Elementary Differential Geometry

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1 Calculus on Euclidean Space

Definition 1.1. The *Euclidean 3-space*, denoted $\mathbb{R}^3$, is the set of ordered triples of the form $p = (p_1, p_2, p_3)$, where $p_i \in \mathbb{R}$. An element of $\mathbb{R}^3$ is called a *point*.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. Define the addition to be $p + q = (p_i + q_i)$ and define the scalar multiplication to be $ap = (ap_i)$. The additive identity $0 = (0, 0, 0)$ is called the *origin* of $\mathbb{R}^3$. It is trivial that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Definition 1.2. Let x, y, and z be real-valued functions on $\mathbb{R}^3$ such that for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $x(p) = p_1$. $y(p) = p_2$, and $z(p) = p_3$. We call x, y, and z the *natural coordinate functions* of $\mathbb{R}^3$.

Let x, y, and z be the natural coordinate functions, rewrite $x = x_1$, $y = x_2$, and $z = x_3$. Then we have $p = (p_i) = (x_i(p))$.

Definition 1.3. A real-valued function f on $\mathbb{R}^3$ is *differentiable* if all partial derivatives exist and continuous.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$. Recall that the dot product is defined to be $p \cdot q = \sum p_i q_i$ and the norm is defined to be $\|p\| = \sqrt{p \cdot p} = \sqrt{\sum p_i^2}$.

Definition 1.4. A subset $O \subset \mathbb{R}^3$ is *open* if for all $p \in O$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$.

Let $f : O \to \mathbb{R}$ be a function defined on an open set. The differetiability of f at p can be determined entirely from values of f on O. This means that differetiation is a local operation. We will discuss this later.

Definition 1.5. A *tangent vector* v_p is an ordered pair $v_p = (v, p)$, where $v, p \in \mathbb{R}^3$. Here v is called the *vector part* and p is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

Definition 1.6. Let $p \in \mathbb{R}^3$. The *tangent space* at p, denoted $T_p(\mathbb{R}^3)$, is the set of all tangent vectors that have p as point of application.

Fix a tangent space $T_p(\mathbb{R}^3)$ and let $T_p(\mathbb{R}^3)$ adapt the operations from $\mathbb{R}^3 \times \mathbb{R}^3$. We have a natural linear map $f : T_p(\mathbb{R}^3) \to \mathbb{R}^3$ defined by $v_p \to v$ and it is trivially an isomorphism.

Definition 1.7. A *vector field* V on $\mathbb{R}^3$ is a function $V : \mathbb{R}^3 \to \bigsqcup_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3)$ such that for all $p \in \mathbb{R}^3$, $V(p) \subset T_p(\mathbb{R}^3)$.

Let V and W be vector field. Let f be a real-valued function. For all $p \in \mathbb{R}^3$, define $V + W$ by $(V + W)(p) = V(p) + W(p)$ and $(fV)(p) = f(p)V(p)$.

Definition 1.8. Let U_1, U_2, and U_3 be vector fields on $\mathbb{R}^3$ such that $U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, and $U_3(p) = (0, 0, 1)_p$ for all $p \in \mathbb{R}^3$. We call (U_1, U_2, U_3) the *natural frame field* on $\mathbb{R}^3$.

Proposition. Let V be a vector field on $\mathbb{R}^3$. There are three uniquely determined real-valued functions v_1, v_2, and v_3 on $\mathbb{R}^3$ such that $V = v_1U_1 + v_2U_2 + v_3U_3$.

Proof. For all $p \in \mathbb{R}^3$, $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1,0,0)_p + v_2(p)(0,1,0)_p + v_3(p)(0,0,1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3U_3(p)$, hence $V = \sum v_iU_i$. □

The functions v_1, v_2, and v_3 are called the *Euclidean coordinate functions* on V.

Definition 1.9. A vector field V is *differentiable* if its Euclidean coordinate functions are differentiable.

Definition 1.10. Let f be a differentiable real-valued function on $\mathbb{R}^3$ and let v_p be a tangent vector on $\mathbb{R}^3$. The *directional derivative* of f with respect to v_p, denoted $v_p[f]$, is defined to be $(\mathrm{d}/\mathrm{d}t)f(p+tv)$ at $t = 0$.

Remark. We will not write the restriction every time for convenience.

Proposition. Let $v_p = (v_1, v_2, v_3)_p$ be a tangent vector, then $v_p[f] = \sum v_i(\partial f/\partial x_i)(p)$.

Proof. Let $p = (p_1, p_2, p_3)$. Then $v_p[f] = (\mathrm{d}/\mathrm{d}t)f(p+tv)|_{t=0} = \sum(\partial f/\partial z)(p)\cdot(\mathrm{d}/\mathrm{d}t)(p_i+tv_i) = \sum(\partial f/\partial x_i)(p)v_i$. □

Example. Consider $f = x^2yz$ with $p = (1,1,0)$ and $v = (1,0,-3)$. By the definition, $p + tv = (1+t, 1, -3t)$, so $v_p[f] = (\mathrm{d}/\mathrm{d}t)(-3t^3 - 6t^2 - 3t) = -3$. Since $(\partial f/\partial x) = 2xyz$, $(\partial f/\partial y) = x^2z$, and $(\partial f/\partial z) = x^2y$, we have $(\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0$ and $(\partial f/\partial z)(p) = 1$, so $v_p[f] = -3$.

Proposition. Let f and g be functions on $\mathbb{R}^3$. Let v_p and w_p be tangent vectors. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.
2. $v_p[af + bg] = av_p[f] + bv_p[g]$.
3. $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$.

Proof. (i) We have $(av_p+bw_p)[f] = \sum(av_i+bw_i)(\partial f/\partial x_i)(p) = \sum av_i(\partial f/\partial x_i)+\sum bw_i(\partial f/\partial x_i)(p) = av_p[f]+bw_p[f]$. (ii) We have $v_p[af+bg] = \sum v_i(\partial(af+bg)/\partial x_i)(p) = \sum v_i(\partial(af)/\partial x_i)(p)+\sum v_i(\partial(bg)/\partial x_i)(p) = av_p[f]+bv_p[g]$. (iii) We have $v_p[fg] = \sum v_i(\partial(fg)/\partial x_i)(p) = \sum v_i(\partial f/\partial x_i)(p)g(p) + f(p)\sum v_i(\partial g/\partial x_i)(p) = v_p[f]g(p) + f(p)v_p[g]$. □

Let V be a vector field, we define $V[f]$ at $p \in \mathbb{R}^3$ to be $V(p)[f]$. By the convention, $U_i(p)[f] = (\partial f/\partial x_i)(p)$.

Proposition. Let V and W be vector fields. Let f, g, and h be real-valued functions. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(fV + gW)[h] = fV[h] + gW[h]$.
2. $V[af + bg] = aV[f] + bV[g]$.
3. $V[fg] = V[f]g + fV[g]$.

Proof. (i) For all $p \in \mathbb{R}^3$, $(fV + gW)(p)[h] = (f(p)V(p) + g(p)W(p))[h] = fV[h] + gW[h]$. (ii) For all $p \in \mathbb{R}^3$, $V(p)[af + bg] = aV(p)[f] + bV(p)[g]$. (iii) For all $p \in \mathbb{R}^3$, $V(p)[f]g(p) + f(p)V(p)[g] = V[f](p)g(p) + f(p)V[g](p) = (V[f]g + fV[g])(p)$. □

Example. Let $V = xU_1 - y^2U_3$ and let $f = x^2y + z^3$. Then $V[f] = xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3] = 2x^2y - 3y^2z^2$.

