In [1]:
```
from sage.misc.html import latex
%display latex
```

In [2]:
```
var("z epsilon tau")
```

Out[2]: (z, ϵ, τ)

In [3]:
```
ep = epsilon
t = tau
ep
```

Out[3]: ϵ

In [4]:
```
rhs = -1/(1 + ep * z)**2
rhs
```

Out[4]: $-\frac{1}{(\epsilon z+1)^2}$

In [5]:
```
rhs_taylor = rhs.taylor(ep, 0, 4)
latex(rhs_taylor)
```

Out[5]: $-5\,\epsilon^4 z^4 + 4\,\epsilon^3 z^3 - 3\,\epsilon^2 z^2 + 2\,\epsilon z - 1$

In [6]:
```
z_expanded = 0
for i in range(5):
    z_expanded += function(f"z{i}")(t) * ep**i
z_expanded
```

Out[6]: $\epsilon^4 z_4(\tau) + \epsilon^3 z_3(\tau) + \epsilon^2 z_2(\tau) + \epsilon z_1(\tau) + z_0(\tau)$

In [7]:
```
rhs_expanded = rhs_taylor.substitute(z=z_expanded).expand().collect(ep)
```

In [8]:
```
lhs = 0
for i in range(5):
    lhs += derivative(function(f"z{i}")(t), t, 2) * ep**i
lhs
```

Out[8]:
$$\epsilon^4 \frac{\partial^2}{(\partial\tau)^2} z_4(\tau) + \epsilon^3 \frac{\partial^2}{(\partial\tau)^2} z_3(\tau) + \epsilon^2 \frac{\partial^2}{(\partial\tau)^2} z_2(\tau) + \epsilon \frac{\partial^2}{(\partial\tau)^2} z_1(\tau) + \frac{\partial^2}{(\partial\tau)^2} z_0(\tau)$$

In [9]:
```
equations = (rhs_expanded - lhs).collect(ep).coefficients(ep, sparse=False)[:6]
[e == 0 for e in equations]
```

Out[9]:
$$\left[-\frac{\partial^2}{(\partial\tau)^2} z_0(\tau) - 1 = 0, 2\,z_0(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_1(\tau) = 0, -3\,z_0(\tau)^2 + 2\,z_1(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_2(\tau) = 0, 4\,z_0(\tau)^3 - 6\,z_0(\tau)\,z_1(\tau) + 2\,z_2(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_3(\tau) = 0, -5\,z_0(\tau)^4 + 12\,z_0(\tau)^2 z_1(\tau) - 3\,z_1(\tau)^2 - 6\,z_0(\tau)\,z_2(\tau) + 2\,z_3(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_4(\tau) = 0, -20\,z_0(\tau)^3 z_1(\tau) + 12\,z_0(\tau)\,z_1(\tau)^2 + 12\,z_0(\tau)^2 z_2(\tau) - 6\,z_1(\tau)\,z_2(\tau) - 6\,z_0(\tau)\,z_3(\tau) + 2\,z_4(\tau) = 0\right]$$

In [10]:
```
i = 0

eqn = integrate(equations[i] == 0, t)
eqn = eqn - eqn.left().operands()[0]
eqn *= -1
eqn
```

Out[10]: $\frac{\partial}{\partial\tau} z_0(\tau) = -c_1 - \tau$

In [11]:
```
term_at_t0 = eqn.right().substitute(t==0)
constant = solve(term_at_t0 == 1, term_at_t0.variables()[0])
constant
```

Out[11]: $[c_1 = (-1)]$

In [12]:
```
eqn = eqn.substitute(constant)
```

In [13]:
```
eqn = integrate(eqn, t)
eqn
```

Out[13]: $z_0(\tau) = -\frac{1}{2}\tau^2 + c_2 + \tau$

In [14]:
```
term_at_t0 = eqn.right().substitute(t==0)
constant = solve(term_at_t0 == 0, term_at_t0.variables()[0])
constant
```

Out[14]: $[c_2 = 0]$

In [15]:
```
eqn = eqn.substitute(constant)
eqn
```

Out[15]: $z_0(\tau) = -\frac{1}{2}\tau^2 + \tau$

In [16]:
```
eqns = [None for _ in range(4)]
eqns[0] = eqn
```

In [17]:
```
def solve_fn(eqn, function):
    eqn = eqn.substitute({function: var("tempy")})
    eqn = eqn.solve(var("tempy"))[0]
    eqn = eqn.substitute({var("tempy"):function})
    return eqn
```

