

# Elementary Differential Geometry

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## 1 Calculus on Euclidean Space

**Definition 1.1.** The *Euclidean 3-space*, denoted  $\mathbb{R}^3$ , is the set of ordered triples of the form  $p = (p_1, p_2, p_3)$ , where  $p_i \in \mathbb{R}$ . An element of  $\mathbb{R}^3$  is called a *point*.

Let  $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$  and let  $a \in \mathbb{R}$ . Define the addition to be  $p + q = (p_i + q_i)$  and define the scalar multiplication to be  $ap = (ap_i)$ . The additive identity  $0 = (0, 0, 0)$  is called the *origin* of  $\mathbb{R}^3$ . It is trivial that  $\mathbb{R}^3$  is a vector space over  $\mathbb{R}$ .

**Definition 1.2.** Let  $x, y$ , and  $z$  be real-valued functions on  $\mathbb{R}^3$  such that for all  $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ ,  $x(p) = p_1$ ,  $y(p) = p_2$ , and  $z(p) = p_3$ . We call  $x, y$ , and  $z$  the *natural coordinate functions* of  $\mathbb{R}^3$ .

Let  $x, y$ , and  $z$  be the natural coordinate functions, rewrite  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . Then we have  $p = (p_i) = (x_i(p))$ .

**Definition 1.3.** A real-valued function  $f$  on  $\mathbb{R}^3$  is *differentiable* if all partial derivatives exist and continuous.

Let  $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ . Recall that the dot product is defined to be  $p \cdot q = \sum p_i q_i$  and the norm is defined to be  $\|p\| = \sqrt{p \cdot p} = \sqrt{\sum p_i^2}$ .

**Definition 1.4.** A subset  $O \subset \mathbb{R}^3$  is *open* if for all  $p \in O$ , there exists  $\varepsilon > 0$  such that  $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$ .

Let  $f : O \rightarrow \mathbb{R}$  be a function defined on an open set. The differentiability of  $f$  at  $p$  can be determined entirely from values of  $f$  on  $O$ . This means that differentiation is a local operation. We will discuss this later.

**Definition 1.5.** A *tangent vector*  $v_p$  is an ordered pair  $v_p = (v, p)$ , where  $v, p \in \mathbb{R}^3$ . Here  $v$  is called the *vector part* and  $p$  is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

**Definition 1.6.** Let  $p \in \mathbb{R}^3$ . The *tangent space* at  $p$ , denoted  $T_p(\mathbb{R}^3)$ , is the set of all tangent vectors that have  $p$  as point of application.

Fix a tangent space  $T_p(\mathbb{R}^3)$  and let  $T_p(\mathbb{R}^3)$  adapt the operations from  $\mathbb{R}^3 \times \mathbb{R}^3$ . We have a natural linear map  $f : T_p(\mathbb{R}^3) \rightarrow \mathbb{R}^3$  defined by  $v_p \rightarrow v$  and it is trivially an isomorphism.

**Definition 1.7.** A *vector field*  $V$  on  $\mathbb{R}^3$  is a function  $V : \mathbb{R}^3 \rightarrow \bigsqcup_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3)$  such that for all  $p \in \mathbb{R}^3$ ,  $V(p) \subset T_p(\mathbb{R}^3)$ .

Let  $V$  and  $W$  be vector field. Let  $f$  be a real-valued function. For all  $p \in \mathbb{R}^3$ , define  $V + W$  by  $(V + W)(p) = V(p) + W(p)$  and  $(fV)(p) = f(p)V(p)$ .

**Definition 1.8.** Let  $U_1, U_2$ , and  $U_3$  be vector fields on  $\mathbb{R}^3$  such that  $U_1(p) = (1, 0, 0)_p$ ,  $U_2(p) = (0, 1, 0)_p$ , and  $U_3(p) = (0, 0, 1)_p$  for all  $p \in \mathbb{R}^3$ . We call  $(U_1, U_2, U_3)$  the *natural frame field* on  $\mathbb{R}^3$ .

**Proposition.** Let  $V$  be a vector field on  $\mathbb{R}^3$ . There are three uniquely determined real-valued functions  $v_1, v_2$ , and  $v_3$  on  $\mathbb{R}^3$  such that  $V = v_1U_1 + v_2U_2 + v_3U_3$ .

*Proof.* For all  $p \in \mathbb{R}^3$ ,  $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1, 0, 0)_p + v_2(p)(0, 1, 0)_p + v_3(p)(0, 0, 1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3(p)U_3(p)$ , hence  $V = \sum v_i U_i$ .  $\square$

The functions  $v_1, v_2$ , and  $v_3$  are called the *Euclidean coordinate functions* on  $V$ .

**Definition 1.9.** A vector field  $V$  is *differentiable* if its Euclidean coordinate functions are differentiable.

**Definition 1.10.** Let  $f$  be a differentiable real-valued function on  $\mathbb{R}^3$  and let  $v_p$  be a tangent vector on  $\mathbb{R}^3$ . The *directional derivative* of  $f$  with respect to  $v_p$ , denoted  $v_p[f]$ , is defined to be  $(d/dt)f(p + tv)$  at  $t = 0$ .

**Remark.** We will not write the restriction every time for convenience.

**Proposition.** Let  $v_p = (v_1, v_2, v_3)_p$  be a tangent vector, then  $v_p[f] = \sum v_i(\partial f/\partial x_i)(p)$ .

*Proof.* Let  $p = (p_1, p_2, p_3)$ . Then  $v_p[f] = (d/dt)f(p + tv)|_{t=0} = \sum (\partial f/\partial x_i)(p) \cdot (d/dt)(p_i + tv_i) = \sum (\partial f/\partial x_i)(p)v_i$ .  $\square$

**Example.** Consider  $f = x^2yz$  with  $p = (1, 1, 0)$  and  $v = (1, 0, -3)$ . By the definition,  $p + tv = (1 + t, 1, -3t)$ , so  $v_p[f] = (d/dt)(-3t^3 - 6t^2 - 3t) = -3$ . Since  $(\partial f/\partial x) = 2xyz$ ,  $(\partial f/\partial y) = x^2z$ , and  $(\partial f/\partial z) = x^2y$ , we have  $(\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0$  and  $(\partial f/\partial z)(p) = 1$ , so  $v_p[f] = -3$ .

**Proposition.** Let  $f$  and  $g$  be functions on  $\mathbb{R}^3$ . Let  $v_p$  and  $w_p$  be tangent vectors. For all  $a, b \in \mathbb{R}$ , the following properties hold.

1.  $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$ .
2.  $v_p[af + bg] = av_p[f] + bv_p[g]$ .
3.  $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$ .

*Proof.* (i) We have  $(av_p + bw_p)[f] = \sum (av_i + bw_i)(\partial f/\partial x_i)(p) = \sum av_i(\partial f/\partial x_i)(p) + \sum bw_i(\partial f/\partial x_i)(p) = av_p[f] + bw_p[f]$ . (ii) We have  $v_p[af + bg] = \sum v_i(\partial(af + bg)/\partial x_i)(p) = \sum v_i(\partial(af)/\partial x_i)(p) + \sum v_i(\partial(bg)/\partial x_i)(p) = av_p[f] + bv_p[g]$ . (iii) We have  $v_p[fg] = \sum v_i(\partial(fg)/\partial x_i)(p) = \sum v_i(\partial f/\partial x_i)(p)g(p) + f(p)\sum v_i(\partial g/\partial x_i)(p) = v_p[f]g(p) + f(p)v_p[g]$ .  $\square$

Let  $V$  be a vector field, we define  $V[f]$  at  $p \in \mathbb{R}^3$  to be  $V(p)[f]$ . By the convention,  $U_i(p)[f] = (\partial f/\partial x_i)(p)$ .

**Proposition.** Let  $V$  and  $W$  be vector fields. Let  $f, g$ , and  $h$  be real-valued functions. For all  $a, b \in \mathbb{R}$ , the following properties hold.

