A Collection of Quick Tours

Hassium

1 Burnside Group of Odd Exponents: Results by Arkarskaya, A. et al.

2 Derived Categories

1 Burnside Group of Odd Exponents: Results by Arkarskaya, A. et al.

This note is entirely based on [1] and I will only rephrase the proofs to make them more readable.

Definition 1.1. Let $F = \langle x_1, \ldots, x_m \rangle$ be the free group F_m, where $m \geq 2$. The *free Burnside group* $B(m, n) = F/\langle\langle x_1, \ldots, x_m \mid w^n, w \in F \rangle\rangle$. We say $B(m, n)$ has *rank* m and *exponent* n.

Definition 1.2. Let F_m be a free group of rank m, then elements of $\{x_1, \ldots, x_m\} \cup \{{x_1}^{-1}, \ldots, {x_m}^{-1}\}$ are called *letters*. A sequence of words is called a *word*. A word without cancellation is called a *reduced word*. The *length* of w, denoted $|w|$, is the number of letters in w.

Definition 1.3. We say a word A is *cyclically contained* in a word w if A is a subword of any cyclic shift of w.

Example. Let $w = abcde$ and let $A = dea$. Shift w cyclically, we obtain $bcdea$, hence A is cyclically contained in w.

Definition 1.4. Let w be a reduced word. A *prefix* of w is any intial segment of w. A *suffix* of w is any final segement of w.

Definition 1.5. Let w be a non-empty reduced word. We say w is *primitive* if there does not exist $k \geq 2$ such that $w = a^k$.

Definition 1.6. Let $\mathrm{Rel}_i \subset \{w^n \mid w \text{ primitive}\}$ be a set of relators such that $\bigcup_{i \in I} \mathrm{Rel}_i = \{w^n \mid w \text{ primitive}\}$ and $\mathrm{Rel}_i \cap \mathrm{Rel}_j = \varnothing$.

Theorem 1.1. Induct on i, then the following properties hold.

1. $\mathrm{Can}_i \subset \mathrm{Can}_{i-1}$;
2. if $L_1 A^\tau R_1, L_2 A^\tau R_2 \in \mathrm{Can}_i$ for some primitive A and $A^n \notin$

If the induction works, we will have the following results immediately.

Theorem 1.2.

2 Derived Categories

This note is based on my presentation given in Math 501.

Definition 2.1. Let C be a category. A category D is a *subcategory* of C if the $\mathrm{Ob}(\mathsf{D})$ and the $\mathrm{Hom}_{\mathsf{D}}(X,Y)$ are subcollections of $\mathrm{Ob}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{C}}(X,Y)$, respectively, for all objects X and Y in C. The subcatgeory D is said to be *full* if for all objects X and Y in D, $\mathrm{Hom}_{\mathsf{D}}(X,Y)$ is exactly $\mathrm{Hom}_{\mathsf{C}}(X,Y)$.

Definition 2.2. Let $\mathtt{C}$ be a category. A morphism $f : A \to B$ is a *monomorphism* if for all $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$. A morphism $f : A \to B$ is called an *epimorphism* if for all $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$.

Definition 2.3. An *initial object* 0 of a category $\mathtt{C}$ is an object in $\mathtt{C}$ such that for any object A, there is a unique morphism $0 \to A$. A *terminal object* 1 of a category $\mathtt{C}$ is an object of $\mathtt{C}$ such that for any object B, there is a unique morphism $A \to 1$. We say an object is the *zero object* if it is both an initial object and a terminal object.

Definition 2.4. Let C be a category with zero object 0. The *zero morphism* $0_{A,B}$ between objects A and B is the unique morphism that factors through 0.

Definition 2.5. Let C be an object and let $f : X \to Y$ be a morphism in C. An object $\ker(f)$ is said to be the *kernel* of f if for every object Z and $h : Z \to X$ such that $f \circ g = 0$, where 0 is the zero morphism, there is a unique morphism $\varphi : Z \to \ker(f)$ such that $h = p \circ \varphi$.

