Elementary Differential Geometry

Hassium

1 Calculus on Euclidean Space

Definition 1.1. The *Euclidean 3-space*, denoted $\mathbb{R}^3$, is the set of ordered triples of the form $p = (p_1, p_2, p_3)$, where $p_i \in \mathbb{R}$. An element of $\mathbb{R}^3$ is called a *point*.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. Define the addition to be $p + q = (p_i + q_i)$ and define the scalar multiplication to be $ap = (ap_i)$. The additive identity $0 = (0, 0, 0)$ is called the *origin* of $\mathbb{R}^3$. It is trivial that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Definition 1.2. Let x, y, and z be real-valued functions on $\mathbb{R}^3$ such that for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $x(p) = p_1$. $y(p) = p_2$, and $z(p) = p_3$. We call x, y, and z the *natural coordinate functions* of $\mathbb{R}^3$.

Let x, y, and z be the natural coordinate functions, rewrite $x = x_1$, $y = x_2$, and $z = x_3$. Then we have $p = (p_i) = (x_i(p))$.

Definition 1.3. A real-valued function f on $\mathbb{R}^3$ is *differetiable* if all partial derivatives exist and continuous.

Definition 1.4. A subset $O \subset \mathbb{R}^3$ is *open* if for all $p \in O$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$.

Let $f : O \to \mathbb{R}$ be a function defined on an open set. The differetiability of f at p can be determined entirely from values of f on O. This means that differetiation is a local operation. We will give a proof of this later.

Definition 1.5. A *tangent vector* v_p is an ordered pair $v_p = (v, p)$, where $v, p \in \mathbb{R}^3$. Here v is called the *vector part* and p is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

Definition 1.6. Let $p \in \mathbb{R}^3$. The *tangent space* at p, denoted $T_p(\mathbb{R}^3)$, is the set of all tangent vectors that have p as point of application.

Fix a tangent space $T_p(\mathbb{R}^3)$ and let $T_p(\mathbb{R}^3)$ adapt the operations from $\mathbb{R}^3 \times \mathbb{R}^3$. We have a natural linear map $f : T_p(\mathbb{R}^3) \to \mathbb{R}^3$ defined by $v_p \to v$ and it is trivially an isomorphism.

Definition 1.7. A *vector field* V on $\mathbb{R}^3$ is a function $V : \mathbb{R}^3 \to \mathbb{R}^3$ such that for all $p \in \mathbb{R}^3$, $V(p) \subset T_p(\mathbb{R}^3)$.

Let V and W be vector field. Let f be a real-valued function. For all $p \in \mathbb{R}^3$, define $V + W$ by $(V + W)(p) = V(p) + W(p)$ and $(fV)(p) = f(p)V(p)$.

Definition 1.8. Let U_1, U_2, and U_3 be vector fields on $\mathbb{R}^3$ such that $U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, and $U_3(p) = (0, 0, 1)_p$ for all $p \in \mathbb{R}^3$. We call (U_1, U_2, U_3) the *natural frame field* on $\mathbb{R}^3$.

Proposition. Let V be a vector field on $\mathbb{R}^3$. There are three uniquely determined real-valued functions v_1, v_2, and v_3 on $\mathbb{R}^3$ such that $V = v_1U_1 + v_2U_2 + v_3U_3$.

Proof. For all $p \in \mathbb{R}^3$, $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1,0,0)_p + v_2(p)(0,1,0)_p + v_3(p)(0,0,1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3U_3(p)$, hence $V = \sum v_iU_i$. $\square$

The functions v_1, v_2, and v_3 are called the *Euclidean coordinate functions* on V.

Definition 1.9. A vecotr field V is *differetiable* if its Euclidean coordinate functions are differetiable.

2 Frame Fields

3 Euclidean Geometry

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6 Geometry of Surfaces in $\mathbb{R}^3$

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Exercises and Proofs

Exercise 1.1.1. Let $f = x^2y$ and $g = y\sin z$ be functions on $\mathbb{R}^3$. Express the following functions in terms of x, y, and z.

1. fg^2;
2. $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f$;
3. $\frac{\partial^2(fg)}{\partial y \partial z}$;
4. $\frac{\partial}{\partial y}(\sin f)$.

Proof. (i) We have $fg^2 = x^2yy^2\text{sin}^2 z = x^2y^3\text{sin}^2 z$. (ii) We have $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial g}{\partial y} = \sin z$, then $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = 2xy^2\sin z + x^2y\sin z$. (iii) We have $fg = x^2y^2\sin z$, then $\frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos z$. (iv) We have $\sin f = \sin(x^2y)$, then $\frac{\partial}{\partial y}(\sin f) = x^2\cos(x^2y)$. □

Exercise 1.1.3. Express $\frac{\partial f}{\partial x}$ in terms of x, y, and z for the following functions.

1. $f = x\sin(xy) + y\cos(xz)$;
2. $f = \sin g$, $g = e^h$, and $h = x^2 + y^2 + z^2$.

Proof. (i) We have $\frac{\partial f}{\partial x} = \frac{x\sin(xy)}{\partial x} + \frac{\partial y\cos(xz)}{\partial x} = \sin(xy) + xy\cos(xy) - yz\sin(xz)$. (ii) We have $f = \sin(e^{x^2+y^2+z^2})$, then $\frac{\partial f}{\partial x} = 2x\cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}$. □

Exercise 1.2.1. Let $v = (-2, 1, -1)$ and $w = (0, 1, 3)$. At an arbitrary point p, express the tangent vector $3v_p - 2w_p$ as a linear combination of $U_1(p)$, $U_2(p)$, and $U_3(p)$.

Proof. We have $3v_p - 2w_p = (-6, 1, -9)_p = -6U_1(p) + U_2(p) - 9U_3(p)$. □

Exercise 1.2.5. Let $V_1 = U_1 - xU_3$, $V_2 = U_2$, and $V_3 = xU_1 + U_3$. Prove that the vectors $V_1(p)$, $V_2(p)$, $V_3(p)$ are linearly indepnedent at each $p \in \mathbb{R}^3$. Express the vector field $xU_1 + yU_2 + zU_3$ as a linear combination of V_i.

Proof. For all $p \in \mathbb{R}^3$, we have $V_1(p) = U_1(p) - xU_3(p) = (1, 0, -x)$. Similarly, $V_2(p) = (0, 1, 0)$ and $V_3 = (x, 0, 1)$. Consider $aV_1(p) + bV_2(p) + cV_3(p) = 0$, where $a, b, c \in \mathbb{R}$. Solve for (a, b, c), then $c(x^2 + 1) = 0$, so $c = 0$. Now $(a, b, c) = (0, 0, 0)$, hence $V_i(p)$ are linearly indepnedent. □

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