Let $I \subset \mathbb{R}$ be an open interval. Let $\alpha : I \to \mathbb{R}^3$ be a function. We can rewrite $\alpha(t)$ as $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where $\alpha_i : I \to \mathbb{R}$. We say α is *differentiable* if α_i are differentiable.

Definition 1.11. A *curve* in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an open interval.

Example. A curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = p + tq$, where $\alpha(0) = p$ and $q \neq 0$, is called a *straight line*.

Example. Here are some examples of curves.

1. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (a\cos t, a\sin t, bt)$.
2. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$.
3. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$.
4. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$.

Definition 1.12. Let $\alpha : I \to \mathbb{R}^3$ be a curve with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For all $t \in I$, the *velocity vector* of α at t is the tangent vector $\alpha'(t) = ((\mathrm{d}\alpha_1/\mathrm{d}t)(t), (\mathrm{d}\alpha_2/\mathrm{d}t)(t), (\mathrm{d}\alpha_3/\mathrm{d}t)(t))_{\alpha(t)}$ at the point $\alpha(t) \in \mathbb{R}^3$. The curve α is said to be *regular* if $\alpha_i \neq 0$ for all i.

Consider the velocity vector $\alpha'(t)$, we can rewrite it by the natural frame fields, so $\alpha'(t) = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t)U_i(\alpha(t))$.

Definition 1.13. Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $h : J \to I$ be differentiable, where J is an open interval of $\mathbb{R}$. The *reparametrization* of α by h is the composition $\alpha \circ h : J \to \mathbb{R}^3$.

The composition of differentiable functions is differentiable, so any reparametrization is differentiable, which means it is a curve.

Proposition. Let β be the reparametrization of α by h, then $\beta'(s) = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$.

Proof. Rewrite $\beta(s) = \alpha(h(s))$, then we have $\beta'(s) = (\mathrm{d}(\alpha_i h_i)/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}\alpha_i/\mathrm{d}s)(h(s)) \cdot (\mathrm{d}h/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$. □

Proposition. Let α be a curve and let f be a differentiable function on $\mathbb{R}^3$, then $\alpha'(t)[f] = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$.

Proof. We have $\alpha'(t)[f] = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t) \cdot (\partial f/\partial x_i)(\alpha(t)) = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$ by the chain rule. □

Now we show a general idea of parametrizations. The proofs will be included in other sections when we have enough tools. Assume every result is correct for now.

Definition 1.14. A *1-form* φ on $\mathbb{R}^3$ is a function $\varphi : \coprod_{p\in\mathbb{R}^3} T_p(\mathbb{R}^3) \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ and $v, w \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$, $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$.

Given a 1-form φ, for any point p, denote the restriction $\varphi|_{T_p(\mathbb{R}^3)} : T_p(\mathbb{R}^3) \to \mathbb{R}$ by φ_p, then φ_p is linear. Let φ and ψ be 1-forms. Define the addition and scalar multiplication by $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$ and $(f\varphi)(v_p) = f(p)\varphi(v_p)$. Given any 1-form φ and point p, φ_p is a linear functional in $T_p^*(\mathbb{R}^3)$, the dual space of $T_p(\mathbb{R}^3)$.

Definition 1.15. Let φ be a 1-form and let V be a vector field. For all $p \in \mathbb{R}^3$, define $\varphi(V)(p) = \varphi_p(V(p))$. We say φ is *differentiable* if for every differentiable vector field V, the function $\varphi(V)$ is differentiable.

Now let V and W be vector fields, we have $\varphi(fV + gW)(p) = \varphi((fV + gW)(p)) = \varphi(fV(p) + gW(p)) = (f\varphi(V) + g\varphi(W))(p)$. Similarly, $(f\varphi + g\psi)(V) = f\varphi(V) + g\psi(V)$.

Definition 1.16. If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable. The *differential* of f, denoted $\mathrm{d}f$, is the function $\mathrm{d}f(v_p) = v_p[f]$ for all tangent vectors v_p.

Let $v_p, w_p \in T_p(\mathbb{R}^3)$ and let $a, b \in \mathbb{R}$, then $\mathrm{d}f(av_p + bw_p) = (av_p + bw_p)[f] = av_p[f] + bw_p[f] = a\mathrm{d}f(v_p) + b\mathrm{d}f(w_p)$. Hence $\mathrm{d}f$ is a 1-form.

Example. Consider the natural coordinate functions x_i. We have $\mathrm{d}x_i(v_p) = v_p[x_i] = \sum v_i(\partial x_i/\partial x_j)(p) = v_i$.

Proposition. If φ is a 1-form on $\mathbb{R}^3$, then $\varphi = \sum f_i \mathrm{d}x_i$, where $f_i = \varphi(U_i)$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\varphi(v_p) = \varphi(\sum v_i U_i(p)) = \sum v_i\varphi(U_i(p)) = \sum v_i f_i(p) = \sum f_i(p)\mathrm{d}x_i(v_p) = (\sum f_i \mathrm{d}x_i)(v_p)$, hence $\varphi = \sum f_i \mathrm{d}x_i$. □

The functions f_1, f_2, and f_3 are called the *Euclidean coordinate functions* of the 1-form φ.

Proposition. Let f be a differentiable function on $\mathbb{R}^3$, then $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\mathrm{d}f(v_p) = v_p[f] = \sum v_i(\partial f/\partial x_i)(p) = \sum(\partial f/\partial x_i)(p)\mathrm{d}x_i(v_p) = (\sum(\partial f/\partial x_i)\mathrm{d}x_i)(v_p)$, hence $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$. □

Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(f+g) = \sum(\partial(f+g)/\partial x_i)\mathrm{d}x_i = \sum(\partial f/\partial x_i)\mathrm{d}x_i + \sum(\partial g/\partial x_i)\mathrm{d}x_i = \mathrm{d}f + \mathrm{d}g$. Now we denote the multiplication to be fg.

Proposition. Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$.

Proof. We have $\mathrm{d}(fg) = \sum(\partial(fg)/\partial x_i)\mathrm{d}x_i = \sum((\partial f/\partial x_i)g + (\partial g/\partial x_i)f)\mathrm{d}x_i = g\mathrm{d}f + f\mathrm{d}g$. □

Proposition. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable, then $\mathrm{d}(h(f)) = (\mathrm{d}h(f)/\mathrm{d}x)\mathrm{d}f$.

Proof. We have $\mathrm{d}(h(f)) = \sum(\partial h(f)/\partial x_i)\mathrm{d}x_i$, by the chain rule, $(\partial h(f)/\partial x_i)\mathrm{d}x_i = (\mathrm{d}h(f)/\mathrm{d}f)(\partial x/\partial x_i)$, so $\mathrm{d}(h(f)) = (\mathrm{d}f(h)/\mathrm{d}f)\mathrm{d}f$. □

Example. Consider the function $f = (x^2-1)y + (y^2+2)z$. We have $\mathrm{d}f = \mathrm{d}((x^2-1)y) + \mathrm{d}((y^2+2)z) = y\mathrm{d}(x^2-1) + (x^2+1)\mathrm{d}y + z\mathrm{d}(y^2+2) + (y^2+2)\mathrm{d}z = 2xy\mathrm{d}x + (x^2+2yz-1)\mathrm{d}y + (y^2+2)\mathrm{d}z$. Since $v_p[f] = \mathrm{d}f(v_p)$, $v_p[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_2^2+2)v_3$.

