A Collection of Quick Tours

Hassium

1 Derived Categories and Derived Functors

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1 Derived Categories and Derived Functors

We shall first review some basic definitions in category theory.

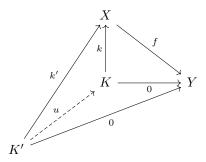
Definition 1.1. Let C be a category. A category D is a *subcategory* of C if the Ob(D) and the $Hom_D(X,Y)$ are subcollections of Ob(C) and $Hom_C(X,Y)$, respectively, for all objects X and Y in C. The subcategory D is said to be *full* if for all objects X and Y in D, $Hom_D(X,Y)$ is exactly $Hom_C(X,Y)$.

Definition 1.2. Let C be a category. A morphism $f: A \to B$ is a monomorphism if for all $g, h: C \to A$, $f \circ g = g \circ h$ implies g = h. A morphism $f: A \to B$ is called an epimorphism if for all $i, j: B \to D$, $i \circ f = j \circ f$ implies i = j.

Definition 1.3. Let C be a category. A morphism $f: X \to Y$ in C is a *constant morphism* if for all object Z and morphisms $g, h: Z \to X$, fg = fh. A morphism is a *zero morphism* if it is both a constant morphism and a coconstant morphism.

Definition 1.4. Let C be a category. A *kernel* of a morphism $f: X \to Y$ is a pair (K, k), where K is an object and $k: K \to X$ is a morphism, such that

- 1. $fk = 0_{KY}$.
- 2. For all (K', k') such that $k'f = 0_{K'Y}$, there exists a unique $u: K' \to K$.



Definition 1.5. A category C is an *additive category* if every hom-set of C is an abelian group, composition of morphisms is bilinear, and C admits finite coproduct.

Definition 1.6. A functor $f: A \to B$ between additive categories is an additive functor if it preserves the finite coproduct.

Definition 1.7. Let A be an additive category. A *cochain complex* is a collection X^* of objects X^n in A with maps $d^n: X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n: X^n \to Y^n$ such that $f^{n+1}d^n_X = d^n_Y f^n$. This defines the category of cochain complexes, denoted C(A).

Definition 1.8. Let X^* and Y^* be cochain complexes. The maps $f, g: X^* \to Y^*$ are homotopic if there exists a collection of maps $k^n: X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 1.9. Let A be an additive category. The objects of the homotopy category, denoted K(A), are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

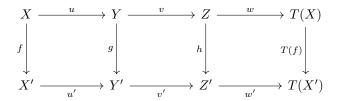
Proposition. Homotopy category is additive.

Proof. Consider a lemma: C(A) is additive.

Definition 1.10. A category C is a *pre-abelian category* if C is an additive category and every morphism in C has a kernel and a cokernel. A pre-abelian category C is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.11. A triangulated category is an additive category C with:

- 1. a translation functor $T: \mathbb{C} \to \mathbb{C}$ that is fully-faithful,
- 2. a collection of triangles (X, Y, Z, u, v, w), where X, Y, and Z are objects of C and $u: X \to Y, v: Y \to Z$, and $w: Z \to T(X)$ are morphisms,
- 3. morphisms $(f, g, h): (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.



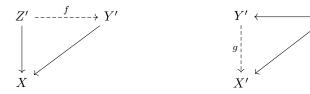
The data subject to the following rules:

- 1. The sextuple $(X, X, 0, id_X, 0, 0)$ is a triangle and for all $f: A \to B$, there exists a triangle (A, B, C, f, g, h).
- 2. A sextuple (A, B, C, f, g, h) is a triangle if and only if (B, C, T(A), g, h, -T(f)) is a triangle.
- 3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f: X \to X'$ and $g: Y \to Y'$ be morphisms in C that commutes with u and u', then there exists $h: Z \to Z'$ such that (f, g, h) is a morphism between triangles.

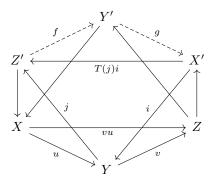
$$X \xrightarrow{u} Y \xrightarrow{} Z \xrightarrow{} T(X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

4. Let (X, Y, Z', u, j,), (Y, Z, X', v, , i), and (X, Z, Y', vu, ,) be triangles, then there exists morphisms $f: Z' \to Y'$ and $g: Y' \to X'$ such that (Z', Y', X', f, g, T(j)i) is a triangle and the following diagram commutes.



This is called the *octohedral axiom*.



Definition 1.12. Let C and D be additive categories. An additive functor $f: C \to D$ is a *covariant* ∂ -functor if it commutes with the translation functor and it preserves triangles. A *contravariant* ∂ -functor takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

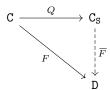
Now we define the triangulated category over a homotopy category.

Proposition. Let $T(X^*)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u: X^* \to Y^*$, we define Z^* to be the mapping cone $T(X^*) \oplus Y^*$. The differential maps in Z^* are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v: Y^* \to Z^*$ and $w: Z^* \to X^*$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u: X^* \to Y^*$. This defines a triangulated category.

Proof. (TR1) Take $\mathrm{id}_X: X^* \to X^*$, then $Z^* = T(X^*) \oplus X^*$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \mathrm{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n: Z^n \to Z^{n-1}$ by $k^n((a,b)) = (0,a)$. Then $k^{n+1}d_Z^n((a,b)) = k^{n+1}((-d_X^n(a)+b,d_X^n(b))) = (0,-d_X^n(a)+b)$ and $d_Z^{n-1}k^n((a,b)) = d_Z^{n-1}((0,a)) = (a,d_X^n(a))$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$, hence $\mathrm{id}_Z \sim 0$. The sextuple $(X,X,0,\mathrm{id}_X,0,0)$ is the triangle induced by $\mathrm{id}_X: X^* \to X^*$. \square (TR2) Let (X^*,Y^*,Z^*,u,v,w) be a triangle. Denote $T(Y^*) \oplus Z^*$ by A^* . Consider the sextuple (Y^*,Z^*,A^*,v,s,t) ,

Definition 1.13. Let C be a category and let S be a collection of morphisms in C, then the *localization* of C with respect to S is a category C_S together with a functor $Q: C \to C_S$ such that

- 1. Q(s) is an isomorphism for every $s \in S$.
- 2. Any functor $F: \mathbb{C} \to \mathbb{D}$ such that F(s) is an isomorphism for all $s \in S$ factors uniquely through Q.



Definition 1.14. Let A be an abelian category. A *quasi-isomorphism* is a morphism $f: X^* \to Y^*$ in K(A) which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.15. The derived category D(A) of an abelian category A is the localization K(A) Qis.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 1.16. Let

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