm{d}h(f)/\mathrm{d}x)\mathrm{d}f$.

Proof. We have $\mathrm{d}(h(f)) = \sum(\partial h(f)/\partial x_i)\mathrm{d}x_i$, by the chain rule, $(\partial h(f)/\partial x_i)\mathrm{d}x_i = (\mathrm{d}h(f)/\mathrm{d}f)(\partial x/\partial x_i)$, so $\mathrm{d}(h(f)) = (\mathrm{d}f(h)/\mathrm{d}f)\mathrm{d}f$. □

Example. Consider the function $f = (x^2-1)y + (y^2+2)z$. We have $\mathrm{d}f = \mathrm{d}((x^2-1)y) + \mathrm{d}((y^2+2)z) = y\mathrm{d}(x^2 - 1) + (x^2+1)\mathrm{d}y + z\mathrm{d}(y^2+2) + (y^2+2)\mathrm{d}z = 2xy\mathrm{d}x + (x^2+2yz-1)\mathrm{d}y + (y^2+2)\mathrm{d}z$. Since $v_p[f] = \mathrm{d}f(v_p)$, $v_p[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_2^2+2)v_3$.

Definition 1.17. Let V be the vector space $\mathbb{R}^3$ and denote the space of all p-linear forms on V by $\Lambda^p(V^*)$. Every element of Λ^p is called a *p-form*. Define the *wedge product* to be a function $\wedge : \Lambda^a(V^*) \times \Lambda^b(V^*) \to \Lambda^{a+b}(V^*)$ such that for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $v_1, \ldots, v_{m+n} \in V$, the following properties hold.

1. $(\omega \wedge \eta)(v_1, \ldots, v_{m+n}) = (\sum_{\sigma\in\mathfrak{S}_{m+n}} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(m)})\eta(v_{\sigma(m+1)}, \ldots, v_{\sigma(m+n)}))/(m!n!)$.
2. $\omega \wedge \eta = (-1)^{mn}\eta \wedge \omega$.

Generally, a p-form is of the form $\sum f(x,y,z)\mathrm{d}x^i \wedge \cdots \mathrm{d}y^j \wedge \cdots \mathrm{d}z^k \wedge \cdots$. We have $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. If $i = j$, then $\mathrm{d}x_i \wedge \mathrm{d}x_i = -\mathrm{d}x_i \wedge \mathrm{d}x_i$, so $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. It is trivial that $\wedge$ is bilinear and associative, that is,

1. for $\omega_1, \omega_2 \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $a, b \in \mathbb{R}$,$(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta)$ and $\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2)$;
2. for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $\theta \in \Lambda^l(V^*)$, $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta$.

Now given a space of p-forms $\Lambda^p(V^*)$ with basis $\{e_1, e_2, e_3\}$, the basis of its dual space is denoted by $\{e^1, e^2, e^3\}$. The basis of $\Lambda^k(V^*)$ is of the form $e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $1 \le i_1 \le \cdots \le i_k \le 3$. In this case, the dimension of $\Lambda^p(V^*)$ is $3!/(p!(3-p)!)$. If $p > 4$, then $\dim(\Lambda^p(V^*)) = 0$, so there are no p-forms on $\mathbb{R}^3$ if $p \ge 4$.

Example. Let $\varphi = x\mathrm{d}x - y\mathrm{d}y$, $\psi = z\mathrm{d}x + x\mathrm{d}z$, $\theta = z\mathrm{d}y$, and $\eta = y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z$.