Definition 1.17. Let V be the vector space $\mathbb{R}^3$ and denote the space of all p-linear forms on V by $\Lambda^p(V^*)$. Every element of Λ^p is called a *p-form*. Define the *wedge product* to be a function $\wedge : \Lambda^a(V^*) \times \Lambda^b(V^*) \to \Lambda^{a+b}(V^*)$ such that for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $v_1, \dots, v_{m+n} \in V$, the following properties hold.

1. $(\omega \wedge \eta)(v_1, \dots, v_{m+n}) = (\sum_{\sigma \in \mathfrak{S}_{m+n}} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(m)})\eta(v_{\sigma(m+1)}, \dots, v_{\sigma(m+n)}))/(m!n!)$.
2. $\omega \wedge \eta = (-1)^{mn}\eta \wedge \omega$.

Generally, a p-form is of the form $\sum f(x,y,z)\mathrm{d}x^i \wedge \cdots \mathrm{d}y^j \wedge \cdots \mathrm{d}z^k \wedge \cdots$. We have $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. If $i = j$, then $\mathrm{d}x_i \wedge \mathrm{d}x_i = -\mathrm{d}x_i \wedge \mathrm{d}x_i$, so $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. It is trivial that $\wedge$ is bilinear and associative, that is,

1. for $\omega_1, \omega_2 \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $a, b \in \mathbb{R}$,$(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta)$ and $\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2)$;
2. for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $\theta \in \Lambda^l(V^*)$, $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta$.

Now given a space of p-forms $\Lambda^p(V^*)$ with basis $\{e_1, e_2, e_3\}$, the basis of its dual space is denoted by $\{e^1, e^2, e^3\}$. The basis of $\Lambda^k(V^*)$ is of the form $e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $1 \le i_1 \le \cdots \le i_k \le 3$. In this case, the dimension of $\Lambda^p(V^*)$ is $3!/(p!(3-p)!)$. If $p > 4$, then $\dim(\Lambda^p(V^*)) = 0$, so there are no p-forms on $\mathbb{R}^3$ if $p \ge 4$.

Example. Let $\varphi = x\mathrm{d}x - y\mathrm{d}y$, $\psi = z\mathrm{d}x + x\mathrm{d}z$, $\theta = z\mathrm{d}y$, and $\eta = y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z$.

1. $\varphi \wedge \psi = xz\mathrm{d}x \wedge \mathrm{d}x + x^2\mathrm{d}x \wedge \mathrm{d}z - yz\mathrm{d}y \wedge \mathrm{d}x - yx\mathrm{d}y \wedge \mathrm{d}z = yz\mathrm{d}x \wedge \mathrm{d}y + x^2\mathrm{d}x \wedge \mathrm{d}z - yx\mathrm{d}y \wedge \mathrm{d}z$
2. $\theta \wedge (\varphi \wedge \psi) = yz^2\mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}y) + x^2z\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y - xyz\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}y = -x^2z\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
3. $\varphi \wedge \eta = xy\mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}z + x^2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - y^2\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z - xy\mathrm{d}y \wedge \mathrm{d}y \wedge \mathrm{d}z = (x^2+y^2)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Proposition. Let φ and ψ be 1-forms, then $\varphi \wedge \psi = -\psi \wedge \varphi$.

Proof. Rewrite $\varphi = \sum f_i\mathrm{d}x_i$ and $\psi = \sum g_i\mathrm{d}x_i$, then $\varphi \wedge \psi = \sum f_ig_i\mathrm{d}x_i\mathrm{d}x_j = \sum -g_if_i\mathrm{d}x_j\mathrm{d}x_i = -\psi \wedge \varphi$. □

Definition 1.18. Let $\varphi = \sum f_i\mathrm{d}x_i$ be a 1-form on $\mathbb{R}^3$. The *exterior derivative* of φ is the 2-form $\mathrm{d}\varphi = \sum \mathrm{d}f_i \wedge \mathrm{d}x_i$. Let $\psi = \sum f_{i,j}\mathrm{d}x_i \wedge \mathrm{d}x_j$ be a 2-form. The *exterior derivative* of ψ is the 3-form $\mathrm{d}\psi = \sum \mathrm{d}f_{i,j} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j$.

Let $a, b \in \mathbb{R}$. Let $\varphi = \sum f_i\mathrm{d}x_i$ and $\psi = \sum g_i\mathrm{d}x_i$ be 1-forms. Then $\mathrm{d}(a\varphi + b\psi) = \mathrm{d}(\sum(af_i + bg_i)\mathrm{d}x_i) = \sum \mathrm{d}(af_i + bg_i) \wedge \mathrm{d}x_i$, since the differential is linear, the exterior derivative is linear.

Proposition. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions and let φ and ψ be 1-forms. Then $\mathrm{d}(f\varphi) = \mathrm{d}f \wedge \varphi + f\mathrm{d}\varphi$ and $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$.

Proof. (i) Let $\varphi = \sum g_i \mathrm{d}x_i$, then $f\varphi = \sum f g_i \mathrm{d}x_i$, so $\mathrm{d}(f\varphi) = \sum(f\mathrm{d}g_i + g_i\mathrm{d}f) \wedge \mathrm{d}x_i = \sum f\mathrm{d}g_i \wedge \mathrm{d}x_i + \sum g_i \mathrm{d}f \wedge \mathrm{d}x_i = f\mathrm{d}\varphi + \mathrm{d}f \wedge \varphi$. (ii) Since $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$, without lose of generality, let $\varphi = f\mathrm{d}x$ and let $\psi = g\mathrm{d}y$. Then $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}(fg\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}(fg) \wedge \mathrm{d}x \wedge \mathrm{d}y = (f\mathrm{d}g + g\mathrm{d}f) \wedge \mathrm{d}x \wedge \mathrm{d}y = f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y + g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$. For the right hand side, $\mathrm{d}\varphi \wedge \psi = \mathrm{d}f \wedge \mathrm{d}x \wedge g\mathrm{d}y = g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$ and $\varphi \wedge \mathrm{d}\psi = f\mathrm{d}x \wedge \mathrm{d}g \wedge \mathrm{d}y = -f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y$, hence $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$. □

Definition 1.19. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and let $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ such that $F(p) = (f_1(p), \dots, f_m(p))$ for all $p \in \mathbb{R}^n$. The functions f_i are called the *Euclidean coordinate functions* of F and we denote $F = (f_1, \dots, f_m)$.

Definition 1.20. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $F = (f_1, \dots, f_m)$, we say F is *differentiable* if all f_i are differentiable. If F is differentiable, we say F is a *mapping* from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 1.21. Let $\alpha : I \to \mathbb{R}^n$ be a curve and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then the composite function $\beta = F(\alpha) : I \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ called the *image* of α under F.

To examine the effect of a mapping, it suffices to take a proper α and check the image of it.

Example. The function $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F = (x - y, x + y, 2z)$ is a mapping. Trivially, F is a linear map, so F is determined by $F(u_i)$.

Example. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F = (u^2 - v^2, 2uv)$. Let $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (r\cos t, r\sin t)$, where $0 \le t \le 2\pi$. The image is $\beta(t) = (r^2 \cos 2t, r^2 \sin 2t)$. This curve takes two counterclockwise trips around the circle of radius r^2 centered at the origin. Therefore, F wraps $\mathbb{R}^2$ around itself twice.

Definition 1.22. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let $v_p \in T_p(\mathbb{R}^n)$. The *tangent map* of F, denoted $F_*(v_p)$, is defined to be $(\mathrm{d}/\mathrm{d}t)F(p + tv)$ at $t = 0$.