Definition 2.6. Let F and G be functors between C and D. A *natural transformation* η from F to G is a family of morphisms such that

1. For all object X in C, there exists a morphism $\eta_X : F(X) \to G(X)$, called the *component* at X.
2. For all $f : X \to Y$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A functor $F : \mathsf{C} \to \mathsf{D}$ is a *natural functor* if for all $G : \mathsf{C} \to \mathsf{D}$, there exists a natueal transformation $\eta : F \to G$.

Definition 2.7. A category C is an *additive category* if the following properties hold.

1. C has an zero object 0.
2. Every hom-set of C is an abelian group.
3. Composition of morphisms is bilinear.
4. C admits finite coproduct.

Definition 2.8. A functor $f : \mathsf{A} \to \mathsf{B}$ between additive categories is an *additive functor* if it preserves the finite coproduct.

Definition 2.9. A category C is a *pre-abelian category* if C is an additive category and every morphism in C has a kernel and a cokernel. A pre-abelian category C is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 2.10. Let A be an additive category. A *cochain complex* is a collection X^* of objects X^n in A with maps $d^n : X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n : X^n \to Y^n$ such that $f^{n+1}d_X^n = d_Y^n f^n$. This defines the category of cochain complexes, denoted $\mathsf{C}(\mathsf{A})$.

Definition 2.11. Let X^* and Y^* be cochain complexes. The maps $f, g : X^* \to Y^*$ are *homotopic* if there exists a collection of maps $k^n : X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 2.12. Let A be an additive category. The objects of the *homotopy category*, denoted $\mathsf{K}(\mathsf{A})$, are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Let A be an additive category and $\mathsf{K}(\mathsf{A})$ be the homotopy category. Define the finite coproduct to be the direct sum $(X^* \oplus Y^*)^n = X^n \oplus Y^n$. Define the composition to be pairwise addition. Since composition in A is bilinear, it is bilinear in $\mathsf{K}(\mathsf{A})$. It is not hard to check that $\mathsf{K}(\mathsf{A})$ is additive.

Definition 2.13. A *triangulated category* is an additive category C with:

1. an additive automorphism $T : \mathsf{C} \to \mathsf{C}$ called the *translation functor*,
2. a collection of *triangles* (X, Y, Z, u, v, w), where X, Y, and Z are objects of C and $u : X \to Y$, $v : Y \to Z$, and $w : Z \to T(X)$ are morphisms,
3. morphisms $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X')
\end{tikzcd}$$

The data subject to the following rules:

1. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle and for all $f : A \to B$, there exists a triangle (A, B, C, f, g, h).
2. A sextuple (A, B, C, f, g, h) is a triangle if and only if $(B, C, T(A), g, h, -T(f))$ is a triangle.
3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms in C that commutes with u and u', then there exists $h : Z \to Z'$ such that (f, g, h) is a morphism between triangles.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dashed, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

4. Let $(X, Y, Z', u, j, -)$, $(Y, Z, X', v, -, i)$, and $(X, Z, Y', vu, -, -)$ be triangles, then there exists morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that $(Z', Y', X', f, g, T(j)i)$ is a triangle and the following diagram commutes.

$$\begin{tikzcd}
Z' \arrow[r, dashed, "f"] \arrow[d] & Y' \arrow[dl] \\
X &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y' \arrow[d, dashed, "g"'] & Z \arrow[l] \arrow[dl] \\
X' &
\end{tikzcd}$$

This is called the *octohedral axiom*.

$$\begin{tikzcd}
 & & Y' \arrow[drr, dashed, "g"] \arrow[dll, shift left=0] \arrow[ddll] & & \\
Z' \arrow[urr, dashed, "f"] \arrow[d] & & & & X' \arrow[llll, "T(j)i"'] \arrow[ddll, "i"'] \\
X \arrow[rrrr, "vu"'] \arrow[drr, "u"'] & & & & Z \arrow[u] \arrow[uull] \\
 & & Y \arrow[urr, "v"'] \arrow[uull, "j"] & &
\end{tikzcd}$$

Definition 2.14. Let C and D be additive categories. An additive functor $f : \mathsf{C} \to \mathsf{D}$ is a *covariant ∂-functor* if it commutes with the translation functor and it preserves triangles. A *contravariant ∂-functor* takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy categories can be triangulated.