1. $\varphi \wedge \psi = xz\mathrm{d}x \wedge \mathrm{d}x + x^2\mathrm{d}x \wedge \mathrm{d}z - yz\mathrm{d}y \wedge \mathrm{d}x - yx\mathrm{d}y \wedge \mathrm{d}z = yz\mathrm{d}x \wedge \mathrm{d}y + x^2\mathrm{d}x \wedge \mathrm{d}z - yx\mathrm{d}y \wedge \mathrm{d}z$
2. $\theta \wedge (\varphi \wedge \psi) = yz^2\mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}y) + x^2z\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y - xyz\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}y = -x^2z\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
3. $\varphi \wedge \eta = xy\mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}z + x^2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - y^2\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z - xy\mathrm{d}y \wedge \mathrm{d}y \wedge \mathrm{d}z = (x^2+y^2)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Proposition. Let φ and ψ be 1-forms, then $\varphi \wedge \psi = -\psi \wedge \varphi$.

Proof. Rewrite $\varphi = \sum f_i\mathrm{d}x_i$ and $\psi = \sum g_i\mathrm{d}x_i$, then $\varphi \wedge \psi = \sum f_ig_i\mathrm{d}x_i\mathrm{d}x_j = \sum -g_if_i\mathrm{d}x_j\mathrm{d}x_i = -\psi \wedge \varphi$. □

Definition 1.18. Let $\varphi = \sum f_i\mathrm{d}x_i$ be a 1-form on $\mathbb{R}^3$. The *exterior derivative* of φ is the 2-form $\mathrm{d}\varphi = \sum \mathrm{d}f_i \wedge \mathrm{d}x_i$. Let $\psi = \sum f_{i,j}\mathrm{d}x_i \wedge \mathrm{d}x_j$ be a 2-form. The *exterior derivative* of ψ is the 3-form $\mathrm{d}\psi = \sum \mathrm{d}f_{i,j} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j$.

Let $a, b \in \mathbb{R}$. Let $\varphi = \sum f_i\mathrm{d}x_i$ and $\psi = \sum g_i\mathrm{d}x_i$ be 1-forms. Then $\mathrm{d}(a\varphi + b\psi) = \mathrm{d}(\sum(af_i + bg_i)\mathrm{d}x_i) = \sum \mathrm{d}(af_i + bg_i) \wedge \mathrm{d}x_i$, since the differential is linear, the exterior derivative is linear.

Proposition. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions and let φ and ψ be 1-forms. Then $\mathrm{d}(f\varphi) = \mathrm{d}f \wedge \varphi + f\mathrm{d}\varphi$ and $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$.

Proof. (i) Let $\varphi = \sum g_i \mathrm{d}x_i$, then $f\varphi = \sum f g_i \mathrm{d}x_i$, so $\mathrm{d}(f\varphi) = \sum(f\mathrm{d}g_i + g_i\mathrm{d}f) \wedge \mathrm{d}x_i = \sum f\mathrm{d}g_i \wedge \mathrm{d}x_i + \sum g_i \mathrm{d}f \wedge \mathrm{d}x_i = f\mathrm{d}\varphi + \mathrm{d}f \wedge \varphi$. (ii) Since $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$, without lose of generality, let $\varphi = f\mathrm{d}x$ and let $\psi = g\mathrm{d}y$. Then $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}(fg\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}(fg) \wedge \mathrm{d}x \wedge \mathrm{d}y = (f\mathrm{d}g + g\mathrm{d}f) \wedge \mathrm{d}x \wedge \mathrm{d}y = f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y + g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$. For the right hand side, $\mathrm{d}\varphi \wedge \psi = \mathrm{d}f \wedge \mathrm{d}x \wedge g\mathrm{d}y = g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$ and $\varphi \wedge \mathrm{d}\psi = f\mathrm{d}x \wedge \mathrm{d}g \wedge \mathrm{d}y = -f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y$, hence $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$. □

Definition 1.19. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and let $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ such that $F(p) = (f_1(p), \dots, f_m(p))$ for all $p \in \mathbb{R}^n$. The functions f_i are called the *Euclidean coordinate functions* of F and we denote $F = (f_1, \dots, f_m)$.

Definition 1.20. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $F = (f_1, \dots, f_m)$, we say F is *differentiable* if all f_i are differentiable. If F is differentiable, we say F is a *mapping* from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 1.21. Let $\alpha : I \to \mathbb{R}^n$ be a curve and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then the composite function $\beta = F(\alpha) : I \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ called the *image* of α under F.

