A Collection of Quick Tours

Hassium

1 Derived Categories

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This note is based on my presentation given in Math 501.

Definition 1.1. Let C be a category. A category D is a *subcategory* of C if the Ob(D) and the $Hom_D(X,Y)$ are subcollections of Ob(C) and $Hom_C(X,Y)$, respectively, for all objects X and Y in C. The subcategory D is said to be *full* if for all objects X and Y in D, $Hom_D(X,Y)$ is exactly $Hom_C(X,Y)$.

Definition 1.2. Let C be a category. A morphism $f: A \to B$ is a monomorphism if for all $g, h: C \to A$, $f \circ g = f \circ h$ implies g = h. A morphism $f: A \to B$ is called an epimorphism if for all $i, j: B \to D$, $i \circ f = j \circ f$ implies i = j.

Definition 1.3. An *initial object* 0 of a category C is an object in C such that for any object A, there is a unique morphism $0 \to A$. A *terminal object* 1 of a category C is an object of C such that for any object B, there is a unique morphism $A \to 1$. We say an object is the *zero object* if it is both an initial object and a terminal object.

Definition 1.4. Let C be a category with zero object 0. The zero morphism $0_{A,B}$ between objects A and B is the unique morphism that factors through 0.

Definition 1.5. Let C be an object and let $f: X \to Y$ be a morphism in C. An object $\ker(f)$ is said to be the *kernel* of f if for every object Z and $h: Z \to X$ such that $f \circ g = 0$, where 0 is the zero morphism, there is a unique morphism $\varphi: Z \to \ker(f)$ such that $h = p \circ \varphi$.

Definition 1.6. Let F and G be functors between C and D. A natural transformation η from F to G is a family of morphisms such that

- 1. For all object X in C, there exists a morphism $\eta_X: F(X) \to G(X)$, called the component at X.
- 2. For all $f: X \to Y$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A functor $F: \mathbb{C} \to \mathbb{D}$ is a natural functor if for all $G: \mathbb{C} \to \mathbb{D}$, there exists a natural transformation $\eta: F \to G$.

Definition 1.7. A category C is an additive category if the following properties hold.

- 1. C has an zero object 0.
- 2. Every hom-set of C is an abelian group.
- 3. Composition of morphisms is bilinear.
- 4. C admits finite coproduct.

Definition 1.8. A functor $f: A \to B$ between additive categories is an additive functor if it preserves the finite coproduct.

Definition 1.9. A category C is a *pre-abelian category* if C is an additive category and every morphism in C has a kernel and a cokernel. A pre-abelian category C is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.10. Let A be an additive category. A cochain complex is a collection X^* of objects X^n in A with maps $d^n: X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n: X^n \to Y^n$ such that $f^{n+1}d^n_X = d^n_Y f^n$. This defines the category of cochain complexes, denoted C(A).

Definition 1.11. Let X^* and Y^* be cochain complexes. The maps $f, g: X^* \to Y^*$ are homotopic if there exists a collection of maps $k^n: X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 1.12. Let A be an additive category. The objects of the homotopy category, denoted K(A), are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Let A be an additive category and K(A) be the homotopy category. Define the finite coproduct to be the direct sum $(X^* \oplus Y^*)^n = X^n \oplus Y^n$. Define the composition to be pairwise addition. Since composition in A is bilinear, it is bilinear in K(A). It is not hard to check that K(A) is additive.

Definition 1.13. A *triangulated category* is an additive category C with:

- 1. an additive automorphism $T: \mathbb{C} \to \mathbb{C}$ called the translation functor,
- 2. a collection of triangles (X, Y, Z, u, v, w), where X, Y, and Z are objects of C and $u: X \to Y, v: Y \to Z$, and $w: Z \to T(X)$ are morphisms,
- 3. morphisms $(f, g, h): (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow$$

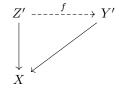
The data subject to the following rules:

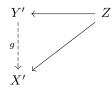
- 1. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle and for all $f: A \to B$, there exists a triangle (A, B, C, f, g, h).
- 2. A sextuple (A, B, C, f, g, h) is a triangle if and only if (B, C, T(A), g, h, -T(f)) is a triangle.
- 3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f: X \to X'$ and $g: Y \to Y'$ be morphisms in C that commutes with u and u', then there exists $h: Z \to Z'$ such that (f, g, h) is a morphism between triangles.

$$X \xrightarrow{u} Y \xrightarrow{} Z \xrightarrow{} T(X)$$

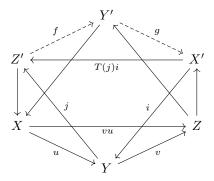
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

4. Let (X,Y,Z',u,j,-), (Y,Z,X',v,-,i), and (X,Z,Y',vu,-,-) be triangles, then there exists morphisms $f:Z'\to Y'$ and $g:Y'\to X'$ such that (Z',Y',X',f,g,T(j)i) is a triangle and the following diagram commutes.





This is called the octohedral axiom.



Definition 1.14. Let C and D be additive categories. An additive functor $f: C \to D$ is a *covariant* ∂ -functor if it commutes with the translation functor and it preserves triangles. A *contravariant* ∂ -functor takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy categories can be triangulated.

Proposition. Let $T(X)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u: X \to Y$, we define Z to be the mapping cone $T(X) \oplus Y$. The differential maps in Z are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v: Y \to Z$ and $w: Z \to X$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u: X \to Y$. This defines a triangulated category.

Proof. (TR1) Take $\mathrm{id}_X:X\to X$, then $Z=T(X)\oplus X$ and the differentials are $d_Z^n=\begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}}\\ 0 & d_X^n \end{pmatrix}$. Define $k^n:Z^n\to Z^{n-1}$ by $k^n((a,b))=(0,a)$. We have $(k^{n+1}d_Z^n+d_Z^{n-1}k^n)((a,b))=(a,b)$, hence $\mathrm{id}_Z\sim 0$. The sextuple $(X,X,0,\mathrm{id}_X,0,0)$ is the triangle induced by $\mathrm{id}_X:X\to X$. \square (TR2) Let (X,Y,Z,u,v,w) be a triangle. Denote $T(Y)\oplus Z$ by A. Consider the sextuple (Y,Z,A,v,s,t). Notice that $(T(X))^i=X^{i+1}$, so we pick $x^{i+1}\in (T(X))^i$. Define $\theta^i:(T(X))^i\to A^i$ by $x^{i+1}\mapsto (-u^{i+1}(x^{i+1}),x^{i+1},0)$.

Define $h^i: Z^i \to A^{i-1}$ by $h^i(x^{i+1}, y^i) = (y^i, 0, 0)$. Then two maps from Z to A are homotopic. Let $\psi: A \to T(X)$ be the natural projection, then $\psi \circ \theta = \mathrm{id}_{T(X)}$. We have $\theta \circ \psi^i: A^i \to A^i$ with $\theta \circ \psi^i: (y^{i+1}, x^{i+1}, y^i) \mapsto (-f^{i+1}(x^{i+1}), x^{i+1}, 0)$. Pick $k^i: A^i \to A^{i-1}$ to be the map $(y^{i+1}, x^{i+1}, y^i) \mapsto (y^i, 0, 0)$, then $\theta \circ \psi \sim \mathrm{id}_A$. \square (TR3) We have $Z = T(X) \oplus Y$ and $Z' = T(X') \oplus Y'$, so the complementing morphism is exactly $h = T(f) \oplus g: Z \to Z'$. \square (TR4) We say an exact sequence of complexes $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is *semi-split* if for all i, there exists a morphism $w^i: Z^i \to Y^i$ such that $v^i w^i = \mathrm{id}_{Z^i}$. Consider a lemma: any semi-split exact sequence in $\mathbb{C}(A)$ can induce a triangle (X, Y, Z, u, v, -) in $\mathbb{K}(A)$. (lemma)

Definition 1.15. Let C be a category and let S be a collection of morphisms in C, then the *localization* of C with respect to S is a category C_S together with a functor $Q: C \to C_S$ such that

- 1. Q(s) is an isomorphism for every $s \in S$.
- 2. Any functor $F: \mathbb{C} \to \mathbb{D}$ such that F(s) is an isomorphism for all $s \in S$ factors uniquely through Q.

If $Q': C \to C_S$ is another functor satisfies these properties, then the two localizations are considered to be equivalent.

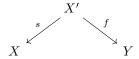
Proposition. The localization C_S exists.

Proof. Let $\mathrm{Ob}(\mathtt{C_S}) = \mathrm{Ob}(\mathtt{C})$ and let Q be the identity on objects. For all $s \in S$, introduce some variable x_s , then construct an oriented graph Γ as follows: let the vertices be objects in \mathtt{C} , let the edges be morphisms in \mathtt{C} union $\{x_s\}$, the edge has the direction $X \to Y$ if there exists a morphism $X \to Y$ in \mathtt{C} , and the edge x_s has the same vertices as the edges but the opposite orientation. Define a morphism to be an equivalence class of paths that share the same initial and terminal in Γ . Two paths are equivalent if they can be joined by a chain of elementary equivalences that two consecutive arrows in a path can be replaced by their composition and arrows $X \xrightarrow{s} Y \xrightarrow{s_x} X$ can be replaced by $X \xrightarrow{id_X} X$. The composition of morphisms is induced by the conjunction of paths. The functor Q takes a morphism to a corresponding path of length 1. The inverse of a morphism $s \in S$ is x_s .

Definition 1.16. Let A be an abelian category. A *quasi-isomorphism* is a morphism $f: X^* \to Y^*$ in K(A) which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.17. The derived category D(A) of an abelian category A is the localization K(A)_{0is}.

Let $s \in \text{Qis}$ and let $f: X' \to Y$ be a morphism in A. A roof is a diagram (s, f) of the form



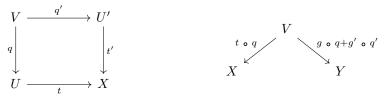
The morphisms in D(A) are equivalence classes of roofs. One could check this is exactly the same as we described in the previous proposition. For a proof, check [GM03] III.2.

Proposition. Derived categories are additive.

Proof. Let $\varphi, \varphi': X \to Y$ be morphisms in D(A).



We want the roofs to have common top and left side. Since t and t' are quasi-isomorphisms, we can construct V by homotopy pullback such that the following diagram commutes up to homotopy. We define the addition $\varphi + \varphi'$ to be the equivalence class of roofs of the form below.



This defines an additive category.

Definition 1.18. Let C be a triangulated category. A collection S of morphisms in C is said to be *compatible with* triangulation if it satisfies the following property.

- 1. $s \in S$ if and only if $T(s) \in S$.
- 2. consider the diagram (TR3), if $f, g \in S$, then the complementing morphism $h \in S$.

Derived categories can be triangulated.

Proposition. Let C be a triangulated category and consider some localization C_S that is compatible with triangulation. Define T to be the localizing functor and define $\mathrm{Ob}(C_S) = \mathrm{Ob}(C)$. A triangle in C_S is a image of a triangle in C under the localization $C \to C_S$. This defines a triangulated category.

It is not hard to check that Qis is compatible with K(A), hence D(A) is triangulated.

Definition 1.19. Let A be an abelian category and let $K^*(A)$ be a triangulated subcategory of K(A). If the natural functor $K^*(A)_{K^*(A)\cap Qis} \to D(A)$ is fully-faithful, then $D^*(A)$ is called a *localizing subcategory* and we denote it by $D^*(A)$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 1.20. Let A and B be abelian categories. Let $F : K^*(A) \to K(B)$ be a ∂ -functor. The right derived covariant functor of F is a ∂ -functor $R^*F : D^*(A) \to D(B)$ together with a natural transformation $\eta : Q \circ F \to R^*F \circ Q$ from $K^*(A) \to D(B)$ such that if $G : D^*(A) \to D(B)$ is a ∂ -functor and $\xi : Q \circ F \to G \circ Q$ is a natural transformation, then there exists unique natural transformation $\eta' : R^*F \to G$ such that $\xi = (\eta' \circ Q) \circ \eta$.

References

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