Elementary Differential Geometry

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1 Calculus on Euclidean Space

Definition 1.1. The *Euclidean 3-space*, denoted $\mathbb{R}^3$, is the set of ordered triples of the form $p = (p_1, p_2, p_3)$, where $p_i \in \mathbb{R}$. An element of $\mathbb{R}^3$ is called a *point*.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. Define the addition to be $p + q = (p_i + q_i)$ and define the scalar multiplication to be $ap = (ap_i)$. The additive identity $0 = (0, 0, 0)$ is called the *origin* of $\mathbb{R}^3$. It is trivial that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Definition 1.2. Let x, y, and z be real-valued functions on $\mathbb{R}^3$ such that for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $x(p) = p_1$. $y(p) = p_2$, and $z(p) = p_3$. We call x, y, and z the *natural coordinate functions* of $\mathbb{R}^3$.

Let x, y, and z be the natural coordinate functions, rewrite $x = x_1$, $y = x_2$, and $z = x_3$. Then we have $p = (p_i) = (x_i(p))$.

Definition 1.3. A real-valued function f on $\mathbb{R}^3$ is *differentiable* if all partial derivatives exist and continuous.

Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3$. Recall that the dot product is defined to be $p \cdot q = \sum p_i q_i$ and the norm is defined to be $\|p\| = \sqrt{p \cdot p} = \sqrt{\sum p_i^2}$.

Definition 1.4. A subset $O \subset \mathbb{R}^3$ is *open* if for all $p \in O$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^3 \mid \|x - p\| < \varepsilon\} \subset O$.

Let $f : O \to \mathbb{R}$ be a function defined on an open set. The differetiability of f at p can be determined entirely from values of f on O. This means that differetiation is a local operation. We will discuss this later.

Definition 1.5. A *tangent vector* v_p is an ordered pair $v_p = (v, p)$, where $v, p \in \mathbb{R}^3$. Here v is called the *vector part* and p is called its *point of application*. Two tangent vectors are said to be *parallel* if they have the same vector part and different points of application.

Definition 1.6. Let $p \in \mathbb{R}^3$. The *tangent space* at p, denoted $T_p(\mathbb{R}^3)$, is the set of all tangent vectors that have p as point of application.

Fix a tangent space $T_p(\mathbb{R}^3)$ and let $T_p(\mathbb{R}^3)$ adapt the operations from $\mathbb{R}^3 \times \mathbb{R}^3$. We have a natural linear map $f : T_p(\mathbb{R}^3) \to \mathbb{R}^3$ defined by $v_p \to v$ and it is trivially an isomorphism.

Definition 1.7. A *vector field* V on $\mathbb{R}^3$ is a function $V : \mathbb{R}^3 \to \bigsqcup_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3)$ such that for all $p \in \mathbb{R}^3$, $V(p) \subset T_p(\mathbb{R}^3)$.

Let V and W be vector field. Let f be a real-valued function. For all $p \in \mathbb{R}^3$, define $V + W$ by $(V + W)(p) = V(p) + W(p)$ and $(fV)(p) = f(p)V(p)$.

Definition 1.8. Let U_1, U_2, and U_3 be vector fields on $\mathbb{R}^3$ such that $U_1(p) = (1,0,0)_p$, $U_2(p) = (0,1,0)_p$, and $U_3(p) = (0,0,1)_p$ for all $p \in \mathbb{R}^3$. We call (U_1, U_2, U_3) the *natural frame field* on $\mathbb{R}^3$.

Proposition. Let V be a vector field on $\mathbb{R}^3$. There are three uniquely determined real-valued functions v_1, v_2, and v_3 on $\mathbb{R}^3$ such that $V = v_1U_1 + v_2U_2 + v_3U_3$.

Proof. For all $p \in \mathbb{R}^3$, $V(p) = (v_1(p), v_2(p), v_3(p))_p = v_1(p)(1,0,0)_p + v_2(p)(0,1,0)_p + v_3(p)(0,0,1)_p = v_1(p)U_1(p) + v_2(p)U_2(p) + v_3U_3(p)$, hence $V = \sum v_iU_i$. ☐

The functions v_1, v_2, and v_3 are called the *Euclidean coordinate functions* on V.

Definition 1.9. A vector field V is *differentiable* if its Euclidean coordinate functions are differentiable.

Definition 1.10. Let f be a differentiable real-valued function on $\mathbb{R}^3$ and let v_p be a tangent vector on $\mathbb{R}^3$. The *directional derivative* of f with respect to v_p, denoted $v_p[f]$, is defined to be $(\mathrm{d}/\mathrm{d}t)f(p+tv)$ at $t = 0$.

Remark. We will not write the restriction every time for convenience.

Proposition. Let $v_p = (v_1, v_2, v_3)_p$ be a tangent vector, then $v_p[f] = \sum v_i(\partial f/\partial x_i)(p)$.

Proof. Let $p = (p_i)$. Then $v_p[f] = (\mathrm{d}/\mathrm{d}t)f(p+tv)|_{t=0} = \sum(\partial f/\partial z)(p) \cdot (\mathrm{d}/\mathrm{d}t)(p_i + tv_i) = \sum(\partial f/\partial x_i)(p)v_i$. ☐

Example. Consider $f = x^2yz$ with $p = (1,1,0)$ and $v = (1,0,-3)$. By the definition, $p + tv = (1+t, 1, -3t)$, so $v_p[f] = (\mathrm{d}/\mathrm{d}t)(-3t^3 - 6t^2 - 3t) = -3$. Since $(\partial f/\partial x) = 2xyz$, $(\partial f/\partial y) = x^2z$, and $(\partial f/\partial z) = x^2y$, we have $(\partial f/\partial x)(p) = (\partial f/\partial y)(p) = 0$ and $(\partial f/\partial z)(p) = 1$, so $v_p[f] = -3$.

Proposition. Let f and g be functions on $\mathbb{R}^3$. Let v_p and w_p be tangent vectors. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.
2. $v_p[af + bg] = av_p[f] + bv_p[g]$.
3. $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$.

Proof. (i) We have $(av_p + bw_p)[f] = \sum(av_i + bw_i)(\partial f/\partial x_i)(p) = \sum av_i(\partial f/\partial x_i) + \sum bw_i(\partial f/\partial x_i)(p) = av_p[f] + bw_p[f]$. (ii) We have $v_p[af + bg] = \sum v_i(\partial(af+bg)/\partial x_i)(p) = \sum v_i(\partial(af)/\partial x_i)(p) + \sum v_i(\partial(bg)/\partial x_i)(p) = av_p[f] + bv_p[g]$. (iii) We have $v_p[fg] = \sum v_i(\partial(fg)/\partial x_i)(p) = \sum v_i(\partial f/\partial x_i)(p)g(p) + f(p)\sum v_i(\partial g/\partial x_i)(p) = v_p[f]g(p) + f(p)v_p[g]$. ☐

Let V be a vector field, we define $V[f]$ at $p \in \mathbb{R}^3$ to be $V(p)[f]$. By the convention, $U_i(p)[f] = (\partial f/\partial x_i)(p)$.

Proposition. Let V and W be vector fields. Let f, g, and h be real-valued functions. For all $a, b \in \mathbb{R}$, the following properties hold.

1. $(fV + gW)[h] = fV[h] + gW[h]$.
2. $V[af + bg] = aV[f] + bV[g]$.
3. $V[fg] = V[f]g + fV[g]$.

Proof. (i) For all $p \in \mathbb{R}^3$, $(fV + gW)(p)[h] = (f(p)V(p) + g(p)W(p))[h] = fV[h] + gW[h]$. (ii) For all $p \in \mathbb{R}^3$, $V(p)[af + bg] = aV(p)[f] + bV(p)[g]$. (iii) For all $p \in \mathbb{R}^3$, $V(p)[f]g(p) + f(p)V(p)[g] = V[f](p)g(p) + f(p)V[g](p) = (V[f]g + fV[g])(p)$. ☐

Example. Let $V = xU_1 - y^2U_3$ and let $f = x^2y + z^3$. Then $V[f] = xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3] = 2x^2y - 3y^2z^2$.

Let $I \subset \mathbb{R}$ be an open interval. Let $\alpha : I \to \mathbb{R}^3$ be a function. We can rewrite $\alpha(t)$ as $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where $\alpha_i : I \to \mathbb{R}$. We say α is *differentiable* if α_i are differentiable.

Definition 1.11. A *curve* in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an open interval.

Example. A curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = p + tq$, where $\alpha(0) = p$ and $q \neq 0$, is called a *straight line*.

Example. Here are some examples of curves.

1. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (a\cos t, a\sin t, bt)$.
2. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$.
3. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$.
4. The cruve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$.

Definition 1.12. Let $\alpha : I \to \mathbb{R}^3$ be a curve with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For all $t \in I$, the *velocity vector* of α at t is the tangent vector $\alpha'(t) = ((\mathrm{d}\alpha_1/\mathrm{d}t)(t), (\mathrm{d}\alpha_2/\mathrm{d}t)(t), (\mathrm{d}\alpha_3/\mathrm{d}t)(t))_{\alpha(t)}$ at the point $\alpha(t) \in \mathbb{R}^3$. The curve α is said to be *regular* if $\alpha_i \neq 0$ for all i.