Fix some mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. For every $p \in \mathbb{R}^n$, it induces a tangent map of F at p, denoted F_{*p}.

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $v_p \in T_p(\mathbb{R}^n)$, then $F_{*p}(v_p) = (v[f_1], \dots, v[f_m])_{F(p)}$.

Proof. Fix $v_p \in T_p(\mathbb{R}^n)$. We have $F_{*p} = (\mathrm{d}/\mathrm{d}t)F(p + tv)|_{t=0} = (\mathrm{d}/\mathrm{d}t)(f_i(p + tv))|_{t=0} = (v_p[f_1], \dots, v_p[f_m])_{F(p)}$. □

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. For all $p \in T_p(\mathbb{R}^n)$, the tangent map $F_{*p} : T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is a linear map.

Proof. Fix $p \in \mathbb{R}^n$. Let $a, b \in \mathbb{R}$ and let $v_p, w_p \in T_p(\mathbb{R}^n)$. We have $F_{*p}(av_p + bw_p) = ((av_p + bw_p)[f_i])_{F(p)} = (av_p[f_i])_{F(p)} + (bw_p[f_i])_{F(p)} = aF_{*p}(v_p) + bF_{*p}(w_p)$. □

Proposition. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let β be the image of some curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$.

Proof. Let $F = (f_1, \dots, f_m)$. We have $F_*(\alpha'(t)) = (\alpha'(t)[f_i])_{F(\alpha(t))} = (\mathrm{d}f_i(\alpha(t))/\mathrm{d}t)_{F(\alpha(t))} = \beta'(t)$. □

Let $\{U_j\}$ and $\{\overline{U_i}\}$ be the natural frame fields of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Proposition. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then $F_*(U_j(p)) = \sum_{i=1}^m (\partial f_i/\partial x_j)(p)\overline{U_i}(F(p))$, where $1 \le j \le n$.

Proof. Recall that $U_j[f_i] = \partial f_i/\partial x_j$, so the proposition trivially holds. □

Definition 1.23. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. The *Jacobian matrix* of F at $x \in \mathbb{R}^n$ is the matrix

$$J_F(x) = \begin{pmatrix} \partial f_1/\partial x_1(x) & \cdots & \partial f_1/\partial x_n(x) \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1(x) & \cdots & \partial f_m/\partial x_n(x) \end{pmatrix}.$$

Definition 1.24. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. We say F is *regular* if for all $p \in \mathbb{R}^n$, F_{*p} is injective.

Notice that $J_F(p) \cdot v = F_{*p}$, so $J_F(p)$ is the matrix representation of F_{*p}.

Definition 1.25. A mapping is a *diffeomorphism* if it has a differentiable inverse mapping.

Definition 1.26. A *topological space* $(X, \mathcal{T})$ consists of two sets X and $\mathcal{T}$, where $\mathcal{T} \subset \mathscr{P}(X)$, that satisfies the following properties.

1. $\varnothing, X \in \mathcal{T}$.
2. Any union of elements in $\mathcal{T}$ is also in $\mathcal{T}$.
3. Any finite intersection of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

Definition 1.27. Let $(X, \mathcal{T})$ be a topological space. A subset $U \subset X$ is said to be *open* if $U \in \mathcal{T}$. Let $x \in X$, a *neighborhood* of x is an open set U_x that contains x.

Let $U \subset \mathbb{R}$. We say U is open in the standard topology $\mathcal{T}$ on $\mathbb{R}$ if for every $x \in U$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. Trivially, $\varnothing, \mathbb{R} \in \mathcal{T}$. Let $\{U_i\}_{i \in I}$ be open sets, then for each U_i and $x \in U_i$, there exists a corresponding $\varepsilon_{i,x}$. For any $x \in \bigcup_{i \in I} U_i$, $x \in U_i$ for some $i \in I$. Pick $\varepsilon = \varepsilon_{i,x}$, then $(x - \varepsilon, x + \varepsilon) \subset U_i \subset \bigcup_{i \in I} U_i$. For any $x \in \bigcap_{i=1}^n U_i$, pick $\varepsilon = \min\{\varepsilon_{i,x}\}$, then $(x - \varepsilon, x + \varepsilon) \subset U_i$ for $1 \le i \le n$, so $(x - \varepsilon, x + \varepsilon) \subset \bigcap_{i=1}^n U_i$. The standard topology on $\mathbb{R}$ is indeed a topology.

Definition 1.28. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A subset $W \subset X \times Y$ is open in the *product topology* on $X \times Y$ if for all $(x, y) \in W$, there exist neighborhoods $U_x \in \mathcal{T}_X$ and $V_y \in \mathcal{T}_Y$ such that $U_x \times V_y \subset W$.

Denote the product topology by $\mathcal{T}$. We have $\varnothing \in \mathcal{T}$ vacuously. For all $(x, y) \in X \times Y$, $U_x \subset X$, and $V_Y \subset Y$, $U_X \times V_Y \subset X \times Y$, so $X \times Y \in \mathcal{T}$. Let $\{W_i\}_{i \in I}$ be open sets. For all $(x, y) \in \bigcup_{i \in I} W_i$, there exist W_i and W_j such that $x \in W_i$ and $y \in W_j$. Pick the corresponding neighborhood in each set, then $U_x \times V_y \subset W_i \cup W_j \subset \bigcup_{i \in I} W_i$. For all $(x, y) \in \bigcap_{i=1}^n W_i$, $(x, y) \in W_i$. For each W_i, we have a corresponding pair $(U_{i,x}, V_{i,y})$. Now consider $U = \bigcap_{i=1}^n U_{i,x} \in \mathcal{T}_X$ and $V = \bigcap_{i=1}^n V_{i,y} \in \mathcal{T}_Y$, we have $U \times V \subset W_i$, so $U \times V \subset \bigcap_{i=1}^n W_i$. The standard topology on $\mathbb{R}^n$ is the product topology of n copies of the standard topology on $\mathbb{R}$.

Theorem 1.1 (inverse function theorem)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. If F_{*p} is injective at some $p \in \mathbb{R}^n$, then there exists a neighborhood U of p such that $F|_U : U \to V$, where V is open, is a diffeomorphism.

We will discuss more on the proof of this theorem and its application later.

2 Frame Fields

Definition 2.1. Let $p, q \in \mathbb{R}^3$. The *Euclidean distance* from p to q is the number $d(p, q) = \|p - q\|$.

Definition 2.2. Let $v_p, w_p \in T_p(\mathbb{R}^3)$ be tangent vectors. The *dot product* of v_p and w_p is defined to be $v_p \cdot w_p = v \cdot w$.

Equivalently, the norm on every tangent space $T_p(\mathbb{R}^3)$ is the composition of the canonical isomorphism $T_p(\mathbb{R}^3) \to \mathbb{R}^3$ with the norm on $\mathbb{R}^3$.

Definition 2.3. A set of three pairwise orthogonal unit vectors tangent to $\mathbb{R}^3$ at p is called a *frame* at p.

By the definition, $\{e_1, e_2, e_3\}$ is a frame at p if and only if $e_i \in T_p(\mathbb{R}^3)$ and $e_i \cdot e_j = \delta_{i,j}$.

Proposition. Let $\{e_1, e_2, e_3\}$ be a frame at $p \in \mathbb{R}^3$. If $v_p \in T_p(\mathbb{R}^3)$, then $v_p = \sum (v \cdot e_i) e_i$.

