A Collection of Quick Tours

Hassium

1 Derived Categories

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In this note, we will sketch some proofs but not in detailed.

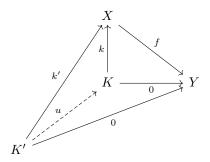
Definition 1.1. Let C be a category. A category D is a *subcategory* of C if the Ob(D) and the $Hom_D(X,Y)$ are subcollections of Ob(C) and $Hom_C(X,Y)$, respectively, for all objects X and Y in C. The subcategory D is said to be *full* if for all objects X and Y in D, $Hom_D(X,Y)$ is exactly $Hom_C(X,Y)$.

Definition 1.2. Let C be a category. A morphism $f: A \to B$ is a monomorphism if for all $g, h: C \to A$, $f \circ g = g \circ h$ implies g = h. A morphism $f: A \to B$ is called an *epimorphism* if for all $i, j: B \to D$, $i \circ f = j \circ f$ implies i = j.

Definition 1.3. Let C be a category. A morphism $f: X \to Y$ in C is a *constant morphism* if for all object Z and morphisms $g, h: Z \to X$, fg = fh. A morphism is a *zero morphism* if it is both a constant morphism and a coconstant morphism.

Definition 1.4. Let C be a category. A *kernel* of a morphism $f: X \to Y$ is a pair (K, k), where K is an object and $k: K \to X$ is a morphism, such that

- 1. $fk = 0_{KY}$.
- 2. For all (K', k') such that $k'f = 0_{K'Y}$, there exists a unique $u: K' \to K$.



Definition 1.5. Let F and G be functors between C and D. A natural transformation η from F to G is a family of morphisms such that

- 1. For all object X in C, there exists a morphism $\eta_X : F(X) \to G(X)$, called the *component* at X.
- 2. For all $f: X \to Y$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A functor $F: \mathbb{C} \to \mathbb{D}$ is a natural functor if for all $G: \mathbb{C} \to \mathbb{D}$, there exists a natural transformation η .

Definition 1.6. A category C is an *additive category* if every hom-set of C is an abelian group, composition of morphisms is bilinear, and C admits finite coproduct.

Definition 1.7. A functor $f: A \to B$ between additive categories is an additive functor if it preserves the finite coproduct.

Definition 1.8. A category C is a *pre-abelian category* if C is an additive category and every morphism in C has a kernel and a cokernel. A pre-abelian category C is an *abelian category* if every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 1.9. Let A be an additive category. A cochain complex is a collection X^* of objects X^n in A with maps $d^n: X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$. The morphism between complexes X^* and Y^* is a collection of maps $f^n: X^n \to Y^n$ such that $f^{n+1}d^n_X = d^n_Y f^n$. This defines the category of cochain complexes, denoted C(A).

Definition 1.10. Let X^* and Y^* be cochain complexes. The maps $f, g: X^* \to Y^*$ are homotopic if there exists a collection of maps $k^n: X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1}k^n + k^{n+1}d_X^n$.

It is trivial that homotopy is an equivalence relation.

Definition 1.11. Let A be an additive category. The objects of the homotopy category, denoted K(A), are cochain complexes and the morphisms are homotopic class of morphisms of cochain complexes.

Let A be an additive category and K(A) be the homotopy category. Define the finite coproduct to be the direct sum $(X^* \oplus Y^*)^n = X^n \oplus Y^n$. Define the composition to be pairwise addition. Since composition in A is bilinear, it is bilinear in K(A). It is not hard to check that K(A) is additive.

Definition 1.12. A triangulated category is an additive category C with:

- 1. an additive automorphism $T: \mathbb{C} \to \mathbb{C}$ called the translation functor,
- 2. a collection of triangles (X, Y, Z, u, v, w), where X, Y, and Z are objects of C and $u: X \to Y$, $v: Y \to Z$, and $w: Z \to T(X)$ are morphisms,
- 3. morphisms $(f, g, h): (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ such that the following diagram commutes.