Consider the velocity vector $\alpha'(t)$, we can rewrite it by the natural frame fields, so $\alpha'(t) = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t)U_i(\alpha(t))$.

Definition 1.13. Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $h : J \to I$ be differentiable, where J is an open interval of $\mathbb{R}$. The *reparametrization* of α by h is the composition $\alpha \circ h : J \to \mathbb{R}^3$.

The composition of differentiable functions is differentiable, so any reparametrization is differentiable, which means it is a curve.

Proposition. Let β be the reparametrization of α by h, then $\beta'(s) = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$.

Proof. Rewrite $\beta(s) = \alpha(h(s))$, then we have $\beta'(s) = (\mathrm{d}(\alpha_i h_i)/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}\alpha_i/\mathrm{d}s)(h(s)) \cdot (\mathrm{d}h/\mathrm{d}s)(s)_{\alpha(h(s))} = (\mathrm{d}h/\mathrm{d}s)(s)\alpha'(h(s))$. □

Proposition. Let α be a curve and let f be a differentiable function on $\mathbb{R}^3$, then $\alpha'(t)[f] = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$.

Proof. We have $\alpha'(t)[f] = \sum(\mathrm{d}\alpha_i/\mathrm{d}t)(t) \cdot (\partial f/\partial x_i)(\alpha(t)) = (\mathrm{d}(f\alpha)/\mathrm{d}t)(t)$ by the chain rule. □

Now we show a general idea of parametrizations. The proofs will be included in other sections when we have enough tools. Assume every result is correct for now.

Definition 1.14. A *1-form* φ on $\mathbb{R}^3$ is a function $\varphi : \coprod_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3) \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ and $v, w \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$, $\varphi(av + bw) = a\varphi(v) + b\varphi(w)$.

Given a 1-form φ, for any point p, denote the restriction $\varphi|_{T_p(\mathbb{R}^3)} : T_p(\mathbb{R}^3) \to \mathbb{R}$ by φ_p, then φ_p is linear. Let φ and ψ be 1-forms. Define the addition and scalar multiplication by $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$ and $(f\varphi)(v_p) = f(p)\varphi(v_p)$. Given any 1-form φ and point p, φ_p is a linear functional in $T_p^*(\mathbb{R}^3)$, the dual space of $T_p(\mathbb{R}^3)$.

Definition 1.15. Let φ be a 1-form and let V be a vector field. For all $p \in \mathbb{R}^3$, define $\varphi(V)(p) = \varphi_p(V(p))$. We say φ is *differentiable* if for every differentiable vector field V, the function $\varphi(V)$ is differentiable.

Now let V and W be vector fields, we have $\varphi(fV + gW)(p) = \varphi((fV + gW)(p)) = \varphi(fV(p) + gW(p)) = (f\varphi(V) + g\varphi(W))(p)$. Similarly, $(f\varphi + g\psi)(V) = f\varphi(V) + g\psi(V)$.

Definition 1.16. If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable. The *differential* of f, denoted $\mathrm{d}f$, is the function $\mathrm{d}f(v_p) = v_p[f]$ for all tangent vectors v_p.

Let $v_p, w_p \in T_p(\mathbb{R}^3)$ and let $a, b \in \mathbb{R}$, then $\mathrm{d}f(av_p + bw_p) = (av_p + bw_p)[f] = av_p[f] + bw_p[f] = a\mathrm{d}f(v_p) + b\mathrm{d}f(w_p)$. Hence $\mathrm{d}f$ is a 1-form.

Example. Consider the natural coordinate functions x_i. We have $\mathrm{d}x_i(v_p) = v_p[x_i] = \sum v_i(\partial x_i/\partial x_j)(p) = v_i$.

Proposition. If φ is a 1-form on $\mathbb{R}^3$, then $\varphi = \sum f_i \mathrm{d}x_i$, where $f_i = \varphi(U_i)$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\varphi(v_p) = \varphi(\sum v_i U_i(p)) = \sum v_i \varphi(U_i(p)) = \sum v_i f_i(p) = \sum f_i(p)\mathrm{d}x_i(v_p) = (\sum f_i \mathrm{d}x_i)(v_p)$, hence $\varphi = \sum f_i \mathrm{d}x_i$. □

The functions f_1, f_2, and f_3 are called the *Euclidean coordinate functions* of the 1-form φ.

Proposition. Let f be a differentiable function on $\mathbb{R}^3$, then $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$.

Proof. Let $v_p \in T_p(\mathbb{R}^3)$, then $\mathrm{d}f(v_p) = v_p[f] = \sum v_i(\partial f/\partial x_i)(p) = \sum(\partial f/\partial x_i)(p)\mathrm{d}x_i(v_p) = (\sum(\partial f/\partial x_i)\mathrm{d}x_i)(v_p)$, hence $\mathrm{d}f = \sum(\partial f/\partial x_i)\mathrm{d}x_i$. □

Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(f+g) = \sum(\partial(f+g)/\partial x_i)\mathrm{d}x_i = \sum(\partial f/\partial x_i)\mathrm{d}x_i + \sum(\partial g/\partial x_i)\mathrm{d}x_i = \mathrm{d}f + \mathrm{d}g$. Now we denote the multiplication to be fg.

Proposition. Let f and g be differentiable functions on $\mathbb{R}^3$, then $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$.

Proof. We have $\mathrm{d}(fg) = \sum(\partial(fg)/\partial x_i)\mathrm{d}x_i = \sum((\partial f/\partial x_i)g + (\partial g/\partial x_i)f)\mathrm{d}x_i = g\mathrm{d}f + f\mathrm{d}g$. □

Proposition. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable, then $\mathrm{d}(h(f)) = (\mathrm{d}h(f)/\mathrm{d}x)\mathrm{d}f$.

Proof. We have $\mathrm{d}(h(f)) = \sum(\partial h(f)/\partial x_i)\mathrm{d}x_i$, by the chain rule, $(\partial h(f)/\partial x_i)\mathrm{d}x_i = (\mathrm{d}h(f)/\mathrm{d}f)(\partial x/\partial x_i)$, so $\mathrm{d}(h(f)) = (\mathrm{d}f(h)/\mathrm{d}f)\mathrm{d}f$. □

Example. Consider the function $f = (x^2-1)y + (y^2+2)z$. We have $\mathrm{d}f = \mathrm{d}((x^2-1)y) + \mathrm{d}((y^2+2)z) = y\mathrm{d}(x^2-1) + (x^2+1)\mathrm{d}y + z\mathrm{d}(y^2+2) + (y^2+2)\mathrm{d}z = 2xy\mathrm{d}x + (x^2+2yz-1)\mathrm{d}y + (y^2+2)\mathrm{d}z$. Since $v_p[f] = \mathrm{d}f(v_p)$, $v_p[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_2^2+2)v_3$.

Definition 1.17. Let V be the vector space $\mathbb{R}^3$ and denote the space of all p-linear forms on V by $\Lambda^p(V^*)$. Every element of Λ^p is called a *p-form*. Define the *wedge product* to be a function $\wedge : \Lambda^a(V^*) \times \Lambda^b(V^*) \to \Lambda^{a+b}(V^*)$ such that for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $v_1, \ldots, v_{m+n} \in V$, the following properties hold.

1. $(\omega \wedge \eta)(v_1, \ldots, v_{m+n}) = (\sum_{\sigma \in \mathfrak{S}_{m+n}} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(m)})\eta(v_{\sigma(m+1)}, \ldots, v_{\sigma(m+n)}))/(m!n!)$.
2. $\omega \wedge \eta = (-1)^{mn}\eta \wedge \omega$.

Generally, a p-form is of the form $\sum f(x,y,z)\mathrm{d}x^i \wedge \cdots \mathrm{d}y^j \wedge \cdots \mathrm{d}z^k \wedge \cdots$. We have $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. If $i = j$, then $\mathrm{d}x_i \wedge \mathrm{d}x_i = -\mathrm{d}x_i \wedge \mathrm{d}x_i$, so $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. It is trivial that $\wedge$ is bilinear and associative, that is,

1. for $\omega_1, \omega_2 \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $a, b \in \mathbb{R}$,$(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta)$ and $\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2)$;
2. for $\omega \in \Lambda^m(V^*)$, $\eta \in \Lambda^n(V^*)$, and $\theta \in \Lambda^l(V^*)$, $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta$.

Now given a space of p-forms $\Lambda^p(V^*)$ with basis $\{e_1, e_2, e_3\}$, the basis of its dual space is denoted by $\{e^1, e^2, e^3\}$. The basis of $\Lambda^k(V^*)$ is of the form $e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $1 \le i_1 \le \cdots \le i_k \le 3$. In this case, the dimension of $\Lambda^p(V^*)$ is $3!/(p!(3-p)!)$. If $p > 4$, then $\dim(\Lambda^p(V^*)) = 0$, so there are no p-forms on $\mathbb{R}^3$ if $p \ge 4$.