1.  $(fV + gW)[h] = fV[h] + gW[h]$ .
2.  $V[af + bg] = aV[f] + bV[g]$ .
3.  $V[fg] = V[f]g + fV[g]$ .

*Proof.* (i) For all  $p \in \mathbb{R}^3$ ,  $(fV + gW)(p)[h] = (f(p)V(p) + g(p)W(p))[h] = fV[h] + gW[h]$ . (ii) For all  $p \in \mathbb{R}^3$ ,  $V(p)[af + bg] = aV(p)[f] + bV(p)[g]$ . (iii) For all  $p \in \mathbb{R}^3$ ,  $V(p)[fg] = V(p)[f]g(p) + f(p)V(p)[g] = V[f](p)g(p) + f(p)V[g](p) = (V[f]g + fV[g])(p)$ .  $\square$

**Example.** Let  $V = xU_1 - y^2U_3$  and let  $f = x^2y + z^3$ . Then  $V[f] = xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3] = 2x^2y - 3y^2z^2$ .

Let  $I \subset \mathbb{R}$  be an open interval. Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a function. We can rewrite  $\alpha(t)$  as  $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$ , where  $\alpha_i : I \rightarrow \mathbb{R}$ . We say  $\alpha$  is *differentiable* if  $\alpha_i$  are differentiable.

**Definition 1.11.** A *curve* in  $\mathbb{R}^3$  is a differentiable function  $\alpha : I \rightarrow \mathbb{R}^3$ , where  $I \subset \mathbb{R}$  is an open interval.

**Example.** A curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = p + tq$ , where  $\alpha(0) = p$  and  $q \neq 0$ , is called a *straight line*.

**Example.** Here are some examples of curves.

1. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (a \cos t, a \sin t, bt)$ .
2. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (1 + \cos t, \sin t, 2 \sin(t/2))$ .
3. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$ .
4. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ .

**Definition 1.12.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve with  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ . For all  $t \in I$ , the *velocity vector* of  $\alpha$  at  $t$  is the tangent vector  $\alpha'(t) = ((d\alpha_1/dt)(t), (d\alpha_2/dt)(t), (d\alpha_3/dt)(t))_{\alpha(t)}$  at the point  $\alpha(t) \in \mathbb{R}^3$ . The curve  $\alpha$  is said to be *regular* if  $\alpha_i \neq 0$  for all  $i$ .

Consider the velocity vector  $\alpha'(t)$ , we can rewrite it by the natural frame fields, so  $\alpha'(t) = \sum (d\alpha_i/dt)(t) U_i(\alpha(t))$ .

**Definition 1.13.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve and let  $h : J \rightarrow I$  be differentiable, where  $J$  is an open interval of  $\mathbb{R}$ . The *reparametrization* of  $\alpha$  by  $h$  is the composition  $\alpha \circ h : J \rightarrow \mathbb{R}^3$ .

The composition of differentiable functions is differentiable, so any reparametrization is differentiable, which means it is a curve.

**Proposition.** Let  $\beta$  be the reparametrization of  $\alpha$  by  $h$ , then  $\beta'(s) = (dh/ds)(s) \alpha'(h(s))$ .

*Proof.* Rewrite  $\beta(s) = \alpha(h(s))$ , then we have  $\beta'(s) = (d(\alpha_i h_i)/ds)(s)_{\alpha(h(s))} = (d\alpha_i/ds)(h(s)) \cdot (dh/ds)(s)_{\alpha(h(s))} = (dh/ds)(s) \alpha'(h(s))$ .  $\square$

**Proposition.** Let  $\alpha$  be a curve and let  $f$  be a differentiable function on  $\mathbb{R}^3$ , then  $\alpha'(t)[f] = (d(f\alpha)/dt)(t)$ .

*Proof.* We have  $\alpha'(t)[f] = \sum (d\alpha_i/dt)(t) \cdot (\partial f/\partial x_i)(\alpha(t)) = (d(f\alpha)/dt)(t)$  by the chain rule.  $\square$

Now we show a general idea of parametrizations. The proofs will be included in other sections when we have enough tools. Assume every result is correct for now.

**Definition 1.14.** A *1-form*  $\varphi$  on  $\mathbb{R}^3$  is a function  $\varphi : \coprod_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3) \rightarrow \mathbb{R}$  such that for all  $a, b \in \mathbb{R}$  and  $v, w \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ ,  $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$ .

Given a 1-form  $\varphi$ , for any point  $p$ , denote the restriction  $\varphi|_{T_p(\mathbb{R}^3)} : T_p(\mathbb{R}^3) \rightarrow \mathbb{R}$  by  $\varphi_p$ , then  $\varphi_p$  is linear. Let  $\varphi$  and  $\psi$  be 1-forms. Define the addition and scalar multiplication by  $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$  and  $(f\varphi)(v_p) = f(p)\varphi(v_p)$ . Given any 1-form  $\varphi$  and point  $p$ ,  $\varphi_p$  is a linear functional in  $T_p^*(\mathbb{R}^3)$ , the dual space of  $T_p(\mathbb{R}^3)$ .

**Definition 1.15.** Let  $\varphi$  be a 1-form and let  $V$  be a vector field. For all  $p \in \mathbb{R}^3$ , define  $\varphi(V)(p) = \varphi_p(V(p))$ . We say  $\varphi$  is *differentiable* if for every differentiable vector field  $V$ , the function  $\varphi(V)$  is differentiable.

Now let  $V$  and  $W$  be vector fields, we have  $\varphi(fV + gW)(p) = \varphi((fV + gW)(p)) = \varphi(fV(p) + gW(p)) = (f\varphi(V) + g\varphi(W))(p)$ . Similarly,  $(f\varphi + g\psi)(V) = f\varphi(V) + g\psi(V)$ .

**Definition 1.16.** If  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is differentiable. The *differential* of  $f$ , denoted  $df$ , is the function  $df(v_p) = v_p[f]$  for all tangent vectors  $v_p$ .

Let  $v_p, w_p \in T_p(\mathbb{R}^3)$  and let  $a, b \in \mathbb{R}$ , then  $df(av_p + bw_p) = (av_p + bw_p)[f] = av_p[f] + bw_p[f] = a df(v_p) + b df(w_p)$ . Hence  $df$  is a 1-form.

**Example.** Consider the natural coordinate functions  $x_i$ . We have  $dx_i(v_p) = v_p[x_i] = \sum v_i(\partial x_i/\partial x_j)(p) = v_i$ .

**Proposition.** If  $\varphi$  is a 1-form on  $\mathbb{R}^3$ , then  $\varphi = \sum f_i dx_i$ , where  $f_i = \varphi(U_i)$ .

*Proof.* Let  $v_p \in T_p(\mathbb{R}^3)$ , then  $\varphi(v_p) = \varphi(\sum v_i U_i(p)) = \sum v_i \varphi(U_i(p)) = \sum v_i f_i(p) = \sum f_i(p) dx_i(v_p) = (\sum f_i dx_i)(v_p)$ , hence  $\varphi = \sum f_i dx_i$ .  $\square$

The functions  $f_1$ ,  $f_2$ , and  $f_3$  are called the *Euclidean coordinate functions* of the 1-form  $\varphi$ .

**Proposition.** Let  $f$  be a differentiable function on  $\mathbb{R}^3$ , then  $df = \sum(\partial f/\partial x_i)dx_i$ .

*Proof.* Let  $v_p \in T_p(\mathbb{R}^3)$ , then  $df(v_p) = v_p[f] = \sum v_i(\partial f/\partial x_i)(p) = \sum(\partial f/\partial x_i)(p)dx_i(v_p) = (\sum(\partial f/\partial x_i)dx_i)(v_p)$ , hence  $df = \sum(\partial f/\partial x_i)dx_i$ .  $\square$

Let  $f$  and  $g$  be differentiable functions on  $\mathbb{R}^3$ , then  $d(f+g) = \sum(\partial(f+g)/\partial x_i)dx_i = \sum(\partial f/\partial x_i)dx_i + \sum(\partial g/\partial x_i)dx_i = df + dg$ . Now we denote the multiplication to be  $fg$ .