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

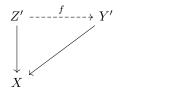
The data subject to the following rules:

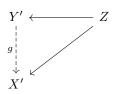
- 1. The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle and for all $f: A \to B$, there exists a triangle (A, B, C, f, g, h).
- 2. A sextuple (A, B, C, f, g, h) is a triangle if and only if (B, C, T(A), g, h, -T(f)) is a triangle.
- 3. Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $f: X \to X'$ and $g: Y \to Y'$ be morphisms in C that commutes with u and u', then there exists $h: Z \to Z'$ such that (f, g, h) is a morphism between triangles.

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow T(X)$$

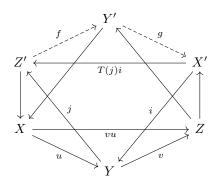
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

4. Let (X, Y, Z', u, j,), (Y, Z, X', v, ,i), and (X, Z, Y', vu, ,) be triangles, then there exists morphisms $f: Z' \to Y'$ and $g: Y' \to X'$ such that (Z', Y', X', f, g, T(j)i) is a triangle and the following diagram commutes.





This is called the octohedral axiom.



Definition 1.13. Let C and D be additive categories. An additive functor $f: C \to D$ is a *covariant \partial-functor* if it commutes with the translation functor and it preserves triangles. A *contravariant \partial-functor* takes triangles into triangles with the arrows reversed, and sends the translation functor into its inverse.

Homotopy categories can be triangulated.

Proposition. Let $T(X^*)^p = X^{p+1}$ and $d_{T(X)} = -d_X$. Given a morphism $u: X^* \to Y^*$, we define Z^* to be the mapping cone $T(X^*) \oplus Y^*$. The differential maps in Z^* are matrices $\begin{pmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{pmatrix}$. We pick $v: Y^* \to Z^*$ and $w: Z^* \to X^*$ to be the natural inclusion and projection. A triangle is defined to be a sextuple that induces by a morphism $u: X^* \to Y^*$. This defines a triangulated category.

Proof. (TR1) Take $\operatorname{id}_X: X^* \to X^*$, then $Z^* = T(X^*) \oplus X^*$ and the differentials are $d_Z^n = \begin{pmatrix} -d_X^n & \operatorname{id}_{X^{n+1}} \\ 0 & d_X^n \end{pmatrix}$. Define $k^n: Z^n \to Z^{n-1}$ by $k^n((a,b)) = (0,a)$. Then $k^{n+1}d_Z^n((a,b)) = k^{n+1}((-d_X^n(a)+b,d_X^n(b))) = (0,-d_X^n(a)+b)$ and $d_Z^{n-1}k^n((a,b)) = d_Z^{n-1}((0,a)) = (a,d_X^n(a))$. We have $(k^{n+1}d_Z^n + d_Z^{n-1}k^n)((a,b)) = (a,b)$, hence $\operatorname{id}_Z \sim 0$. The sextuple $(X,X,0,\operatorname{id}_X,0,0)$ is the triangle induced by $\operatorname{id}_X: X^* \to X^*$. \square (TR2) Let (X^*,Y^*,Z^*,u,v,w) be a triangle. Denote $T(Y^*) \oplus Z^*$ by A^* . Consider the sextuple (Y^*,Z^*,A^*,v,s,t) ,

Definition 1.14. Let C be a category and let S be a collection of morphisms in C, then the *localization* of C with respect to S is a category C_S together with a functor $Q: C \to C_S$ such that

- 1. Q(s) is an isomorphism for every $s \in S$.
- 2. Any functor $F: \mathbb{C} \to \mathbb{D}$ such that F(s) is an isomorphism for all $s \in S$ factors uniquely through Q.

If $Q': C \to C_S$ is another functor satisfies these properties, then the two localizations are considered to be equivalent.

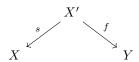
Definition 1.15. Let A be an abelian category. A *quasi-isomorphism* is a morphism $f: X^* \to Y^*$ in K(A) which induces an isomorphism on cohomology. The collection of all quasi-isomorphisms is denoted by Qis.

