BINOMIAL EXPANSION

For expanding brackets with high exponents from the form: $(a+b)^n$

Introduction:

One common method of expanding brackets is using

PASCAL'S TRIANGLE

1 : Start on $(x+y)^0$

1 1 ($1+1=2$)

1 2 1

1 3 3 1 ($3+3=6$)

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$5+10=15$

* Pascal's triangle works by adding the above two number together each time.
* the rows go with the exponents of the brackets you want to expand, starting with $(x+y)^0$.

* Pascal's triangle tells us about the coefficients of each term in the expanded form

EXAMPLE 1:

Expand using pascal's triangle:

(a) $(x+2y)^3$

ANSWER:

The exponent is 3, therefore we use row 4 of the triangle (row 1 is 0)

row 4: 1 3 3 1

STEPS:

1. coefficients = ?
2. a of each term = ?
3. b of each term = ?
4. final expantion = ?

from form $(a+b)^n$

ANSWER (using steps)

1. coefficients = ?

our coefficients are 1 3 3 1

2. a of each term = ?

the powers of a go in descending order, starting with n from the bracket $(a+b)^n$

in this example $n=3$

a: ∴ x^3 x^2 x^1 x^0

③ b of each term = ?

the powers of b go in ascending order and end on n from the bracket $(a+b)^n$

b: $\therefore (2y)^0 \quad (2y)^1 \quad (2y)^2 \quad (2y)^3$

④ final expansion = ?

for the final expantion, each term has a constant coefficient, and an a value, and a b value.

We times these together each time.

then the terms are added in the final expansion.

Term 1 Term 2 Term 3 Term 4

$$(1)(x)^3(2y)^0 + (3)(x)^2(2y)^1 + (3)(x)^1(2y)^2 + (1)(x)^0(2y)^3$$

$$\therefore \quad (x+2y)^3$$

$$= x^3 + 3x^2(2y) + 3x(4y^2) + (2y)^3$$

$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

NOTE: this may seem complicated at first but it gets easy really quickly with practice.

A QUICKER METHOD FOR FINDING THE COEFFICIENT

${}^nc_r$ is read as 'n choose r'.

↳ this is on most calculators and will give you your coefficient without Pascal's amazing triangle.

about ${}^nc_r$:

$${}^nc_r = \frac{n!}{r!(n-r)!}$$

this formula will work if your calculator does not have ${}^nc_r$ option

$n!$ is n factorial
eg $3! = 3 \times 2 \times 1$

using ${}^nc_r$ in binomial expansion:

EXAMPLE 2: what is the coefficient of the 3rd term when you expand $(a+b)^4$?

ANSWER: ${}^nc_r$

n: 4 (given)

r: the position of the term minus 1

$\therefore 3-1$

$\therefore r = 2$

$$\therefore {}^4c_2 = \frac{4!}{2!(4-2)!}$$

$$= 6$$

or just directly stick ${}^4c_2$ into your calculator.

A QUICK METHOD FOR FINDING EACH TERM IN THE BINOMIAL EXPANSION

To find a term, use this: (I call this the AnyTermFormula)

Binomial expansion AnyTermFormula

$$\boxed{{}^nC_r\, a^{n-r} b^r}$$

taken from $(a+b)^n$

r is the position of the term minus 1

also known as the general term

So: if we use the AnyTermFormula and apply it to the entire expansion, we have a formula for the Binomial expansion in full, which looks like this:

$$\boxed{(a+b)^n = {}^nC_r\, a^{n-r} b^r + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_r\, a^{n-r} b^r}$$

where the number of terms will be $n+1$

n is a positive whole number

but most textbooks use this long thing:

$$(x+b)^n = a^n + {}^nC_r\, a^{n-1} b + {}^nC_r\, a^{n-2} b^2 + \ldots + b^n$$

both work.