In [18]:
```
for i in range(1, 4):

    eqns[i] = integrate(equations[i] == 0, t)
    eqns[i].show()
    for j in range(i):
        eqns[i] = eqns[i].substitute({eqns[j].left():eqns[j].right()})
    eqns[i].show()

    subterm = (eqns[i].substitute({derivative(function(f"z{i}")(t), t):var("dzidt")}).solve(dzidt)[0].right() ==
0).substitute(t==0)

    subterm.show()

    subterm = subterm.solve(subterm.variables()[0])[0]

    subterm.show()

    eqns[i] = eqns[i].substitute(subterm)

    eqns[i] = integrate(eqns[i], t)
    eqns[i].show()

    eqns[i] = solve_fn(eqns[i], function(f"z{i}")(t))
    eqns[i].show()

    subterm = (eqns[i].right().substitute(t==0)==0)
    subterm = subterm.solve(subterm.variables()[0])[0]
    subterm.show()

    eqns[i] = eqns[i].substitute(subterm)

    eqns[i].show()

eqns
```

$\int 2\,z_0(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_1(\tau)\,d\tau = c_3$

$-\frac{1}{3}\tau^3 + \tau^2 - \frac{\partial}{\partial\tau} z_1(\tau) = c_3$

$-c_3 = 0$

$c_3 = 0$

$-\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3 - z_1(\tau) = c_4$

$z_1(\tau) = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3 - c_4$

$c_4 = 0$

$z_1(\tau) = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3$

$\int -3\,z_0(\tau)^2 + 2\,z_1(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_2(\tau)\,d\tau = c_5$

$-\frac{11}{60}\tau^5 + \frac{11}{12}\tau^4 - \tau^3 - \frac{\partial}{\partial\tau} z_2(\tau) = c_5$

$-c_5 = 0$

$c_5 = 0$

$-\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4 - z_2(\tau) = c_6$

$z_2(\tau) = -\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4 - c_6$

$c_6 = 0$

$z_2(\tau) = -\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4$

$\int 4\,z_0(\tau)^3 - 6\,z_0(\tau)\,z_1(\tau) + 2\,z_2(\tau) - \frac{\partial^2}{(\partial\tau)^2} z_3(\tau)\,d\tau = c_7$

$-\frac{73}{630}\tau^7 + \frac{73}{90}\tau^6 - \frac{17}{10}\tau^5 + \tau^4 - \frac{\partial}{\partial\tau} z_3(\tau) = c_7$

$-c_7 = 0$

$c_7 = 0$

$-\frac{73}{5040}\tau^8 + \frac{73}{630}\tau^7 - \frac{17}{60}\tau^6 + \frac{1}{5}\tau^5 - z_3(\tau) = c_8$

$z_3(\tau) = -\frac{73}{5040}\tau^8 + \frac{73}{630}\tau^7 - \frac{17}{60}\tau^6 + \frac{1}{5}\tau^5 - c_8$

$c_8 = 0$

$z_3(\tau) = -\frac{73}{5040}\tau^8 + \frac{73}{630}\tau^7 - \frac{17}{60}\tau^6 + \frac{1}{5}\tau^5$

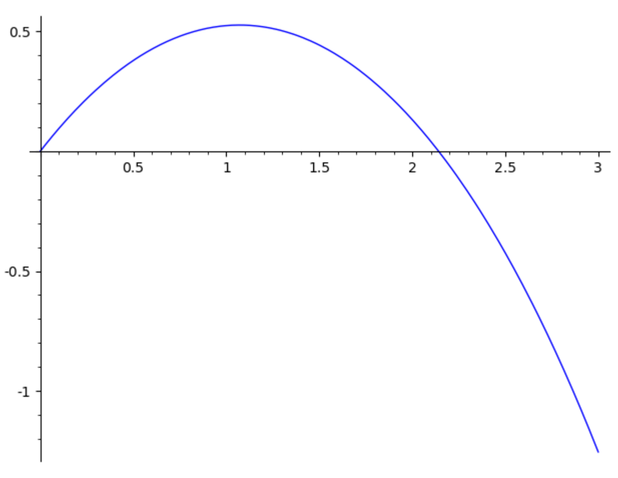
Out[18]:
$$\left[z_0(\tau) = -\frac{1}{2}\tau^2 + \tau, z_1(\tau) = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3, z_2(\tau) = -\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4, z_3(\tau) = -\frac{73}{5040}\tau^8 + \frac{73}{630}\tau^7 - \frac{17}{60}\tau^6 + \frac{1}{5}\tau^5\right]$$

In [19]:
```
z_series = sum(eqns[i].right() * ep**i for i in range(4))
z_series
```

