

# Elementary Differential Geometry

Hassium

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## 1 Calculus on Euclidean Space

**Definition 1.1.** The *Euclidean 3-space*, denoted  $\mathbb{R}^3$ , is the set of ordered triples of the form  $p = (p_1, p_2, p_3)$ , where  $p_i \in \mathbb{R}$ . An element of  $\mathbb{R}^3$  is called a *point*.

Let  $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$  and let  $a \in \mathbb{R}$ . Define the addition to be  $p + q = (p_i + q_i)$  and define the scalar multiplication to be  $ap = (ap_i)$ . The additive identity  $0 = (0, 0, 0)$  is called the *origin* of  $\mathbb{R}^3$ . It is trivial that  $\mathbb{R}^3$  is a vector space over  $\mathbb{R}$ .

**Definition 1.2.** Let  $x, y$ , and  $z$  be real-valued functions on  $\mathbb{R}^3$  such that for all  $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ ,  $x(p) = p_1$ ,  $y(p) = p_2$ , and  $z(p) = p_3$ . We call  $x, y$ , and  $z$  the *natural coordinate functions* of  $\mathbb{R}^3$ .

Let  $x, y$ , and  $z$  be the natural coordinate functions, rewrite  $x = x_1, y = x_2$ , and  $z = x_3$ . Then we have  $p = (p_i) = (x_i(p))$ .

**Definition 1.3.** A real-valued function  $f$  on  $\mathbb{R}^3$  is *differentiable* if all partial derivatives exist and continuous.

Let  $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ . Recall that the dot product is defined to be  $p \cdot q = \sum p_i q_i$  and the norm is defined to be  $\|p\| = \sqrt{p \cdot p} = \sqrt{\sum p_i^2}$ .

**Definition 1.4.** A subset  $O \subset \mathbb{R}^3$  is *open* if for all  $p \in O$ , there exists  $\varepsilon > 0$  such that  $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$ .

Let  $f : O \rightarrow \mathbb{R}$  be a function defined on an open set. The differentiability of  $f$  at  $p$  can be determined entirely from values of  $f$  on  $O$ . This means that differentiation is a local operation. We will discuss this later.

**Definition 1.5.** A *tangent vector*  $v_p$  is an ordered pair  $v_p = (v, p)$ , where  $v, p \in \mathbb{R}^3$ . Here  $v$  is called the *vector part* and  $p$  is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

**Definition 1.6.** Let  $p \in \mathbb{R}^3$ . The *tangent space* at  $p$ , denoted  $T_p(\mathbb{R}^3)$ , is the set of all tangent vectors that have  $p$  as point of application.

Fix a tangent space  $T_p(\mathbb{R}^3)$  and let  $T_p(\mathbb{R}^3)$  adapt the operations from  $\mathbb{R}^3 \times \mathbb{R}^3$ . We have a natural linear map  $f : T_p(\mathbb{R}^3) \rightarrow \mathbb{R}^3$  defined by  $v_p \rightarrow v$  and it is trivially an isomorphism.

**Definition 1.7.** A *vector field*  $V$  on  $\mathbb{R}^3$  is a function  $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that for all  $p \in \mathbb{R}^3$ ,  $V(p) \in T_p(\mathbb{R}^3)$ .

Let  $V$  and  $W$  be vector field. Let  $f$  be a real-valued function. For all  $p \in \mathbb{R}^3$ , define  $V + W$  by  $(V + W)(p) = V(p) + W(p)$  and  $(fV)(p) = f(p)V(p)$ .

**Definition 1.8.** Let  $U_1, U_2$ , and  $U_3$  be vector fields on  $\mathbb{R}^3$  such that  $U_1(p) = (1, 0, 0)_p$ ,  $U_2(p) = (0, 1, 0)_p$ , and  $U_3(p) = (0, 0, 1)_p$  for all  $p \in \mathbb{R}^3$ . We call  $(U_1, U_2, U_3)$  the *natural frame field* on  $\mathbb{R}^3$ .

**Proposition.** Let  $V$  be a vector field on  $\mathbb{R}^3$ . There are three uniquely determined real-valued functions  $v_1$ ,  $v_2$ , and  $v_3$  on  $\mathbb{R}^3$  such that  $V = v_1U_1 + v_2U_2 + v_3U_3$ .

*Proof.* For all  $p \in \mathbb{R}^3$ ,  $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1, 0, 0)_p + v_2(p)(0, 1, 0)_p + v_3(p)(0, 0, 1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3(p)U_3(p)$ , hence  $V = \sum v_iU_i$ .  $\square$

The functions  $v_1$ ,  $v_2$ , and  $v_3$  are called the *Euclidean coordinate functions* on  $V$ .

**Definition 1.9.** A vector field  $V$  is *differentiable* if its Euclidean coordinate functions are differentiable.

**Definition 1.10.** Let  $f$  be a differentiable real-valued function on  $\mathbb{R}^3$  and let  $v_p$  be a tangent vector on  $\mathbb{R}^3$ . The *directional derivative* of  $f$  with respect to  $v_p$ , denoted  $v_p[f]$ , is defined to be  $(d/dt)f(p + tv)$  at  $t = 0$ .