Proof. Let $c_1, c_2, c_3 \in \mathbb{R}$ such that $\sum c_i e_i = 0$. For all $1 \le j \le 3$, $0 = (\sum c_i e_i) \cdot e_j = \sum c_i (e_i \cdot e_j) = c_j$, so $\{e_1, e_2, e_3\}$ is a basis of $T_p(\mathbb{R}^3)$. Rewrite $v_p = \sum a_i e_i$. For all $1 \le j \le 3$, $v_p \cdot e_j = \sum a_i e_i \cdot e_j = a_j$. Hence $v_p = \sum (v_p \cdot e_i) e_i$. $\square$

For any frame $\{e_1, e_2, e_3\}$ at p and $a, b \in T_p(\mathbb{R}^3)$, if $a = \sum a_i e_i$ and $b = \sum b_i e_i$, we always have $a \cdot b = \sum a_i b_i$.

Definition 2.4. Let $\{e_1, e_2, e_3\}$ be a frame at $p \in \mathbb{R}^3$ with $e_i = (a_{i,1}, a_{i,2}, a_{i,3})_p$, then the *attitude matrix* of the frame is defined to be the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Consider the transpose $A^\top$ of A, for each column of $A^\top A$, we have $e_i e_i = 1$, so $A^\top A = I$ and A is orthogonal.

Definition 2.5. Let $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. The *cross product* of v_p and w_p, denoted $v_p \times w_p$, is the tangent vector

$$v_p \times w_p = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example. Let $v_p = (1, 0, -1)_p$ and let $w_p = (2, 2, -7)_p$, then $v_p \times w_p = 2U_1(p) + 5U_2(p) + 2U_3(p) = (2, 5, 2)_p$.

It is trivial that $\times$ is linear and $v_p \times w_p = -w_p \times v_p$.

Proposition. Let $v_p, w_p \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then $v_p \times w_p$ is orthogonal to both v_p and w_p. Moreover, $\|v_p \times w_p\|^2 = (v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2$.

Proof. Let $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p$. Then $(v_p \times w_p) \cdot v_p = v_1(v_2w_3 - v_2w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) = 0$. Similarly, $(v_p \times w_p) \cdot w_p = 0$. We have $(v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2 = (\sum v_i^2)(\sum w_i^2) - (\sum v_i w_i)^2 = \sum v_i^2 w_j^2 - \sum v_i^2 w_i^2 - 2\sum_{i<j} v_i w_i v_j - w_j = (v_2w_3 - v_2w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 = \|v_p \times w_p\|^2$. □

Definition 2.6. Let $\alpha : I \to \mathbb{R}^3$ be a curve. The *speed* of α at t is the tangent vector $\|\alpha'(t)\|$. The *arc length* of α from $t = a$ to $t = b$ is defined to be $\int_a^b \|\alpha'(t)\| \mathrm{d}t$.

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, then there exists a reparametrization β of α such that $\|\beta'\| = 1$.

Proof. Fix some $\alpha \in \mathbb{R}$ and consider the function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(t) = \int_a^t \|\alpha'(x)\| \mathrm{d}x$. Since α is regular, $\|\alpha'(x)\| > 0$ for all x. By the inverse function theorem, $s(t)$ has an inverse $t(s)$. Define $\beta(s) = \alpha(t(s))$, then $\|\beta'\| = \|(\mathrm{d}t/\mathrm{d}s)(s)\alpha'(t(s))\| = (\mathrm{d}t/\mathrm{d}s)(s)\|\alpha'(t(s))\| = (\mathrm{d}t/\mathrm{d}s)(s) \cdot (\mathrm{d}s/\mathrm{d}t)(t(s)) = 1$. □

Such reparametrization β of α is called the *arc-length reparametrization* of α.

Example. Consider the curve $\alpha : I \to \mathbb{R}^3$ defined by $\alpha(t) = (a\cos t, a\sin t, bt)$ for some $a, b \in \mathbb{R}$. We have $\|\alpha'\| = \sqrt{a^2\sin^2 t + a^2\sin^2 t + b^2} = c$, where $c^2 = a^2 + b^2$. Now measure the arc length from $t = 0$, then $s(t) = \int_0^c c\mathrm{d}u = ct$, so $t(s) = s/c$, the arc-length reparametrization is therefore $\beta(s) = \alpha(t(s)) = (a\cos(s/c), a\sin(s/c), bs/c)$.

Definition 2.7. A *vector field* Y on a curve $\alpha : I \to \mathbb{R}^3$ is a function $Y : I \to \bigsqcup_{p \in \mathrm{ran}(\alpha)} T_p(\mathbb{R}^3)$ such that for all $t \in I$, $Y(t) \in T_{\alpha(t)}(\mathbb{R}^3)$.

Fix $t \in I$, then we can rewrite $Y(t)$ as $\sum y_i(t)U_i(\alpha(t))$. The functions y_1, y_2, and y_3 are called the *Euclidean coordinate functions* on Y.

We define the addition, scalar multiplication, dot multiplication, and cross product on vector fields pointwisely. For a vector $Y = \sum y_i U_i$ on α, the derivative of Y is defined to be $Y' = \sum(\mathrm{d}y_i/\mathrm{d}t)U_i$. Let Y and Z be vector fields on a curve α. Fix t, rewrite $Y = (y_1, y_2, y_3)_{\alpha(t)}$ and $Z = (z_1, z_2, z_3)_{\alpha(t)}$. Consider Y as a function $Y_t : I \to T_{\alpha(t)}(\mathbb{R}^3) \approx \mathbb{R}^3$, then for $a, b \in \mathbb{R}$ and a differentiable $f : \mathbb{R} \to \mathbb{R}$, $(aY + bZ)' = aY' + bZ'$ and $(fY)' = (\mathrm{d}f/\mathrm{d}t)Y + Y'f$. We also have $(Y \cdot Z)' = (\sum y_i z_i)' = \sum y_i z_i' + \sum y_i' z_i = Y \cdot Z' + Y' \cdot Z$.

Definition 2.8. Let Y be a vector field on a curve $\alpha : I \to \mathbb{R}^3$. We say Y is *parallel* if for all $t \in I$, $Y(t)$ have the same vector part.

Proposition. A curve α is constant if and only if $\alpha' = 0$. A nonconstant curve α is a straight line if and only if $\alpha'' = 0$. A vector field Y on α is parallel if and only if $Y' = 0$.

Proof. Rewrite $\alpha : I \to \mathbb{R}^3$ as $\alpha = (\alpha_i)$. (i) The velocity $\alpha' = (\alpha_i')$, then $\alpha' = 0$ if and only if $\alpha_i' = 0$. Hence $\alpha' = 0$ if and only if α_i is a constant function. (ii) We have $\alpha'' = (\alpha_i'')$, so $\alpha'' = 0$ if and only if $\alpha_i = p_i t + q_i$ for some $p_i, q_i \in \mathbb{R}$. Hence $\alpha'' = 0$ if and only if $\alpha = pt + q$, where $p = (p_i)$ and $q = (q_1)$. (iii) Fix t and let $Y = (y_i)_{\alpha(t)}$, then $Y' = \sum y_i' U_i = 0$, which means y_i are constant functions. Hence Y is parallel if and only if $Y' = 0$. $\square$

Definition 2.9. Let $\beta : I \to \mathbb{R}^3$ be a curve. Then we call $T = \beta'$ the *unit tangent field* of β. The function $\kappa(s) = \|T'(s)\|$ is called the *curvature* of β.

Remark. We shall only consider the cases where $\kappa \neq 0$.