NOTE THAT $\binom{n}{r} = {}^nC_r$

BINOMIAL EXPANSION: TYPES of QUESTIONS

1 (a) finding a specific term in the expansion or the coefficient of
(see EXAMPLE 2 above)

(b) Finding the term independent of x (constant term)

2 expanding the binomial

3 finding the coefficient of a specific x → when r is not given
(see example 3 below)

4 finding a random unknown in the original binomial, eg: $(x+qx)^3 \to q$ is an unknown.

4 find n of the binomial

5 Two bracket questions.

(a) when r is not given

(b) finding an unknown q / p etc

6 Binomial estimation questions
(applying the expansion to approximate stuff)

EXAMPLES:

1. a) finding the coefficient of a specific term in the expansion
(see EXAMPLE 2 above)

EXAMPLE 3 → (when r is not given)

Find the coefficient of x^3 in the following expansion. $(2x+4)^8$

ANSWER:

STEPS:
1. $n=?$ $r=?$ $a=?$ $b=?$
2. sub in what you have to AnyTermFormula
3. $r=?$
   → notice we don't have r yet, find r
4. sub r into AnyTermFormula
5. solve and answer the question.

1. $n=8$ $r=?$ $a=2x$ $b=4$
2. $^8c_r \times (2x)^{8-r} \times (4)^r$
3. $r=?$

from question $\;\;{}^8c_r \times 2^8 \times 2^{-r} \times \underbrace{x^8 \times x^{-r}}$ $\times 4^r$

→ this is what we need to find r

$x^3 = x^8 . x^{-r}$ → this says 'what r gives the coefficient of x^3

$3 = 8-r$

$r = 5$

(4) $^8C_5 \times (2x)^3 \times (4)^5$

(5) $^8C_5 \times 4^5 \times 2^3 \times x^3$ → check! is this the correct x?

$458752x^3$

$\therefore$ 458752 is the coefficient

(b) Finding the term independent of x (constant term)

STEPS: (1) Put all known values into AnyTermFormula
(2) let x-values $= x^0$ to find r
(3) sub in r and solve.

EXAMPLE 4

Give the constant term for the following binomial:
$(2x+3)^3$

ANSWER:

(1) $n=3$ $r=?$ $a=2x$ $b=3$

$^3C_r \times (2x)^{3-r} \times (3)^r$

(2) $x^0 = x^3 \times x^{-r}$

$0 = 3-r$
$r = 3$

(3) $^3C_3 \times (2x)^0 \times 3^3 = 27$

② expanding the binomial

EXAMPLE 5

Expand the following: $(x+5)^4$

ANSWER:

STEPS:
1. find coefficients with ${}^nC_r$
2. find all a's (a's exponent decreases)
3. find all b's (b's exponent increases)
4. write completed expansion and simplify

	T_1	T_2	T_3	T_4	T_5
①	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$
②	x^4	$\cancel{x^3}$	x^2	x^1	x^0
③	5^0	5^1	5^2	5^3	5^4

these 3 steps can be done at once.

④ ${}^4C_0x^4 5^0 + {}^4C_1 x^3 5^1 + {}^4C_2 x^2 5^2 + {}^4C_3 x^1 5^3 + {}^4C_4 x^0 5^4$

$= 1x^4 1 + 4x^3 5 + 6x^2 25 + 4x\,125 + 1 \times 1 \times 625$

$= x^4 + 20x^3 + 150x^2 + 500x + 625$

③ finding the coefficient of a specific x ↳ when r is not given

(see EXAMPLE 3 above)

④ finding a random unknown in the original binomial, eg: $(x+qx)^3 \rightarrow q$ is an unknown.

EXAMPLE 6

The coefficient of x^3 is equal to 15 in the binomial expansion of

$(1+kx)^{10}$ where k is a constant.

Find the value of k.

ANSWER:

STEPS: ① $a=?$ $b=?$ $r=?$ $n=?$

② sub known values into AnyTermFormula

③ use x's to find r

④ $15x^3 = x^3$ term and solve.