**Proposition.** Let  $f$  and  $g$  be differentiable functions on  $\mathbb{R}^3$ , then  $d(fg) = gdf + f dg$ .

*Proof.* We have  $d(fg) = \sum(\partial(fg)/\partial x_i)dx_i = \sum((\partial f/\partial x_i)g + (\partial g/\partial x_i)f)dx_i = gdf + f dg$ .  $\square$

**Proposition.** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable, then  $d(h(f)) = (dh(f)/dx)df$ .

*Proof.* We have  $d(h(f)) = \sum(\partial h(f)/\partial x_i)dx_i$ , by the chain rule,  $(\partial h(f)/\partial x_i)dx_i = (dh(f)/df)(\partial x/\partial x_i)$ , so  $d(h(f)) = (df(h)/df)df$ .  $\square$

**Example.** Consider the function  $f = (x^2 - 1)y + (y^2 + 2)z$ . We have  $df = d((x^2 - 1)y) + d((y^2 + 2)z) = yd(x^2 - 1) + (x^2 + 1)dy + zd(y^2 + 2) + (y^2 + 2)dz = 2xydx + (x^2 + 2yz - 1)dy + (y^2 + 2)dz$ . Since  $v_p[f] = df(v_p)$ ,  $v_p[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_2^2 + 2)v_3$ .

**Definition 1.17.** Let  $V$  be the vector space  $\mathbb{R}^3$  and denote the space of all  $p$ -linear forms on  $V$  by  $\Lambda^p(V^*)$ . Every element of  $\Lambda^p$  is called a  $p$ -form. Define the *wedge product* to be a function  $\wedge : \Lambda^a(V^*) \times \Lambda^b(V^*) \rightarrow \Lambda^{a+b}(V^*)$  such that for  $\omega \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $v_1, \dots, v_{m+n} \in V$ , the following properties hold.

1.  $(\omega \wedge \eta)(v_1, \dots, v_{m+n}) = (\sum_{\sigma \in \mathfrak{S}_{m+n}} \text{sgn}(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(m)})\eta(v_{\sigma(m+1)}, \dots, v_{\sigma(m+n)}))/ (m!n!)$ .
2.  $\omega \wedge \eta = (-1)^{mn}\eta \wedge \omega$ .

Generally, a  $p$ -form is of the form  $\sum f(x, y, z)dx^i \wedge \dots \wedge dy^j \wedge \dots \wedge dz^k \wedge \dots$ . We have  $dx_i \wedge dx_j = -dx_j \wedge dx_i$ . If  $i = j$ , then  $dx_i \wedge dx_i = -dx_i \wedge dx_i$ , so  $dx_i \wedge dx_i = 0$ . It is trivial that  $\wedge$  is bilinear and associative, that is,

1. for  $\omega_1, \omega_2 \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $a, b \in \mathbb{R}$ ,  $(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta)$  and  $\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2)$ ;
2. for  $\omega \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $\theta \in \Lambda^l(V^*)$ ,  $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta$ .

Now given a space of  $p$ -forms  $\Lambda^p(V^*)$  with basis  $\{e^1, e^2, e^3\}$ , the basis of its dual space is denoted by  $\{e^1, e^2, e^3\}$ . The basis of  $\Lambda^k(V^*)$  is of the form  $e^{i_1} \wedge \dots \wedge e^{i_k}$ , where  $1 \leq i_1 \leq \dots \leq i_k \leq 3$ . In this case, the dimension of  $\Lambda^p(V^*)$  is  $3!/(p!(3-p)!)$ . If  $p > 4$ , then  $\dim(\Lambda^p(V^*)) = 0$ , so there are no  $p$ -forms on  $\mathbb{R}^3$  if  $p \geq 4$ .

**Example.** Let  $\varphi = xdx - ydy$ ,  $\psi = zdx + xdz$ ,  $\theta = zdy$ , and  $\eta = ydx \wedge dz + xdy \wedge dz$ .

1.  $\varphi \wedge \psi = xzdx \wedge dx + x^2dx \wedge dz - yzdy \wedge dx - yxdy \wedge dz = yzdx \wedge dy + x^2dx \wedge dz - yxdy \wedge dz$
2.  $\theta \wedge (\varphi \wedge \psi) = yz^2dx \wedge (dy \wedge dy) + x^2zdx \wedge dz \wedge dy - xyzdy \wedge dz \wedge dy = -x^2zdx \wedge dy \wedge dz$
3.  $\varphi \wedge \eta = xydx \wedge dx \wedge dz + x^2dx \wedge dy \wedge dz - y^2dy \wedge dx \wedge dz - xydy \wedge dy \wedge dz = (x^2 + y^2)dx \wedge dy \wedge dz$

**Proposition.** Let  $\varphi$  and  $\psi$  be 1-forms, then  $\varphi \wedge \psi = -\psi \wedge \varphi$ .

*Proof.* Rewrite  $\varphi = \sum f_i dx_i$  and  $\psi = \sum g_i dx_i$ , then  $\varphi \wedge \psi = \sum f_i g_i dx_i dx_j = \sum -g_i f_i dx_j dx_i = -\psi \wedge \varphi$ .  $\square$

**Definition 1.18.** Let  $\varphi = \sum f_i dx_i$  be a 1-form on  $\mathbb{R}^3$ . The *exterior derivative* of  $\varphi$  is the 2-form  $d\varphi = \sum df_i \wedge dx_i$ . Let  $\psi = \sum f_{i,j} dx_i \wedge dx_j$  be a 2-form. The *exterior derivative* of  $\psi$  is the 3-form  $d\psi = \sum df_{i,j} \wedge dx_i \wedge dx_j$ .

Let  $a, b \in \mathbb{R}$ . Let  $\varphi = \sum f_i dx_i$  and  $\psi = \sum g_i dx_i$  be 1-forms. Then  $d(a\varphi + b\psi) = d(\sum (af_i + bg_i)dx_i) = \sum d(af_i + bg_i) \wedge dx_i$ , since the differential is linear, the exterior derivative is linear.

**Proposition.** Let  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$  be functions and let  $\varphi$  and  $\psi$  be 1-forms. Then  $d(f\varphi) = df \wedge \varphi + f d\varphi$  and  $d(\varphi \wedge \psi) = d\varphi \wedge \psi - \varphi \wedge d\psi$ .

*Proof.* (i) Let  $\varphi = \sum g_i dx_i$ , then  $f\varphi = \sum f g_i dx_i$ , so  $d(f\varphi) = \sum (f dg_i + g_i df) \wedge dx_i = \sum f dg_i \wedge dx_i + \sum g_i df \wedge dx_i = f d\varphi + d\varphi \wedge f$ . (ii) Since  $dx_i \wedge dx_i = 0$ , without loss of generality, let  $\varphi = f dx$  and let  $\psi = g dy$ . Then  $d(\varphi \wedge \psi) = d(f g dx \wedge dy) = d(fg) \wedge dx \wedge dy = (f dg + g df) \wedge dx \wedge dy = f dg \wedge dx \wedge dy + g df \wedge dx \wedge dy$ . For the right hand side,  $d\varphi \wedge \psi = df \wedge dx \wedge g dy = g df \wedge dx \wedge dy$  and  $\varphi \wedge d\psi = f dx \wedge dg \wedge dy = -f dg \wedge dx \wedge dy$ , hence  $d(\varphi \wedge \psi) = d\varphi \wedge \psi - \varphi \wedge d\psi$ .  $\square$

**Definition 1.19.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and let  $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $F(p) = (f_1(p), \dots, f_m(p))$  for all  $p \in \mathbb{R}^n$ . The functions  $f_i$  are called the *Euclidean coordinate functions* of  $F$  and we denote  $F = (f_1, \dots, f_m)$ .

**Definition 1.20.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $F = (f_1, \dots, f_m)$ , we say  $F$  is *differentiable* if all  $f_i$  are differentiable. If  $F$  is differentiable, we say  $F$  is a *mapping* from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .

**Definition 1.21.** Let  $\alpha : I \rightarrow \mathbb{R}^n$  be a curve and let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. Then the composite function  $\beta = F(\alpha) : I \rightarrow \mathbb{R}^m$  is a curve in  $\mathbb{R}^m$  called the *image* of  $\alpha$  under  $F$ .

To examine the effect of a mapping, it suffices to take a proper  $\alpha$  and check the image of it.

**Example.** The function  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $F = (x - y, x + y, 2z)$  is a mapping. Trivially,  $F$  is a linear map, so  $F$  is determined by  $F(u_i)$ .

**Example.** Consider the mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $F = (u^2 - v^2, 2uv)$ . Let  $\alpha : I \rightarrow \mathbb{R}^2$  defined by  $\alpha(t) = (r \cos t, r \sin t)$ , where  $0 \leq t \leq 2\pi$ . The image is  $\beta(t) = (r^2 \cos 2t, r^2 \sin 2t)$ . This curve takes two counterclockwise trips around the circle of radius  $r^2$  centered at the origin. Therefore,  $F$  wraps  $\mathbb{R}^2$  around itself twice.

**Definition 1.22.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping and let  $v_p \in T_p(\mathbb{R}^n)$ . The *tangent map* of  $F$ , denoted  $F_*(v_p)$ , is defined to be  $(d/dt)F(p + tv)$  at  $t = 0$ .

Fix some mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . For every  $p \in \mathbb{R}^n$ , it induces a tangent map of  $F$  at  $p$ , denoted  $F_{*p}$ .

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. If  $v_p \in T_p(\mathbb{R}^n)$ , then  $F_{*p}(v_p) = (v[f_1], \dots, v[f_m])_{F(p)}$ .

*Proof.* Fix  $v_p \in T_p(\mathbb{R}^n)$ . We have  $F_{*p} = (d/dt)F(p + tv)|_{t=0} = (d/dt)(f_i(p + tv))|_{t=0} = (v_p[f_1], \dots, v_p[f_m])_{F(p)}$ .  $\square$

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. For all  $p \in T_p(\mathbb{R}^n)$ , the tangent map  $F_{*p} : T_p(\mathbb{R}^n) \rightarrow T_{F(p)}(\mathbb{R}^m)$  is a linear map.

*Proof.* Fix  $p \in \mathbb{R}^n$ . Let  $a, b \in \mathbb{R}$  and let  $v_p, w_p \in T_p(\mathbb{R}^n)$ . We have  $F_{*p}(av_p + bw_p) = ((av_p + bw_p)[f_i])_{F(p)} = (av_p[f_i])_{F(p)} + (bw_p[f_i])_{F(p)} = aF_{*p}(v_p) + bF_{*p}(w_p)$ .  $\square$

**Proposition.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping and let  $\beta$  be the image of some curve  $\alpha$  in  $\mathbb{R}^n$ , then  $\beta' = F_*(\alpha')$ .

*Proof.* Let  $F = (f_1, \dots, f_m)$ . We have  $F_*(\alpha'(t)) = (\alpha'(t)[f_i])_{F(\alpha(t))} = (df_i(\alpha(t))/dt)_{F(\alpha(t))} = \beta'(t)$ .  $\square$

Let  $\{U_j\}$  and  $\{\overline{U}_i\}$  be the natural frame fields of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively.

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. Then  $F_*(U_j(p)) = \sum_{i=1}^m (\partial f_i / \partial x_j)(p) \overline{U}_i(F(p))$ , where  $1 \leq j \leq n$ .

*Proof.* Recall that  $U_j[f_i] = \partial f_i / \partial x_j$ , so the proposition trivially holds.  $\square$

**Definition 1.23.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. The *Jacobian matrix* of  $F$  at  $x \in \mathbb{R}^n$  is the matrix

$$J_F(x) = \begin{pmatrix} \partial f_1 / \partial x_1(x) & \cdots & \partial f_1 / \partial x_n(x) \\ \vdots & \ddots & \vdots \\ \partial f_m / \partial x_1(x) & \cdots & \partial f_m / \partial x_n(x) \end{pmatrix}.$$

**Definition 1.24.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. We say  $F$  is *regular* if for all  $p \in \mathbb{R}^n$ ,  $F_{*p}$  is injective.

Notice that  $J_F(p) \cdot v = F_{*p}$ , so  $J_F(p)$  is the matrix representation of  $F_{*p}$ .

**Definition 1.25.** A mapping is a *diffeomorphism* if it has a differentiable inverse mapping.

**Definition 1.26.** A *topological space*  $(X, \mathcal{T})$  consists of two sets  $X$  and  $\mathcal{T}$ , where  $\mathcal{T} \subset \mathcal{P}(X)$ , that satisfies the following properties.

1.  $\emptyset, X \in \mathcal{T}$ .
2. Any union of elements in  $\mathcal{T}$  is also in  $\mathcal{T}$ .
3. Any finite intersection of elements in  $\mathcal{T}$  is also in  $\mathcal{T}$ .

The collection  $\mathcal{T}$  is called a *topology* on  $X$ .

**Definition 1.27.** Let  $(X, \mathcal{T})$  be a topological space. A subset  $U \subset X$  is said to be *open* if  $U \in \mathcal{T}$ . Let  $x \in X$ , a *neighborhood* of  $x$  is an open set  $U_x$  that contains  $x$ .

Let  $U \subset \mathbb{R}$ . We say  $U$  is open in the standard topology  $\mathcal{T}$  on  $\mathbb{R}$  if for every  $x \in U$ , there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subset U$ . Trivially,  $\emptyset, \mathbb{R} \in \mathcal{T}$ . Let  $\{U_i\}_{i \in I}$  be open sets, then for each  $U_i$  and  $x \in U_i$ , there exists a corresponding  $\varepsilon_{i,x}$ . For any  $x \in \bigcup_{i \in I} U_i$ ,  $x \in U_i$  for some  $i \in I$ . Pick  $\varepsilon = \varepsilon_{i,x}$ , then  $(x - \varepsilon, x + \varepsilon) \subset U_i \subset \bigcup_{i \in I} U_i$ . For any  $x \in \bigcap_{i=1}^n U_i$ , pick  $\varepsilon = \min\{\varepsilon_{i,x}\}$ , then  $(x - \varepsilon, x + \varepsilon) \subset U_i$  for  $1 \leq i \leq n$ , so  $(x - \varepsilon, x + \varepsilon) \subset \bigcap_{i=1}^n U_i$ . The standard topology on  $\mathbb{R}$  is indeed a topology.

**Definition 1.28.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A subset  $W \subset X \times Y$  is open in the *product topology* on  $X \times Y$  if for all  $(x, y) \in W$ , there exist neighborhoods  $U_x \in \mathcal{T}_X$  and  $V_y \in \mathcal{T}_Y$  such that  $U_x \times V_y \subset W$ .

Denote the product topology by  $\mathcal{T}$ . We have  $\emptyset \in \mathcal{T}$  vacuously. For all  $(x, y) \in X \times Y$ ,  $U_x \subset X$ , and  $V_y \subset Y$ ,  $U_x \times V_y \subset X \times Y$ , so  $X \times Y \in \mathcal{T}$ . Let  $\{W_i\}_{i \in I}$  be open sets. For all  $(x, y) \in \bigcup_{i \in I} W_i$ , there exist  $W_i$  and  $W_j$  such that  $x \in W_i$  and  $y \in W_j$ . Pick the corresponding neighborhood in each set, then  $U_x \times V_y \subset W_i \cup W_j \subset \bigcup_{i \in I} W_i$ . For all  $(x, y) \in \bigcap_{i=1}^n W_i$ ,  $(x, y) \in W_i$ . For each  $W_i$ , we have a corresponding pair  $(U_{i,x}, V_{i,y})$ . Now consider  $U = \bigcap_{i=1}^n U_{i,x} \in \mathcal{T}_X$  and  $V = \bigcap_{i=1}^n V_{i,y} \in \mathcal{T}_Y$ , we have  $U \times V \subset W_i$ , so  $U \times V \subset \bigcap_{i=1}^n W_i$ . The standard topology on  $\mathbb{R}^n$  is the product topology of  $n$  copies of the standard topology on  $\mathbb{R}$ .