To examine the effect of a mapping, it suffices to take a proper α and check the image of it.

Example. The function $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F = (x - y, x + y, 2z)$ is a mapping. Trivially, F is a linear map, so F is determined by $F(u_i)$.

Example. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F = (u^2 - v^2, 2uv)$. Let $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (r\cos t, r\sin t)$, where $0 \le t \le 2\pi$. The image is $\beta(t) = (r^2 \cos 2t, r^2 \sin 2t)$. This curve takes two counterclockwise trips around the circle of radius r^2 centered at the origin. Therefore, F wraps $\mathbb{R}^2$ around itself twice.

Definition 1.22. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let $v_p \in T_p(\mathbb{R}^n)$. The *tangent map* of F, denoted $F_*(v_p)$, is defined to be $(\mathrm{d}/\mathrm{d}t)F(p + tv)$ at $t = 0$.

Fix some mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. For every $p \in \mathbb{R}^n$, it induces a tangent map of F at p, denoted F_{*p}.

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $v_p \in T_p(\mathbb{R}^n)$, then $F_{*p}(v_p) = (v[f_1], \dots, v[f_m])_{F(p)}$.

Proof. Fix $v_p \in T_p(\mathbb{R}^n)$. We have $F_{*p} = (\mathrm{d}/\mathrm{d}t)F(p + tv)|_{t=0} = (\mathrm{d}/\mathrm{d}t)(f_i(p + tv))|_{t=0} = (v_p[f_1], \dots, v_p[f_m])_{F(p)}$. □

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. For all $p \in T_p(\mathbb{R}^n)$, the tangent map $F_{*p} : T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is a linear map.

Proof. Fix $p \in \mathbb{R}^n$. Let $a, b \in \mathbb{R}$ and let $v_p, w_p \in T_p(\mathbb{R}^n)$. We have $F_{*p}(av_p + bw_p) = ((av_p + bw_p)[f_i])_{F(p)} = (av_p[f_i])_{F(p)} + (bw_p[f_i])_{F(p)} = aF_{*p}(v_p) + bF_{*p}(w_p)$. □

Proposition. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let β be the image of some curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$.

Proof. Let $F = (f_1, \dots, f_m)$. We have $F_*(\alpha'(t)) = (\alpha'(t)[f_i])_{F(\alpha(t))} = (\mathrm{d}f_i(\alpha(t))/\mathrm{d}t)_{F(\alpha(t))} = \beta'(t)$. □

Let $\{U_j\}$ and $\{\overline{U_i}\}$ be the natural frame fields of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then $F_*(U_j(p)) = \sum_{i=1}^m (\partial f_i/\partial x_j)(p)\overline{U_i}(F(p))$, where $1 \le j \le n$.

Proof. Recall that $U_j[f_i] = \partial f_i/\partial x_j$, so the proposition trivially holds. □

Definition 1.23. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. The *Jacobian matrix* of F at $x \in \mathbb{R}^n$ is the matrix

$$J_F(x) = \begin{pmatrix} \partial f_1/\partial x_1(x) & \cdots & \partial f_1/\partial x_n(x) \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1(x) & \cdots & \partial f_m/\partial x_n(x) \end{pmatrix}.$$

Definition 1.24. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. We say F is *regular* if for all $p \in \mathbb{R}^n$, F_{*p} is injective.

Notice that $J_F(p) \cdot v = F_{*p}$, so $J_F(p)$ is the matrix representation of F_{*p}.

Definition 1.25. A mapping is a *diffeomorphism* if it has a differentiable inverse mapping.