① $a=1$ $b=kx$ $r=?$ $n=10$

② ${}^{10}C_r \times (1)^{10-r} \times (kx)^r$

③ $x^r = x^3$

$r=3$

④ ${}^{10}C_3 \times (1)^7 \times (kx)^3 = 15x^3$

${}^{10}C_3 \times k^3 \times x^3 = 15x^3$

${}^{10}C_3 k^3 = 15$

$k^3 = \dfrac{15}{{}^{10}C_3}$

$k = \sqrt[3]{\dfrac{15}{{}^{10}C_3}}$

$k = \dfrac{1}{2}$

5 Two bracket questions.

a) when r is not given

General note for two bracket questions:

* Expand both brackets
  (if the exponent is 1, n=1, then the bracket is already expanded)
* For the final answer, multiply the answers from each bracket together
* Only find terms that are needed if it does not ask you to fully expand the binomial.

EXAMPLE 7

Find the independent term of the expansion of the following: $(2x+7)^8(4x+3)^3$

STEPS: for both brackets:

1. write what you know into AnyTermFormula
2. For independent term $x^0 = x$ term to solve for r
3. sub r's

4. times the two answers together for final answer.

ANSWER:

① $(2x+7)^8$

$n=8 \quad a=2x \quad b=7$

$^8C_r \times (2x)^{8-r} \times 7^r$

② $x^{8-r} = x^0$

$x^8 \times x^{-r} = x^0$

$8 - r = 0$

$r = 8$

③ $^8C_8 \times (2x)^{8-8} \times 7^8$

$= 1 \times 7^8$

$= 7^8$

$(4x+3)^3$

$n=3 \quad a=4x \quad b=7$

$^3C_r \times (4x)^{3-r} \times 3^r$

$x^{3-r} = x^0$

$r = 3$

$^3C_3 \times (4x)^{3-3} \times 3^3$

$= 1 \times 3^3$

$= 3^3$

④ $7^8 \times 3^3 = 155649627$

EXAMPLE 8

Find the term in x and x^2 in the following expansion:

$(4-x)(2-4x)^6$

STEPS: ① Check which bracket(s) need to be expanded

② For two bracket questions, the final answer will be multiplied, therefore to be safe, check for all terms in x up to the highest power asked for. eg. x^0, x^1, x^2 for x^2.

3) Write what you know into AnyTermFormula
4) $x^0 = x$ terms
$x^1 = x$ terms
$x^2 = x$ terms
etc
to solve for r in each case
(or by inspection if possible)
5) sub r's to get terms
6) multiply for final answer.

ANSWER:

1) $(4-x)$ already expanded

$(2-4x)^6$ needs to be expanded

2) for x^2 need x^0, x^1, x^2

3) ${}^6C_r \times (2)^{6-r} \times (-4x)^r$

4) x term: x^r

$\therefore x^0 = x^r$
$0 = r$

$\therefore x^1 = x^r$
$1 = r$

$\therefore x^2 = x^r$
$2 = r$

5) x^0: ${}^6C_0 \times (2)^{6-0} \times (-4x)^0$
$= 1 \times 2^6 \times 1$
$= 2^6$

x^1: ${}^6C_1 \times (2)^{6-1} \times (-4x)^1$
$= 6 \times 2^5 \times -4x$
$= -768x$

x^2: ${}^6C_2 \times (2)^{6-2} \times (-4x)^2$
$= 15 \times 2^4 \times 16x^2$
$= 3840x^2$

(6) $(3840x^2 - 768x + 2^6)(4 - x)$

all possible x^1 and x^2

$3840x^2 \times 4 + (-768x \times 4) + (-768x) \times (-x) + (2^6) \times (-x)$

$= 15360x^2 - 3072x + 768x^2 - 64x$

$= 16128x^2 - 3136x$

EXAMPLE 9

(a) Find the first 3 terms in the binomial expansion
$\left(x - \frac{2}{x}\right)^5$

these types of questions follow on themselves. The answer for (a) is likely to be used in (b).