**Theorem 1.1** (inverse function theorem). Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a mapping. If  $F_{*p}$  is injective at some  $p \in \mathbb{R}^n$ , then there exists a neighborhood  $U$  of  $p$  such that  $F|_U : U \rightarrow V$ , where  $V$  is open, is a diffeomorphism.

We will discuss more on the proof of this theorem and its application later.

## 2 Frame Fields

**Definition 2.1.** Let  $p, q \in \mathbb{R}^3$ . The *Euclidean distance* from  $p$  to  $q$  is the number  $d(p, q) = \|p - q\|$ .

**Definition 2.2.** Let  $v_p, w_p \in T_p(\mathbb{R}^3)$  be tangent vectors. The *dot product* of  $v_p$  and  $w_p$  is defined to be  $v_p \cdot w_p = v \cdot w$ .

Equivalently, the norm on every tangent space  $T_p(\mathbb{R}^3)$  is the composition of the canonical isomorphism  $T_p(\mathbb{R}^3) \rightarrow \mathbb{R}^3$  with the norm on  $\mathbb{R}^3$ .

**Definition 2.3.** A set of three pairwise orthogonal unit vectors tangent to  $\mathbb{R}^3$  at  $p$  is called a *frame* at  $p$ .

By the definition,  $\{e_1, e_2, e_3\}$  is a frame at  $p$  if and only if  $e_i \in T_p(\mathbb{R}^3)$  and  $e_i \cdot e_j = \delta_{i,j}$ .

**Proposition.** Let  $\{e_1, e_2, e_3\}$  be a frame at  $p \in \mathbb{R}^3$ . If  $v_p \in T_p(\mathbb{R}^3)$ , then  $v_p = \sum (v \cdot e_i) e_i$ .

*Proof.* Let  $c_1, c_2, c_3 \in \mathbb{R}$  such that  $\sum c_i e_i = 0$ . For all  $1 \leq j \leq 3$ ,  $0 = (\sum c_i e_i) \cdot e_j = \sum c_i (e_i \cdot e_j) = c_j$ , so  $\{e_1, e_2, e_3\}$  is a basis of  $T_p(\mathbb{R}^3)$ . Rewrite  $v_p = \sum a_i e_i$ . For all  $1 \leq j \leq 3$ ,  $v_p \cdot e_j = \sum a_i e_i \cdot e_j = a_j$ . Hence  $v_p = \sum (v_p \cdot e_i) e_i$ .  $\square$

For any frame  $\{e_1, e_2, e_3\}$  at  $p$  and  $a, b \in T_p(\mathbb{R}^3)$ , if  $a = \sum a_i e_i$  and  $b = \sum b_i e_i$ , we always have  $a \cdot b = \sum a_i b_i$ .

**Definition 2.4.** Let  $\{e_1, e_2, e_3\}$  be a frame at  $p \in \mathbb{R}^3$  with  $e_i = (a_{i,1}, a_{i,2}, a_{i,3})_p$ , then the *attitude matrix* of the frame is defined to be the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Consider the transpose  $A^\top$  of  $A$ , for each column of  $A^\top A$ , we have  $e_i e_i = 1$ , so  $A^\top A = I$  and  $A$  is orthogonal.

**Definition 2.5.** Let  $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ . The *cross product* of  $v_p$  and  $w_p$ , denoted  $v_p \times w_p$ , is the tangent vector

$$v_p \times w_p = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

**Example.** Let  $v_p = (1, 0, -1)_p$  and let  $w_p = (2, 2, -7)_p$ , then  $v_p \times w_p = 2U_1(p) + 5U_2(p) + 2U_3(p) = (2, 5, 2)_p$ .

It is trivial that  $\times$  is linear and  $v_p \times w_p = -w_p \times v_p$ .

**Proposition.** Let  $v_p, w_p \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ . Then  $v_p \times w_p$  is orthogonal to both  $v_p$  and  $w_p$ . Moreover,  $\|v_p \times w_p\|^2 = (v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2$ .

*Proof.* Let  $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p$ . Then  $(v_p \times w_p) \cdot v_p = v_1(v_2 w_3 - v_3 w_2) + v_2(v_3 w_1 - v_1 w_3) + v_3(v_1 w_2 - v_2 w_1) = 0$ . Similarly,  $(v_p \times w_p) \cdot w_p = 0$ . We have  $(v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2 = (\sum v_i^2)(\sum w_i^2) - (\sum v_i w_i)^2 = \sum v_i^2 w_j^2 - \sum v_i^2 w_i^2 - 2\sum_{i < j} v_i w_i w_j - w_j = (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2 = \|v_p \times w_p\|^2$ .  $\square$

**Definition 2.6.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve. The *speed* of  $\alpha$  at  $t$  is the tangent vector  $\|\alpha'(t)\|$ . The *arc length* of  $\alpha$  from  $t = a$  to  $t = b$  is defined to be  $\int_a^b \|\alpha'(t)\| dt$ .

**Proposition.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a regular curve, then there exists a reparametrization  $\beta$  of  $\alpha$  such that  $\|\beta'\| = 1$ .

*Proof.* Fix some  $\alpha \in \mathbb{R}$  and consider the function  $s : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $s(t) = \int_a^t \|\alpha'(x)\| dx$ . Since  $\alpha$  is regular,  $\|\alpha'(x)\| > 0$  for all  $x$ . By the inverse function theorem,  $s(t)$  has an inverse  $t(s)$ . Define  $\beta(s) = \alpha(t(s))$ , then  $\|\beta'\| = \|(dt/ds)(s)\alpha'(t(s))\| = (dt/ds)(s)\|\alpha'(t(s))\| = (dt/ds)(s) \cdot (ds/dt)(t(s)) = 1$ .  $\square$

Such reparametrization  $\beta$  of  $\alpha$  is called the *arc-length reparametrization* of  $\alpha$ .

**Example.** Consider the curve  $\alpha : I \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (a \cos t, a \sin t, bt)$  for some  $a, b \in \mathbb{R}$ . We have  $\|\alpha'\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = c$ , where  $c^2 = a^2 + b^2$ . Now measure the arc length from  $t = 0$ , then  $s(t) = \int_0^t c du = ct$ , so  $t(s) = s/c$ , the arc-length reparametrization is therefore  $\beta(s) = \alpha(t(s)) = (a \cos(s/c), a \sin(s/c), bs/c)$ .

**Definition 2.7.** A *vector field*  $Y$  on a curve  $\alpha : I \rightarrow \mathbb{R}^3$  is a function  $Y : I \rightarrow \bigsqcup_{p \in \text{ran}(\alpha)} T_p(\mathbb{R}^3)$  such that for all  $t \in I$ ,  $Y(t) \in T_{\alpha(t)}(\mathbb{R}^3)$ .

Fix  $t \in I$ , then we can rewrite  $Y(t)$  as  $\sum y_i(t)U_i(\alpha(t))$ . The functions  $y_1, y_2$ , and  $y_3$  are called the *Euclidean coordinate functions* on  $Y$ .

We define the addition, scalar multiplication, dot multiplication, and cross product on vector fields pointwisely. For a vector  $Y = \sum y_i U_i$  on  $\alpha$ , the derivative of  $Y$  is defined to be  $Y' = \sum (dy_i/dt)U_i$ . Let  $Y$  and  $Z$  be vector fields on a curve  $\alpha$ . Fix  $t$ , rewrite  $Y = (y_1, y_2, y_3)_{\alpha(t)}$  and  $Z = (z_1, z_2, z_3)_{\alpha(t)}$ . Consider  $Y$  as a function  $Y_t : I \rightarrow T_{\alpha(t)}(\mathbb{R}^3) \approx \mathbb{R}^3$ , then for  $a, b \in \mathbb{R}$  and a differentiable  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $(aY + bZ)' = aY' + bZ'$  and  $(fY)' = (df/dt)Y + Y'f$ . We also have  $(Y \cdot Z)' = (\sum y_i z_i)' = \sum y_i z_i' + \sum y_i' z_i = Y \cdot Z' + Y' \cdot Z$ .