Example. Let $\varphi = x\mathrm{d}x - y\mathrm{d}y$, $\psi = z\mathrm{d}x + x\mathrm{d}z$, $\theta = z\mathrm{d}y$, and $\eta = y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z$.

1. $\varphi \wedge \psi = xz\mathrm{d}x \wedge \mathrm{d}x + x^2\mathrm{d}x \wedge \mathrm{d}z - yz\mathrm{d}y \wedge \mathrm{d}x - yx\mathrm{d}y \wedge \mathrm{d}z = yz\mathrm{d}x \wedge \mathrm{d}y + x^2\mathrm{d}x \wedge \mathrm{d}z - yx\mathrm{d}y \wedge \mathrm{d}z$
2. $\theta \wedge (\varphi \wedge \psi) = yz^2\mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}y) + x^2z\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y - xyz\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}y = -x^2z\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
3. $\varphi \wedge \eta = xy\mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}z + x^2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - y^2\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z - xy\mathrm{d}y \wedge \mathrm{d}y \wedge \mathrm{d}z = (x^2+y^2)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Proposition. Let φ and ψ be 1-forms, then $\varphi \wedge \psi = -\psi \wedge \varphi$.

Proof. Rewrite $\varphi = \sum f_i \mathrm{d}x_i$ and $\psi = \sum g_i \mathrm{d}x_i$, then $\varphi \wedge \psi = \sum f_i g_i \mathrm{d}x_i \mathrm{d}x_j = \sum -g_i f_i \mathrm{d}x_j \mathrm{d}x_i = -\psi \wedge \varphi$. □

Definition 1.18. Let $\varphi = \sum f_i \mathrm{d}x_i$ be a 1-form on $\mathbb{R}^3$. The *exterior derivative* of φ is the 2-form $\mathrm{d}\varphi = \sum \mathrm{d}f_i \wedge \mathrm{d}x_i$. Let $\psi = \sum f_{i,j}\mathrm{d}x_i \wedge \mathrm{d}x_j$ be a 2-form. The *exterior derivative* of ψ is the 3-form $\mathrm{d}\psi = \sum \mathrm{d}f_{i,j} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j$.

Let $a, b \in \mathbb{R}$. Let $\varphi = \sum f_i \mathrm{d}x_i$ and $\psi = \sum g_i \mathrm{d}x_i$ be 1-forms. Then $\mathrm{d}(a\varphi + b\psi) = \mathrm{d}(\sum(af_i + bg_i)\mathrm{d}x_i) = \sum \mathrm{d}(af_i + bg_i) \wedge \mathrm{d}x_i$, since the differential is linear, the exterior derivative is linear.

Proposition. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions and let φ and ψ be 1-forms. Then $\mathrm{d}(f\varphi) = \mathrm{d}f \wedge \varphi + f\mathrm{d}\varphi$ and $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$.

Proof. (i) Let $\varphi = \sum g_i \mathrm{d}x_i$, then $f\varphi = \sum fg_i \mathrm{d}x_i$, so $\mathrm{d}(f\varphi) = \sum(f\mathrm{d}g_i + g_i\mathrm{d}f) \wedge \mathrm{d}x_i = \sum f\mathrm{d}g_i \wedge \mathrm{d}x_i + \sum g_i \mathrm{d}f \wedge \mathrm{d}x_i = f\mathrm{d}\varphi + \mathrm{d}f \wedge \varphi$. (ii) Since $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$, without lose of generality, let $\varphi = f\mathrm{d}x$ and let $\psi = g\mathrm{d}y$. Then $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}(fg\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}(fg) \wedge \mathrm{d}x \wedge \mathrm{d}y = (f\mathrm{d}g + g\mathrm{d}f) \wedge \mathrm{d}x \wedge \mathrm{d}y = f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y + g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$. For the right hand side, $\mathrm{d}\varphi \wedge \psi = \mathrm{d}f \wedge \mathrm{d}x \wedge g\mathrm{d}y = g\mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y$ and $\varphi \wedge \mathrm{d}\psi = f\mathrm{d}x \wedge \mathrm{d}g \wedge \mathrm{d}y = -f\mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}y$, hence $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$. $\square$

Definition 1.19. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ such that $F(p) = (f_1(p), \ldots, f_m(p))$ for all $p \in \mathbb{R}^n$. The functions f_i are called the *Euclidean coordinate functions* of F and we denote $F = (f_1, \ldots, f_m)$.

Definition 1.20. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $F = (f_1, \ldots, f_m)$, we say F is *differentiable* if all f_i are differentiable. If F is differentiable, we say F is a *mapping* from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 1.21. Let $\alpha : I \to \mathbb{R}^n$ be a curve and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then the composite function $\beta = F(\alpha) : I \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ called the *image* of α under F.

Example. The function $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F = (x - y, x + y, 2z)$ is a mapping. Trivially, F is a linear map, so F is determined by $F(u_i)$.

Example. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F = (u^2 - v^2, 2uv)$. Let $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (r\cos t, r\sin t)$, where $0 \le t \le 2\pi$. The image is $\beta(t) = (r^2\cos 2t, r^2\sin 2t)$. This curve takes two counterclockwise trips around the circle of radius r^2 centered at the origin. Therefore, F wraps $\mathbb{R}^2$ around itself twice.

Definition 1.22. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let $v_p \in T_p(\mathbb{R}^n)$. The *tangent map* of F, denoted $F_*(v_p)$, is defined to be $(\mathrm{d}/\mathrm{d}t)F(p + tv)$ at $t = 0$.

Fix some mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. For every $p \in \mathbb{R}^n$, it induces a tangent map of F at p, denoted F_{*p}.

Proposition. Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $v_p \in T_p(\mathbb{R}^n)$, then $F_{*p}(v_p) = (v[f_1], \ldots, v[f_m])_{F(p)}$.

Proof. Fix $v_p \in T_p(\mathbb{R}^n)$. We have $F_{*p} = (\mathrm{d}/\mathrm{d}t)F(p+tv)|_{t=0} = (\mathrm{d}/\mathrm{d}t)(f_i(p+tv))|_{t=0} = (v_p[f_1], \ldots, v_p[f_m])_{F(p)}$. $\square$

Proposition. Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. For all $p \in T_p(\mathbb{R}^n)$, the tangent map $F_{*p} : T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is a linear map.

Proof. Fix $p \in \mathbb{R}^n$. Let $a, b \in \mathbb{R}$ and let $v_p, w_p \in T_p(\mathbb{R}^n)$. We have $F_{*p}(av_p + bw_p) = ((av_p + bw_p)[f_i])_{F(p)} = (av_p[f_i])_{F(p)} + (bw_p[f_i])_{F(p)} = aF_{*p}(v_p) + bF_{*p}(w_p)$. $\square$

Proposition. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and let β be the image of some curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$.

Proof. Let $F = (f_1, \ldots, f_m)$. We have $F_*(\alpha'(t)) = (\alpha'(t)[f_i])_{F(\alpha(t))} = (\mathrm{d}f_i(\alpha(t))/\mathrm{d}t)_{F(\alpha(t))} = \beta'(t)$. $\square$

Let $\{U_j\}$ and $\{\overline{U_i}\}$ be the natural frame fields of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Proposition. Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then $F_*(U_j(p)) = \sum_{i=1}^m (\partial f_i/\partial x_j)(p)\overline{U_i}(F(p))$, where $1 \le j \le n$.

Proof. Recall that $U_j[f_i] = \partial f_i/\partial x_j$, so the proposition trivially holds. $\square$

Definition 1.23. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. The *Jacobian matrix* of F at $x \in \mathbb{R}^n$ is the matrix

$$J_F(x) = \begin{pmatrix} \partial f_1/\partial x_1(x) & \cdots & \partial f_1/\partial x_n(x) \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1(x) & \cdots & \partial f_m/\partial x_n(x) \end{pmatrix}.$$

Definition 1.24. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. We say F is *regular* if for all $p \in \mathbb{R}^n$, F_{*p} is injective.

Notice that $J_F(p) \cdot v = F_{*p}$, so $J_F(p)$ is the matrix representation of F_{*p}.

Definition 1.25. A mapping is a *diffeomorphism* if it has a differentiable inverse mapping.

Definition 1.26. A *topological space* $(X, \mathcal{T})$ consists of two sets X and $\mathcal{T}$, where $\mathcal{T} \subset \mathscr{P}(X)$, that satisfies the following properties.

1. $\varnothing, X \in \mathcal{T}$.
2. Any union of elements in $\mathcal{T}$ is also in $\mathcal{T}$.
3. Any finite intersection of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

Definition 1.27. Let $(X, \mathcal{T})$ be a topological space. A subset $U \subset X$ is said to be *open* if $U \in \mathcal{T}$. Let $x \in X$, a *neighborhood* of x is an open set U_x that contains x.