Definition 2.10. Let $\beta : I \to \mathbb{R}^3$ be a curve. Then we call $N = T'/\kappa$ the *principal normal vector field* of β. The vector field $B = T \times N$ on β is called the *binormal vector field* of β.

Proposition. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then the three vector fields T, N, and B on β are unit vector fields that are mutually orthogonal at each point.

Proof. Since $T = \beta'$, $\|T\| = \sqrt{T \cdot T} = 1$, then $(T \cdot T)' = T \cdot T' + T' \cdot T = 2T' \cdot T = 0$, so $T \cdot T' = 0$. For all $s \in \operatorname{dom}(\beta)$, $\|N\| = \|T'(s)\|/\kappa(s) = \|T'(s)\|/\|T'(s)\| = 1$. Since $B = T \times N$, B is orthogonal to T and N. Moreover, $\|B\|^2 = \|T\|\|N\| - (T \cdot N)^2 = 1 - 0 = 1$. $\square$

Definition 2.11. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then (T, N, B) is called the *Frenet frame field* of β.

Now we claim that B' can be written as a scalar multiple of N. Consider $B' = B' \cdot NN + B' \cdot TT + B' \cdot BB$. Differentiate $B \cdot T$, $B' \cdot T + T' \cdot B = 0$, then $B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0$. Similarly, $B \cdot B' = 0$, so $B' = B' \cdot NN$.

Definition 2.12. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then the *torsion* of β is a function $\tau : I \to \mathbb{R}$ such that $B' = -\tau N$.

Theorem 2.1 (Frenet formulas)**.** Let $\beta : I \to \mathbb{R}^3$ be a curve with $\kappa > 0$ and $\|\beta'\| = 1$. Then $T' = \kappa N$, $N' = -\kappa T + \tau B$, and $B' = -\tau N$.

Proof. Rewrite $N' = N' \cdot TT + N' \cdot NN + N' \cdot BB$. Differentiate $T \cdot N$, we have $T' \cdot N + N' \cdot T = 0$, so $N' \cdot T = -T' \cdot N = -(\kappa N) \cdot N = -\kappa$. Similarly, $N' \cdot B = -B' \cdot N = -(-\tau N) \cdot N = \tau$. Hence $N' = -\kappa T + \tau B$. $\square$

Example. Consider the curve $\beta : \mathbb{R} \to \mathbb{R}^3$ defined by $\beta(s) = (a\cos(s/c), a\sin(s/c), bs/c)$, where $a > 0$ and $c = \sqrt{a^2 + b^2}$. It is trivial that $\|\beta'\| = 1$. Here $T(s) = \beta'(s) = (-a\sin(s/c)/c, a\cos(s/c)/c, b/c)$, then $T'(s) = (-a\cos(s/c)/c^2, -a\sin(s/c)/c^2, 0)$, so $\kappa(s) = \|T'(s)\| = a/c^2 > 0$. We also have $N(s) = (-\cos(s/c), -\sin(s/c), 0)$. Now $\beta(s) = T(s) \times N(s) = (b\sin(s/c)/c, -b\cos(s/c)/c, a/c)$, then $B'(s) = (b/\cos(s/c)/c^2, b\sin(s/c)/c^2, 0)$, so $\tau(s) = -B'(s)/N(s) = (-b/\cos(s/c)/c^2, -b\sin(s/c)/c^2, 0)/(-\cos(s/c), -\sin(s/c), 0) = b/c^2$.

Definition 2.13. Let $p, q \in \mathbb{R}^3$ with $q \neq 0$. The *plane* through p orthogonal to q is the set $\{r \in \mathbb{R}^3 \mid (r - p) \cdot q = 0\}$. A curve $\beta : I \to \mathbb{R}^3$ is said to be a *plane curve* if $\operatorname{ran}(\beta) \subset P$, where P is a plane in $\mathbb{R}^3$.

Proposition. Let $\beta : I \to \mathbb{R}^3$ be a curve with $\|\beta'\| = 1$ and $\kappa > 0$. Then β is a plane curve if and only if $\tau = 0$.

Proof. $(\Rightarrow)$ Let β be a plane curve, then there exists $p, q \in \mathbb{R}^3$ such that for all $s \in I$, $(\beta(s) - p) \cdot q = 0$. Consider q as a constant vector field, so $((\beta - p) \cdot q)' = (\beta - p)' \cdot q + q' \cdot (\beta - p) = \beta' \cdot q - p' \cdot q = \beta' \cdot q + q' \cdot \beta = \beta' \cdot q = \beta'' \cdot q = 0$. Rewrite $q = q \cdot TT + q \cdot NN + q \cdot BB$, then $q = q \cdot BB$. We have $B = B \cdot BB = (B \cdot B)/(q \cdot B)q = (B \cdot B)/(q \cdot B)\|q\|$. Since $\|B\| = 1$, $B = \pm q/\|q\|$, which is a point, then $B' = 0$, hence $\tau = 0$. $(\Leftarrow)$ Let $\tau = 0$, then $B' = 0$, so B is constant. Define $f : I \to \mathbb{R}$ by $f(s) = (\beta(s) - \beta(0)) \cdot B$, then $\mathrm{d}f/\mathrm{d}s = T(s) \cdot 0 = 0$ and $f(0) = 0 \cdot B = 0$, so $f(s) = 0$ for all s. Hence $\operatorname{ran}(\beta) \subset \{r \mid (r - \beta(0)) \cdot B = 0\}$. $\square$

Proposition. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$, $\kappa' = 0$, $\|\beta'\| = 1$, and $\tau = 0$. Then β lies in a circle of radius $1/\kappa$.

Proof. Define $\gamma : I \to \mathbb{R}^3$ by $\gamma(s) = \beta(s) + N(s)/\kappa$, then $\gamma' = T + N'/\kappa$. By the Frenet formulas, $\gamma' = T + (-\kappa T + \tau B)/\kappa = T - T = 0$. Fix $t \in I$. For any $s \in I$, the distance $\|\beta(s) - (\beta(t) + N(t)/\kappa)\| = \|\beta(s) - (\beta(s) + N(s)/\kappa)\| = \|N(s)\|/\kappa = 1/\kappa$. $\square$

We have shown the properties of curves with unit speed. Now given a curve $\alpha : I \to \mathbb{R}^3$ with $\|\alpha'\| \neq 1$, let $\bar{\alpha}$ be the arc-length reparametrization of α, so $\|\bar{\alpha}'\| = 1$. Let $\overline{T}$, $\bar{k}$, $\overline{N}$, $\overline{B}$, and $\bar{\tau}$ be the corresponding functions of $\bar{\alpha}$. Define the T, λ, N, B, and τ of α to be those of $\bar{\alpha}$. We denote the speed of α by v.

Theorem 2.2 (Frenet formulas)**.** Let α be a regular curve on $\mathbb{R}^3$ with $\kappa > 0$, then $T' = \kappa v N$, $N' = -\kappa v T + \tau v B$, and $B' = -\tau v N$.

Proof. Apply the Frenet formulas on $\bar{\alpha}$, then $\overline{T}' = \bar{\kappa}\overline{N}$. Since $T' = \overline{T}'\mathrm{d}s/\mathrm{d}t = \overline{T}'v$, $T' = \bar{\kappa}v\overline{N}$. Similarly, $N' = -\kappa v T + \tau v B$ and $B' = -\tau v N$. $\square$

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be regular. Then $\alpha' = vT$ and $\alpha'' = (\mathrm{d}v/\mathrm{d}t)T + \kappa v^2 N$.

