Proof.

$$\begin{aligned}\psi^{-1}(\mathfrak{m}_p) &= (\varphi^{\#})^{-1}(\mathfrak{m}_p) = \{g \in K[Y] \mid \varphi^{\#}(g) \in \mathfrak{m}_p\} \\ &= \{g \in K[Y] \mid g \circ \varphi \in \mathfrak{m}_p\} = \{g \in K[Y] \mid g(\varphi(p)) = 0 \in \mathfrak{m}_p\} \\ &= \mathfrak{m}_{\varphi(p)} \subseteq K[Y]\end{aligned}$$

□

Vocabulary 4.4 (isomorphism)**.** A regular map $\varphi : X \to Y$ is an isomorphism if there exists a regular $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.

Corollary. $X \simeq Y$ if and only if $K[X] \simeq K[Y]$ as K-algebras.

Proof. The bijection $\hom(X, Y) \leftrightarrow \hom_{K\text{-alg}}(K[Y], K[X])$ sends isomorphisms to isomorphisms. To see this, suppose $X \simeq Y$ is realized by $f : X \to Y$ and $g : Y \to X$. Consider a regular function $h : X \to K$. We get a regular function on Y by $g^{\#}(h) = h \circ g$. This is a ring map and $f^{\#}g^{\#}(h) = h \circ f \circ g = h$. This shows $f^{\#}g^{\#} = \mathrm{id}_{K[X]}$. We can show the other order of composition in the same way. Moreover, we can reverse this argument to show a K-algebra isomorphism is sent to an isomorphism between X and Y. □

Example. A regular bijection is not necessarily an isomorphism. Consider $Z = Z(y^2 - x^3) \subseteq \mathbb{A}^2_K$. Let $\varphi : \mathbb{A}^1_K \to Z$ be a map given by $t \mapsto (t^2, t^3)$. φ is surjective because for any $(x_0, y_0) \in Z$, let $\pm t$ be square roots of x_0: $x_0 = t^2$. So $x_0^3 = t^6$ tells us $\pm t^3$ are the two square roots of x_0^3. Conclude that either $y_0 = t^3$ or $y_0 = -t^3$. φ is injective because if $(t_1^2, t_1^3) = (t_2^2, t_2^3)$, $t_1^2 = t_2^2$ and $t_1^3 = t_2^3$. It follows that $t_1 = \pm t_2$ and $t_1 = t_2$. But φ is not an isomorphism. This is because the map

$$\varphi^{\#} : K[Z] = K[x, y]/\left\langle y^2 - x^3 \right\rangle \to K[t]$$

brings $x \mapsto t^2$ and $y \mapsto t^3$. But $\varphi^{\#}(K[Z]) = K\left\langle t^2, t^3 \right\rangle \subsetneq K[t]$. This is, indeed, not an isomorphism.

Vocabulary 4.5 (closed immersion)**.** A map $i : X \to Y$ is a closed immersion if $i(X)$ is closed in Y and $i : X \to i(X)$ is an isomorphism.

Remark. The map in the example above is a regular injection but not isomorphic onto its image.

Proposition 7. Let $\varphi : X \to Y$ be a regular map. $\ker(\varphi^{\#}) = I(\overline{\varphi(X)})$.

Proof.

$$f \in \ker(\varphi^{\#}) \Leftrightarrow f \circ \varphi = 0 \Leftrightarrow f|_{\varphi(X)} = 0 \Leftrightarrow f\big|_{\overline{\varphi(X)}} = 0$$

The above holds if and only if $f \in I(\overline{\varphi(X)})$. □

Proposition 8. $i : X \to Y$ is a closed immersion if and only if $i^{\#} : K[Y] \to K[X]$ is surjective.

Proof. Let $i : X \to Y$ be a closed immersion. So $i : X \to i(X) \subseteq Y$. Recall that if a set is larger, the corresponding ideal is smaller. This gives us the following maps

$$\begin{tikzcd} K[X] & K[i(X)] = K[t_1, ..., t_n]/I(i(X)) \arrow[l, "\sim"'] & K[Y] = K[t_1, ..., t_n]/I(Y) \arrow[l, two heads] \arrow[ll, bend left, "i^{\#}"] \end{tikzcd}$$

Let $i : X \to Y$ be such that $i^{\#} : K[Y] \to K[X]$ is surjective. By Proposition 7, $\ker(i^{\#}) = I(\overline{i(X)}) \subseteq K[Y]$. So $i^{\#} : K[Y]/I(\overline{i(X)}) \xrightarrow{\sim} K[X]$ is an isomorphism. Note that

$$\begin{aligned}K[Y]/I(\overline{i(X)}) &= {}^{K[t_1, ..., t_n]/I(Y)}\Big/_{I(\overline{i(x)})} = K[t_1, ..., t_n]/(I(Y) + I(\overline{i(x)})) \\ &= K[t_1, ..., t_n]/(I(Y \cap \overline{i(x)})) = K[t_1, ..., t_n]/I(\overline{i(x)}) = K[\overline{i(X)}]\end{aligned}$$

□

Vocabulary 4.6 (dominant)**.** $\varphi : X \to Y$ is dominant if $\overline{\varphi(X)} = Y$.