## A Collection of Quick Tours

Hassium

1 Derived Categories and Derived Functors

## 1 Derived Categories and Derived Functors

We shall first review some basic definitions in category theory.

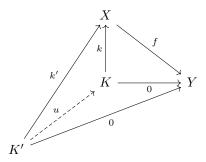
**Definition 1.1.** Let C be a category. A category D is a *subcategory* of C if the Ob(D) and the  $Hom_D(X,Y)$  are subcollections of Ob(C) and  $Hom_C(X,Y)$ , respectively, for all objects X and Y in C. The subcategory D is said to be *full* if for all objects X and Y in D,  $Hom_D(X,Y)$  is exactly  $Hom_C(X,Y)$ .

**Definition 1.2.** Let C be a category. A morphism  $f: A \to B$  is a monomorphism if for all  $g, h: C \to A$ ,  $f \circ g = g \circ h$  implies g = h. A morphism  $f: A \to B$  is called an epimorphism if for all  $i, j: B \to D$ ,  $i \circ f = j \circ f$  implies i = j.

**Definition 1.3.** Let C be a category. A morphism  $f: X \to Y$  in C is a constant morphism if for all object Z and morphisms  $g, h: Z \to X$ , fg = fh. A morphism is a zero morphism if it is both a constant morphism and a coconstant morphism.

**Definition 1.4.** Let C be a category. A *fernel* of a morphism  $f: X \to Y$  is a pair (K, k), where K is an object and  $k: K \to X$  is a morphism, such that

- 1.  $fk = 0_{KY}$ .
- 2. For all (K', k') such that  $k'f = 0_{K'Y}$ , there exists a unique  $u: K' \to K$ .



**Definition 1.5.** A category C is an *additive category* if every hom-set of C is an abelian group, composition of morphisms is bilinear, and C admits finite coproduct.

**Definition 1.6.** A functor  $f: A \to B$  between additive categories is an additive functor if it preserves the finite coproduct.

**Definition 1.7.** Let A be an additive category. A *cochain complex* is a collection  $X^*$  of objects  $X^n$  in A with maps  $d^n: X^n \to X^{n+1}$  such that  $d^{n+1}d^n = 0$ . The morphism between complexes  $X^*$  and  $Y^*$  is a collection of maps  $f^n: X^n \to Y^n$  such that  $f^{n+1}d^n_X = d^n_Y f^n$ . This defines the category of cochain complexes, denoted C(A).

**Definition 1.8.** Let  $X^*$  and  $Y^*$  be cochain complexes. The maps  $f, g: X^* \to Y^*$  are homotopic if there exists a collection of maps  $k^n: X^n \to Y^{n-1}$  such that  $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$ .

It is trivial that homotopy is an equivalence relation.

**Definition 1.9.** Let A be an additive category. The objects of the homotopy category, denoted K(A), are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

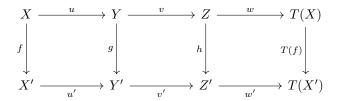
**Proposition.** Homotopy category is additive.

*Proof.* Consider a lemma: C(A) is additive.

**Definition 1.10.** A category C is a *pre-abelian category* if C is an additive category and every morphism in C has a kernel and a cokernel. A pre-abelian category C is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

**Definition 1.11.** A triangulated category is an additive category C with:

- 1. a translation functor  $T: \mathbb{C} \to \mathbb{C}$  that is fully-faithful,
- 2. a collection of triangles (X, Y, Z, u, v, w), where X, Y, and Z are objects of C and  $u: X \to Y, v: Y \to Z$ , and  $w: Z \to T(X)$  are morphisms,
- 3. morphisms  $(f, g, h): (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$  such that the following diagram commutes.



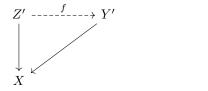
The data subject to the following rules:

- 1. The sextuple  $(X, X, 0, id_X, 0, 0)$  is a triangle and for all  $f: A \to B$ , there exists a triangle (A, B, C, f, g, h).
- 2. A sextuple (A, B, C, f, g, h) is a triangle if and only if (B, C, T(A), g, h, -T(f)) is a triangle.
- 3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let  $f: X \to X'$  and  $g: Y \to Y'$  be morphisms in C that commutes with u and u', then there exists  $h: Z \to Z'$  such that (f, g, h) is a morphism between triangles.

$$X \xrightarrow{u} Y \xrightarrow{} Z \xrightarrow{} T(X)$$

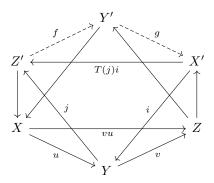
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

4. Let (X, Y, Z', u, j, ), (Y, Z, X', v, , i), and (X, Z, Y', vu, , ) be triangles, then there exists morphisms  $f: Z' \to Y'$  and  $g: Y' \to X'$  such that (Z', Y', X', f, g, T(j)i) is a triangle and the following diagram commutes.



 $Y' \leftarrow Z$ 

This is called the *octohedral axiom*.



**Definition 1.12.** Let C and D be additive categories. An additive functor  $f: C \to D$  is a *covariant*  $\partial$ -functor if it commutes with the translation functor and it preserves triangles. A *contravariant*  $\partial$ -functor takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

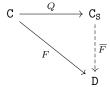
Homotopy category can be triangulated.

**Proposition.** Let  $T(X^*)^p = X^{p+1}$  and  $d_{T(X)} = -d_X$ . Given a morphism  $u: X^* \to Y^*$ , we define  $Z^*$  to be the mapping cone  $T(X^*) \oplus Y^*$ . The differential maps in  $Z^*$  are matrices  $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$ . We pick  $v: Y^* \to Z^*$  and  $w: Z^* \to X^*$  to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism  $u: X^* \to Y^*$ . This defines a triangulated category.

Proof. (TR1) Take  $\mathrm{id}_X: X^* \to X^*$ , then  $Z^* = T(X^*) \oplus X^*$  and the differentials are  $d_Z^n = \begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$ . Define  $k^n: Z^n \to Z^{n-1}$  by  $k^n((a,b)) = (0,a)$ . Then  $k^{n+1}d_Z^n((a,b)) = k^{n+1}((-d_X^n(a)+b,d_X^n(b))) = (0,-d_X^n(a)+b)$  and  $d_Z^{n-1}k^n((a,b)) = d_Z^{n-1}((0,a)) = (a,d_X^n(a))$ . We have  $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$ , hence  $\mathrm{id}_Z \sim 0$ . The sextuple  $(X,X,0,\mathrm{id}_X,0,0)$  is the triangle induced by  $\mathrm{id}_X: X^* \to X^*$ .  $\square$  (TR2) Let  $(X^*,Y^*,Z^*,u,v,w)$  be a triangle. Denote  $T(Y^*) \oplus Z^*$  by  $A^*$ . Consider the sextuple  $(Y^*,Z^*,A^*,v,s,t)$ ,

**Definition 1.13.** Let C be a category and let S be a collection of morphisms in C, then the *localization* of C with respect to S is a category  $C_S$  together with a functor  $Q: C \to C_S$  such that

- 1. Q(s) is an isomorphism for every  $s \in S$ .
- 2. Any functor  $F: \mathbb{C} \to \mathbb{D}$  such that F(s) is an isomorphism for all  $s \in S$  factors uniquely through Q.



**Definition 1.14.** Let A be an abelian category. A *quasi-isomorphism* is a morphism  $f: X^* \to Y^*$  in K(A) which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.15. The derived category D(A) of an abelian category A is the localization K(A) Qis.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.