**Definition 2.8.** Let  $Y$  be a vector field on a curve  $\alpha : I \rightarrow \mathbb{R}^3$ . We say  $Y$  is *parallel* if for all  $t \in I$ ,  $Y(t)$  have the same vector part.

**Proposition.** A curve  $\alpha$  is constant if and only if  $\alpha' = 0$ . A nonconstant curve  $\alpha$  is a straight line if and only if  $\alpha'' = 0$ . A vector field  $Y$  on  $\alpha$  is parallel if and only if  $Y' = 0$ .

*Proof.* Rewrite  $\alpha : I \rightarrow \mathbb{R}^3$  as  $\alpha = (\alpha_i)$ . (i) The velocity  $\alpha' = (\alpha'_i)$ , then  $\alpha' = 0$  if and only if  $\alpha'_i = 0$ . Hence  $\alpha' = 0$  if and only if  $\alpha_i$  is a constant function. (ii) We have  $\alpha'' = (\alpha''_i)$ , so  $\alpha'' = 0$  if and only if  $\alpha_i = p_i t + q_i$  for some  $p_i, q_i \in \mathbb{R}$ . Hence  $\alpha'' = 0$  if and only if  $\alpha = pt + q$ , where  $p = (p_i)$  and  $q = (q_i)$ . (iii) Fix  $t$  and let  $Y = (y_i)_{\alpha(t)}$ , then  $Y' = \sum y'_i U_i = 0$ , which means  $y_i$  are constant functions. Hence  $Y$  is parallel if and only if  $Y' = 0$ .  $\square$

**Definition 2.9.** Let  $\beta : I \rightarrow \mathbb{R}^3$  be a curve. Then we call  $T = \beta'$  the *unit tangent field* of  $\beta$ . The function  $\kappa(s) = \|T'(s)\|$  is called the *curvature* of  $\beta$ .

**Remark.** We shall only consider the cases where  $\kappa \neq 0$ .

**Definition 2.10.** Let  $\beta : I \rightarrow \mathbb{R}^3$  be a curve. Then we call  $N = T'/\kappa$  the *principal normal vector field* of  $\beta$ . The vector field  $B = T \times N$  on  $\beta$  is called the *binormal vector field* of  $\beta$ .

**Proposition.** Let  $\beta$  be a curve in  $\mathbb{R}^3$  with  $\kappa > 0$  and  $\|\beta'\| = 1$ . Then the three vector fields  $T$ ,  $N$ , and  $B$  on  $\beta$  are unit vector fields that are mutually orthogonal at each point.

*Proof.* Since  $T = \beta'$ ,  $\|T\| = \sqrt{T \cdot T} = 1$ , then  $(T \cdot T)' = T \cdot T' + T' \cdot T = 2T' \cdot T = 0$ , so  $T \cdot T' = 0$ . For all  $s \in \text{dom}(\beta)$ ,  $\|N\| = \|T'(s)\|/\kappa(s) = \|T'(s)\|/\|T'(s)\| = 1$ . Since  $B = T \times N$ ,  $B$  is orthogonal to  $T$  and  $N$ . Moreover,  $\|B\|^2 = \|T\|\|N\| - (T \cdot N)^2 = 1 - 0 = 1$ .  $\square$

**Definition 2.11.** Let  $\beta$  be a curve in  $\mathbb{R}^3$  with  $\kappa > 0$  and  $\|\beta'\| = 1$ . Then  $(T, N, B)$  is called the *Frenet frame field* of  $\beta$ .

Now we claim that  $B'$  can be written as a scalar multiple of  $N$ . Consider  $B' = B' \cdot NN + B' \cdot TT + B' \cdot BB$ . Differentiate  $B \cdot T$ ,  $B' \cdot T + T' \cdot B = 0$ , then  $B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0$ . Similarly,  $B \cdot B' = 0$ , so  $B' = B' \cdot NN$ .

**Definition 2.12.** Let  $\beta$  be a curve in  $\mathbb{R}^3$  with  $\kappa > 0$  and  $\|\beta'\| = 1$ . Then the *torsion* of  $\beta$  is a function  $\tau : I \rightarrow \mathbb{R}$  such that  $B' = -\tau N$ .

**Theorem 2.1** (Frenet formulas). Let  $\beta : I \rightarrow \mathbb{R}^3$  be a curve with  $\kappa > 0$  and  $\|\beta'\| = 1$ . Then  $T' = \kappa N$ ,  $N' = -\kappa T + \tau B$ , and  $B' = -\tau N$ .

*Proof.* Rewrite  $N' = N' \cdot TT + N' \cdot NN + N' \cdot BB$ . Differentiate  $T \cdot N$ , we have  $T' \cdot N + N' \cdot T = 0$ , so  $N' \cdot T = -T' \cdot N = -(\kappa N) \cdot N = -\kappa$ . Similarly,  $N' \cdot B = -B' \cdot N = -(-\tau N) \cdot N = \tau$ . Hence  $N' = -\kappa T + \tau B$ .  $\square$

**Example.** Consider the curve  $\beta : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\beta(s) = (a \cos(s/c), a \sin(s/c), bs/c)$ , where  $a > 0$  and  $c = \sqrt{a^2 + b^2}$ . It is trivial that  $\|\beta'\| = 1$ . Here  $T(s) = \beta'(s) = (-a \sin(s/c)/c, a \cos(s/c)/c, b/c)$ , then  $T'(s) = (-a \cos(s/c)/c^2, -a \sin(s/c)/c^2, 0)$ , so  $\kappa(s) = \|T'(s)\| = a/c^2 > 0$ . We also have  $N(s) = (-\cos(s/c), -\sin(s/c), 0)$ . Now  $\beta(s) = T(s) \times N(s) = (b \sin(s/c)/c, -b \cos(s/c)/c, a/c)$ , then  $B'(s) = (b/\cos(s/c)/c^2, b \sin(s/c)/c^2, 0)$ , so  $\tau(s) = -B'(s)/N(s) = (-b/\cos(s/c)/c^2, -b \sin(s/c)/c^2, 0)/(-\cos(s/c), -\sin(s/c), 0) = b/c^2$ .

**Definition 2.13.** Let  $p, q \in \mathbb{R}^3$  with  $q \neq 0$ . The *plane* through  $p$  orthogonal to  $q$  is the set  $\{r \in \mathbb{R}^3 \mid (r - p) \cdot q = 0\}$ . A curve  $\beta : I \rightarrow \mathbb{R}^3$  is said to be a *plane curve* if  $\text{ran}(\beta) \subset P$ , where  $P$  is a plane in  $\mathbb{R}^3$ .

**Proposition.** Let  $\beta : I \rightarrow \mathbb{R}^3$  be a curve with  $\|\beta'\| = 1$  and  $\kappa > 0$ . Then  $\beta$  is a plane curve if and only if  $\tau = 0$ .

