A Collection of Quick Tours

Hassium

1 Derived Categories

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In this note, we will sketch some proofs but not in detailed.

Definition 1.1. Let $\mathtt{C}$ be a category. A category $\mathtt{D}$ is a *subcategory* of $\mathtt{C}$ if the $\mathrm{Ob}(\mathtt{D})$ and the $\mathrm{Hom}_{\mathtt{D}}(X, Y)$ are subcollections of $\mathrm{Ob}(\mathtt{C})$ and $\mathrm{Hom}_{\mathtt{C}}(X, Y)$, respectively, for all objects X and Y in $\mathtt{C}$. The subcatgeory $\mathtt{D}$ is said to be *full* if for all objects X and Y in $\mathtt{D}$, $\mathrm{Hom}_{\mathtt{D}}(X, Y)$ is exactly $\mathrm{Hom}_{\mathtt{C}}(X, Y)$.

Definition 1.2. Let $\mathtt{C}$ be a category. A morphism $f : A \to B$ is a *monomorphism* if for all $g, h : C \to A$, $f \circ g = g \circ h$ implies $g = h$. A morphism $f : A \to B$ is called an *epimorphism* if for all $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$.

Definition 1.3. Let $\mathtt{C}$ be a category. A morphism $f : X \to Y$ in $\mathtt{C}$ is a *constant morphism* if for all object Z and morphisms $g, h : Z \to X$, $fg = fh$. A morphism is a *zero morphism* if it is both a constant morphism and a coconstant morphism.

Definition 1.4. Let $\mathtt{C}$ be a category. A *kernel* of a morphism $f : X \to Y$ is a pair (K, k), where K is an object and $k : K \to X$ is a morphism, such that

1. $fk = 0_{KY}$.
2. For all (K', k') such that $k'f = 0_{K'Y}$, there exists a unique $u : K' \to K$.

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & K \arrow[u, "k"] \arrow[r, "0"] & Y \\
K' \arrow[uur, "k'"] \arrow[ur, dashed, "u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

Definition 1.5. Let F and G be functors between $\mathtt{C}$ and $\mathtt{D}$. A *natural transformation* η from F to G is a family of morphisms such that

1. For all object X in $\mathtt{C}$, there exists a morphism $\eta_X : F(X) \to G(X)$, called the *component* at X.
2. For all $f : X \to Y$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A functor $F : \mathtt{C} \to \mathtt{D}$ is a *natural functor* if for all $G : \mathtt{C} \to \mathtt{D}$, there exists a natueal transformation η.

Definition 1.6. A category $\mathtt{C}$ is an *additive category* if every hom-set of $\mathtt{C}$ is an abelian group, composition of morphisms is bilinear, and $\mathtt{C}$ admits finite coproduct.

Definition 1.7. A functor $f : \mathtt{A} \to \mathtt{B}$ between additive categories is an *additive functor* if it preserves the finite coproduct.

Definition 1.8. A category $\mathtt{C}$ is a *pre-abelian category* if $\mathtt{C}$ is an additive category and every morphism in $\mathtt{C}$ has a kernel and a cokernel. A pre-abelian category $\mathtt{C}$ is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.9. Let $\mathtt{A}$ be an additive category. A *cochain complex* is a collection X^* of objects X^n in $\mathtt{A}$ with maps $d^n : X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n : X^n \to Y^n$ such that $f^{n+1}d_X^n = d_Y^n f^n$. This defines the category of cochain complexes, denoted $\mathtt{C(A)}$.

Definition 1.10. Let X^* and Y^* be cochain complexes. The maps $f, g : X^* \to Y^*$ are *homotopic* if there exists a collection of maps $k^n : X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 1.11. Let `A` be an additive category. The objects of the *homotopy category*, denoted `K(A)`, are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Let `A` be an additive category and `K(A)` be the homotopy category. Define the finite coproduct to be the direct sum $(X^* \oplus Y^*)^n = X^n \oplus Y^n$. Define the composition to be pairwise addition. Since composition in `A` is bilinear, it is bilinear in `K(A)`. It is not hard to check that `K(A)` is additive.

Definition 1.12. A *triangulated category* is an additive category `C` with:

1. an additive automorphism $T : \mathtt{C} \to \mathtt{C}$ called the *translation functor*,
2. a collection of *triangles* (X, Y, Z, u, v, w), where X, Y, and Z are objects of `C` and $u : X \to Y$, $v : Y \to Z$, and $w : Z \to T(X)$ are morphisms,
3. morphisms $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X')
\end{tikzcd}$$

The data subject to the following rules:

1. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle and for all $f : A \to B$, there exists a triangle (A, B, C, f, g, h).
2. A sextuple (A, B, C, f, g, h) is a triangle if and only if $(B, C, T(A), g, h, -T(f))$ is a triangle.
3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms in `C` that commutes with u and u', then there exists $h : Z \to Z'$ such that (f, g, h) is a morphism between triangles.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dashed, "h"'] & T(X) \arrow[d, "T(f)"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

4. Let $(X, Y, Z', u, j, \ )$, $(Y, Z, X', v, \ , i)$, and $(X, Z, Y', vu, \ , \ )$ be triangles, then there exists morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that $(Z', Y', X', f, g, T(j)i)$ is a triangle and the following diagram commutes.

$$\begin{tikzcd}
Z' \arrow[r, dashed, "f"] \arrow[d] & Y' \arrow[dl] \\
X &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y' \arrow[d, dashed, "g"'] & Z \arrow[l] \arrow[dl] \\
X' &
\end{tikzcd}$$

This is called the *octohedral axiom*.

$$\begin{tikzcd}
 & & Y' \arrow[drr, dashed, "g"] \arrow[ddll] & & \\
Z' \arrow[urr, dashed, "f"] \arrow[d] & & & & X' \arrow[llll, "T(j)i"] \arrow[ddll, "i"'] \\
X \arrow[rrrr, "vu"'] \arrow[drr, "u"'] & & & & Z \arrow[u] \arrow[uull] \\
 & & Y \arrow[uull, "j"] \arrow[urr, "v"'] & &
\end{tikzcd}$$

Definition 1.13. Let $\mathtt{C}$ and $\mathtt{D}$ be additive categories. An additive functor $f : \mathtt{C} \to \mathtt{D}$ is a *covariant ∂-functor* if it commutes with the translation functor and it preserves triangles. A *contravariant ∂-functor* takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy categories can be triangulated.

Proposition. Let $T(X^*)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u : X^* \to Y^*$, we define Z^* to be the *mapping cone* $T(X^*) \oplus Y^*$. The differential maps in Z^* are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v : Y^* \to Z^*$ and $w : Z^* \to X^*$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u : X^* \to Y^*$. This defines a triangulated category.

Proof. (TR1) Take $\mathrm{id}_X : X^* \to X^*$, then $Z^* = T(X^*) \oplus X^*$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n : Z^n \to Z^{n-1}$ by $k^n((a,b)) = (0,a)$. Then $k^{n+1}d_Z^n((a,b)) = k^{n+1}((-d_X^n(a)+b, d_X^n(b))) = (0, -d_X^n(a)+b)$ and $d_Z^{n-1}k^n((a,b)) = d_Z^{n-1}((0,a)) = (a, d_X^n(a))$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$, hence $\mathrm{id}_{\mathbb{Z}} \sim 0$. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is the triangle induced by $\mathrm{id}_X : X^* \to X^*$. $\square$ (TR2) Let (X^*, Y^*, Z^*, u, v, w) be a triangle. Denote $T(Y^*) \oplus Z^*$ by A^*. Consider the sextuple (Y^*, Z^*, A^*, v, s, t), $\square$

Definition 1.14. Let $\mathtt{C}$ be a category and let S be a collection of morphisms in $\mathtt{C}$, then the *localization* of $\mathtt{C}$ with respect to S is a category $\mathtt{C}_\mathtt{S}$ together with a functor $Q : \mathtt{C} \to \mathtt{C}_\mathtt{S}$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$.
2. Any functor $F : \mathtt{C} \to \mathtt{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

If $Q' : \mathtt{C} \to \mathtt{C}_\mathtt{S}$ is another functor satisfies these properties, then the two localizations are considered to be equivalent.

Definition 1.15. Let $\mathtt{A}$ be an abelian category. A *quasi-isomorphism* is a morphism $f : X^* \to Y^*$ in $\mathtt{K(A)}$ which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.16. The *derived category* $\mathtt{D(A)}$ of an abelian category $\mathtt{A}$ is the localization $\mathtt{K(A)}_{\mathtt{Qis}}$.

Proposition. Given an abelian category $\mathtt{A}$, there exists a derived category $\mathtt{D(A)}$.

Proof. Consider the generalization: any localization $\mathtt{C}_\mathtt{S}$ exists. Let $\mathrm{Ob}(\mathtt{C}_\mathtt{S}) = \mathrm{Ob}(\mathtt{C})$ and let Q be the identity on objects. For all $s \in S$, introduce some variable x_s, then construct an oriented graph Γ as follows: let the vertices be objects in $\mathtt{C}$, let the edges be morphisms in $\mathtt{C}$ union $\{x_s\}$, the edge has the direction $X \to Y$ if there exists a morphism $X \to Y$ in $\mathtt{C}$, and the edge x_s has the same vertices as the edges but the opposite orientation. Define a morphism to be an equivalence class of paths that share the same initial and terminal in Γ. Two paths are equivalent if they can be joined by a chain of elementary equivalences that two consecutive arrows in a path can be replaced by their composition and arrows $X \xrightarrow{s} Y \xrightarrow{s_x} X$ can be replaced by $X \xrightarrow{\mathrm{id}_X} X$. The composition of morphisms is induced by the conjunction of paths. The functor Q takes a morphism to a corresponding path of length 1. The inverse of a morphism $s \in S$ is x_s. Suppose we have another functor $\mathtt{C} \to \mathtt{C}'$ satisfies the universal property. We construct $G : \mathtt{C}_\mathtt{S} \to \mathtt{C}'$ by $G(X) = F(X)$, $G(f) = F(f)$, and $G((x_s)) = F(s)^{-1}$ for all $X \in \mathrm{Ob}(\mathtt{C})$, $f \in \mathrm{Mor}_\mathtt{A}$, and $s \in S$. This defines a localization but we will not check the axioms here. $\square$

Let $s \in S$ and let $f : X' \to Y$ be a morphism in $\mathtt{A}$. A *roof* is a diagram (s, f) of the form

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

The morphisms in $\mathtt{C}_\mathtt{S}$ are equivalence classes of roofs. One could check this is exactly the same as we described in the previous proposition.

Proposition. Derived categories are additive.

Proof. Let $\varphi, \varphi' : X \to Y$ be morphisms in $\mathtt{D(A)}$.

$$\begin{tikzcd} & U \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & U' \arrow[dl, "t'"'] \arrow[dr, "g'"] & \\ X & & Y \end{tikzcd}$$

Since t and t' are quasi-isomorphisms, we can construct V by homotopy pullback such that the following diagram commutes up to homotopy. We define the addition $\varphi + \varphi'$ to be the equivalence class of roofs of the form below.

$$\begin{tikzcd} V \arrow[r, "q'"] \arrow[d, "q"'] & U' \arrow[d, "t'"] \\ U \arrow[r, "t"'] & X \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "t \circ q"'] \arrow[dr, "g \circ q + g' \circ q'"] & \\ X & & Y \end{tikzcd}$$

It is left to check that addition does define an additive structure, but we will skip the details. $\square$

Definition 1.17. Let $\mathtt{C}$ be a triangulated category. A collection S of morphisms in $\mathtt{C}$ is said to be *compatible with triangulation* if

1. $s \in S$ if and only if $T(s) \in S$.
2. consider the diagram (TR3), if $f, g \in S$, then the complementing morphism $h \in S$.

Derived categories can be triangulated. Let $\mathtt{C}$ be a triangulated category and consider some localization $\mathtt{C_S}$ that is compatible with triangulation. Define T to be the localizing functor and define $\mathrm{Ob}(\mathtt{C_S}) = \mathrm{Ob}(\mathtt{C})$. A triangle in $\mathtt{C_S}$ is a image of a triangle in $\mathtt{C}$ under the localization $\mathtt{C} \to \mathtt{C_S}$. This defines a triangulated category. See [Gel03] IV.2 for a detailed proof. It is not hard to check that Qis is compatible with $\mathtt{K(A)}$, hence $\mathtt{D(A)}$ is triangulated.

Definition 1.18. Let $\mathtt{A}$ be an abelian category and let $\mathtt{K^*(A)}$ be a triangulated subcategory of $\mathtt{K(A)}$. If the natural functor $\mathtt{K^*(A)}_{\mathtt{K^*(A)} \cap \mathtt{Qis}} \to \mathtt{D(A)}$ is fully-faithful, then $\mathtt{D^*(A)}$ is called a *localizing subcategory* and we denote it by $\mathtt{D^*(A)}$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 1.19. Let $\mathtt{A}$ and $\mathtt{B}$ be abelian categories. Let $F : \mathtt{K^*(A)} \to \mathtt{K(B)}$ be a ∂-functor. The *right derived covariant functor* of F is a ∂-functor $R^*F : \mathtt{D^*(A)} \to \mathtt{D(B)}$ together with a natural transformation $\eta : Q \circ F \to R^*F \circ Q$ from $\mathtt{K^*(A)} \to \mathtt{D(B)}$ such that if $G : \mathtt{D^*(A)} \to \mathtt{D(B)}$ is a ∂-functor and $\xi : Q \circ F \to G \circ Q$ is a natural transformation, then there exists unique natural transformation $\eta' : R^*F \to G$ such that $\xi = (\eta' \circ Q) \circ \eta$.

References

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