Proof. We have $\alpha' = \bar{\alpha}'\mathrm{d}s/\mathrm{d}t = v\overline{T} = vT$, then $\alpha'' = (\mathrm{d}v/\mathrm{d}t)T + T'v = (\mathrm{d}v/\mathrm{d}t)T + (\kappa v N)\kappa = (\mathrm{d}v/\mathrm{d}t)T + v^2\kappa N$. $\square$

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve. Then $T = \alpha'/\|\alpha'\|$, $N = B \times T$, $B = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$, $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3$, and $\tau = (\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2$.

Proof. Since $\|\alpha'\| = v$, $\alpha'/\|\alpha'\| = vT/v = T$. We have $\alpha' \times \alpha'' = vT \times ((\mathrm{d}v/\mathrm{d}t)T + \kappa v^2 N) = vT \times (\mathrm{d}v/\mathrm{d}t)T + vT \times \kappa v^2 N$, since $T \times T = 0$, $\alpha' \times \alpha'' = \kappa v^3 T \times N = \kappa v^3 B$. The norm $\|\alpha' \times \alpha''\| = \|\kappa v^3\| = \kappa v^3$, hence $B = \kappa v^3 B/(\kappa v^3) = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$. Consider a lemma: for $w_p, v_p, u_p \in T_p(\mathbb{R}^3)$, $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$. (lemma) Rewrite $u = \sum u_i$, $v = \sum v_i$, and $w = \sum w_i$, then $(u \cdot w)v - (v \cdot w)u = (\sum u_i w_j)\sum v_i - (\sum v_i w_j)\sum u_i = u \times v \times w$. By the lemma, $B \times T = T \times N \times T = (T \cdot T)N - (N \cdot T)T = N - 0 = N$. We have shown $\|\alpha' \times \alpha''\| = \kappa v^3$, so $\|\alpha' \times \alpha''\|/\|\alpha'\|^3 = \kappa v^3/v^3 = \kappa$. Differentiate α'', then $\alpha''' = (\mathrm{d}v/\mathrm{d}t)T' + (\mathrm{d}^2v/\mathrm{d}t^2)T + 2\kappa(\mathrm{d}v/\mathrm{d}t)vN + N'\kappa v^2$. Since $B \cdot T = B \cdot N = 0$, $\kappa v^3 B \cdot \alpha''' = \kappa v^3 B \cdot ((\mathrm{d}v/\mathrm{d}t)T' + N'\kappa v^2)$, by the Frenet formulas, T' term becomes 0 and N' term becomes $\tau v B$. Now $(\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2 v^6 \tau B \cdot B = \kappa^2 v^6 \tau$. Hence $(\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2 = \kappa^2 v^6 \tau/(\kappa^2 v^6) = \tau$. $\square$

Example. Consider the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$, then $\alpha' = (3 - 3t^2, 6t, 3 + 3t^2)$, $\alpha'' = (-6t, 6, 6t)$, and $\alpha''' = (-6, 0, 6)$. We have $\alpha' \cdot \alpha' = 18(1 + t^2)^2$, so $\|\alpha'\| = 3\sqrt{2}(1 + t^2)$ and $T = (1 - t^2, 2t, 1 + t^2)/(\sqrt{2}(1 + t^2))$. The cross product $\alpha' \times \alpha'' = (18t^2 - 18, -36t, 18t^2 + 18) = 18(t^2 - 1, -2t, t^2 + 1)$ and its norm $\|\alpha' \times \alpha''\| = 18\sqrt{2}(1 + t^2)$, hence $B = (t^2 - 1, -2t, t^2 + 1)/(\sqrt{2}(1 + t^2))$. Now $N = B \times T = (-2t, 1 - t^2, 0)/(1 + t^2)$. By our computation, $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3 = 1/(3(1 + t^2)^2)$ and $\tau = (\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2 = 18(t^2 - 1, -2t, t^2 + 1) \cdot 6(-1, 0, 1)/(18\sqrt{2}(1 + t^2))^2 = 216/(648(1 + t^2)^2) = 1/(3(1 + t^2)^2)$.

Definition 2.14. Let $\alpha : I \to \mathbb{R}^3$ be regular. We say α is a *cylindrical helix* if there exists $u \in \mathbb{R}^3$ such that for all $t \in I$, $T(t) \cdot u = \cos\theta$, where θ is a constant angle.

If α has $\|\alpha'\| \neq 1$, take the arc-length reparametrization $\bar{\alpha}$ of α, by our definition, (T, N, B), κ, and τ are all invariant under this reparametrization, so it suffices to consider a curve with unit speed.

Proposition. A regular curve α on $\mathbb{R}^3$ with $\kappa > 0$ is a cylindrical helix if and only if τ/κ is constant.

Proof. Let α be a curve with $\|\alpha'\| = 1$. ($\Rightarrow$) Let α be a cylindrical helix with unit vector u and constant angle θ, then $T \cdot u = \cos\theta$, so $0 = (T \cdot u)' = T' \cdot u = \kappa N \cdot u$. Since $\kappa > 0$, $u \cdot N = 0$, then $u = (u \cdot T)T + (u \cdot B)B = \cos\theta T + (u \cdot B)B$. Since $\|u\| = 1$, $u \cdot B = \sin\theta$. Now $0 = u' = \cos\theta\kappa N - \sin\theta\tau N$, so $\cos\theta\kappa = \sin\theta\tau$, which implies $\tau/\kappa = \cot\theta$. ($\Leftarrow$) Let τ/κ be a constant and let $\cot\theta = \tau/\kappa$ for some angle θ. Consider $U = \cos\theta T + \sin\theta B$, here $U' = \cos\theta T' + \sin\theta B' = (\cos\theta\kappa - \sin\theta\tau)N$, since $\tau/\kappa = \cot\theta$, $U' = 0$. For all $s_1, s_2 \in I$, $U(s_1) = U(s_2)$, so pick $U(0) = u$. Now $T \cdot u = T \cdot U = T \cdot (\cos\theta T + \sin\theta B) = \cos\theta$. $\square$

Let $\tau = 0$, $\kappa > 0$, and $\kappa' = 0$ for some α with $\|\alpha'\| = 1$, then α lies in a circle of radius $1/\kappa$ as we proved before. Consider a circle α in $\mathbb{R}^3$, define $R : I \to \mathbb{R}^3$ by $R(s) = \alpha(s) - c$, where c is the center. We have $(R \cdot R)' = 2R' \cdot R = 2T \cdot R = 0$. Since $T \cdot N = 0$, $R = nN$. We have $T = R' = n'N + N'n = n'N + n(-\kappa T + \tau B)$, then $(-\kappa n - 1)T + \tau n B + n'N = 0$, so $-\kappa n - 1 = \tau n = n' = 0$. Since $n = -1/\kappa$, $n \neq 0$, this implies $\tau = 0$. Moreover, since $n' = 0$, n is a constant, then κ is a constant.

Definition 2.15. Let $u \in \mathbb{R}^3$ be a point with $\|u\| = 1$ and let V be a plane orthogonal to u. The *projection map* of $p \in \mathbb{R}^3$ onto V is a function $\mathrm{proj} : \mathbb{R}^3 \to V$ defined by $\mathrm{proj}(p) = p - (p \cdot u)u$.

Let α be a curve and let u be a unit vector. Let β be the curve $\beta = \mathrm{proj} \circ \alpha$. Then $\beta' = \alpha' - (\alpha' \cdot u)u$, $\beta'' = \alpha'' - (\alpha'' \cdot u)u$, and $\beta''' = \alpha''' - (\alpha''' \cdot u)u$. We have $\beta' \cdot u = \alpha' \cdot u - (\alpha' \cdot u)(u \cdot u) = 0$, similarly, $\beta'' \cdot u = 0 = \beta''' \cdot u$, then $(\beta' \times \beta'') \cdot u = (\beta' \cdot u)\beta'' - (\beta'' \cdot u)\beta' = 0$, it suffices to rewrite $\beta' \times \beta'' = nu$ for some $n \in \mathbb{R}$. Now $(\beta' \times \beta'') \cdot \beta''' = nu \cdot \beta''' = 0$. Hence every curve under a projection map has $\tau = 0$.

Definition 2.16. Let $\alpha : I \to \mathbb{R}^3$ be a cylindrical helix with axis direction given by the unit vector u. We call α a *circular helix* if for every plane orthogonal to u, the projection of α onto that plane is a circle.

Let α be a circular helix, then $\tau/\kappa = 0$ and there exists a unit vector u such that $T \cdot u = \cos\theta$. Consider the projection $\beta = \mathrm{proj} \circ \alpha$ with $\beta = \alpha - (\alpha \cdot u)u$. We have $\beta' = T - (T \cdot u)u = T - \cos\theta u$ and $\beta'' = T' - (T' \cdot u)u = \kappa N - (\kappa N \cdot u)u = \kappa N$, then $\beta' \times \beta'' = (T - \cos\theta u) \times \kappa N = \kappa(T - \cos\theta u) \times N = \kappa(B - \cos\theta(u \times N))$. Since $u = \cos\theta T + \sin\theta B$, $u \times N = \cos\theta(T \times N) + \sin\theta(B \times N) = \cos\theta B - \sin\theta T$, then $\|\beta' \times \beta''\| = \kappa\sqrt{\sin^4\theta + \cos^2\theta \sin^2\theta} = \kappa \sin\theta\sqrt{\sin^2\theta + \cos^2\theta} = \kappa\sin\theta$. Similarly, $\|\beta'\| = \sqrt{\sin^2\theta T + \sin\theta\cos\theta B} = \sin\theta$. The curvature $\kappa_\beta = \kappa/\sin^2\theta$, and κ_β is a constant if and only if κ is a constant. Hence $\kappa, \tau > 0$ and $\kappa' = 0 = \tau'$ if and only if α is a circular helix.

Definition 2.17. Let W be a vector field on $\mathbb{R}^3$ and let $v \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then the *covariant derivative* of W with respect to v is the tangent vector $\nabla_v W = W(p + tv)'(0)$ at p.

Proposition. Let $W = \sum w_i U_i$ be a vector field on $\mathbb{R}^3$ and let $v \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then $\nabla_v W = \sum v[w_i]U_i(p)$.

Proof. We have $W(p + tv) = \sum w_i(p + tv)U_i$. Since $(\mathrm{d}/\mathrm{d}t)w_i(p + tv)$ at $t = 0$ is $v[f]$, $\nabla_v W = \sum v[w_i]U_i(p)$. ☐

Proposition. Let $v, w \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Let Y and Z be vector fields on $\mathbb{R}^3$. For any $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, the following properties hold.

1. $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$.
2. $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$.
3. $\nabla_v(fY) = v[f]Y(p) + f(p)\nabla_v Y$.
4. $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$.

Proof. Rewrite $Y = \sum y_i U_i$ and $Z = \sum z_i U_i$. (i) We have $\nabla_{av+bw} Y = \sum(av + bw)[y_i]U_i(p) = a\sum v[y_i]U_i + b\sum w[y_i]U_i = a\nabla_v Y + b\nabla_w Y$. (ii) Similarly, $\nabla_v(aY + bZ) = \sum v[ay_i + bz_i]U_i = a\sum v[y_i]U_i + b\sum v[z_i]U_i = a\nabla_v Y + b\nabla_v Z$. (iii) Similarly, $\nabla_v(fY) = \sum v[fy_i]U_i = \sum(v[f]y_i + fv[y_i])U_i = v[f]Y(p) + f(p)\nabla_v Y$. (iv) We have $Y \cdot Z = \sum y_i z_i$, then $v[Y \cdot Z] = \sum v[y_i]z_i U_i + \sum v[z_i]y_i U_i = \nabla_v Y \cdot Z + Y \cdot \nabla_v Z$. ☐

Let V and $W = \sum w_i$ be vector fields. We define $\nabla_V W$ at some $p \in \mathbb{R}^3$ to be $\nabla_{V(p)} W$, hence $\nabla_V W = \sum V[w_i]U_i$.

Proposition. Let V, W, Y, and Z be vector fields on $\mathbb{R}^3$. For all functions $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$. Then the following properties hold.

1. $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$.
2. $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$.
3. $\nabla_V(fY) = V[f]Y + f\nabla_V Y$.
4. $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$.

Proof. Since $\nabla_V W(p) = \nabla_{V(p)} W$, those properties are direct consequences of the previous proposition. $\square$

Definition 2.18. Let E_1, E_2, and E_3 be vector fields on $\mathbb{R}^3$. We say $\{E_1, E_2, E_3\}$ is a *frame field* on $\mathbb{R}^3$ if $E_i \cdot E_j = \delta_{i,j}$.

If $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$, then for all $p \in \mathbb{R}^3$, $\{E_1(p), E_2(p), E_3(p)\}$ is trivially a frame at p.

Example. Consider a cylindrical coordinate system with coordinates (r, θ, z). Define $E_1 = \cos\theta U_1 + \sin\theta U_2$, $E_2 = -\sin\theta U_1 + \cos\theta U_2$, and $E_3 = U_3$. For any p, $E_1(p) \cdot E_2(p) = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$, $E_1(p) \cdot E_3(p) = 0 = E_2(p) \cdot E_3(p)$, $E_1(p) \cdot E_1(p) = \cos^2\theta + \sin^2\theta = E_2(p) \cdot E_2(p) = 1 = E_3(p) \cdot E_3(p) = 1$. Hence $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$, known as the *cylindrical frame field*.

Example. Consider a spherical coordinate system with coordinates (ρ, θ, φ). Define $F_1 = \cos\varphi E_1 + \sin\varphi E_3$, $F_2 = E_2$, and $F_3 = -\sin\varphi E_1 + \cos\varphi E_3$, where $\{E_1, E_2, E_3\}$ is the cylindrical frame field. It is trivial that $F_1 \cdot F_1 = F_2 \cdot F_2 = F_3 \cdot F_3 = 1$ and $F_1 \cdot F_2 = 0 = F_2 \cdot F_3$. Now $F_1 \cdot F_3 = \cos\varphi E_1 \cdot (-\sin\varphi E_1 + \cos\varphi E_3) + \sin\varphi E_3 \cdot (-\sin\varphi E_1 + \cos\varphi E_3) = -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = 0$. Hence $\{F_1, F_2, F_3\}$ is a frame field on $\mathbb{R}^3$, called the *spherical frame field*.

Proposition. Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$. If V is a vector field on $\mathbb{R}^3$, then $V = \sum f_i E_i$, where $f_i = V \cdot E_i$. If $V = \sum f_i E_i$ and $W = g_i E_i$, then $V \cdot W = \sum f_i g_i$.

Proof. (i) For any point p, $\{E_1(p), E_2(p), E_3(p)\}$ is a frame at p, so $V(p) = \sum (V(p) \cdot E_i(p)) E_i(p)$. (ii) For any p, $\{E_1(p), E_2(p), E_3(p)\}$ is a frame at p, so $V \cdot W = \sum f_i g_i$ as we shown before. $\square$

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