*Proof.* ( $\Rightarrow$ ) Let  $\beta$  be a plane curve, then there exists  $p, q \in \mathbb{R}^3$  such that for all  $s \in I$ ,  $(\beta(s) - p) \cdot q = 0$ . Consider  $q$  as a constant vector field, so  $((\beta - p) \cdot q)' = (\beta - p)' \cdot q + q' \cdot (\beta - p) = \beta' \cdot q - p' \cdot q = \beta' \cdot q + q' \cdot \beta = \beta' \cdot q = \beta'' \cdot q = 0$ . Rewrite  $q = q \cdot TT + q \cdot NN + q \cdot BB$ , then  $q = q \cdot BB$ . We have  $B = B \cdot BB = (B \cdot B)/(q \cdot B)q = (B \cdot B)/(q \cdot B)\|q\|$ . Since  $\|B\| = 1$ ,  $B = \pm q/\|q\|$ , which is a point, then  $B' = 0$ , hence  $\tau = 0$ . ( $\Leftarrow$ ) Let  $\tau = 0$ , then  $B' = 0$ , so  $B$  is constant. Define  $f : I \rightarrow \mathbb{R}$  by  $f(s) = (\beta(s) - \beta(0)) \cdot B$ , then  $df/ds = T(s) \cdot B = 0$  and  $f(0) = 0 \cdot B = 0$ , so  $f(s) = 0$  for all  $s$ . Hence  $\text{ran}(\beta) \subset \{r \mid (r - \beta(0)) \cdot B = 0\}$ .  $\square$



**Proposition.** Let  $\beta$  be a curve in  $\mathbb{R}^3$  with  $\kappa > 0$ ,  $\kappa' = 0$ ,  $\|\beta'\| = 1$ , and  $\tau = 0$ . Then  $\beta$  lies in a circle of radius  $1/\kappa$ .

*Proof.* Define  $\gamma : I \rightarrow \mathbb{R}^3$  by  $\gamma(s) = \beta(s) + N(s)/\kappa$ , then  $\gamma' = T + N'/\kappa$ . By the Fernet formulas,  $\gamma' = T + (-\kappa T + \tau B)/\kappa = T - T = 0$ . Fix  $t \in I$ . For any  $s \in I$ , the distance  $\|\beta(s) - (\beta(t) + N(t)/\kappa)\| = \|\beta(s) - (\beta(s) + N(s)/\kappa)\| = \|N(s)\|/\kappa = 1/\kappa$ .  $\square$

We have shown the properties of curves with unit speed. Now given a curve  $\alpha : I \rightarrow \mathbb{R}^3$  with  $\|\alpha'\| \neq 1$ , let  $\bar{\alpha}$  be the arc-length reparametrization of  $\alpha$ , so  $\|\bar{\alpha}'\| = 1$ . Let  $\bar{T}$ ,  $\bar{k}$ ,  $\bar{N}$ ,  $\bar{B}$ , and  $\bar{\tau}$  be the corresponding functions of  $\bar{\alpha}$ . Define the  $T$ ,  $\lambda$ ,  $N$ ,  $B$ , and  $\tau$  of  $\alpha$  to be those of  $\bar{\alpha}$ . We denote the speed of  $\alpha$  by  $v$ .

**Theorem 2.2** (Fernet formulas). Let  $\alpha$  be a regular curve on  $\mathbb{R}^3$  with  $\kappa > 0$ , then  $T' = \kappa v N$ ,  $N' = -\kappa v T + \tau v B$ , and  $B' = -\tau v N$ .

*Proof.* Apply the Fernet formulas on  $\bar{\alpha}$ , then  $\bar{T}' = \bar{\kappa} \bar{N}$ . Since  $T' = \bar{T}' ds/dt = \bar{T}' v$ ,  $T' = \bar{\kappa} v \bar{N}$ . Similarly,  $N' = -\kappa v T + \tau v B$  and  $B' = -\tau v N$ .  $\square$

**Proposition.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be regular. Then  $\alpha' = vT$  and  $\alpha'' = (dv/dt)T + \kappa v^2 N$ .

*Proof.* We have  $\alpha' = \bar{\alpha}' ds/dt = v\bar{T} = vT$ , then  $\alpha'' = (dv/dt)T + T'v = (dv/dt)T + (\kappa v N)\kappa = (dv/dt)T + v^2 \kappa N$ .  $\square$

**Proposition.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a regular curve. Then  $T = \alpha'/\|\alpha'\|$ ,  $N = B \times T$ ,  $B = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$ ,  $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3$ , and  $\tau = (\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2$ .

*Proof.* Since  $\|\alpha'\| = v$ ,  $\alpha'/\|\alpha'\| = vT/v = T$ . We have  $\alpha' \times \alpha'' = vT \times ((dv/dt)T + \kappa v^2 N) = vT \times (dv/dt)T + vT \times \kappa v^2 N$ , since  $T \times T = 0$ ,  $\alpha' \times \alpha'' = \kappa v^3 T \times N = \kappa v^3 B$ . The norm  $\|\alpha' \times \alpha''\| = \|\kappa v^3 B\| = \kappa v^3$ , hence  $B = \kappa v^3 B/(\kappa v^3) = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$ . Consider a lemma: for  $w_p, v_p, u_p \in T_p(\mathbb{R}^3)$ ,  $(u \cdot v) \times w = (u \cdot w)v - (v \cdot w)u$ . (lemma) Rewrite  $u = \sum u_i$ ,  $v = \sum v_i$ , and  $w = \sum w_i$ , then  $(u \cdot w)v - (v \cdot w)u = (\sum u_i w_j) \sum v_i - (\sum v_i w_j) \sum u_i = u \times v \times w$ .  $\square$  By the lemma,  $B \times T = T \times N \times T = (T \cdot T)N - (N \cdot T)T = N - 0 = N$ . We have shown  $\|\alpha' \times \alpha''\| = \kappa v^3$ , so  $\|\alpha' \times \alpha''\|/\|\alpha'\|^3 = \kappa v^3/v^3 = \kappa$ . Differentiate  $\alpha''$ , then  $\alpha''' = (dv/dt)T' + (d^2v/dt^2)T + 2\kappa(dv/dt)vN + N'\kappa v^2$ . Since  $B \cdot T = B \cdot N = 0$ ,  $\kappa v^3 B \cdot \alpha''' = \kappa v^3 B \cdot ((dv/dt)T' + N'\kappa v^2)$ , by the Fernet formulas,  $T'$  term becomes 0 and  $N'$  term becomes  $\tau v B$ . Now  $(\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2 v^6 \tau B \cdot B = \kappa^2 v^6 \tau$ . Hence  $(\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2 = \kappa^2 v^6 \tau/(\kappa^2 v^6) = \tau$ .  $\square$

**Example.** Consider the curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ , then  $\alpha' = (3 - 3t^2, 6t, 3 + 3t^2)$ ,  $\alpha'' = (-6t, 6, 6t)$ , and  $\alpha''' = (-6, 0, 6)$ . We have  $\alpha' \cdot \alpha' = 18(1 + t^2)^2$ , so  $\|\alpha'\| = 3\sqrt{2}(1 + t^2)$  and  $T = (1 - t^2, 2t, 1 + t^2)/(\sqrt{2}(1 + t^2))$ . The cross product  $\alpha' \times \alpha'' = (18t^2 - 18, -36t, 18t^2 + 18) = 18(t^2 - 1, -2t, t^2 + 1)$  and its norm  $\|\alpha' \times \alpha''\| = 18\sqrt{2}(1 + t^2)$ , hence  $B = (t^2 - 1, -2t, t^2 + 1)/(\sqrt{2}(1 + t^2))$ . Now  $N = B \times T = (-2t, 1 - t^2, 0)/(1 + t^2)$ . By our computation,  $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3 = 1/(3(1 + t^2)^2)$  and  $\tau = (\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2 = 18(t^2 - 1, -2t, t^2 + 1) \cdot 6(-1, 0, 1)/(18\sqrt{2}(1 + t^2))^2 = 216/(648(1 + t^2)^2) = 1/(3(1 + t^2)^2)$ .

**Definition 2.14.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be regular. We say  $\alpha$  is a *cylindrical helix* if there exists  $u \in \mathbb{R}^3$  such that for all  $t \in I$ ,  $T(t) \cdot u = \cos \theta$ , where  $\theta$  is a constant angle.

If  $\alpha$  has  $\|\alpha'\| \neq 1$ , take the arc-length reparametrization  $\bar{\alpha}$  of  $\alpha$ , by our definition,  $(T, N, B)$ ,  $\kappa$ , and  $\tau$  are all invariant under this reparametrization, so it suffices to consider a curve with unit speed.

**Proposition.** A regular curve  $\alpha$  on  $\mathbb{R}^3$  with  $\kappa > 0$  is a cylindrical helix if and only if  $\tau/\kappa$  is constant.

