Elementary Differential Geometry

Hassium

1 Calculus on Euclidean Space

Definition 1.1. The *Euclidean 3-space*, denoted $\mathbb{R}^3$, is the set of ordered triples of the form $p = (p_1, p_2, p_3)$, where $p_i \in \mathbb{R}$. An element of $\mathbb{R}^3$ is called a *point*.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. Define the addition to be $p + q = (p_i + q_i)$ and define the scalar multiplication to be $ap = (ap_i)$. The additive identity $0 = (0, 0, 0)$ is called the *origin* of $\mathbb{R}^3$. It is trivial that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Definition 1.2. Let x, y, and z be real-valued functions on $\mathbb{R}^3$ such that for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $x(p) = p_1$. $y(p) = p_2$, and $z(p) = p_3$. We call x, y, and z the *natural coordinate functions* of $\mathbb{R}^3$.

Let x, y, and z be the natural coordinate functions, rewrite $x = x_1$, $y = x_2$, and $z = x_3$. Then we have $p = (p_i) = (x_i(p))$.

Definition 1.3. A real-valued function f on $\mathbb{R}^3$ is *differetiable* if all partial derivatives exist and continuous.

Definition 1.4. A subset $O \subset \mathbb{R}^3$ is *open* if for all $p \in O$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$.

Let $f : O \to \mathbb{R}$ be a function defined on an open set. The differetiability of f at p can be determined entirely from values of f on O. This means that differetiation is a local operation. We will give a proof of this later.

Definition 1.5. A *tangent vector* v_p is an ordered pair $v_p = (v, p)$, where $v, p \in \mathbb{R}^3$. Here v is called the *vector part* and p is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

Definition 1.6. Let $p \in \mathbb{R}^3$. The *tangent space* at p, denoted $T_p(\mathbb{R}^3)$, is the set of all tangent vectors that have p as point of application.

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Exercises and Proofs

Exercise 1.1.1. Let $f = x^2y$ and $g = y\sin z$ be functions on $\mathbb{R}^3$. Express the following functions in terms of x, y, and z.

1. fg^2;
2. $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f$;
3. $\frac{\partial^2 (fg)}{\partial y \partial z}$;
4. $\frac{\partial}{\partial y}(\sin f)$.

Proof. (i) We have $fg^2 = x^2yy^2\sin^2 z = x^2y^3\sin^2 z$. (ii) We have $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial g}{\partial y} = \sin z$, then $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = 2xy^2\sin z + x^2y\sin z$. (iii) We have $fg = x^2y^2\sin z$, then $\frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos z$. (iv) We have $\sin f = \sin(x^2y)$, then $\frac{\partial}{\partial y}(\sin f) = x^2\cos(x^2y)$. □

Exercise 1.1.3. Express $\frac{\partial f}{\partial x}$ in terms of x, y, and z for the following functions.

1. $f = x\sin(xy) + y\cos(xz)$;
2. $f = \sin g$, $g = e^h$, and $h = x^2 + y^2 + z^2$.

Proof. (i) We have $\frac{\partial f}{\partial x} = \frac{x\sin(xy)}{\partial x} + \frac{\partial y\cos(xz)}{\partial x} = \sin(xy) + xy\cos(xy) - yz\sin(xz)$. (ii) We have $f = \sin(e^{x^2+y^2+z^2})$, then $\frac{\partial f}{\partial x} = 2x\cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}$. □

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