Let $U \subset \mathbb{R}$. We say U is open in the standard topology $\mathcal{T}$ on $\mathbb{R}$ if for every $x \in U$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. Trivially, $\varnothing, \mathbb{R} \in \mathcal{T}$. Let $\{U_i\}_{i \in I}$ be open sets, then for each U_i and $x \in U_i$, there exists a corresponding $\varepsilon_{i,x}$. For any $x \in \bigcup_{i \in I} U_i$, $x \in U_i$ for some $i \in I$. Pick $\varepsilon = \varepsilon_{i,x}$, then $(x - \varepsilon, x + \varepsilon) \subset U_i \subset \bigcup_{i \in I} U_i$. For any $x \in \bigcap_{i=1}^n U_i$, pick $\varepsilon = \min\{\varepsilon_{i,x}\}$, then $(x - \varepsilon, x + \varepsilon) \subset U_i$ for $1 \le i \le n$, so $(x - \varepsilon, x + \varepsilon) \subset \bigcap_{i=1}^n U_i$. The standard topology on $\mathbb{R}$ is indeed a topology.

Definition 1.28. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A subset $W \subset X \times Y$ is open in the *product topology* on $X \times Y$ if for all $(x, y) \in W$, there exist neighborhoods $U_x \in \mathcal{T}_X$ and $V_y \in \mathcal{T}_Y$ such that $U_x \times V_y \subset W$.

Denote the product topology by $\mathcal{T}$. We have $\varnothing \in \mathcal{T}$ vacuously. For all $(x, y) \in X \times Y$, $U_x \subset X$, and $V_Y \subset Y$, $U_X \times V_Y \subset X \times Y$, so $X \times Y \in \mathcal{T}$. Let $\{W_i\}_{i \in I}$ be open sets. For all $(x, y) \in \bigcup_{i \in I} W_i$, there exist W_i and W_j such that $x \in W_i$ and $y \in W_j$. Pick the corresponding neighborhood in each set, then $U_x \times V_y \subset W_i \cup W_j \subset \bigcup_{i \in I} W_i$. For all $(x, y) \in \bigcap_{i=1}^n W_i$, $(x, y) \in W_i$. For each W_i, we have a corresponding pair $(U_{i,x}, V_{i,y})$. Now consider $U = \bigcap_{i=1}^n U_{i,x} \in \mathcal{T}_X$ and $V = \bigcap_{i=1}^n V_{i,y} \in \mathcal{T}_Y$, we have $U \times V \subset W_i$, so $U \times V \subset \bigcap_{i=1}^n W_i$. The standard topology on $\mathbb{R}^n$ is the product topology of n copies of the standard topology on $\mathbb{R}$.

Theorem 1.1 (inverse function theorem)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. If F_{*p} is injective at some $p \in \mathbb{R}^n$, then there exists a neighborhood U of p such that $F|_U : U \to V$, where V is open, is a diffeomorphism.

We will discuss more on the proof of this theorem and its application later.

2 Frame Fields

Definition 2.1. Let $p, q \in \mathbb{R}^3$. The *Euclidean distance* from p to q is the number $d(p, q) = \|p - q\|$.

Definition 2.2. Let $v_p, w_p \in T_p(\mathbb{R}^3)$ be tangent vectors. The *dot product* of v_p and w_p is defined to be $v_p \cdot w_p = v \cdot w$.

Equivalently, the norm on every tangent space $T_p(\mathbb{R}^3)$ is the composition of the canonical isomorphism $T_p(\mathbb{R}^3) \to \mathbb{R}^3$ with the norm on $\mathbb{R}^3$.

Definition 2.3. A set of three pairwise orthogonal unit vectors tangent to $\mathbb{R}^3$ at p is called a *frame* at p.

By the definition, $\{e_1, e_2, e_3\}$ is a frame at p if and only if $e_i \in T_p(\mathbb{R}^3)$ and $e_i \cdot e_j = \delta_{i,j}$.

Proposition. Let $\{e_1, e_2, e_3\}$ be a frame at $p \in \mathbb{R}^3$. If $v_p \in T_p(\mathbb{R}^3)$, then $v_p = \sum (v \cdot e_i) e_i$.

Proof. Let $c_1, c_2, c_3 \in \mathbb{R}$ such that $\sum c_i e_i = 0$. For all $1 \le j \le 3$, $0 = (\sum c_i e_i) \cdot e_j = \sum c_i (e_i \cdot e_j) = c_j$, so $\{e_1, e_2, e_3\}$ is a basis of $T_p(\mathbb{R}^3)$. Rewrite $v_p = \sum a_i e_i$. For all $1 \le j \le 3$, $v_p \cdot e_j = \sum a_i e_i \cdot e_j = a_j$. Hence $v_p = \sum (v_p \cdot e_i) e_i$. □

For any frame $\{e_1, e_2, e_3\}$ at p and $a, b \in T_p(\mathbb{R}^3)$, if $a = \sum a_i e_i$ and $b = \sum b_i e_i$, we always have $a \cdot b = \sum a_i b_i$.

Definition 2.4. Let $\{e_1, e_2, e_3\}$ be a frame at $p \in \mathbb{R}^3$ with $e_i = (a_{i,1}, a_{i,2}, a_{i,3})_p$, then the *attitude matrix* of the frame is defined to be the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Consider the transpose $A^\top$ of A, for each column of $A^\top A$, we have $e_i e_i = 1$, so $A^\top A = I$ and A is orthogonal.

Definition 2.5. Let $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. The *cross product* of v_p and w_p, denoted $v_p \times w_p$, is the tangent vector

$$v_p \times w_p = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example. Let $v_p = (1, 0, -1)_p$ and let $w_p = (2, 2, -7)_p$, then $v_p \times w_p = 2U_1(p) + 5U_2(p) + 2U_3(p) = (2, 5, 2)_p$.

It is trivial that $\times$ is linear and $v_p \times w_p = -w_p \times v_p$.

Proposition. Let $v_p, w_p \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then $v_p \times w_p$ is orthogonal to both v_p and w_p. Moreover, $\|v_p \times w_p\|^2 = (v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2$.

Proof. Let $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p$. Then $(v_p \times w_p) \cdot v_p = v_1(v_2 w_3 - v_2 w_2) + v_2(v_3 w_1 - v_1 w_3) + v_3(v_1 w_2 - v_2 w_1) = 0$. Similarly, $(v_p \times w_p) \cdot w_p = 0$. We have $(v_p \cdot v_p)(w_p \cdot w_p) - (v_p \cdot w_p)^2 = (\sum v_i^2)(\sum w_i^2) - (\sum v_i w_i)^2 = \sum v_i^2 w_j^2 - \sum v_i^2 w_i^2 - 2\sum_{i<j} v_i w_i v_j - w_j = (v_2 w_3 - v_2 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2 = \|v_p \times w_p\|^2$. □

Definition 2.6. Let $\alpha : I \to \mathbb{R}^3$ be a curve. The *speed* of α at t is the tangent vector $\|\alpha'(t)\|$. The *arc length* of α from $t = a$ to $t = b$ is defined to be $\int_a^b \|\alpha'(t)\| \mathrm{d}t$.

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, then there exists a reparametrization β of α such that $\|\beta'\| = 1$.

Proof. Fix some $\alpha \in \mathbb{R}$ and consider the function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(t) = \int_a^t \|\alpha'(x)\| \mathrm{d}x$. Since α is regular, $\|\alpha'(x)\| > 0$ for all x. By the inverse function theorem, $s(t)$ has an inverse $t(s)$. Define $\beta(s) = \alpha(t(s))$, then $\|\beta'\| = \|(\mathrm{d}t/\mathrm{d}s)(s)\alpha'(t(s))\| = (\mathrm{d}t/\mathrm{d}s)(s)\|\alpha'(t(s))\| = (\mathrm{d}t/\mathrm{d}s)(s) \cdot (\mathrm{d}s/\mathrm{d}t)(t(s)) = 1$. □

Such reparametrization β of α is called the *arc-length reparametrization* of α.

Example. Consider the curve $\alpha : I \to \mathbb{R}^3$ defined by $\alpha(t) = (a \cos t, a \sin t, bt)$ for some $a, b \in \mathbb{R}$. We have $\|\alpha'\| = \sqrt{a^2 \sin^2 t + a^2 \sin^2 t + b^2} = c$, where $c^2 = a^2 + b^2$. Now measure the arc length from $t = 0$, then $s(t) = \int_0^c c \mathrm{d}u = ct$, so $t(s) = s/c$, the arc-length reparametrization is therefore $\beta(s) = \alpha(t(s)) = (a \cos(s/c), a \sin(s/c), bs/c)$.

Definition 2.7. A *vector field* Y on a curve $\alpha : I \to \mathbb{R}^3$ is a function $Y : I \to \bigsqcup_{p \in \mathrm{ran}(\alpha)} T_p(\mathbb{R}^3)$ such that for all $t \in I$, $Y(t) \in T_{\alpha(t)}(\mathbb{R}^3)$.

Fix $t \in I$, then we can rewrite $Y(t)$ as $\sum y_i(t)U_i(\alpha(t))$. The functions y_1, y_2, and y_3 are called the *Euclidean coordinate functions* on Y.