*Proof.* Let  $\alpha$  be a curve with  $\|\alpha'\| = 1$ . ( $\Rightarrow$ ) Let  $\alpha$  be a cylindrical helix with unit vector  $u$  and constant angle  $\theta$ , then  $T \cdot u = \cos \theta$ , so  $0 = (T \cdot u)' = T' \cdot u = \kappa N \cdot u$ . Since  $\kappa > 0$ ,  $u \cdot N = 0$ , then  $u = (u \cdot T)T + (u \cdot B)B = \cos \theta T + (u \cdot B)B$ . Since  $\|u\| = 1$ ,  $u \cdot B = \sin \theta$ . Now  $0 = u' = \cos \theta \kappa N - \sin \theta \tau N$ , so  $\cos \theta \kappa = \sin \theta \tau$ , which implies  $\tau/\kappa = \cot \theta$ . ( $\Leftarrow$ ) Let  $\tau/\kappa$  be a constant and let  $\cot \theta = \tau/\kappa$  for some angle  $\theta$ . Consider  $U = \cos \theta T + \sin \theta B$ , here  $U' = \cos \theta T' + \sin \theta B' = (\cos \theta \kappa - \sin \theta \tau)N$ , since  $\tau/\kappa = \cot \theta$ ,  $U' = 0$ . For all  $s_1, s_2 \in I$ ,  $U(s_1) = U(s_2)$ , so pick  $U(0) = u$ . Now  $T \cdot u = T \cdot U = T \cdot (\cos \theta T + \sin \theta B) = \cos \theta$ .  $\square$

Let a regular curve  $\alpha$  on  $\mathbb{R}^3$  with  $\|\alpha'\| = 1$  be a circle. Let  $C$  be the center and let  $R : I \rightarrow \mathbb{R}^3$  be the function defined by  $R(s) = \alpha(s) - C$ .

Let  $\tau = 0$ ,  $\kappa > 0$ , and  $\kappa' = 0$  for some  $\alpha$  with  $\|\alpha'\| = 1$ . By the Fernet formulas,  $B' = 0$  and  $N' = -\kappa T$ . Define  $C(s) = \alpha(s) + N(s)/\kappa$ , then  $C'(s) = T(s) + N'(s)/\kappa = T(s) - \kappa T(s)/\kappa = 0$ , so  $C$  is constant. Let  $C$  be the center, then for all  $s$ ,  $\|\alpha(s) - C\| = 1/\kappa$ .

**Definition 2.15.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a cylindrical helix. We say  $\alpha$  is a *circular helix* if its projection onto any plane orthogonal to the axis direction  $u$  is a circle.

**Definition 2.16.** Let  $W$  be a vector field on  $\mathbb{R}^3$  and let  $v \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ . Then the *covariant derivative* of  $W$  with respect to  $v$  is the tangent vector  $\nabla_v W = W(p + tv)'(0)$  at  $p$ .

**Proposition.** Let  $W = \sum w_i U_i$  be a vector field on  $\mathbb{R}^3$  and let  $v \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ . Then  $\nabla_v W = \sum v[w_i]U_i(p)$ .

*Proof.* We have  $W(p + tv) = \sum w_i(p + tv)U_i$ . Since  $(d/dt)w_i(p + tv)$  at  $t = 0$  is  $v[w_i]$ ,  $\nabla_v W = \sum v[w_i]U_i(p)$ .  $\square$

**Proposition.** Let  $v, w \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ . Let  $Y$  and  $Z$  be vector fields on  $\mathbb{R}^3$ . For any  $a, b \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the following properties hold.

1.  $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$ .
2.  $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$ .
3.  $\nabla_v(fY) = v[f]Y(p) + f(p)\nabla_v Y$ .
4.  $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$ .

*Proof.* Rewrite  $Y = \sum y_i U_i$  and  $Z = \sum z_i U_i$ . (i) We have  $\nabla_{av+bw} Y = \sum (av + bw)[y_i]U_i(p) = a \sum v[y_i]U_i + b \sum w[y_i]U_i = a\nabla_v Y + b\nabla_w Y$ . (ii) Similarly,  $\nabla_v(aY + bZ) = \sum v[ay_i + bz_i]U_i = a \sum v[y_i]U_i + b \sum v[z_i]U_i = a\nabla_v Y + b\nabla_v Z$ . (iii) Similarly,  $\nabla_v(fY) = \sum v[fy_i]U_i = \sum (v[f]y_i + f v[y_i])U_i = v[f]Y(p) + f(p)\nabla_v Y$ . (iv) We have  $Y \cdot Z = \sum y_i z_i$ , then  $v[Y \cdot Z] = \sum v[y_i]z_i U_i + \sum v[z_i]y_i U_i = \nabla_v Y \cdot Z + Y \cdot \nabla_v Z$ .  $\square$

Let  $V$  and  $W = \sum w_i$  be vector fields. We define  $\nabla_V W$  at some  $p \in \mathbb{R}^3$  to be  $\nabla_{V(p)} W$ , hence  $\nabla_V W = \sum V[w_i]U_i$ .

**Proposition.** Let  $V, W, Y$ , and  $Z$  be vector fields on  $\mathbb{R}^3$ . For all functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  and  $a, b \in \mathbb{R}$ . Then the following properties hold.

1.  $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$ .
2.  $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ .
3.  $\nabla_V(fY) = V[f]Y + f\nabla_V Y$ .
4.  $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$ .

*Proof.* Since  $\nabla_V W(p) = \nabla_{V(p)} W$ , those properties are direct consequences of the previous proposition.  $\square$

**Definition 2.17.** Let  $E_1, E_2$ , and  $E_3$  be vector fields on  $\mathbb{R}^3$ . We say  $\{E_1, E_2, E_3\}$  is a *frame field* on  $\mathbb{R}^3$  if  $E_i \cdot E_j = \delta_{i,j}$ .

If  $\{E_1, E_2, E_3\}$  is a frame field on  $\mathbb{R}^3$ , then for all  $p \in \mathbb{R}^3$ ,  $\{E_1(p), E_2(p), E_3(p)\}$  is trivially a frame at  $p$ .

**Example.** Consider a cylindrical coordinate system with coordinates  $(r, \theta, z)$ . Define  $E_1 = \cos \theta U_1 + \sin \theta U_2$ ,  $E_2 = -\sin \theta U_1 + \cos \theta U_2$ , and  $E_3 = U_3$ . For any  $p$ ,  $E_1(p) \cdot E_2(p) = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0$ ,  $E_1(p) \cdot E_3(p) = 0 = E_2(p) \cdot E_3(p)$ ,  $E_1(p) \cdot E_1(p) = \cos^2 \theta + \sin^2 \theta = E_2(p) \cdot E_2(p) = 1 = E_3(p) \cdot E_3(p) = 1$ . Hence  $\{E_1, E_2, E_3\}$  is a frame field on  $\mathbb{R}^3$ , known as the *cylindrical frame field*.

**Example.** Consider a spherical coordinate system with coordinates  $(\rho, \theta, \varphi)$ . Define  $F_1 = \cos \varphi E_1 + \sin \varphi E_3$ ,  $F_2 = E_2$ , and  $F_3 = -\sin \varphi E_1 + \cos \varphi E_3$ , where  $\{E_1, E_2, E_3\}$  is the cylindrical frame field. It is trivial that  $F_1 \cdot F_1 = F_2 \cdot F_2 = F_3 \cdot F_3 = 1$ . We have

Hence  $\{F_1, F_2, F_3\}$  is a frame field on  $\mathbb{R}^3$ , called the *spherical frame field*.

**Proposition.** Let  $\{E_1, E_2, E_3\}$  be a frame field on  $\mathbb{R}^3$ . If  $V$  is a vector field on  $\mathbb{R}^3$ , then  $V = \sum f_i E_i$ , where  $f_i = V \cdot E_i$ . If  $V = \sum f_i E_i$  and  $W = \sum g_i E_i$ , then  $V \cdot W = \sum f_i g_i$ .

*Proof.*

□

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