Definition 1.26. A *topological space* $(X, \mathcal{T})$ consists of two sets X and $\mathcal{T}$, where $\mathcal{T} \subset \mathscr{P}(X)$, that satisfies the following properties:

1. $\varnothing, X \in \mathcal{T}$;
2. any union of elements in $\mathcal{T}$ is also in $\mathcal{T}$;
3. any finite intersection of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

Definition 1.27. Let $(X, \mathcal{T})$ be a topological space. A subset $U \subset X$ is said to be *open* if $U \in \mathcal{T}$. Let $x \in X$, a *neighborhood* of x is an open set U_x that contains x.

Theorem 1.1 (inverse function theorem)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. If F_{*p} is injective at some $p \in \mathbb{R}^n$, then there exists a neighborhood U of p such that $F|_U : U \to V$, where V is open, is a diffeomorphism.

2 Frame Fields

3 Euclidean Geometry

4 Calculus on a Surface

5 Shape Operators

6 Geometry of Surfaces in $\mathbb{R}^3$

7 Riemannian Geometry

8 Global Structure of Surfaces

Exercises and Proofs

Exercise 1.1.1. Let $f = x^2y$ and $g = y\sin z$ be functions on $\mathbb{R}^3$. Express the following functions in terms of x, y, and z: fg^2, $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f$, $\frac{\partial^2(fg)}{\partial y \partial z}$, and $\frac{\partial}{\partial y}(\sin f)$.

Proof. (i) We have $fg^2 = x^2yy^2\sin^2 z = x^2y^3\sin^2 z$. (ii) We have $\partial f/\partial x = 2xy$ and $\partial g/\partial y = \sin z$, then $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = 2xy^2\sin z + x^2y\sin z$. (iii) We have $fg = x^2y^2\sin z$, then $\frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos z$. (iv) We have $\sin f = \sin(x^2y)$, then $\frac{\partial}{\partial y}(\sin f) = x^2\cos(x^2y)$. □

Exercise 1.1.3. Express $\partial f/\partial x$ in terms of x, y, and z for the following functions.

1. $f = x\sin(xy) + y\cos(xz)$;
2. $f = \sin g$, $g = e^h$, and $h = x^2 + y^2 + z^2$.

Proof. (i) We have $\frac{\partial f}{\partial x} = \frac{\partial(x\sin(xy))}{\partial x} + \frac{\partial(y\cos(xz))}{\partial x} = \sin(xy) + xy\cos(xy) - yz\sin(xz)$. (ii) We have $f = \sin(e^{x^2+y^2+z^2})$, then $\frac{\partial f}{\partial x} = 2x\cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}$. □

Exercise 1.2.1. Let $v = (-2, 1, -1)$ and $w = (0, 1, 3)$. At an arbitrary point p, express the tangent vector $3v_p - 2w_p$ as a linear combination of $U_1(p)$, $U_2(p)$, and $U_3(p)$.

Proof. We have $3v_p - 2w_p = (-6, 1, -9)_p = -6U_1(p) + U_2(p) - 9U_3(p)$. □

Exercise 1.2.3. Let $p = (p_1, p_2, p_3)$. In each case, express the given vector field V in the standard form $\sum v_iU_i$.

1. $2z^2U_1 = 7V + xyU_3$.
2. $V(p) = (p_1, p_3 - p_1, 0)_p$ for all p.
3. $V = 2(xU_1 + yU_2) - x(U_1 - y^2U_3)$.
4. For all $p \in \mathbb{R}^3$, $V(p)$ is the vector from (p_1, p_2, p_3) to $(1 + p_1, p_2p_3, p_2)$.
5. For all $p \in \mathbb{R}^3$, $V(p)$ is the vector from p to 0.

Proof. (i) We have $V = (2z^2U_1 - xyU_3)/7$. For all $p \in \mathbb{R}^3$, $V(p) = ((2z^2, 0, 0) - (0, 0, xy))/7 = (2z^2/7, 0, -xy/7)$, so $(v_i) = (2z^2/7, 0, -xy/7)$. (ii) Here $V(p) = xU_1 + (z - x)U_2 + 0U_3$. □

Exercise 1.2.5. Let $V_1 = U_1 - xU_3$, $V_2 = U_2$, and $V_3 = xU_1 + U_3$. Prove that the vectors $V_1(p)$, $V_2(p)$, $V_3(p)$ are linearly indepnedent at each $p \in \mathbb{R}^3$. Express the vector field $xU_1 + yU_2 + zU_3$ as a linear combination of V_i.

