A Collection of Quick Tours

Hassium

1 Derived Categories and Derived Functors

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We shall first review some basic definitions in category theory.

Definition 1.1. Let $\mathtt{C}$ be a category. A category $\mathtt{D}$ is a *subcategory* of $\mathtt{C}$ if the $\mathrm{Ob}(\mathtt{D})$ and the $\mathrm{Hom}_{\mathtt{D}}(X, Y)$ are subcollections of $\mathrm{Ob}(\mathtt{C})$ and $\mathrm{Hom}_{\mathtt{C}}(X, Y)$, respectively, for all objects X and Y in $\mathtt{C}$. The subcatgeory $\mathtt{D}$ is said to be *full* if for all objects X and Y in $\mathtt{D}$, $\mathrm{Hom}_{\mathtt{D}}(X, Y)$ is exactly $\mathrm{Hom}_{\mathtt{C}}(X, Y)$.

Definition 1.2. Let $\mathtt{C}$ be a category. A morphism $f : A \to B$ is a *monomorphism* if for all $g, h : C \to A$, $f \circ g = g \circ h$ implies $g = h$. A morphism $f : A \to B$ is called an *epimorphism* if for all $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$.

Definition 1.3. Let $\mathtt{C}$ be a category. A morphism $f : X \to Y$ in $\mathtt{C}$ is a *constant morphism* if for all object Z and morphisms $g, h : Z \to X$, $fg = fh$. A morphism is a *zero morphism* if it is both a constant morphism and a coconstant morphism.

Definition 1.4. Let $\mathtt{C}$ be a category. A *kernel* of a morphism $f : X \to Y$ is a pair (K, k), where K is an object and $k : K \to X$ is a morphism, such that

1. $fk = 0_{KY}$.
2. For all (K', k') such that $k'f = 0_{K'Y}$, there exists a unique $u : K' \to K$.

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
& K \arrow[u, "k"] \arrow[r, "0"'] & Y \\
K' \arrow[uur, "k'"] \arrow[ur, dashed, "u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

Definition 1.5. A category $\mathtt{C}$ is an *additive category* if every hom-set of $\mathtt{C}$ is an abelian group, composition of morphisms is bilinear, and $\mathtt{C}$ admits finite coproduct.

Definition 1.6. A functor $f : \mathtt{A} \to \mathtt{B}$ between additive categories is an *additive functor* if it preserves the finite coproduct.

Definition 1.7. Let $\mathtt{A}$ be an additive category. A *cochain complex* is a collection X^* of objects X^n in $\mathtt{A}$ with maps $d^n : X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n : X^n \to Y^n$ such that $f^{n+1}d^n_X = d^n_Y f^n$. This defines the category of cochain complexes, denoted $\mathtt{C(A)}$.

Definition 1.8. Let X^* and Y^* be cochain complexes. The maps $f, g : X^* \to Y^*$ are *homotopic* if there exists a collection of maps $k^n : X^n \to Y^{n-1}$ such that $f^n - g^n = d^{n-1}_Y k^n + k^{n+1}d^n_X$.

It is trivial that homotopy is an equivalence relation.

Definition 1.9. Let $\mathtt{A}$ be an additive category. The objects of the *homotopy category*, denoted $\mathtt{K(A)}$, are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Proposition. Homotopy category is additive.

Proof. Consider a lemma: $\mathtt{C(A)}$ is additive. □

Definition 1.10. A category $\mathtt{C}$ is a *pre-abelian category* if $\mathtt{C}$ is an additive category and every morphism in $\mathtt{C}$ has a kernel and a cokernel. A pre-abelian category $\mathtt{C}$ is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.11. A *triangulated category* is an additive category $\mathtt{C}$ with:

1. a *translation functor* $T : \mathtt{C} \to \mathtt{C}$ that is fully-faithful,
2. a collection of *triangles* (X, Y, Z, u, v, w), where X, Y, and Z are objects of $\mathtt{C}$ and $u : X \to Y$, $v : Y \to Z$, and $w : Z \to T(X)$ are morphisms,
3. morphisms $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X')
\end{tikzcd}$$