(b) Find the coefficient of x in the following binomial expansion: $\left(4 + \frac{1}{x^2}\right)\left(x - \frac{2}{x}\right)^5$

ANSWER:

(a) STEPS: (1) for the first three terms, sub all info into Any Term Formula and solve

(1) ${}^5C_0 \times (x)^{5-0} \times \left(-\frac{2}{x}\right)^0$ $T_1 : r = 0$

$= 1 \times x^5 \times 1$
$= x^5$

$^5C_1 \times (x)^{5-1} \times \left(-\frac{2}{x}\right)^1$ T_2 : $r=1$

$= 5 \times x^4 \times -\frac{2}{x}$

$= -10x^3$

$^5C_2 \times (x)^{5-2} \times \left(-\frac{2}{x}\right)^2$ T_3 $r=2$

$= 10 \times x^3 \times \frac{4}{x^2}$

$= 10 \times \frac{4x^3}{x^2}$

$= 40x$

$\therefore x^5 - 10x^3 + 40x$

(b) STEPS: (1) use the previous expansion (check it works)
(2) multiply the brackets to find terms in x

(1) $(x^5 - 10x^3 + 40x)\left(4 + \frac{1}{x^2}\right)$ ✓

$(x^5 - 10x^3 + 40x)(4 + x^{-2})$

2 $-10x^3 \times x^{-2} + 40x \times 4$

$= -10x + 160x$

$= 150x$

$\therefore$ 150 is the coefficient of x

5 Two bracket questions.

b finding an unknown q/p etc

EXAMPLE 10

Find the value of k for which there is no term in x^2 in the expansion of

$$(1+kx)(2-x)^6$$

ANSWER:

STEPS: for 'no term in x^2', find the x^2, then equal to zero

1. Expand first 3 terms of second bracket
2. times brackets to find x^2 terms
3. let x^2 terms $= 0x^2$ and solve for k

1. $(2-x)^6$

T_1: ${}^6C_0 \times (2)^{6-0} \times (-x)^0 = 1 \times 2^6 \times 1 = 64$

T_2: ${}^6C_1 \times (2)^{6-1} \times (-x)^1 = 6 \times 2^5 \times -x = -192x$

T_3: ${}^6C_2 \times (2)^{6-2} \times (-x)^2 = 15 \times 2^4 \times x^2 = 240x^2$

$\therefore (64 - 192x + 240x^2)$

2. $(64 - 192x + 240x^2)(1+kx)$

$240x^2 \times 1 + (-192x \times kx) = 0x^2$

$240x^2 - 192kx^2 = 0x^2$

$240 - 192k = 0$

$240 = 192k$

$k = 1.25$

6 Binomial estimation questions
(applying the expansion to approximate stuff)

EXAMPLE 11

(a) Find the first four terms of the binomial expansion

$$\left(1-\frac{x}{4}\right)^{10}$$

(b) Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

ANSWER:

(a) T_1: $^{10}C_0 \times 1 \times \left(-\frac{x}{4}\right)^0 = 1$

T_2: $^{10}C_1 \times 1 \times \left(-\frac{x}{4}\right)^1 = -\frac{x}{4} \times 10 = -\frac{10}{4}x$

T_3: $^{10}C_2 \times 1 \times \left(-\frac{x}{4}\right)^2 = 45 \times 1 \times \frac{x^2}{16} = \frac{45}{16}x^2$

T_4: $^{10}C_3 \times 1 \times \left(-\frac{x}{4}\right)^3 = 120 \times 1 \times -\frac{x^3}{64} = -\frac{120}{64}x^3$

$\therefore\ 1 - 2.5x + 2.8125x^2 - 1.875x^3$

(b) **STEPS:** 1) for estimation, let the original given bracket equal the given value and solve for x

2) sub the answer of x into the expanded binomial from question (a)

① let $(1-\frac{x}{4})^{10} = 0.975^{10}$

$1-\frac{x}{4} = 0.975$

$x = -(0.975-1)\times 4$

$x = \frac{1}{10} = 0.1$

② sub $x=0.1$ into

$1-2.5x+2.8125x^2-1.875x^3$

$\therefore 1-2.5(0.1)+2.8125(0.1)^2-1.875(0.1)^3$

$= 0.77625$

$= 0.7763$ to four decimal places.