Proposition. Let $T(X)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u : X \to Y$, we define Z to be the *mapping cone* $T(X) \oplus Y$. The differential maps in Z are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v : Y \to Z$ and $w : Z \to X$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u : X \to Y$. This defines a triangulated category.

Proof. (TR1) Take $\mathrm{id}_X : X \to X$, then $Z = T(X) \oplus X$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n : Z^n \to Z^{n-1}$ by $k^n((a,b)) = (0,a)$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$, hence $\mathrm{id}_{\mathbb{Z}} \sim 0$. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is the triangle induced by $\mathrm{id}_X : X \to X$. $\square$ (TR2) Let (X, Y, Z, u, v, w) be a triangle. Denote $T(Y) \oplus Z$ by A. Consider the sextuple (Y, Z, A, v, s, t). Notice that $(T(X))^i = X^{i+1}$, so we pick $x^{i+1} \in (T(X))^i$. Define $\theta^i : (T(X))^i \to A^i$ by $x^{i+1} \mapsto (-u^{i+1}(x^{i+1}), x^{i+1}, 0)$.

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "\mathrm{id}_Y"'] & T(X) \oplus Y \arrow[r, "w"] \arrow[d, "\mathrm{id}_{T(X)\oplus Y}"'] & T(X) \arrow[r, "T(v)"] \arrow[d, "\theta"'] & T(Y) \arrow[d, "\mathrm{id}_{T(Y)}"'] \\
Y \arrow[r, "v"'] & T(X) \oplus Y \arrow[r] & T(Y) \oplus T(X) \oplus Y \arrow[r] & T(Y)
\end{tikzcd}$$

Define $h^i : Z^i \to A^{i-1}$ by $h^i(x^{i+1}, y^i) = (y^i, 0, 0)$. Then two maps from Z to A are homotopic. Let $\psi : A \to T(X)$ be the natural projection, then $\psi \circ \theta = \mathrm{id}_{T(X)}$. We have $\theta \circ \psi^i : A^i \to A^i$ with $\theta \circ \psi^i : (y^{i+1}, x^{i+1}, y^i) \mapsto (-f^{i+1}(x^{i+1}), x^{i+1}, 0)$. Pick $k^i : A^i \to A^{i-1}$ to be the map $(y^{i+1}, x^{i+1}, y^i) \mapsto (y^i, 0, 0)$, then $\theta \circ \psi \sim \mathrm{id}_A$. $\square$ (TR3) We have $Z = T(X) \oplus Y$ and $Z' = T(X') \oplus Y'$, so the complementing morphism is exactly $h = T(f) \oplus g : Z \to Z'$. $\square$ (TR4) We say an exact sequence of complexes $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is *semi-split* if for all i, there exists a morphism $w^i : Z^i \to Y^i$ such that $v^i w^i = \mathrm{id}_{Z^i}$. Consider a lemma: any semi-split exact sequence in $\mathsf{C}(\mathsf{A})$ can induce a triangle $(X, Y, Z, u, v, -)$ in $\mathsf{K}(\mathsf{A})$. (lemma)

Now given the upper cap, we want to construct the lower cap by this lemma.

$\square$

Definition 2.15. Let C be a category and let S be a collection of morphisms in C, then the *localization* of C with respect to S is a category $\mathsf{C_S}$ together with a functor $Q : \mathsf{C} \to \mathsf{C_S}$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$.
2. Any functor $F : \mathsf{C} \to \mathsf{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

If $Q' : \mathsf{C} \to \mathsf{C_S}$ is another functor satisfies these properties, then the two localizations are considered to be equivalent.

Proposition. The localization $\mathsf{C_S}$ exists.

Proof. Let $\mathrm{Ob}(\mathsf{C_S}) = \mathrm{Ob}(\mathsf{C})$ and let Q be the identity on objects. For all $s \in S$, introduce some variable x_s, then construct an oriented graph Γ as follows: let the vertices be objects in C, let the edges be morphisms in C union $\{x_s\}$, the edge has the direction $X \to Y$ if there exists a morphism $X \to Y$ in C, and the edge x_s has the same vertices as the edges but the opposite orientation. Define a morphism to be an equivalence class of paths that share the same initial and terminal in Γ. Two paths are equivalent if they can be joined by a chain of elementary equivalences that two consecutive arrows in a path can be replaced by their composition and arrows $X \xrightarrow{s} Y \xrightarrow{s_x} X$ can be replaced by $X \xrightarrow{\mathrm{id}_X} X$. The composition of morphisms is induced by the conjunction of paths. The functor Q takes a morphism to a corresponding path of length 1. The inverse of a morphism $s \in S$ is x_s. $\square$

Definition 2.16. Let A be an abelian category. A *quasi-isomorphism* is a morphism $f : X^* \to Y^*$ in $\mathsf{K}(\mathsf{A})$ which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 2.17. The *derived category* $\mathsf{D}(\mathsf{A})$ of an abelian category A is the localization $\mathsf{K}(\mathsf{A})_{\mathsf{Qis}}$.

Let $s \in \mathrm{Qis}$ and let $f : X' \to Y$ be a morphism in A. A *roof* is a diagram (s, f) of the form

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

The morphisms in $\mathsf{D}(\mathsf{A})$ are equivalence classes of roofs. One could check this is exactly the same as we described in the previous proposition. For a proof, check [GM03] III.2.

Proposition. Derived categories are additive.

Proof. Let $\varphi, \varphi' : X \to Y$ be morphisms in $\mathsf{D}(\mathsf{A})$.

$$\begin{tikzcd} & U \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & U' \arrow[dl, "t'"'] \arrow[dr, "g'"] & \\ X & & Y \end{tikzcd}$$

We want the roofs to have common top and left side. Since t and t' are quasi-isomorphisms, we can construct V by homotopy pullback such that the following diagram commutes up to homotopy. We define the addition $\varphi + \varphi'$ to be the equivalence class of roofs of the form below.

$$\begin{tikzcd} V \arrow[r, "q'"] \arrow[d, "q"'] & U' \arrow[d, "t'"] \\ U \arrow[r, "t"'] & X \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "t \circ q"'] \arrow[dr, "g \circ q + g' \circ q'"] & \\ X & & Y \end{tikzcd}$$

This defines an additive category. □

Definition 2.18. Let C be a triangulated category. A collection S of morphisms in C is said to be *compatible with triangulation* if it satisfies the following property.

1. $s \in S$ if and only if $T(s) \in S$.
2. consider the diagram (TR3), if $f, g \in S$, then the complementing morphism $h \in S$.

Derived categories can be triangulated.

Proposition. Let C be a triangulated category and consider some localization C_S that is compatible with triangulation. Define T to be the localizing functor and define $\mathrm{Ob}(\mathsf{C}_\mathsf{S}) = \mathrm{Ob}(\mathsf{C})$. A triangle in C_S is a image of a triangle in C under the localization $\mathsf{C} \to \mathsf{C}_\mathsf{S}$. This defines a triangulated category.

It is not hard to check that Qis is compatible with $\mathsf{K}(\mathsf{A})$, hence $\mathsf{D}(\mathsf{A})$ is triangulated.

Definition 2.19. Let A be an abelian category and let $\mathsf{K}^*(\mathsf{A})$ be a triangulated subcategory of $\mathsf{K}(\mathsf{A})$. If the natural functor $\mathsf{K}^*(\mathsf{A})_{\mathsf{K}^*(\mathsf{A}) \cap \mathsf{Qis}} \to \mathsf{D}(\mathsf{A})$ is fully-faithful, then $\mathsf{D}^*(\mathsf{A})$ is called a *localizing subcategory* and we denote it by $\mathsf{D}^*(\mathsf{A})$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 2.20. Let A and B be abelian categories. Let $F : \mathsf{K}^*(\mathsf{A}) \to \mathsf{K}(\mathsf{B})$ be a ∂-functor. The *right derived covariant functor* of F is a ∂-functor $R^*F : \mathsf{D}^*(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$ together with a natural transformation $\eta : Q \circ F \to R^*F \circ Q$ from $\mathsf{K}^*(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$ such that if $G : \mathsf{D}^*(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$ is a ∂-functor and $\xi : Q \circ F \to G \circ Q$ is a natural transformation, then there exists unique natural transformation $\eta' : R^*F \to G$ such that $\xi = (\eta' \circ Q) \circ \eta$.

References

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