**Remark.** We will not write the restriction every time for convenience.

**Proposition.** Let  $v_p = (v_1, v_2, v_3)_p$  be a tangent vector, then  $v_p[f] = \sum v_i(\partial f/\partial x_i)(p)$ .

*Proof.* Let  $p = (p_1, p_2, p_3)$ . Then  $v_p[f] = (d/dt)f(p + tv)|_{t=0} = \sum (\partial f/\partial x_i)(p) \cdot (d/dt)(p_i + tv_i) = \sum (\partial f/\partial x_i)(p)v_i$ .  $\square$

**Example.** Consider  $f = x^2yz$  with  $p = (1, 1, 0)$  and  $v = (1, 0, -3)$ . By the definition,  $p + tv = (1 + t, 1, -3t)$ , so  $v_p[f] = (d/dt)(-3t^3 - 6t^2 - 3t) = -3$ . Since  $(\partial f/\partial x) = 2xyz$ ,  $(\partial f/\partial y) = x^2z$ , and  $(\partial f/\partial z) = x^2y$ , we have  $(\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0$  and  $(\partial f/\partial z)(p) = 1$ , so  $v_p[f] = -3$ .

**Proposition.** Let  $f$  and  $g$  be functions on  $\mathbb{R}^3$ . Let  $v_p$  and  $w_p$  be tangent vectors. For all  $a, b \in \mathbb{R}$ , the following properties hold.

1.  $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$ .
2.  $v_p[af + bg] = av_p[f] + bv_p[g]$ .
3.  $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$ .

*Proof.* (i) We have  $(av_p + bw_p)[f] = \sum (av_i + bw_i)(\partial f/\partial x_i)(p) = \sum av_i(\partial f/\partial x_i)(p) + \sum bw_i(\partial f/\partial x_i)(p) = av_p[f] + bw_p[f]$ .

(ii) We have  $v_p[af + bg] = \sum v_i(\partial(af + bg)/\partial x_i)(p) = \sum v_i(\partial(af)/\partial x_i)(p) + \sum v_i(\partial(bg)/\partial x_i)(p) = av_p[f] + bv_p[g]$ . (iii)

We have  $v_p[fg] = \sum v_i(\partial(fg)/\partial x_i)(p) = \sum v_i(\partial f/\partial x_i)(p)g(p) + f(p)\sum v_i(\partial g/\partial x_i)(p) = v_p[f]g(p) + f(p)v_p[g]$ .  $\square$

Let  $V$  be a vector field, we define  $V[f]$  at  $p \in \mathbb{R}^3$  to be  $V(p)[f]$ . By the convention,  $U_i(p)[f] = (\partial f/\partial x_i)(p)$ .

**Proposition.** Let  $V$  and  $W$  be vector fields. Let  $f$ ,  $g$ , and  $h$  be real-valued functions. For all  $a, b \in \mathbb{R}$ , the following properties hold.

1.  $(fV + gW)[h] = fV[h] + gW[h]$ .
2.  $V[af + bg] = aV[f] + bV[g]$ .
3.  $V[fg] = V[f]g + fV[g]$ .

*Proof.* (i) For all  $p \in \mathbb{R}^3$ ,  $(fV + gW)(p)[h] = (f(p)V(p) + g(p)W(p))[h] = fV[h] + gW[h]$ . (ii) For all  $p \in \mathbb{R}^3$ ,  $V(p)[af + bg] = aV(p)[f] + bV(p)[g]$ . (iii) For all  $p \in \mathbb{R}^3$ ,  $V(p)[fg] = V(p)[f]g(p) + f(p)V(p)[g] = V[f](p)g(p) + f(p)V[g](p) = (V[f]g + fV[g])(p)$ .  $\square$

**Example.** Let  $V = xU_1 - y^2U_3$  and let  $f = x^2y + z^3$ . Then  $V[f] = xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3] = 2x^2y - 3y^2z^2$ .

Let  $I \subset \mathbb{R}$  be an open interval. Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a function. We can rewrite  $\alpha(t)$  as  $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$ , where  $\alpha_i : I \rightarrow \mathbb{R}$ . We say  $\alpha$  is *differentiable* if  $\alpha_i$  are differentiable.

**Definition 1.11.** A *curve* in  $\mathbb{R}^3$  is a differentiable function  $\alpha : I \rightarrow \mathbb{R}^3$ , where  $I \subset \mathbb{R}$  is an open interval.

**Example.** A curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = p + tq$ , where  $\alpha(0) = p$  and  $q \neq 0$ , is called a *straight line*.

**Example.** Here are some examples of curves.

1. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (a \cos t, a \sin t, bt)$ .
2. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (1 + \cos t, \sin t, 2 \sin(t/2))$ .
3. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$ .
4. The curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by  $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ .

**Definition 1.12.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve with  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ . For all  $t \in I$ , the *velocity vector* of  $\alpha$  at  $t$  is the tangent vector  $\alpha'(t) = ((d\alpha_1/dt)(t), (d\alpha_2/dt)(t), (d\alpha_3/dt)(t))_{\alpha(t)}$  at the point  $\alpha(t) \in \mathbb{R}^3$ . The curve  $\alpha$  is said to be *regular* if  $\alpha_i \neq 0$  for all  $i$ .

Consider the velocity vector  $\alpha'(t)$ , we can rewrite it by the natural frame fields, so  $\alpha'(t) = \sum (d\alpha_i/dt)(t) U_i(\alpha(t))$ .

**Definition 1.13.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve and let  $h : J \rightarrow I$  be differentiable, where  $J$  is an open interval of  $\mathbb{R}$ . The *reparametrization* of  $\alpha$  by  $h$  is the composition  $\alpha \circ h : J \rightarrow \mathbb{R}^3$ .

The composition of differentiable functions is differentiable, so any reparametrization is differentiable, which means it is a curve.

**Proposition.** Let  $\beta$  be the reparametrization of  $\alpha$  by  $h$ , then  $\beta'(s) = (dh/ds)(s) \alpha'(h(s))$ .

*Proof.* Rewrite  $\beta(s) = \alpha(h(s))$ , then we have  $\beta'(s) = (d(\alpha_i h_i)/ds)(s)_{\alpha(h(s))} = (d\alpha_i/ds)(h(s)) \cdot (dh/ds)(s)_{\alpha(h(s))} = (dh/ds)(s) \alpha'(h(s))$ .  $\square$

**Proposition.** Let  $\alpha$  be a curve and let  $f$  be a differentiable function on  $\mathbb{R}^3$ , then  $\alpha'(t)[f] = (d(f\alpha)/dt)(t)$ .

*Proof.* We have  $\alpha'(t)[f] = \sum (d\alpha_i/dt)(t) \cdot (\partial f/\partial x_i)(\alpha(t)) = (d(f\alpha)/dt)(t)$  by the chain rule.  $\square$

Now we show a general idea of parametrizations. The proofs will be included in other sections when we have enough tools. Assume every result is correct for now.

**Definition 1.14.** A *1-form*  $\varphi$  on  $\mathbb{R}^3$  is a function  $\varphi : \coprod_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3) \rightarrow \mathbb{R}$  such that for all  $a, b \in \mathbb{R}$  and  $v, w \in T_p(\mathbb{R}^3)$  for some  $p \in \mathbb{R}^3$ ,  $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$ .

Given a 1-form  $\varphi$ , for any point  $p$ , denote the restriction  $\varphi|_{T_p(\mathbb{R}^3)} : T_p(\mathbb{R}^3) \rightarrow \mathbb{R}$  by  $\varphi_p$ , then  $\varphi_p$  is linear. Let  $\varphi$  and  $\psi$  be 1-forms. Define the addition and scalar multiplication by  $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$  and  $(f\varphi)(v_p) = f(p)\varphi(v_p)$ . Given any 1-form  $\varphi$  and point  $p$ ,  $\varphi_p$  is a linear functional in  $T_p^*(\mathbb{R}^3)$ , the dual space of  $T_p(\mathbb{R}^3)$ .

**Definition 1.15.** Let  $\varphi$  be a 1-form and let  $V$  be a vector field. For all  $p \in \mathbb{R}^3$ , define  $\varphi(V)(p) = \varphi_p(V(p))$ . We say  $\varphi$  is *differentiable* if for every differentiable vector field  $V$ , the function  $\varphi(V)$  is differentiable.

Now let  $V$  and  $W$  be vector fields, we have  $\varphi(fV + gW)(p) = \varphi((fV + gW)(p)) = \varphi(fV(p) + gW(p)) = (f\varphi(V) + g\varphi(W))(p)$ . Similarly,  $(f\varphi + g\psi)(V) = f\varphi(V) + g\psi(V)$ .

**Definition 1.16.** If  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is differentiable. The *differential* of  $f$ , denoted  $df$ , is the function  $df(v_p) = v_p[f]$  for all tangent vectors  $v_p$ .

Let  $v_p, w_p \in T_p(\mathbb{R}^3)$  and let  $a, b \in \mathbb{R}$ , then  $df(av_p + bw_p) = (av_p + bw_p)[f] = av_p[f] + bw_p[f] = a df(v_p) + b df(w_p)$ . Hence  $df$  is a 1-form.

**Example.** Consider the natural coordinate functions  $x_i$ . We have  $dx_i(v_p) = v_p[x_i] = \sum v_i(\partial x_i/\partial x_j)(p) = v_i$ .

**Proposition.** If  $\varphi$  is a 1-form on  $\mathbb{R}^3$ , then  $\varphi = \sum f_i dx_i$ , where  $f_i = \varphi(U_i)$ .

*Proof.* Let  $v_p \in T_p(\mathbb{R}^3)$ , then  $\varphi(v_p) = \varphi(\sum v_i U_i(p)) = \sum v_i \varphi(U_i(p)) = \sum v_i f_i(p) = \sum f_i(p) dx_i(v_p) = (\sum f_i dx_i)(v_p)$ , hence  $\varphi = \sum f_i dx_i$ .  $\square$

The functions  $f_1$ ,  $f_2$ , and  $f_3$  are called the *Euclidean coordinate functions* of the 1-form  $\varphi$ .

**Proposition.** Let  $f$  be a differentiable function on  $\mathbb{R}^3$ , then  $df = \sum (\partial f / \partial x_i) dx_i$ .

*Proof.* Let  $v_p \in T_p(\mathbb{R}^3)$ , then  $df(v_p) = v_p[f] = \sum v_i(\partial f / \partial x_i)(p) = \sum (\partial f / \partial x_i)(p) dx_i(v_p) = (\sum (\partial f / \partial x_i) dx_i)(v_p)$ , hence  $df = \sum (\partial f / \partial x_i) dx_i$ .  $\square$

Let  $f$  and  $g$  be differentiable functions on  $\mathbb{R}^3$ , then  $d(f + g) = \sum (\partial(f + g) / \partial x_i) dx_i = \sum (\partial f / \partial x_i) dx_i + \sum (\partial g / \partial x_i) dx_i = df + dg$ . Now we denote the multiplication to be  $fg$ .

**Proposition.** Let  $f$  and  $g$  be differentiable functions on  $\mathbb{R}^3$ , then  $d(fg) = gdf + f dg$ .

*Proof.* We have  $d(fg) = \sum (\partial(fg) / \partial x_i) dx_i = \sum ((\partial f / \partial x_i)g + (\partial g / \partial x_i)f) dx_i = gdf + f dg$ .  $\square$

**Proposition.** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable, then  $d(h(f)) = (dh(f)/dx)df$ .

*Proof.* We have  $d(h(f)) = \sum (\partial h(f) / \partial x_i) dx_i$ , by the chain rule,  $(\partial h(f) / \partial x_i) dx_i = (dh(f)/df)(\partial x / \partial x_i)$ , so  $d(h(f)) = (df(h)/df)df$ .  $\square$

**Example.** Consider the function  $f = (x^2 - 1)y + (y^2 + 2)z$ . We have  $df = d((x^2 - 1)y) + d((y^2 + 2)z) = yd(x^2 - 1) + (x^2 + 1)dy + zd(y^2 + 2) + (y^2 + 2)dz = 2xydx + (x^2 + 2yz - 1)dy + (y^2 + 2)dz$ . Since  $v_p[f] = df(v_p)$ ,  $v_p[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_2^2 + 2)v_3$ .

**Definition 1.17.** Let  $V$  be the vector space  $\mathbb{R}^3$  and denote the space of all  $p$ -linear forms on  $V$  by  $\Lambda^p(V^*)$ . Every element of  $\Lambda^p$  is called a  $p$ -form. Define the *wedge product* to be a function  $\wedge : \Lambda^a(V^*) \times \Lambda^b(V^*) \rightarrow \Lambda^{a+b}(V^*)$  such that for  $\omega \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $v_1, \dots, v_{m+n} \in V$ , the following properties hold.

1.  $(\omega \wedge \eta)(v_1, \dots, v_{m+n}) = (\sum_{\sigma \in \mathfrak{S}_{m+n}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(m)}) \eta(v_{\sigma(m+1)}, \dots, v_{\sigma(m+n)})) / (m!n!)$ .
2.  $\omega \wedge \eta = (-1)^{mn} \eta \wedge \omega$ .

Generally, a  $p$ -form is of the form  $\sum f(x, y, z) dx^i \wedge \dots \wedge dy^j \wedge \dots \wedge dz^k \wedge \dots$ . We have  $dx_i \wedge dx_j = -dx_j \wedge dx_i$ . If  $i = j$ , then  $dx_i \wedge dx_i = -dx_i \wedge dx_i$ , so  $dx_i \wedge dx_i = 0$ . It is trivial that  $\wedge$  is bilinear and associative, that is,

1. for  $\omega_1, \omega_2 \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $a, b \in \mathbb{R}$ ,  $(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta)$  and  $\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2)$ ;
2. for  $\omega \in \Lambda^m(V^*)$ ,  $\eta \in \Lambda^n(V^*)$ , and  $\theta \in \Lambda^l(V^*)$ ,  $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta$ .

Now given a space of  $p$ -forms  $\Lambda^p(V^*)$  with basis  $\{e^1, e^2, e^3\}$ , the basis of its dual space is denoted by  $\{e^1, e^2, e^3\}$ . The basis of  $\Lambda^k(V^*)$  is of the form  $e^{i_1} \wedge \dots \wedge e^{i_k}$ , where  $1 \leq i_1 \leq \dots \leq i_k \leq 3$ . In this case, the dimension of  $\Lambda^p(V^*)$  is  $3!/(p!(3-p)!)$ . If  $p > 4$ , then  $\dim(\Lambda^p(V^*)) = 0$ , so there are no  $p$ -forms on  $\mathbb{R}^3$  if  $p \geq 4$ .

**Example.** Let  $\varphi = xdx - ydy$ ,  $\psi = zdx + xdz$ ,  $\theta = zdy$ , and  $\eta = ydx \wedge dz + xdy \wedge dz$ .

1.  $\varphi \wedge \psi = xzdx \wedge dx + x^2dx \wedge dz - yzdy \wedge dx - yxdy \wedge dz = yzdx \wedge dy + x^2dx \wedge dz - yxdy \wedge dz$
2.  $\theta \wedge (\varphi \wedge \psi) = yz^2dx \wedge (dy \wedge dy) + x^2zdx \wedge dz \wedge dy - xyzdy \wedge dz \wedge dy = -x^2zdx \wedge dy \wedge dz$
3.  $\varphi \wedge \eta = xydx \wedge dx \wedge dz + x^2dx \wedge dy \wedge dz - y^2dy \wedge dx \wedge dz - xydy \wedge dy \wedge dz = (x^2 + y^2)dx \wedge dy \wedge dz$

**Proposition.** Let  $\varphi$  and  $\psi$  be 1-forms, then  $\varphi \wedge \psi = -\psi \wedge \varphi$ .

*Proof.* Rewrite  $\varphi = \sum f_i dx_i$  and  $\psi = \sum g_i dx_i$ , then  $\varphi \wedge \psi = \sum f_i g_i dx_i dx_i = \sum -g_i f_i dx_i dx_i = -\psi \wedge \varphi$ .  $\square$

**Definition 1.18.** Let  $\varphi = \sum f_i dx_i$  be a 1-form on  $\mathbb{R}^3$ . The *exterior derivative* of  $\varphi$  is the 2-form  $d\varphi = \sum df_i \wedge dx_i$ . Let  $\psi = \sum f_{i,j} dx_i \wedge dx_j$  be a 2-form. The *exterior derivative* of  $\psi$  is the 3-form  $d\psi = \sum df_{i,j} \wedge dx_i \wedge dx_j$ .

Let  $a, b \in \mathbb{R}$ . Let  $\varphi = \sum f_i dx_i$  and  $\psi = \sum g_i dx_i$  be 1-forms. Then  $d(a\varphi + b\psi) = d(\sum (af_i + bg_i) dx_i) = \sum d(af_i + bg_i) \wedge dx_i$ , since the differential is linear, the exterior derivative is linear.

**Proposition.** Let  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$  be functions and let  $\varphi$  and  $\psi$  be 1-forms. Then  $d(f\varphi) = df \wedge \varphi + f d\varphi$  and  $d(\varphi \wedge \psi) = d\varphi \wedge \psi - \varphi \wedge d\psi$ .

*Proof.* (i) Let  $\varphi = \sum g_i dx_i$ , then  $f\varphi = \sum f g_i dx_i$ , so  $d(f\varphi) = \sum (f dg_i + g_i df) \wedge dx_i = \sum f dg_i \wedge dx_i + \sum g_i df \wedge dx_i = f d\varphi + d\varphi \wedge f$ . (ii) Since  $dx_i \wedge dx_i = 0$ , without loss of generality, let  $\varphi = f dx$  and let  $\psi = g dy$ . Then  $d(\varphi \wedge \psi) = d(f g dx \wedge dy) = d(fg) \wedge dx \wedge dy = (f dg + g df) \wedge dx \wedge dy = f dg \wedge dx \wedge dy + g df \wedge dx \wedge dy$ . For the right hand side,  $d\varphi \wedge \psi = df \wedge dx \wedge g dy = g df \wedge dx \wedge dy$  and  $\varphi \wedge d\psi = f dx \wedge dg \wedge dy = -f dg \wedge dx \wedge dy$ , hence  $d(\varphi \wedge \psi) = d\varphi \wedge \psi - \varphi \wedge d\psi$ .  $\square$

**Definition 1.19.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and let  $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $F(p) = (f_1(p), \dots, f_m(p))$  for all  $p \in \mathbb{R}^n$ . The functions  $f_i$  are called the *Euclidean coordinate functions* of  $F$  and we denote  $F = (f_1, \dots, f_m)$ .

**Definition 1.20.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $F = (f_1, \dots, f_m)$ , we say  $F$  is *differentiable* if all  $f_i$  are differentiable. If  $F$  is differentiable, we say  $F$  is a *mapping* from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .

**Definition 1.21.** Let  $\alpha : I \rightarrow \mathbb{R}^n$  be a curve and let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. Then the composite function  $\beta = F(\alpha) : I \rightarrow \mathbb{R}^m$  is a curve in  $\mathbb{R}^m$  called the *image* of  $\alpha$  under  $F$ .

To examine the effect of a mapping, it suffices to take a proper  $\alpha$  and check the image of it.

**Example.** The function  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $F = (x - y, x + y, 2z)$  is a mapping. Trivially,  $F$  is a linear map, so  $F$  is determined by  $F(u_i)$ .

**Example.** Consider the mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $F = (u^2 - v^2, 2uv)$ . Let  $\alpha : I \rightarrow \mathbb{R}^2$  defined by  $\alpha(t) = (r \cos t, r \sin t)$ , where  $0 \leq t \leq 2\pi$ . The image is  $\beta(t) = (r^2 \cos 2t, r^2 \sin 2t)$ . This curve takes two counterclockwise trips around the circle of radius  $r^2$  centered at the origin. Therefore,  $F$  wraps  $\mathbb{R}^2$  around itself twice.

**Definition 1.22.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping and let  $v_p \in T_p(\mathbb{R}^n)$ . The *tangent map* of  $F$ , denoted  $F_*(v_p)$ , is defined to be  $(d/dt)F(p + tv)$  at  $t = 0$ .

Fix some mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . For every  $p \in \mathbb{R}^n$ , it induces a tangent map of  $F$  at  $p$ , denoted  $F_{*p}$ .

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. If  $v_p \in T_p(\mathbb{R}^n)$ , then  $F_{*p}(v_p) = (v[f_1], \dots, v[f_m])_{F(p)}$ .

*Proof.* Fix  $v_p \in T_p(\mathbb{R}^n)$ . We have  $F_{*p} = (d/dt)F(p + tv)|_{t=0} = (d/dt)(f_i(p + tv))|_{t=0} = (v_p[f_1], \dots, v_p[f_m])_{F(p)}$ .  $\square$

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. For all  $p \in T_p(\mathbb{R}^n)$ , the tangent map  $F_{*p} : T_p(\mathbb{R}^n) \rightarrow T_{F(p)}(\mathbb{R}^m)$  is a linear map.

*Proof.* Fix  $p \in \mathbb{R}^n$ . Let  $a, b \in \mathbb{R}$  and let  $v_p, w_p \in T_p(\mathbb{R}^n)$ . We have  $F_{*p}(av_p + bw_p) = ((av_p + bw_p)[f_i])_{F(p)} = (av_p[f_i])_{F(p)} + (bw_p[f_i])_{F(p)} = aF_{*p}(v_p) + bF_{*p}(w_p)$ .  $\square$

**Proposition.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping and let  $\beta$  be the image of some curve  $\alpha$  in  $\mathbb{R}^n$ , then  $\beta' = F_*(\alpha')$ .

*Proof.* Let  $F = (f_1, \dots, f_m)$ . We have  $F_*(\alpha'(t)) = (\alpha'(t)[f_i])_{F(\alpha(t))} = (df_i(\alpha(t))/dt)_{F(\alpha(t))} = \beta'(t)$ .  $\square$

Let  $\{U_j\}$  and  $\{\overline{U}_i\}$  be the natural frame fields of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively.

**Proposition.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. Then  $F_*(U_j(p)) = \sum_{i=1}^m (\partial f_i / \partial x_j)(p) \overline{U}_i(F(p))$ , where  $1 \leq j \leq n$ .

*Proof.* Recall that  $U_j[f_i] = \partial f_i / \partial x_j$ , so the proposition trivially holds.  $\square$

**Definition 1.23.** Let  $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. The *Jacobian matrix* of  $F$  at  $x \in \mathbb{R}^n$  is the matrix

$$J_F(x) = \begin{pmatrix} \partial f_1 / \partial x_1(x) & \cdots & \partial f_1 / \partial x_n(x) \\ \vdots & \ddots & \vdots \\ \partial f_m / \partial x_1(x) & \cdots & \partial f_m / \partial x_n(x) \end{pmatrix}.$$

**Definition 1.24.** Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. We say  $F$  is *regular* if for all  $p \in \mathbb{R}^n$ ,  $F_{*p}$  is injective.

Notice that  $J_F(p) \cdot v = F_{*p}$ , so  $J_F(p)$  is the matrix representation of  $F_{*p}$ .

**Definition 1.25.** A mapping is a *diffeomorphism* if it has a differentiable inverse mapping.

**Definition 1.26.** A *topological space*  $(X, \mathcal{T})$  consists of two sets  $X$  and  $\mathcal{T}$ , where  $\mathcal{T} \subset \mathcal{P}(X)$ , that satisfies the following properties:

1.  $\emptyset, X \in \mathcal{T}$ ;
2. any union of elements in  $\mathcal{T}$  is also in  $\mathcal{T}$ ;
3. any finite intersection of elements in  $\mathcal{T}$  is also in  $\mathcal{T}$ .

The collection  $\mathcal{T}$  is called a *topology* on  $X$ .

**Definition 1.27.** Let  $(X, \mathcal{T})$  be a topological space. A subset  $U \subset X$  is said to be *open* if  $U \in \mathcal{T}$ . Let  $x \in X$ , a *neighborhood* of  $x$  is an open set  $U_x$  that contains  $x$ .

**Theorem 1.1** (inverse function theorem). Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a mapping. If  $F_{*p}$  is injective at some  $p \in \mathbb{R}^n$ , then there exists a neighborhood  $U$  of  $p$  such that  $F|_U : U \rightarrow V$ , where  $V$  is open, is a diffeomorphism.

## 2 Frame Fields

**Definition 2.1.** Let  $p, q \in \mathbb{R}^3$ . The *Euclidean distance* from  $p$  to  $q$  is the number  $d(p, q) = \|p - q\|$ .

**Definition 2.2.** Let  $v_p, w_p \in T_p(\mathbb{R}^3)$  be tangent vectors. The *dot product* of  $v_p$  and  $w_p$  is defined to be  $v_p \cdot w_p = v \cdot w$ .

Equivalently, the norm on every tangent space  $T_p(\mathbb{R}^3)$  is the composition of the canonical isomorphism  $T_p(\mathbb{R}^3) \rightarrow \mathbb{R}^3$  with the norm on  $\mathbb{R}^3$ .

**Definition 2.3.** A set of three pairwise orthogonal unit vectors tangent to  $\mathbb{R}^3$  at  $p$  is called a *frame* at  $p$ .

By the definition,  $\{e_1, e_2, e_3\}$  is a frame at  $p$  if and only if  $e_i \in T_p(\mathbb{R}^3)$  and  $e_i \cdot e_j = \delta_{i,j}$ .

**Proposition.** Let  $\{e_1, e_2, e_3\}$  be a frame at  $p \in \mathbb{R}^3$ . If  $v \in T_p(\mathbb{R}^3)$ , then  $v = \sum (v \cdot e_i) e_i$ .

*Proof.*

□

- 3 Euclidean Geometry
- 4 Calculus on a Surface
- 5 Shape Operators
- 6 Geometry of Surfaces in  $\mathbb{R}^3$
- 7 Riemannian Geometry
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## Exercises and Proofs

**Exercise 1.1.1.** Let  $f = x^2y$  and  $g = y \sin z$  be functions on  $\mathbb{R}^3$ . Express the following functions in terms of  $x$ ,  $y$ , and  $z$ :  $fg^2$ ,  $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f$ ,  $\frac{\partial^2(fg)}{\partial y \partial z}$ , and  $\frac{\partial}{\partial y}(\sin f)$ .

*Proof.* (i) We have  $fg^2 = x^2yy^2\sin^2z = x^2y^3\sin^2z$ . (ii) We have  $\partial f/\partial x = 2xy$  and  $\partial g/\partial y = \sin z$ , then  $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = 2xy^2\sin z + x^2y\sin z$ . (iii) We have  $fg = x^2y^2\sin z$ , then  $\frac{\partial^2(fg)}{\partial y \partial z} = 2x^2y\cos z$ . (iv) We have  $\sin f = \sin(x^2y)$ , then  $\frac{\partial}{\partial y}(\sin f) = x^2\cos(x^2y)$ .  $\square$

**Exercise 1.1.3.** Express  $\partial f/\partial x$  in terms of  $x$ ,  $y$ , and  $z$  for the following functions.

1.  $f = x \sin(xy) + y \cos(xz)$ ;
2.  $f = \sin g$ ,  $g = e^h$ , and  $h = x^2 + y^2 + z^2$ .

*Proof.* (i) We have  $\frac{\partial f}{\partial x} = \frac{\partial(x \sin(xy))}{\partial x} + \frac{\partial(y \cos(xz))}{\partial x} = \sin(xy) + xy \cos(xy) - yz \sin(xz)$ . (ii) We have  $f = \sin(e^{x^2+y^2+z^2})$ , then  $\frac{\partial f}{\partial x} = 2x \cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}$ .  $\square$

**Exercise 1.2.1.** Let  $v = (-2, 1, -1)$  and  $w = (0, 1, 3)$ . At an arbitrary point  $p$ , express the tangent vector  $3v_p - 2w_p$  as a linear combination of  $U_1(p)$ ,  $U_2(p)$ , and  $U_3(p)$ .

*Proof.* We have  $3v_p - 2w_p = (-6, 1, -9)_p = -6U_1(p) + U_2(p) - 9U_3(p)$ .  $\square$

**Exercise 1.2.3.** Let  $p = (p_1, p_2, p_3)$ . In each case, express the given vector field  $V$  in the standard form  $\sum v_i U_i$ .

1.  $2z^2U_1 = 7V + xyU_3$ .
2.  $V(p) = (p_1, p_3 - p_1, 0)_p$  for all  $p$ .
3.  $V = 2(xU_1 + yU_2) - x(U_1 - y^2U_3)$ .
4. For all  $p \in \mathbb{R}^3$ ,  $V(p)$  is the vector from  $(p_1, p_2, p_3)$  to  $(1 + p_1, p_2p_3, p_2)$ .
5. For all  $p \in \mathbb{R}^3$ ,  $V(p)$  is the vector from  $p$  to 0.

*Proof.* (i) We have  $V = (2z^2U_1 - xyU_3)/7$ . For all  $p \in \mathbb{R}^3$ ,  $V(p) = ((2z^2, 0, 0) - (0, 0, xy))/7 = (2z^2/7, 0, -xy/7)$ , so  $(v_i) = (2z^2/7, 0, -xy/7)$ . (ii) Here  $V(p) = xU_1 + (z - x)U_2 + 0U_3$ .  $\square$

**Exercise 1.2.5.** Let  $V_1 = U_1 - xU_3$ ,  $V_2 = U_2$ , and  $V_3 = xU_1 + U_3$ . Prove that the vectors  $V_1(p)$ ,  $V_2(p)$ ,  $V_3(p)$  are linearly independent at each  $p \in \mathbb{R}^3$ . Express the vector field  $xU_1 + yU_2 + zU_3$  as a linear combination of  $V_i$ .

*Proof.* For all  $p \in \mathbb{R}^3$ , we have  $V_1(p) = U_1(p) - xU_3(p) = (1, 0, -x)$ . Similarly,  $V_2(p) = (0, 1, 0)$  and  $V_3 = (x, 0, 1)$ . Consider  $aV_1(p) + bV_2(p) + cV_3(p) = 0$ , where  $a, b, c \in \mathbb{R}$ . Solve for  $(a, b, c)$ , then  $c(x^2 + 1) = 0$ , so  $c = 0$ . Now  $(a, b, c) = (0, 0, 0)$ , hence  $V_i(p)$  are linearly independent. For all  $p \in \mathbb{R}^3$ ,  $aV_1(p) + bV_2(p) + cV_3(p) = (a + cx, b, c - a)$  and  $xU_1(p) + yU_2(p) + zU_3(p) = (x, y, z)$ . Solve  $(a + cx, b, c - a) = (x, y, z)$ , then  $(a, b, c) = ((x - zx)/(1 + x^2), y, (x^2 + z)/(1 + x^2))$ .  $\square$

**Exercise 1.3.1.** Let  $v_p$  be the tangent vector with  $v = (2, -1, 3)$  and  $p = (2, 0, -1)$ . Use the definition to compute the directional derivative for the following functions.

1.  $f = y^2z$ .
2.  $f = x^7$ .
3.  $f = e^x \cos y$ .

*Proof.* We have  $p + tv = (2 + 2t, -t, 3t - 1)$ . (i) Now  $f(p + tv) = 3t^3 - t^2$ , then  $v_p[f] = 9t^2 - 2t = 0$ . (ii) Now  $f(p + tv) = (2 + 2t)^7$ , then  $v_p[f] = 7(2 + 2t)^6 \cdot 2 = 14(2 + 2t)^6 = 7 \cdot 2^7$ . (iii) Now  $f(p + tv) = e^{2+2t} \cos(-t)$ , then  $v_p[f] = e^{2+2t} \sin(-t) + 2e^{2+2t} \cos(-t) = 2e^2$ .  $\square$

**Exercise 1.3.3.** Let  $V = y^2U_1 - xU_3$ . Let  $f = xy$  and let  $g = z^3$ . Compute the following functions.

1.  $V[f]$ .



2.  $V[g]$ .
3.  $V[fg]$ .
4.  $fV[g] - gV[f]$ .
5.  $V[f^2 + g^2]$ .
6.  $V[V[f]]$ .

*Proof.* (i) We have  $V[f] = y^2U_1[xy] - xU_3[xy] = y^3$ . (ii) We have  $V[g] = y^2U_1[z^3] - xU_3[z^3] = -3xz^2$ . (iii) We have  $V[fg] = V[f]g + fV[g] = y^3z^3 - 3x^2yz^2$ . (iv) We have  $fV[g] - gV[f] = -3x^2yz^2 - y^3z^3$ . (v) We have  $V[f^2 + g^2] = V[f^2] + V[g^2] = V[f]f + fV[f] + V[g]g + gV[g] = 2xy^4 - 6xz^5$ . (vi) We have  $V[V[f]] = V[y^3] = y^2U_1[y^3] - xU_3[y^3] = 0$ .  $\square$

**Exercise 1.3.5.** If  $V[f] = W[f]$  for all  $f$  on  $\mathbb{R}^3$ , prove that  $V = W$ .

*Proof.* Let  $V = \sum a_i U_i$  and let  $W = \sum b_i U_i$ . Since  $V[f] = W[f]$ ,  $(V - W)[f] = \sum (a_i - b_i) \frac{\partial f}{\partial x_i} = 0$ . Pick  $f = x$ , then  $a_1 = b_1$ . Similarly, if we pick  $f = y$  and  $f = z$ , we have  $a_2 = b_2$  and  $a_3 = b_3$ . Hence  $V = W$ .  $\square$

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