The data subject to the following rules:

1. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle and for all $f : A \to B$, there exists a triangle (A, B, C, f, g, h).
2. A sextuple (A, B, C, f, g, h) is a triangle if and only if $(B, C, T(A), g, h, -T(f))$ is a triangle.
3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms in $\mathtt{C}$ that commutes with u and u', then there exists $h : Z \to Z'$ such that (f, g, h) is a morphism between triangles.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dashed, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

4. Let $(X, Y, Z', u, j, \ )$, $(Y, Z, X', v, \ , i)$, and $(X, Z, Y', vu, \ , \ )$ be triangles, then there exists morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that $(Z', Y', X', f, g, T(j)i)$ is a triangle and the following diagram commutes.

$$\begin{tikzcd}
Z' \arrow[r, dashed, "f"] \arrow[d] & Y' \arrow[dl] \\
X &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y' \arrow[d, dashed, "g"'] & Z \arrow[l] \arrow[dl] \\
X' &
\end{tikzcd}$$

This is called the *octohedral axiom*.

$$\begin{tikzcd}
 & & Y' \arrow[drr, dashed, "g"] \arrow[ddll] & & \\
Z' \arrow[urr, dashed, "f"] \arrow[d] & & & & X' \arrow[llll, "T(j)i"] \arrow[ddll, "i"'] \\
X \arrow[rrrr, "vu"] \arrow[drr, "u"'] & & & & Z \arrow[u] \arrow[uull] \\
 & & Y \arrow[urr, "v"'] \arrow[uull, "j"] & &
\end{tikzcd}$$

Definition 1.12. Let $\mathtt{C}$ and $\mathtt{D}$ be additive categories. An additive functor $f : \mathtt{C} \to \mathtt{D}$ is a *covariant ∂-functor* if it commutes with the translation functor and it preserves triangles. A *contravariant ∂-functor* takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy category can be triangulated.

Proposition. Let $T(X^*)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u : X^* \to Y^*$, we define Z^* to be the *mapping cone* $T(X^*) \oplus Y^*$. The differential maps in Z^* are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v : Y^* \to Z^*$ and $w : Z^* \to X^*$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u : X^* \to Y^*$. This defines a triangulated category.

Proof. (TR1) Take $\mathrm{id}_X : X^* \to X^*$, then $Z^* = T(X^*) \oplus X^*$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n : Z^n \to Z^{n-1}$ by $k^n((a,b)) = (0,a)$. Then $k^{n+1}d_Z^n((a,b)) = k^{n+1}((-d_X^n(a)+b, d_X^n(b))) = (0, -d_X^n(a)+b)$ and $d_Z^{n-1}k^n((a,b)) = d_Z^{n-1}((0,a)) = (a, d_X^n(a))$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$, hence $\mathrm{id}_{\mathbb{Z}} \sim 0$. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is the triangle induced by $\mathrm{id}_X : X^* \to X^*$. $\square$ (TR2) Let (X^*, Y^*, Z^*, u, v, w) be a triangle. Denote $T(Y^*) \oplus Z^*$ by A^*. Consider the sextuple (Y^*, Z^*, A^*, v, s, t), $\square$

Definition 1.13. Let $\mathtt{C}$ be a category and let S be a collection of morphisms in $\mathtt{C}$, then the *localization* of $\mathtt{C}$ with respect to S is a category $\mathtt{C_S}$ together with a functor $Q : \mathtt{C} \to \mathtt{C_S}$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$.
2. Any functor $F : \mathtt{C} \to \mathtt{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

$$\begin{tikzcd} \mathtt{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathtt{C_S} \arrow[d, dashed, "\overline{F}"] \\ & \mathtt{D} \end{tikzcd}$$

Definition 1.14. Let $\mathtt{A}$ be an abelian category. A *quasi-isomorphism* is a morphism $f : X^* \to Y^*$ in $\mathtt{K(A)}$ which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.15. The *derived category* $\mathtt{D(A)}$ of an abelian category $\mathtt{A}$ is the localization $\mathtt{K(A)}_{\mathtt{Qis}}$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.