Out[19]:
$$-\frac{1}{5040}\left(73\,\tau^8 - 584\,\tau^7 + 1428\,\tau^6 - 1008\,\tau^5\right)\epsilon^3 - \frac{1}{360}\left(11\,\tau^6 - 66\,\tau^5 + 90\,\tau^4\right)\epsilon^2 - \frac{1}{12}\left(\tau^4 - 4\,\tau^3\right)\epsilon - \frac{1}{2}\tau^2 + \tau$$

In [20]:
```
plot(z_series.substitute(ep==.1), t, 0, 3)
```

Out[20]:



In [21]:
```
velocity = derivative(z_series, t)

velocity
```

Out[21]:
$$-\frac{1}{630}\left(73\,\tau^7 - 511\,\tau^6 + 1071\,\tau^5 - 630\,\tau^4\right)\epsilon^3 - \frac{1}{60}\left(11\,\tau^5 - 55\,\tau^4 + 60\,\tau^3\right)\epsilon^2 - \frac{1}{3}\left(\tau^3 - 3\,\tau^2\right)\epsilon - \tau + 1$$

In [22]:
```
velocity == 0
```

Out[22]:
$$-\frac{1}{630}\left(73\,\tau^7 - 511\,\tau^6 + 1071\,\tau^5 - 630\,\tau^4\right)\epsilon^3 - \frac{1}{60}\left(11\,\tau^5 - 55\,\tau^4 + 60\,\tau^3\right)\epsilon^2 - \frac{1}{3}\left(\tau^3 - 3\,\tau^2\right)\epsilon - \tau + 1 = 0$$

In [23]:
```
apex_time = var("apex_time", latex_name="\\tau_{apex}")
series_apex = apex_time == sum(var(f"a{i}")*ep^i for i in range(5))
series_apex.show()
```

$\tau_{apex} = a_4\epsilon^4 + a_3\epsilon^3 + a_2\epsilon^2 + a_1\epsilon + a_0$

In [24]:
```
expanded_apex_time = (velocity).substitute(t==apex_time).substitute(series_apex).expand().collect(ep).coefficien
ts(ep, sparse=False)
expanded_apex_time[:5]
```

Out[24]:
$$\left[-a_0 + 1, -\frac{1}{3}a_0^3 + a_0^2 - a_1, -\frac{11}{60}a_0^5 + \frac{11}{12}a_0^4 - a_0^3 - a_0^2 a_1 + 2\,a_0 a_1 - a_2, -\frac{73}{630}a_0^7 + \frac{73}{90}a_0^6 - \frac{17}{10}a_0^5 - \frac{11}{12}a_0^4 a_1 + a_0^4 + \frac{11}{3}a_0^3 a_1 - 3\,a_0^2 a_1 - a_0 a_1^2 - a_0^2 a_2 + a_1^2 + 2\,a_0 a_2 - a_3, -\frac{73}{90}a_0^6 a_1 + \frac{73}{15}a_0^5 a_1 - \frac{17}{2}a_0^4 a_1 - \frac{11}{6}a_0^3 a_1^2 - \frac{11}{12}a_0^4 a_2 + 4\,a_0^3 a_1 + \frac{11}{2}a_0^2 a_1^2 + \frac{11}{3}a_0^3 a_2 - 3\,a_0 a_1^2 - \frac{1}{3}a_1^3 - 3\,a_0^2 a_2 - 2\,a_0 a_1 a_2 - a_0^2 a_3 + 2\,a_1 a_2 + 2\,a_0 a_3 - a_4\right]$$

In [25]:
```
apex_eqns = [None for _ in range(4)]

for i in range(4):
    apex_eqns[i] = expanded_apex_time[i] == 0
    apex_eqns[i] = apex_eqns[i].solve(var(f"a{i}"))[0]
    apex_eqns[i].show()
    for j in range(i):
        apex_eqns[i] = apex_eqns[i].substitute(apex_eqns[j])
    apex_eqns[i].show()
```

$a_0 = 1$

$a_0 = 1$

$a_1 = -\frac{1}{3}a_0^3 + a_0^2$

$a_1 = \left(\frac{2}{3}\right)$

$a_2 = -\frac{11}{60}a_0^5 + \frac{11}{12}a_0^4 - a_0^3 - \left(a_0^2 - 2\,a_0\right)a_1$