We define the addition, scalar multiplication, dot multiplication, and cross product on vector fields pointwisely. For a vector $Y = \sum y_i U_i$ on α, the derivative of Y is defined to be $Y' = \sum(\mathrm{d}y_i/\mathrm{d}t)U_i$. Let Y and Z be vector fields on a curve α. Fix t, rewrite $Y = (y_1, y_2, y_3)_{\alpha(t)}$ and $Z = (z_1, z_2, z_3)_{\alpha(t)}$. Consider Y as a function $Y_t : I \to T_{\alpha(t)}(\mathbb{R}^3) \approx \mathbb{R}^3$, then for $a, b \in \mathbb{R}$ and a differentiable $f : \mathbb{R} \to \mathbb{R}$, $(aY + bZ)' = aY' + bZ'$ and $(fY)' = (\mathrm{d}f/\mathrm{d}t)Y + Y'f$. We also have $(Y \cdot Z)' = (\sum y_i z_i)' = \sum y_i z_i' + \sum y_i' z_i = Y \cdot Z' + Y' \cdot Z$.

Definition 2.8. Let Y be a vector field on a curve $\alpha : I \to \mathbb{R}^3$. We say Y is *parallel* if for all $t \in I$, $Y(t)$ have the same vector part.

Proposition. A curve α is constant if and only if $\alpha' = 0$. A nonconstant curve α is a straight line if and only if $\alpha'' = 0$. A vector field Y on α is parallel if and only if $Y' = 0$.

Proof. Rewrite $\alpha : I \to \mathbb{R}^3$ as $\alpha = (\alpha_i)$. (i) The velocity $\alpha' = (\alpha_i')$, then $\alpha' = 0$ if and only if $\alpha_i' = 0$. Hence $\alpha' = 0$ if and only if α_i is a constant function. (ii) We have $\alpha'' = (\alpha_i'')$, so $\alpha'' = 0$ if and only if $\alpha_i = p_i t + q_i$ for some $p_i, q_i \in \mathbb{R}$. Hence $\alpha'' = 0$ if and only if $\alpha = pt + q$, where $p = (p_i)$ and $q = (q_1)$. (iii) Fix t and let $Y = (y_i)_{\alpha(t)}$, then $Y' = \sum y_i' U_i = 0$, which means y_i are constant functions. Hence Y is parallel if and only if $Y' = 0$. $\square$

Definition 2.9. Let $\beta : I \to \mathbb{R}^3$ be a curve. Then we call $T = \beta'$ the *unit tangent field* of β. The function $\kappa(s) = \|T'(s)\|$ is called the *curvature* of β.

Remark. We shall only consider the cases where $\kappa \neq 0$.

Definition 2.10. Let $\beta : I \to \mathbb{R}^3$ be a curve. Then we call $N = T'/\kappa$ the *principal normal vector field* of β. The vector field $B = T \times N$ on β is called the *binormal vector field* of β.

Proposition. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then the three vector fields T, N, and B on β are unit vector fields that are mutually orthogonal at each point.

Proof. Since $T = \beta'$, $\|T\| = \sqrt{T \cdot T} = 1$, then $(T \cdot T)' = T \cdot T' + T' \cdot T = 2T' \cdot T = 0$, so $T \cdot T' = 0$. For all $s \in \mathrm{dom}(\beta)$, $\|N\| = \|T'(s)\|/\kappa(s) = \|T'(s)\|/\|T'(s)\| = 1$. Since $B = T \times N$, B is orthogonal to T and N. Moreover, $\|B\|^2 = \|T\|\|N\| - (T \cdot N)^2 = 1 - 0 = 1$. $\square$

Definition 2.11. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then (T, N, B) is called the *Frenet frame field* of β.

Now we claim that B' can be written as a scalar multiple of N. Consider $B' = B' \cdot NN + B' \cdot TT + B' \cdot BB$. Differentiate $B \cdot T$, $B' \cdot T + T' \cdot B = 0$, then $B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0$. Similarly, $B \cdot B' = 0$, so $B' = B' \cdot NN$.

Definition 2.12. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$ and $\|\beta'\| = 1$. Then the *torsion* of β is a function $\tau : I \to \mathbb{R}$ such that $B' = -\tau N$.

Theorem 2.1 (Frenet formulas)**.** Let $\beta : I \to \mathbb{R}^3$ be a curve with $\kappa > 0$ and $\|\beta'\| = 1$. Then $T' = \kappa N$, $N' = -\kappa T + \tau B$, and $B' = -\tau N$.

Proof. Rewrite $N' = N' \cdot TT + N' \cdot NN + N' \cdot BB$. Differentiate $T \cdot N$, we have $T' \cdot N + N' \cdot T = 0$, so $N' \cdot T = -T' \cdot N = -(\kappa N) \cdot N = -\kappa$. Similarly, $N' \cdot B = -B' \cdot N = -(-\tau N) \cdot N = \tau$. Hence $N' = -\kappa T + \tau B$. $\square$

Example. Consider the curve $\beta : \mathbb{R} \to \mathbb{R}^3$ defined by $\beta(s) = (a\cos(s/c), a\sin(s/c), bs/c)$, where $a > 0$ and $c = \sqrt{a^2 + b^2}$. It is trivial that $\|\beta'\| = 1$. Here $T(s) = \beta'(s) = (-a\sin(s/c)/c, a\cos(s/c)/c, b/c)$, then $T'(s) = (-a\cos(s/c)/c^2, -a\sin(s/c)/c^2, 0)$, so $\kappa(s) = \|T'(s)\| = a/c^2 > 0$. We also have $N(s) = (-\cos(s/c), -\sin(s/c), 0)$. Now $\beta(s) = T(s) \times N(s) = (b\sin(s/c)/c, -b\cos(s/c)/c, a/c)$, then $B'(s) = (b/\cos(s/c)/c^2, b\sin(s/c)/c^2, 0)$, so $\tau(s) = -B'(s)/N(s) = (-b/\cos(s/c)/c^2, -b\sin(s/c)/c^2, 0)/(-\cos(s/c), -\sin(s/c), 0) = b/c^2$.

Definition 2.13. Let $p, q \in \mathbb{R}^3$ with $q \neq 0$. The *plane* through p orthogonal to q is the set $\{r \in \mathbb{R}^3 \mid (r-p)\cdot q = 0\}$. A curve $\beta : I \to \mathbb{R}^3$ is said to be a *plane curve* if $\operatorname{ran}(\beta) \subset P$, where P is a plane in $\mathbb{R}^3$.

Proposition. Let $\beta : I \to \mathbb{R}^3$ be a curve with $\|\beta'\| = 1$ and $\kappa > 0$. Then β is a plane curve if and only if $\tau = 0$.

Proof. ($\Rightarrow$) Let β be a plane curve, then there exists $p, q \in \mathbb{R}^3$ such that for all $s \in I$, $(\beta(s) - p)\cdot q = 0$. Consider q as a constant vector field, so $((\beta-p)\cdot q)' = (\beta-p)'\cdot q + q'\cdot(\beta-p) = \beta'\cdot q - p'\cdot q = \beta'\cdot q + q'\cdot\beta = \beta'\cdot q = \beta''\cdot q = 0$. Rewrite $q = q\cdot TT + q\cdot NN + q\cdot BB$, then $q = q\cdot BB$. We have $B = B\cdot BB = (B\cdot B)/(q\cdot B)q = (B\cdot B)/(q\cdot B)\|q\|$. Since $\|B\| = 1$, $B = \pm q/\|q\|$, which is a point, then $B' = 0$, hence $\tau = 0$. ($\Leftarrow$) Let $\tau = 0$, then $B' = 0$, so B is constant. Define $f : I \to \mathbb{R}$ by $f(s) = (\beta(s) - \beta(0))\cdot B$, then $\mathrm{d}f/\mathrm{d}s = T(s)\cdot 0 = 0$ and $f(0) = 0\cdot B = 0$, so $f(s) = 0$ for all s. Hence $\operatorname{ran}(\beta) \subset \{r \mid (r - \beta(0))\cdot B = 0\}$. □

Proposition. Let β be a curve in $\mathbb{R}^3$ with $\kappa > 0$, $\kappa' = 0$, $\|\beta'\| = 1$, and $\tau = 0$. Then β lies in a circle of radius $1/\kappa$.

Proof. Define $\gamma : I \to \mathbb{R}^3$ by $\gamma(s) = \beta(s) + N(s)/\kappa$, then $\gamma' = T + N'/\kappa$. By the Frenet formulas, $\gamma' = T + (-\kappa T + \tau B)/\kappa = T - T = 0$. Fix $t \in I$. For any $s \in I$, the distance $\|\beta(s) - (\beta(t) + N(t)/\kappa)\| = \|\beta(s) - (\beta(s) + N(s)/\kappa)\| = \|N(s)\|/\kappa = 1/\kappa$. □

We have shown the properties of curves with unit speed. Now given a curve $\alpha : I \to \mathbb{R}^3$ with $\|\alpha'\| \neq 1$, let $\overline{\alpha}$ be the arc-length reparametrization of α, so $\|\overline{\alpha}'\| = 1$. Let $\overline{T}$, $\overline{k}$, $\overline{N}$, $\overline{B}$, and $\overline{\tau}$ be the corresponding functions of $\overline{\alpha}$. Define the T, λ, N, B, and τ of α to be those of $\overline{\alpha}$. We denote the speed of α by v.

Theorem 2.2 (Frenet formulas)**.** Let α be a regular curve on $\mathbb{R}^3$ with $\kappa > 0$, then $T' = \kappa v N$, $N' = -\kappa v T + \tau v B$, and $B' = -\tau v N$.

Proof. Apply the Frenet formulas on $\overline{\alpha}$, then $\overline{T}' = \overline{\kappa}\overline{N}$. Since $T' = \overline{T}'\mathrm{d}s/\mathrm{d}t = \overline{T}'v$, $T' = \overline{\kappa}v\overline{N}$. Similarly, $N' = -\kappa v T + \tau v B$ and $B' = -\tau v N$. □

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be regular. Then $\alpha' = vT$ and $\alpha'' = (\mathrm{d}v/\mathrm{d}t)T + \kappa v^2 N$.

Proof. We have $\alpha' = \overline{\alpha}'\mathrm{d}s/\mathrm{d}t = v\overline{T} = vT$, then $\alpha'' = (\mathrm{d}v/\mathrm{d}t)T + T'v = (\mathrm{d}v/\mathrm{d}t)T + (\kappa v N)\kappa = (\mathrm{d}v/\mathrm{d}t)T + v^2\kappa N$. □

Proposition. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve. Then $T = \alpha'/\|\alpha'\|$, $N = B \times T$, $B = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$, $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3$, and $\tau = (\alpha' \times \alpha'')\cdot\alpha'''/\|\alpha' \times \alpha''\|^2$.

Proof. Since $\|\alpha'\| = v$, $\alpha'/\|\alpha'\| = vT/v = T$. We have $\alpha' \times \alpha'' = vT \times ((\mathrm{d}v/\mathrm{d}t)T + \kappa v^2 N) = vT \times (\mathrm{d}v/\mathrm{d}t)T + vT \times \kappa v^2 N$, since $T \times T = 0$, $\alpha' \times \alpha'' = \kappa v^3 T \times N = \kappa v^3 B$. The norm $\|\alpha' \times \alpha''\| = \|\kappa v^3\| = \kappa v^3$, hence $B = \kappa v^3 B/(\kappa v^3) = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|$. Consider a lemma: for $w_p, v_p, u_p \in T_p(\mathbb{R}^3)$, $(u \times v) \times w = (u\cdot w)v - (v\cdot w)u$. (lemma) Rewrite $u = \sum u_i$, $v = \sum v_i$, and $w = \sum w_i$, then $(u\cdot w)v - (v\cdot w)u = \left(\sum u_i w_j\right)\sum v_i - \left(\sum v_i w_j\right)\sum u_i = u \times v \times w$. By the lemma, $B \times T = T \times N \times T = (T\cdot T)N - (N\cdot T)T = N - 0 = N$. We have shown $\|\alpha' \times \alpha''\| = \kappa v^3$, so $\|\alpha' \times \alpha''\|/\|\alpha'\|^3 = \kappa v^3/v^3 = \kappa$. Differentiate α'', then $\alpha''' = (\mathrm{d}v/\mathrm{d}t)T' + (\mathrm{d}^2v/\mathrm{d}t^2)T + 2\kappa(\mathrm{d}v/\mathrm{d}t)vN + N'\kappa v^2$. Since $B\cdot T = B\cdot N = 0$, $\kappa v^3 B\cdot\alpha''' = \kappa v^3 B\cdot((\mathrm{d}v/\mathrm{d}t)T' + N'\kappa v^2)$, by the Frenet formulas, T' term becomes 0 and N' term becomes $\tau v B$. Now $(\alpha' \times \alpha'')\cdot\alpha''' = \kappa^2 v^6 \tau B\cdot B = \kappa^2 v^6 \tau$. Hence $(\alpha' \times \alpha'')\cdot\alpha'''/\|\alpha' \times \alpha''\|^2 = \kappa^2 v^6 \tau/(\kappa^2 v^6) = \tau$. □

Example. Consider the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$, then $\alpha' = (3 - 3t^2, 6t, 3 + 3t^2)$, $\alpha'' = (-6t, 6, 6t)$, and $\alpha''' = (-6, 0, 6)$. We have $\alpha'\cdot\alpha' = 18(1+t^2)^2$, so $\|\alpha'\| = 3\sqrt{2}(1+t^2)$ and $T = (1-t^2, 2t, 1+t^2)/(\sqrt{2}(1+t^2))$. The cross product $\alpha' \times \alpha'' = (18t^2 - 18, -36t, 18t^2 + 18) = 18(t^2 - 1, -2t, t^2 + 1)$ and its norm $\|\alpha' \times \alpha''\| = 18\sqrt{2}(1+t^2)$, hence $B = (t^2 - 1, -2t, t^2 + 1)/(\sqrt{2}(1+t^2))$. Now $N = B \times T = (-2t, 1 - t^2, 0)/(1+t^2)$. By our computation, $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3 = 1/(3(1+t^2)^2)$ and $\tau = (\alpha' \times \alpha'')\cdot\alpha'''/\|\alpha' \times \alpha''\|^2 = 18(t^2 - 1, -2t, t^2 + 1)\cdot 6(-1, 0, 1)/(18\sqrt{2}(1+t^2))^2 = 216/(648(1+t^2)^2) = 1/(3(1+t^2)^2)$.

Definition 2.14. Let $\alpha : I \to \mathbb{R}^3$ be regular. We say α is a *cylindrical helix* if there exists $u \in \mathbb{R}^3$ such that for all $t \in I$, $T(t) \cdot u = \cos\theta$, where θ is a constant angle.

If α has $\|\alpha'\| \neq 1$, take the arc-length reparametrization $\overline{\alpha}$ of α, by our definition, (T, N, B), κ, and τ are all invariant under this reparametrization, so it suffices to consider a curve with unit speed.

Proposition. A regular curve α on $\mathbb{R}^3$ with $\kappa > 0$ is a cylindrical helix if and only if τ/κ is constant.

Proof. Let α be a curve with $\|\alpha'\| = 1$. ($\Rightarrow$) Let α be a cylindrical helix with unit vector u and constant angle θ, then $T \cdot u = \cos\theta$, so $0 = (T \cdot u)' = T' \cdot u = \kappa N \cdot u$. Since $\kappa > 0$, $u \cdot N = 0$, then $u = (u \cdot T)T + (u \cdot B)B = \cos\theta T + (u \cdot B)B$. Since $\|u\| = 1$, $u \cdot B = \sin\theta$. Now $0 = u' = \cos\theta\kappa N - \sin\theta\tau N$, so $\cos\theta\kappa = \sin\theta\tau$, which implies $\tau/\kappa = \cot\theta$. ($\Leftarrow$) Let τ/κ be a constant and let $\cot\theta = \tau/\kappa$ for some angle θ. Consider $U = \cos\theta T + \sin\theta B$, here $U' = \cos\theta T' + \sin\theta B' = (\cos\theta\kappa - \sin\theta\tau)N$, since $\tau/\kappa = \cot\theta$, $U' = 0$. For all $s_1, s_2 \in I$, $U(s_1) = U(s_2)$, so pick $U(0) = u$. Now $T \cdot u = T \cdot U = T \cdot (\cos\theta T + \sin\theta B) = \cos\theta$. □

Let $\tau = 0$, $\kappa > 0$, and $\kappa' = 0$ for some α with $\|\alpha'\| = 1$, then α lies in a circle of radius $1/\kappa$. Consider a circle α in $\mathbb{R}^3$, define $R : I \to \mathbb{R}^3$ by $R(s) = \alpha(s) - c$, where c is the center. We have $(R \cdot R)' = 2R' \cdot R = 2T \cdot R = 0$. Since $T \cdot N = 0$, $R = nN$. We have $T = R' = n'N + N'n = n'N + n(-\kappa T + \tau B)$, then $(-\kappa n - 1)T + \tau n B + n'N = 0$, so $-\kappa n - 1 = \tau n = n' = 0$. Since $n = -1/\kappa$, $n \neq 0$, then $\tau = 0$. Since $n' = 0$, n is a constant, then κ is a constant.

Definition 2.15. Let $u \in \mathbb{R}^3$ be a point with $\|u\| = 1$ and let V be a plane orthogonal to u. The *projection map* of $p \in \mathbb{R}^3$ onto V is a function $\mathrm{proj} : \mathbb{R}^3 \to V$ defined by $\mathrm{proj}(p) = p - (p \cdot u)u$.

Let α be a curve and let u be a unit vector. Let β be the curve $\beta = \mathrm{proj} \circ \alpha$. Then $\beta' = \alpha' - (\alpha' \cdot u)u$, $\beta'' = \alpha'' - (\alpha'' \cdot u)u$, and $\beta''' = \alpha''' - (\alpha''' \cdot u)u$. We have $\beta' \cdot u = \alpha' \cdot u - (\alpha' \cdot u)(u \cdot u) = 0$, similarly, $\beta'' \cdot u = 0 = \beta''' \cdot u$, then $(\beta' \times \beta'') \cdot u = (\beta' \cdot u)\beta'' - (\beta'' \cdot u)\beta' = 0$, it suffices to rewrite $\beta' \times \beta'' = nu$ for some $n \in \mathbb{R}$. Now $(\beta' \times \beta'') \cdot \beta''' = nu \cdot \beta''' = 0$. Hence every curve under a projection map has $\tau = 0$.

Definition 2.16. Let $\alpha : I \to \mathbb{R}^3$ be a cylindrical helix with axis direction given by the unit vector u. We call α a *circular helix* if for every plane orthogonal to u, the projection of α onto that plane is a circle.

Let α be a circular helix, then $\tau/\kappa = 0$ and there exists a unit vector u such that $T \cdot u = \cos\theta$. Consider the projection $\beta = \mathrm{proj} \circ \alpha$ with $\beta = \alpha - (\alpha \cdot u)u$. We have $\beta' = T - (T \cdot u)u = T - \cos\theta u$ and $\beta'' = T' - (T' \cdot u)u = \kappa N - (\kappa N \cdot u)u = \kappa N$, then $\beta' \times \beta'' = (T - \cos\theta u) \times \kappa N = \kappa(T - \cos\theta u) \times N = \kappa(B - \cos\theta(u \times N))$. Since $u = \cos\theta T + \sin\theta B$, $u \times N = \cos\theta(T \times N) + \sin\theta(B \times N) = \cos\theta B - \sin\theta T$, then $\|\beta' \times \beta''\| = \kappa\sqrt{\sin^4\theta + \cos^2\theta\sin^2\theta} = \kappa\sin\theta\sqrt{\sin^2\theta + \cos^2\theta} = \kappa\sin\theta$. Similarly, $\|\beta'\| = \sin\theta$. The curvature $\kappa_\beta = \kappa/\sin^2\theta$, and κ_β is a constant if and only if κ is a constant. Hence $\kappa, \tau > 0$ and $\kappa' = 0 = \tau'$ if and only if α is a circular helix.

Definition 2.17. Let W be a vector field on $\mathbb{R}^3$ and let $v \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then the *covariant derivative* of W with respect to v is the tangent vector $\nabla_v W = W(p + tv)'(0)$ at p.

Proposition. Let $W = \sum w_i U_i$ be a vector field on $\mathbb{R}^3$ and let $v \in T_p(\mathbb{R}^3)$ for some $p \in \mathbb{R}^3$. Then $\nabla_v W = \sum v[w_i]U_i(p)$.

Proof. We have $W(p + tv) = \sum w_i(p + tv)U_i$. Since $(\mathrm{d}/\mathrm{d}t)w_i(p + tv)$ at $t = 0$ is $v[f]$, $\nabla_v W = \sum v[w_i]U_i(p)$. □

Proposition. Let $v, w \in T_p(\mathbb{R}^3)$. Let Y and Z be vector fields on $\mathbb{R}^3$. For any $f : \mathbb{R} \to \mathbb{R}$, the following holds.

1. $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$.
2. $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$.
3. $\nabla_v(fY) = v[f]Y(p) + f(p)\nabla_v Y$.
4. $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$.

Proof. Rewrite $Y = \sum y_i U_i$ and $Z = \sum z_i U_i$. (i) We have $\nabla_{av+bw} Y = \sum (av+bw)[y_i] U_i(p) = a \sum v[y_i] U_i + b \sum w[y_i] U_i = a\nabla_v Y + b\nabla_w Y$. (ii) Similarly, $\nabla_v (aY+bZ) = \sum v[ay_i + bz_i] U_i = a\sum v[y_i]U_i + b\sum v[z_i]U_i = a\nabla_v Y + b\nabla_v Z$. (iii) Similarly, $\nabla_v(fY) = \sum v[fy_i]U_i = \sum (v[f]y_i + fv[y_i])U_i = v[f]Y(p) + f(p)\nabla_v Y$. (iv) We have $Y\cdot Z = \sum y_i z_i$, then $v[Y\cdot Z] = \sum v[y_i] z_i U_i + \sum v[z_i] y_i U_i = \nabla_v Y \cdot Z + Y \cdot \nabla_v Z$. □

Let V and $W = \sum w_i$ be vector fields. We define $\nabla_V W$ at some $p \in \mathbb{R}^3$ to be $\nabla_{V(p)} W$, hence $\nabla_V W = \sum V[w_i] U_i$.

Proposition. Let V, W, Y, and Z be vector fields on $\mathbb{R}^3$. For all functions $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$. Then the following properties hold.

1. $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$.
2. $\nabla_V (aY+bZ) = a\nabla_V Y + b\nabla_V Z$.
3. $\nabla_V (fY) = V[f]Y + f\nabla_V Y$.
4. $V[Y\cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$.

Proof. Since $\nabla_V W(p) = \nabla_{V(p)} W$, those properties are direct consequences of the previous proposition. □

Definition 2.18. Let E_1, E_2, and E_3 be vector fields on $\mathbb{R}^3$. We say $\{E_1, E_2, E_3\}$ is a *frame field* on $\mathbb{R}^3$ if $E_i \cdot E_j = \delta_{i,j}$.

If $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$, then for all $p \in \mathbb{R}^3$, $\{E_1(p), E_2(p), E_3(p)\}$ is trivially a frame at p.

Example. Consider a cylindrical coordinate system with coordinates (r, θ, z). Define $E_1 = \cos\theta U_1 + \sin\theta U_2$, $E_2 = -\sin\theta U_1 + \cos\theta U_2$, and $E_3 = U_3$. For any p, $E_1(p)\cdot E_2(p) = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$, $E_1(p)\cdot E_3(p) = 0 = E_2(p)\cdot E_3(p)$, $E_1(p)\cdot E_1(p) = \cos^2\theta + \sin^2\theta = E_2(p)\cdot E_2(p) = 1 = E_3(p)\cdot E_3(p) = 1$. Hence $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$, known as the *cylindrical frame field*.

Example. Consider a spherical coordinate system with coordinates (ρ, θ, φ). Define $F_1 = \cos\varphi E_1 + \sin\varphi E_3$, $F_2 = E_2$, and $F_3 = -\sin\varphi E_1 + \cos\varphi E_3$, where $\{E_1, E_2, E_3\}$ is the cylindrical frame field. It is trivial that $F_1\cdot F_1 = F_2\cdot F_2 = F_3\cdot F_3 = 1$ and $F_1\cdot F_2 = 0 = F_2\cdot F_3$. Now $F_1\cdot F_3 = \cos\varphi E_1 \cdot (-\sin\varphi E_1 + \cos\varphi E_3) + \sin\varphi E_3 \cdot (-\sin\varphi E_1 + \cos\varphi E_3) = -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = 0$. Hence $\{F_1, F_2, F_3\}$ is a frame field on $\mathbb{R}^3$, called the *spherical frame field*.

Proposition. Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$. If V is a vector field on $\mathbb{R}^3$, then $V = \sum f_i E_i$, where $f_i = V\cdot E_i$. If $V = \sum f_i E_i$ and $W = g_i E_i$, then $V\cdot W = \sum f_i g_i$.

Proof. (i) For any point p, $\{E_1(p), E_2(p), E_3(p)\}$ is a frame at p, so $V(p) = \sum (V(p)\cdot E_i(p))E_i(p)$. (ii) For any p, $\{E_1(p), E_2(p), E_3(p)\}$ is a frame at p, so $V\cdot W = \sum f_i g_i$ as we shown before. □

Definition 2.19. Let V be a vector field on $\mathbb{R}^3$ and let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$. The functions $f_i = V\cdot E_i$ are called the *coordinate functions* of V with respect to the frame field.

Definition 2.20. Let $\{E_1, E_2, E_3\}$ be a frame field. For any $v_p \in T_p(\mathbb{R}^3)$, $\nabla_v E_i = \sum \omega_{i,j} E_j(p)$, where $\omega_{i,j}(v) = \nabla_v E_i \cdot E_j(p)$. These $\omega_{i,j}$ are called the *connection forms* of the frame field.

Proposition. Each connected form is a 1-form and $\omega_{i,j} = -\omega_{j,i}$.

Proof. Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$, for any $v_p, w_p \in T_p(\mathbb{R}^3)$ and $a, b \in \mathbb{R}$, $\omega_{i,j}(av+bw) = \nabla_{av+bw} E_i \cdot E_j(p) = a\nabla_v E_i \cdot E_j(p) + b\nabla_w E_i \cdot E_j(p)$, so $\omega_{i,j}$ is a 1-form. Moreover, $0 = v[E_i \cdot E_j] = \nabla_v E_i \cdot E_j(p) + E_i(p)\cdot\nabla_v E_j = \omega_{i,j}(v) + \omega_{j,i}(v)$, hence $\omega_{i,j} = -\omega_{j,i}$. □

Let $i = j$, then $\omega_{i,i} = -\omega_{i,i}$, so $\omega_{i,i} = 0$.

Proposition. Let $\omega_{i,j}$ be connection forms of a frame field $\{E_1, E_2, E_3\}$. For any vector field V on $\mathbb{R}^3$, $\nabla_V E_i = \sum_j \omega_{i,j}(V)E_j$.

Proof. Fix i. For all $p \in \mathbb{R}^3$, $V(p) \in T_p(\mathbb{R}^3)$, then the equation is a direct consequence of the definition. $\square$

Let $\{E_1, E_2, E_3\}$ be a frame field, E_i can be rewriten by the natural frame field $E_i = a_{i,1}U_1 + a_{i,2}U_2 + a_{i,3}U_3$.

Proposition. Let A be the attitude matrix of the frame field $\{E_1, E_2, E_3\}$. Let $\omega = (\omega_{i,j})$ be a matrix of the connected forms. Then $\omega = \mathrm{d}AA^\top$, where $\mathrm{d}A$ is the matrix $(\mathrm{d}a_{i,j})$.

Proof. Equivalently, we shall show $\omega_{i,j} = \sum_k (\mathrm{d}a_{i,k})a_{j,k}$. For all $v_p \in T_p(\mathbb{R}^3)$, $\nabla_v E_i = \sum v[a_{i,k}]U_k(p)$ and $E_j(p) = \sum a_{j,k}U_i(p)$, so $\omega_{i,j} = \nabla_v E_i \cdot E_j(p) = \sum_k (\mathrm{d}a_{i,k})a_{j,k}$. $\square$

Definition 2.21. Let $\{E_1, E_2, E_3\}$ be a frame field. The *dual 1-forms* of the frame field $\{E_1, E_2, E_3\}$ are defined to be $\theta_i(v) = v \cdot E_i(p)$ for all $v_p \in T_p(\mathbb{R}^3)$.

For some $v_p, w_p \in T_p(\mathbb{R}^3)$, $\theta_i(av + bw) = (av + bw) \cdot E_i(p) = av \cdot E_i(p) + bw \cdot E_i(p)$, hence θ_i is a 1-form.

Proposition. Let θ_1, θ_2, and θ_3 be dual 1-forms of a frame field $\{E_1, E_2, E_3\}$. For any 1-form φ on $\mathbb{R}^3$, $\varphi = \sum \varphi(E_i)\theta_i$.

Proof. Let V be a vector field on $\mathbb{R}^3$. Then we have $\varphi(V) = \varphi(\sum V \cdot E_i E_i) = \sum \varphi(\theta_i(V)E_i) = \sum \varphi(E_i)\theta_i(V) = (\sum \varphi(E_i)\theta_i)(V)$, hence $\varphi = \sum \varphi(E_i)\theta_i$. $\square$

If $\{E_i\}$ is a frame field on the natural frame field for any $v_p \in T_p(\mathbb{R}^3)$, $\mathrm{d}x_i(v) = v_i = v \cdot U_i(p) = \theta_i$. By the previous proposition, $\theta_i = \sum_j \theta_i(U_j)\mathrm{d}x_j = \sum E_i \cdot U_j = \sum a_{i,j}\mathrm{d}x_j$.

Theorem 2.3 (Cartan structural equartions)**.** Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$ with dual forms θ_i and connected forms $\omega_{i,j}$. The exterior derivatives of these forms satisfy $\mathrm{d}\theta_i = \sum_j \omega_{i,j} \wedge \theta_j$ and $\mathrm{d}\omega_{i,j} = \sum_k \omega_{i,k} \wedge \omega_{k,j}$.

Proof. Consider ω and θ as matrices of connection forms and dual forms. We define the wedge product of two matrices as their matrix product, except that the multiplication of entries is replaced by the wedge product. This wedge product follows the following lemma: let M and N be matrices, if the entries of M and N are functions, then $\mathrm{d}(M \wedge N) = \mathrm{d}M \wedge N + M \wedge \mathrm{d}N$, if the entries are 1-forms, then $\mathrm{d}(M \wedge N) = \mathrm{d}M \wedge N - M \wedge \mathrm{d}N$. (lemma) We have $(\mathrm{d}(M \wedge N))_{i,j} = \mathrm{d}(M \wedge N)_{i,j} = \sum_k \mathrm{d}M_{i,k} \wedge N_{k,j} + \sum_k M_{i,k} \wedge \mathrm{d}N_{k,j}$. Similarly, we can prove the case of 1-forms. 👻 (i) Let x_i be the natural coordinates of $\mathbb{R}^3$ and define ξ to be the matrix $(x_{i,1})$. Here ξ is a 0-form, so $\mathrm{d}(\mathrm{d}\xi) = 0$, then $\mathrm{d}\theta = \mathrm{d}(A \wedge \mathrm{d}\xi) = \mathrm{d}A \wedge \mathrm{d}\xi = \mathrm{d}A \wedge (A^\top A \wedge)\mathrm{d}\xi = (\mathrm{d}AA^\top) \wedge (A\mathrm{d}\xi) = \omega \wedge \theta$. (ii) We have $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}AA^\top) = \mathrm{d}(\mathrm{d}A) \wedge A^\top - \mathrm{d}A \wedge \mathrm{d}A^\top = -\mathrm{d}A \wedge \mathrm{d}A^\top$. Notice that $0 = \mathrm{d}(A^\top A) = \mathrm{d}A^\top \wedge A + A^\top \wedge \mathrm{d}A$, then $\mathrm{d}A^\top = -A^\top \wedge \mathrm{d}A \wedge A^\top$, which implies $\mathrm{d}\omega = -\mathrm{d}A \wedge (-A^\top \wedge \mathrm{d}A \wedge A^\top) = (\mathrm{d}AA^\top) \wedge (\mathrm{d}AA^\top) = \omega \wedge \omega$. $\square$

3 Euclidean Geometry

Definition 3.1. A mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an *isometry* of $\mathbb{R}^3$ if for all $p, q \in \mathbb{R}^3$, $d(F(p), F(q)) = d(p, q)$.

Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(p) = p + a$ for a fixed $a \in \mathbb{R}^3$. For all $p, q \in \mathbb{R}^3$, $d(T(p), T(q)) = \|p + a - (q - a)\| = \|p - q\| = d(p, q)$. Hence T is an isometry, called the *translation* by a.

Example. Fix an angle θ. For all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, the *rotation* of $\mathbb{R}^3$ around z-axis is the map $C : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $C(p) = (p_1 \cos\theta - p_2 \sin\theta, p_1 \sin\theta + p_2 \cos\theta, p_3)$. Now

Proposition. If F and G are isometries on $\mathbb{R}^3$, then $G \circ F$ is an isometry.

Proof. For all $p, q \in \mathbb{R}^3$, $d(G \circ F(p), G \circ F(q)) = d(F(p), F(q)) = d(p, q)$. $\square$

Proposition. If T and S are translations, then $T \circ S = S \circ T$ is a translation. Every translation has an inverse translation. Given two points $p, q \in \mathbb{R}^3$, there exists a unique translation T such that $T(p) = q$.

Proof. For all $p \in \mathbb{R}^3$, let $T(p) = p + a$ and let $S(p) = p + b$. Then $(TS)(p) = T(p+b) = p + b + a = S(p+a) = (ST)(p)$. Define $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ to be $T^{-1}(p) = p - a$, then T^{-1} is the inverse of T. Define $T(p) = p + (q - p)$, here $q - p$ is a fixed point, so $T(p) = q$. $\square$

Definition 3.2. An *orthogonal transformation* $C : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that for all $p, q \in \mathbb{R}^3$, $C(p) \cdot C(q) = p \cdot q$.

Proposition. Any orthogonal transformation is an isometry.

Proof. Let C be an orthogonal transformation. For all $p \in \mathbb{R}^3$, $\|C(p)\| = \sqrt{C(p) \cdot C(p)} = \sqrt{p \cdot p} = \|p\|$. For all $a, b \in \mathbb{R}^3$, $d(C(a), C(b)) = \|C(a) - C(b)\| = \|C(a - b)\| = \|a - b\| = d(a, b)$. $\square$

Proposition. If F is an isometry of $\mathbb{R}^3$ with $F(0) = 0$, then F is an orthogonal transformation.

Proof. For all $p \in \mathbb{R}^3$, $\|p\| = d(0, p) = d(F(0), F(p)) = d(0, F(p)) = \|F(p)\|$. Let $q \in \mathbb{R}^3$, now $\|F(p) - F(q)\| = \|p - q\|$, this implies $(F(p) - F(q)) \cdot (F(p) - F(q)) = (p - q) \cdot (p - q)$. Simplify the equation, we obtain $\|F(p)\|^2 - 2F(p) \cdot F(q) + \|F(q)\|^2 = \|p\|^2 - 2p \cdot q + \|q\|^2$, hence $p \cdot q = F(p) \cdot F(q)$. $\square$

4 Calculus on a Surface

5 Shape Operators

6 Geometry of Surfaces in $\mathbb{R}^3$

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