Definition 1.16. The derived category D(A) of an abelian category A is the localization K(A) Qis.

Proposition. Given an abelian category A, there exists a derived category D(A).

Proof. Consider the generalization: any localization C_S exists. Let $Ob(C_S) = Ob(C)$ and let Q be the identity on objects. For all $s \in S$, introduce some variable x_s , then construct an oriented graph Γ as follows: let the vertices be objects in C, let the edges be morphisms in C union $\{x_s\}$, the edge has the direction $X \to Y$ if there exists a morphism $X \to Y$ in C, and the edge x_s has the same vertices as the edges but the opposite orientation. Define a morphism to be an equivalence class of paths that share the same initial and terminal in Γ . Two paths are equivalent if they can be joined by a chain of elementary equivalences that two consecutive arrows in a path can be replaced by their composition and arrows $X \xrightarrow{s} Y \xrightarrow{s_x} X$ can be replaced by $X \xrightarrow{id_X} X$. The composition of morphisms is induced by the conjunction of paths. The functor Q takes a morphism to a corresponding path of length 1. The inverse of a morphism $s \in S$ is s_s . Suppose we have another functor $C \to C'$ satisfies the universal property. We construct $G: C_S \to C'$ by G(X) = F(X), G(f) = F(f), and $G((s_s)) = F(s)^{-1}$ for all $S \in C$ ob $S \in C$. This defines a localization but we will not check the axioms here.

Let $s \in S$ and let $f: X' \to Y$ be a morphism in A. A roof is a diagram (s, f) of the form



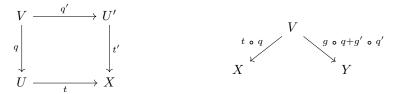
The morphisms in C_S are equivalence classes of roofs. One could check this is exactly the same as we described in the previous proposition.

Proposition. Derived categories are additive.

Proof. Let $\varphi, \varphi': X \to Y$ be morphisms in D(A).



Since t and t' are quasi-isomorphisms, we can construct V by homotopy pullback such that the following diagram commutes up to homotopy. We define the addition $\varphi + \varphi'$ to be the equivalence class of roofs of the form below.



It is left to check that addition does define an additive structure, but we will skip the details.

Definition 1.17. Let C be a triangulated category. A collection S of morphisms in C is said to be compatible with triangulation if

- 1. $s \in S$ if and only if $T(s) \in S$.
- 2. consider the diagram (TR3), if $f, g \in S$, then the complementing morphism $h \in S$.

Derived categories can be triangulated. Let C be a triangulated category and consider some localization C_S that is compatible with triangulation. Define T to be the localizing functor and define $\mathrm{Ob}(C_S) = \mathrm{Ob}(C)$. A triangle in C_S is a image of a triangle in C under the localization $C \to C_S$. This defines a triangulated category. See [Gel03] IV.2 for a detailed proof. It is not hard to check that Qis is compatible with K(A), hence D(A) is triangulated.

Definition 1.18. Let A be an abelian category and let $K^*(A)$ be a triangulated subcategory of K(A). If the natural functor $K^*(A)_{K^*(A)\cap G}$ is fully-faithful, then $D^*(A)$ is called a *localizing subcategory* and we denote it by $D^*(A)$.

We show the definition of a right derived covariant functor and the other definitions can be obtained similarly.

Definition 1.19. Let A and B be abelian categories. Let $F: K^*(A) \to K(B)$ be a ∂ -functor. The right derived covariant functor of F is a ∂ -functor $R^*F: D^*(A) \to D(B)$ together with a natural transformation $\eta: Q \circ F \to R^*F \circ Q$ from $K^*(A) \to D(B)$ such that if $G: D^*(A) \to D(B)$ is a ∂ -functor and $\xi: Q \circ F \to G \circ Q$ is a natural transformation, then there exists unique natural transformation $\eta': R^*F \to G$ such that $\xi = (\eta' \circ Q) \circ \eta$.

References

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