$a_2 = \left(\frac{2}{5}\right)$

$a_3 = -\frac{73}{630}a_0^7 + \frac{73}{90}a_0^6 - \frac{17}{10}a_0^5 + a_0^4 - (a_0 - 1)a_1^2 - \frac{1}{12}\left(11\,a_0^4 - 44\,a_0^3 + 36\,a_0^2\right)a_1 - \left(a_0^2 - 2\,a_0\right)a_2$

$a_3 = \left(\frac{8}{35}\right)$

In [26]:
```
show(series_apex.substitute(apex_eqns))
```

$\tau_{apex} = a_4\epsilon^4 + \frac{8}{35}\epsilon^3 + \frac{2}{5}\epsilon^2 + \frac{2}{3}\epsilon + 1$

In [27]:
```
print(ep)
```

```
epsilon
```

Problem 2

In [28]:
```
eqn = derivative(function("z")(t), t, 2) == rhs.substitute({var("z"):function("z")(t)})
eqn.show()

eqn = eqn * derivative(z(t), t)
eqn.show()

eqn = integrate(eqn, t)
eqn *= 2
eqn.show()



integration_constant = (eqn.right().substitute({z(t):0}) == 1).solve(eqn.right().variables()[0])

integration_constant[0].show()

eqn = eqn.substitute(integration_constant)

eqn.show()

zprime = sqrt(eqn.right().substitute({function("z")(t):var("z")}))

zprime
```

$\frac{\partial^2}{(\partial\tau)^2} z(\tau) = -\frac{1}{(\epsilon z(\tau)+1)^2}$

$\frac{\partial}{\partial\tau} z(\tau)\,\frac{\partial^2}{(\partial\tau)^2} z(\tau) = -\frac{\frac{\partial}{\partial\tau} z(\tau)}{(\epsilon z(\tau)+1)^2}$

$\frac{\partial}{\partial\tau} z(\tau)^2 = 2\,c_9 + \frac{2}{(\epsilon z(\tau)+1)\epsilon}$

$c_9 = \frac{\epsilon - 2}{2\,\epsilon}$

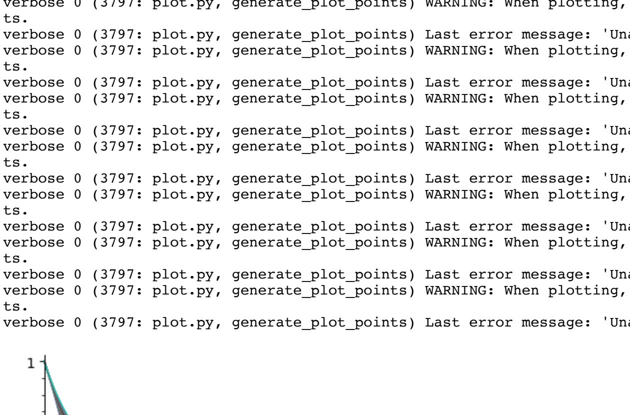
$\frac{\partial}{\partial\tau} z(\tau)^2 = \frac{\epsilon - 2}{\epsilon} + \frac{2}{(\epsilon z(\tau)+1)\epsilon}$

Out[28]: $\sqrt{\frac{\epsilon - 2}{\epsilon} + \frac{2}{(\epsilon z+1)\epsilon}}$

In [29]:
```
plot(zprime.substitute(ep==0.01), z0, 0, 5)
plot([zprime.substitute(ep==epp) for epp in [.01, .2, .4, .6, .8, 1, 1.2, 1.4, 1.6, 1.8, 2, 2.2]], z, 0, 5)
```

```
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 180 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'unable to convert 2.91955639161321*I to fl
oat; use abs() or real_part() as desired'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 180 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 177 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 174 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 171 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 166 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 160 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 150 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 133 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
verbose 0 (3797: plot.py, generate_plot_points) WARNING: When plotting, failed to evaluate function at 100 poin
ts.
verbose 0 (3797: plot.py, generate_plot_points) Last error message: 'Unable to compute f(5.0)'
```

Out[29]:



Problem 3

Problem 4

Solving forward in time from the apex

In [30]:
```
function("v")(t) == -zprime
```

Out[30]: $v(\tau) = -\sqrt{\frac{\epsilon - 2}{\epsilon} + \frac{2}{(\epsilon z+1)\epsilon}}$

solving backwards in time from the apex,

In [31]:
```
function("v")(t) == zprime
```

Out[31]: $v(\tau) = \sqrt{\frac{\epsilon - 2}{\epsilon} + \frac{2}{(\epsilon z+1)\epsilon}}$

By the existence and uniqeness theorem, these solutions will have the same initial conditions and so be the same, just mirrored.

Problem 5

In [ ]: