

A Collection of Quick Tours

Hassium

1 Derived Categories

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This note is based on my presentation given in Math 501.

Definition 1.1. Let \mathcal{C} be a category. A category \mathcal{D} is a *subcategory* of \mathcal{C} if the $\text{Ob}(\mathcal{D})$ and the $\text{Hom}_{\mathcal{D}}(X, Y)$ are subcollections of $\text{Ob}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}}(X, Y)$, respectively, for all objects X and Y in \mathcal{C} . The subcategory \mathcal{D} is said to be *full* if for all objects X and Y in \mathcal{D} , $\text{Hom}_{\mathcal{D}}(X, Y)$ is exactly $\text{Hom}_{\mathcal{C}}(X, Y)$.

Definition 1.2. Let \mathcal{C} be a category. A morphism $f : A \rightarrow B$ is a *monomorphism* if for all $g, h : C \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$. A morphism $f : A \rightarrow B$ is called an *epimorphism* if for all $i, j : B \rightarrow D$, $i \circ f = j \circ f$ implies $i = j$.

Definition 1.3. An *initial object* 0 of a category \mathcal{C} is an object in \mathcal{C} such that for any object A , there is a unique morphism $0 \rightarrow A$. A *terminal object* 1 of a category \mathcal{C} is an object of \mathcal{C} such that for any object B , there is a unique morphism $A \rightarrow 1$. We say an object is the *zero object* if it is both an initial object and a terminal object.

Definition 1.4. Let \mathcal{C} be a category with zero object 0 . The *zero morphism* $0_{A,B}$ between objects A and B is the unique morphism that factors through 0 .

Definition 1.5. Let \mathcal{C} be an object and let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . An object $\ker(f)$ is said to be the *kernel* of f if for every object Z and $h : Z \rightarrow X$ such that $f \circ h = 0$, where 0 is the zero morphism, there is a unique morphism $\varphi : Z \rightarrow \ker(f)$ such that $h = \varphi \circ f$.

Definition 1.6. Let F and G be functors between \mathcal{C} and \mathcal{D} . A *natural transformation* η from F to G is a family of morphisms such that

1. For all object X in \mathcal{C} , there exists a morphism $\eta_X : F(X) \rightarrow G(X)$, called the *component* at X .
2. For all $f : X \rightarrow Y$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural functor* if for all $G : \mathcal{C} \rightarrow \mathcal{D}$, there exists a natural transformation $\eta : F \rightarrow G$.

Definition 1.7. A category \mathcal{C} is an *additive category* if the following properties hold.

1. \mathcal{C} has an zero object 0 .
2. Every hom-set of \mathcal{C} is an abelian group.
3. Composition of morphisms is bilinear.
4. \mathcal{C} admits finite coproduct.

Definition 1.8. A functor $f : \mathcal{A} \rightarrow \mathcal{B}$ between additive categories is an *additive functor* if it preserves the finite coproduct.

Definition 1.9. A category \mathcal{C} is a *pre-abelian category* if \mathcal{C} is an additive category and every morphism in \mathcal{C} has a kernel and a cokernel. A pre-abelian category \mathcal{C} is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.10. Let \mathcal{A} be an additive category. A *cochain complex* is a collection X^* of objects X^n in \mathcal{A} with maps $d^n : X^n \rightarrow X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n : X^n \rightarrow Y^n$ such that $f^{n+1}d_X^n = d_Y^n f^n$. This defines the category of cochain complexes, denoted $\mathcal{C}(\mathcal{A})$.

Definition 1.11. Let X^* and Y^* be cochain complexes. The maps $f, g : X^* \rightarrow Y^*$ are *homotopic* if there exists a collection of maps $k^n : X^n \rightarrow Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 1.12. Let \mathcal{A} be an additive category. The objects of the *homotopy category*, denoted $\mathcal{K}(\mathcal{A})$, are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Let \mathbf{A} be an additive category and $\mathbf{K}(\mathbf{A})$ be the homotopy category. Define the finite coproduct to be the direct sum $(X^* \oplus Y^*)^n = X^n \oplus Y^n$. Define the composition to be pairwise addition. Since composition in \mathbf{A} is bilinear, it is bilinear in $\mathbf{K}(\mathbf{A})$. It is not hard to check that $\mathbf{K}(\mathbf{A})$ is additive.

Definition 1.13. A *triangulated category* is an additive category \mathbf{C} with:

1. an additive automorphism $T : \mathbf{C} \rightarrow \mathbf{C}$ called the *translation functor*,
2. a collection of *triangles* (X, Y, Z, u, v, w) , where X, Y , and Z are objects of \mathbf{C} and $u : X \rightarrow Y$, $v : Y \rightarrow Z$, and $w : Z \rightarrow T(X)$ are morphisms,
3. morphisms $(f, g, h) : (X, Y, Z, u, v, w) \rightarrow (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
 \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\
 X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X')
 \end{array}$$

The data subject to the following rules:

1. The sextuple $(X, X, 0, \text{id}_X, 0, 0)$ is a triangle and for all $f : A \rightarrow B$, there exists a triangle (A, B, C, f, g, h) .
2. A sextuple (A, B, C, f, g, h) is a triangle if and only if $(B, C, T(A), g, h, -T(f))$ is a triangle.
3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be morphisms in \mathbf{C} that commutes with u and u' , then there exists $h : Z \rightarrow Z'$ such that (f, g, h) is a morphism between triangles.

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
 \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\
 X' & \xrightarrow{u'} & Y' & \longrightarrow & Z' & \longrightarrow & T(X')
 \end{array}$$

4. Let $(X, Y, Z', u, j, -)$, $(Y, Z, X', v, -, i)$, and $(X, Z, Y', vu, -, -)$ be triangles, then there exists morphisms $f : Z' \rightarrow Y'$ and $g : Y' \rightarrow X'$ such that $(Z', Y', X', f, g, T(j)i)$ is a triangle and the following diagram commutes.

$$\begin{array}{ccc}
 Z' & \xrightarrow{\quad f \quad} & Y' \\
 \downarrow & \swarrow & \downarrow \\
 X & & X'
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y' & \xleftarrow{\quad g \quad} & Z \\
 \downarrow & \swarrow & \downarrow \\
 X' & & X
 \end{array}$$

This is called the *octohedral axiom*.

$$\begin{array}{ccccc}
 & & Y' & & \\
 & \swarrow f & & \searrow g & \\
 Z' & & & & X' \\
 \downarrow & & \downarrow T(j)i & & \downarrow \\
 X & & & & Z \\
 \swarrow u & & \searrow v & & \\
 & & Y & &
 \end{array}$$

Definition 1.14. Let \mathbf{C} and \mathbf{D} be additive categories. An additive functor $f : \mathbf{C} \rightarrow \mathbf{D}$ is a *covariant ∂ -functor* if it commutes with the translation functor and it preserves triangles. A *contravariant ∂ -functor* takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy categories can be triangulated.

Proposition. Let $T(X)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u : X \rightarrow Y$, we define Z to be the *mapping cone* $T(X) \oplus Y$. The differential maps in Z are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v : Y \rightarrow Z$ and $w : Z \rightarrow T(X)$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u : X \rightarrow Y$. This defines a triangulated category.

Proof. (TR1) Take $\text{id}_X : X \rightarrow X$, then $Z = T(X) \oplus X$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \text{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n : Z^n \rightarrow Z^{n-1}$ by $k^n((a, b)) = (0, a)$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a, b)) = (a, b)$, hence $\text{id}_Z \sim 0$. The sextuple $(X, X, 0, \text{id}_X, 0, 0)$ is the triangle induced by $\text{id}_X : X \rightarrow X$. \square (TR2) Let (X, Y, Z, u, v, w) be a triangle. Denote $T(Y) \oplus Z$ by A . Consider the sextuple (Y, Z, A, v, s, t) . Notice that $(T(X))^i = X^{i+1}$, so we pick $x^{i+1} \in (T(X))^i$. Define $\theta^i : (T(X))^i \rightarrow A^i$ by $x^{i+1} \mapsto (-u^{i+1}(x^{i+1}), x^{i+1}, 0)$.

$$\begin{array}{ccccccc}
 Y & \xrightarrow{v} & T(X) \oplus Y & \xrightarrow{w} & T(X) & \xrightarrow{T(v)} & T(Y) \\
 \text{id}_Y \downarrow & & \downarrow \text{id}_{T(X) \oplus Y} & & \downarrow \theta & & \downarrow \text{id}_{T(Y)} \\
 Y & \xrightarrow{v} & T(X) \oplus Y & \longrightarrow & T(Y) \oplus T(X) \oplus Y & \longrightarrow & T(Y)
 \end{array}$$

Define $h^i : Z^i \rightarrow A^{i-1}$ by $h^i(x^{i+1}, y^i) = (y^i, 0, 0)$. Then two maps from Z to A are homotopic. Let $\psi : A \rightarrow T(X)$ be the natural projection, then $\psi \circ \theta = \text{id}_{T(X)}$. We have $\theta \circ \psi^i : A^i \rightarrow A^i$ with $\theta \circ \psi^i : (y^{i+1}, x^{i+1}, y^i) \mapsto (-f^{i+1}(x^{i+1}), x^{i+1}, 0)$. Pick $k^i : A^i \rightarrow A^{i-1}$ to be the map $(y^{i+1}, x^{i+1}, y^i) \mapsto (y^i, 0, 0)$, then $\theta \circ \psi \sim \text{id}_A$. \square (TR3) We have $Z = T(X) \oplus Y$ and $Z' = T(X') \oplus Y'$, so the complementing morphism is exactly $h = T(f) \oplus g : Z \rightarrow Z'$. \square (TR4) We say an exact sequence of complexes $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$ is *semi-split* if for all i , there exists a morphism $w^i : Z^i \rightarrow Y^i$ such that $v^i w^i = \text{id}_{Z^i}$. Consider a lemma: any semi-split exact sequence in $\mathbf{C}(\mathbf{A})$ can induce a triangle $(X, Y, Z, u, v, -)$ in $\mathbf{K}(\mathbf{A})$. (lemma) \square

Definition 1.15. Let \mathbf{C} be a category and let S be a collection of morphisms in \mathbf{C} , then the *localization* of \mathbf{C} with respect to S is a category \mathbf{C}_S together with a functor $Q : \mathbf{C} \rightarrow \mathbf{C}_S$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$.
2. Any functor $F : \mathbf{C} \rightarrow \mathbf{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q .

If $Q' : \mathbf{C} \rightarrow \mathbf{C}_S$ is another functor satisfies these properties, then the two localizations are considered to be equivalent.

Proposition. The localization \mathbf{C}_S exists.

Proof. Let $\text{Ob}(\mathbf{C}_S) = \text{Ob}(\mathbf{C})$ and let Q be the identity on objects. For all $s \in S$, introduce some variable x_s , then construct an oriented graph Γ as follows: let the vertices be objects in \mathbf{C} , let the edges be morphisms in \mathbf{C} union $\{x_s\}$, the edge has the direction $X \rightarrow Y$ if there exists a morphism $X \rightarrow Y$ in \mathbf{C} , and the edge x_s has the same vertices as the edges but the opposite orientation. Define a morphism to be an equivalence class of paths that share the same initial and terminal in Γ . Two paths are equivalent if they can be joined by a chain of elementary equivalences that two consecutive arrows in a path can be replaced by their composition and arrows $X \xrightarrow{s} Y \xrightarrow{s_x} X$ can be replaced by $X \xrightarrow{\text{id}_X} X$. The composition of morphisms is induced by the conjunction of paths. The functor Q takes a morphism to a corresponding path of length 1. The inverse of a morphism $s \in S$ is x_s . \square

Definition 1.16. Let \mathbf{A} be an abelian category. A *quasi-isomorphism* is a morphism $f : X^* \rightarrow Y^*$ in $\mathbf{K}(\mathbf{A})$ which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis .

Definition 1.17. The *derived category* $\mathbf{D}(\mathbf{A})$ of an abelian category \mathbf{A} is the localization $\mathbf{K}(\mathbf{A})_{\text{Qis}}$.

Let $s \in \text{Qis}$ and let $f : X' \rightarrow Y$ be a morphism in \mathbf{A} . A *roof* is a diagram (s, f) of the form

$$\begin{array}{ccc} & X' & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

The morphisms in $D(\mathbf{A})$ are equivalence classes of roofs. One could check this is exactly the same as we described in the previous proposition. For a proof, check [GM03] III.2.

Proposition. Derived categories are additive.

Proof. Let $\varphi, \varphi' : X \rightarrow Y$ be morphisms in $D(\mathbf{A})$.

$$\begin{array}{ccc} & U & \\ t \swarrow & & \searrow g \\ X & & Y \end{array} \quad \begin{array}{ccc} & U' & \\ t' \swarrow & & \searrow g' \\ X & & Y \end{array}$$

We want the roofs to have common top and left side. Since t and t' are quasi-isomorphisms, we can construct V by homotopy pullback such that the following diagram commutes up to homotopy. We define the addition $\varphi + \varphi'$ to be the equivalence class of roofs of the form below.

$$\begin{array}{ccc} V & \xrightarrow{q'} & U' \\ q \downarrow & & \downarrow t' \\ U & \xrightarrow{t} & X \end{array} \quad \begin{array}{ccc} & V & \\ t \circ q \swarrow & & \searrow g \circ q + g' \circ q' \\ X & & Y \end{array}$$

This defines an additive category. □

Definition 1.18. Let \mathbf{C} be a triangulated category. A collection S of morphisms in \mathbf{C} is said to be *compatible with triangulation* if it satisfies the following property.

1. $s \in S$ if and only if $T(s) \in S$.
2. consider the diagram (TR3), if $f, g \in S$, then the complementing morphism $h \in S$.

Derived categories can be triangulated.

Proposition. Let \mathbf{C} be a triangulated category and consider some localization \mathbf{C}_S that is compatible with triangulation. Define T to be the localizing functor and define $\text{Ob}(\mathbf{C}_S) = \text{Ob}(\mathbf{C})$. A triangle in \mathbf{C}_S is a image of a triangle in \mathbf{C} under the localization $\mathbf{C} \rightarrow \mathbf{C}_S$. This defines a triangulated category.

It is not hard to check that Qis is compatible with $K(\mathbf{A})$, hence $D(\mathbf{A})$ is triangulated.

Definition 1.19. Let \mathbf{A} be an abelian category and let $K^*(\mathbf{A})$ be a triangulated subcategory of $K(\mathbf{A})$. If the natural functor $K^*(\mathbf{A})_{K^*(\mathbf{A}) \cap \text{Qis}} \rightarrow D(\mathbf{A})$ is fully-faithful, then $D^*(\mathbf{A})$ is called a *localizing subcategory* and we denote it by $D^*(\mathbf{A})$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 1.20. Let \mathbf{A} and \mathbf{B} be abelian categories. Let $F : K^*(\mathbf{A}) \rightarrow K(\mathbf{B})$ be a ∂ -functor. The *right derived covariant functor* of F is a ∂ -functor $R^*F : D^*(\mathbf{A}) \rightarrow D(\mathbf{B})$ together with a natural transformation $\eta : Q \circ F \rightarrow R^*F \circ Q$ from $K^*(\mathbf{A}) \rightarrow D(\mathbf{B})$ such that if $G : D^*(\mathbf{A}) \rightarrow D(\mathbf{B})$ is a ∂ -functor and $\xi : Q \circ F \rightarrow G \circ Q$ is a natural transformation, then there exists unique natural transformation $\eta' : R^*F \rightarrow G$ such that $\xi = (\eta' \circ Q) \circ \eta$.

References

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