Proof. For all $p \in \mathbb{R}^3$, we have $V_1(p) = U_1(p) - xU_3(p) = (1, 0, -x)$. Similarly, $V_2(p) = (0, 1, 0)$ and $V_3 = (x, 0, 1)$. Consider $aV_1(p) + bV_2(p) + cV_3(p) = 0$, where $a, b, c \in \mathbb{R}$. Solve for (a, b, c), then $c(x^2 + 1) = 0$, so $c = 0$. Now $(a, b, c) = (0, 0, 0)$, hence $V_i(p)$ are linearly independent. For all $p \in \mathbb{R}^3$, $aV_1(p) + bV_2(p) + cV_3(p) = (a + cx, b, c - a)$ and $xU_1(p) + yU_2(p) + zU_3(p) = (x, y, z)$. Solve $(a + cx, b, c - a) = (x, y, z)$, then $(a, b, c) = ((x - zx)/(1 + x^2), y, (x^2 + z)/(1 + x^2))$. □

Exercise 1.3.1. Let v_p be the tangent vector with $v = (2, -1, 3)$ and $p = (2, 0, -1)$. Use the definition to compute the directional derivative for the following functions.

1. $f = y^2z$.
2. $f = x^7$.
3. $f = e^x\cos y$.

Proof. We have $p + tv = (2 + 2t, -t, 3t - 1)$. (i) Now $f(p + tv) = 3t^3 - t^2$, then $v_p[f] = 9t^2 - 2t = 0$. (ii) Now $f(p + tv) = (2 + 2t)^7$, then $v_p[f] = 7(2 + 2t)^6 \cdot 2 = 14(2 + 2t)^6 = 7 \cdot 2^7$. (iii) Now $f(p + tv) = e^{2+2t}\cos(-t)$, then $v_p[f] = e^{2+2t}\sin(-t) + 2e^{2+2t}\cos(-t) = 2e^2$. □

Exercise 1.3.3. Let $V = y^2U_1 - xU_3$. Let $f = xy$ and let $g = z^3$. Compute the following functions.

1. $V[f]$.

2. $V[g]$.
3. $V[fg]$.
4. $fV[g] - gV[f]$.
5. $V[f^2 + g^2]$.
6. $V[V[f]]$.

Proof. (i) We have $V[f] = y^2U_1[xy] - xU_3[xy] = y^3$. (ii) We have $V[g] = y^2U_1[z^3] - xU_3[z^3] = -3xz^2$. (iii) We have $V[fg] = V[f]g + fV[g] = y^3z^3 - 3x^2yz^2$. (iv) We have $fV[g] - gV[f] = -3x^2yz^2 - y^3z^3$. (v) We have $V[f^2 + g^2] = V[f^2] + V[g^2] = V[f]f + fV[f] + V[g]g + gV[g] = 2xy^4 - 6xz^5$. (vi) We have $V[V[f]] = V[y^3] = y^2U_1[y^3] - xU_3[y^3] = 0$. □

Exercise 1.3.5. If $V[f] = W[f]$ for all f on $\mathbb{R}^3$, prove that $V = W$.

Proof. Let $V = \sum a_iU_i$ and let $W = \sum b_iU_i$. Since $V[f] = W[f]$, $(V - W)[f] = \sum(a_i - b_i)\frac{\partial f}{\partial x_i} = 0$. Pick $f = x$, then $a_1 = b_1$. Similarly, if we pick $f = y$ and $f = z$, we have $a_2 = b_2$ and $a_3 = b_3$. Hence $V = W$. □

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