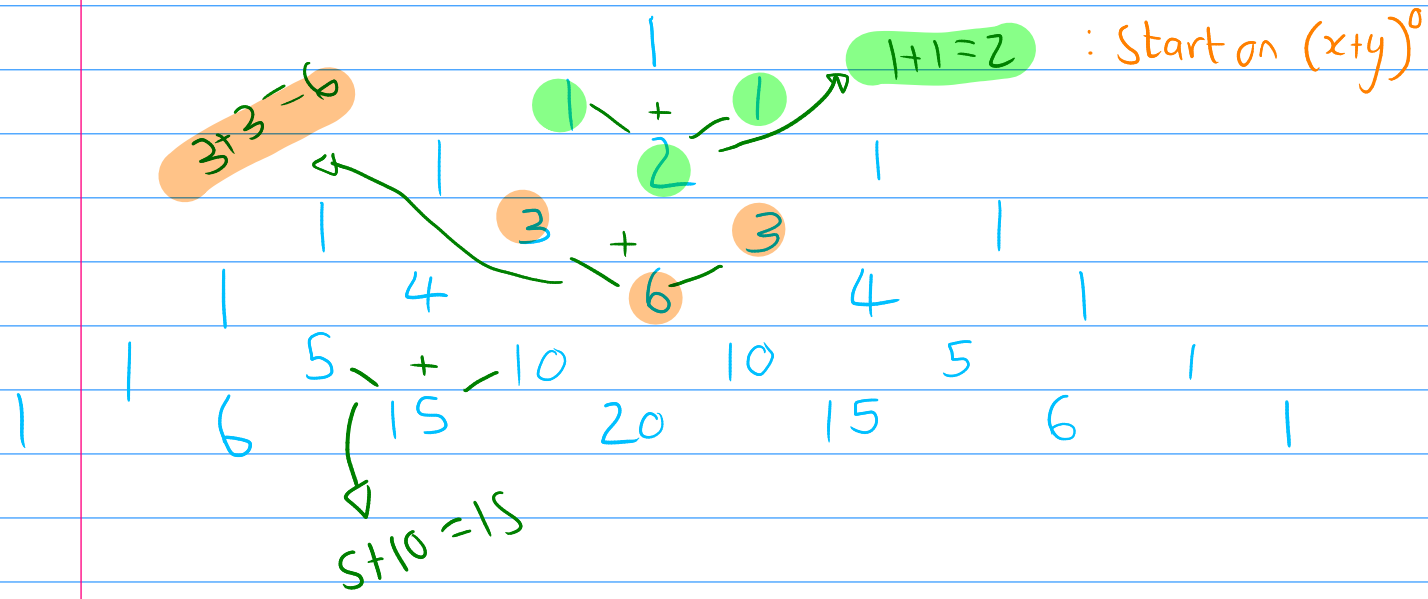
BINOMIAL EXPANSION

for expanding brackets with high exponents from the form: $(a+b)^n$

Introduction:

One common method of expanding brackets is using

PASCAL'S TRIANGLE



* Pascal's triangle works by adding the above two number together each time.
* the rows go with the exponents of the brackets you want to expand, starting with $(x+y)^0$.

* Pascal's triangle tells us about the coefficients of each term in the expanded form

EXAMPLE 1:

Expand using pascal's triangle:

(a) $(x+2y)^3$

ANSWER:

The exponent is 3; therefore we use row 4 of the triangle (row 1 is 0)

row 4: 1 3 3 1

STEPS:

1. coefficients = ?
2. a of each term = ?
3. b of each term = ?
4. final expantion = ?

] from form $(a+b)^n$

ANSWER (using steps)

1. coefficients = ?

our coefficients are 1 3 3 1

2. a of each term = ?

the powers of a go in descending order, starting with n from the bracket $(a+b)^n$

in this example $n=3$

a: $\therefore$ x^3 x^2 x^1 x^0

③ b of each term = ?

the powers of b go in ascending order and end on n from the bracket $(a+b)^n$

b: $\therefore (2y)^0 \quad (2y)^1 \quad (2y)^2 \quad (2y)^3$

④ final expansion = ?

for the final expantion, each term has a constant coefficient, and an a value, and a b value.

We times these together each time.

then the terms are added in the final expansion.

Term 1 Term 2 Term 3 Term 4

$$(1)(x)^3(2y)^0 + (3)(x)^2(2y)^1 + (3)(x)^1(2y)^2 + (1)(x)^0(2y)^3$$

$$\therefore (x+2y)^3$$

$$= x^3 + 3x^2(2y) + 3x(4y^2) + (2y)^3$$

$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

NOTE: this may seem complicated at first but it gets easy really quickly with practice.

A QUICKER METHOD for FINDING THE COEFFICIENT

${}^{n}C_{r}$ is read as 'n choose r'.

↳ this is on most calculators and will give you your coefficient without Pascal's amazing triangle.

about ${}^{n}C_{r}$:

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

→ this formula will work if your calculator does not have ${}^{n}C_{r}$ option

→ n! is n factorial
eg $3! = 3 \times 2 \times 1$

using ${}^{n}C_{r}$ in binomial expansion:

EXAMPLE 2: what is the coefficient of the 3rd term when you expand $(a+b)^4$?

ANSWER: ${}^{n}C_{r}$

- 4 (given)
- the position of the term minus 1
  $\therefore 3-1$
  $\therefore r = 2$

$$\therefore {}^{4}C_{2} = \frac{4!}{2!(4-2)!}$$

$$= 6$$

→ or just directly stick ${}^{4}C_{2}$ into your calculator.

A QUICK METHOD FOR FINDING EACH TERM IN THE BINOMIAL EXPANSION

To find a term, use this: (I call this the AnyTermFormula)

Binomial expansion AnyTermFormula

$$ {}^{n}C_{r}\, a^{n-r} b^{r} $$

→ taken from $(a+b)^n$

→ r is the position of the term minus 1

also known as the general term

So: if we use the AnyTermFormula and apply it to the entire expansion, we have a formula for the Binomial expansion in full, which looks like this:

$$ (a+b)^{n} = {}^{n}C_{r}\, a^{n-r} b^{r} + {}^{n}C_{r}\, a^{n-r} b^{r} + \ldots + {}^{n}C_{r}\, a^{n-r} b^{r} $$

where the number of terms will be $n+1$

→ n is a positive whole number

but most textbooks use this long thing:

$$ (x+b)^{n} = a^{n} + {}^{n}C_{r}\, a^{n-1} b + {}^{n}C_{r}\, a^{n-2} b^{2} + \ldots + b^{n} $$

both work.

NOTE THAT $\binom{n}{r} = {}^{n}C_{r}$

BINOMIAL EXPANSION: TYPES of QUESTIONS

1. a) finding the coefficient of x^r a specific term in the expansion (see EXAMPLE 2 above)
   b) Finding the term independent of x (constant term)
2. expanding the binomial

3. finding the coefficient of a specific x (see example 3 below) → when r is not given
4. finding a random unknown in the original binomial, eg: $(x+qx)^3$ → q is an unknown.
4. find n of the binomial
5. Two bracket questions.
   a) when r is not given
   b) finding an unknown q / p etc
6. Binomial estimation questions
   (applying the expansion to approximate stuff)

EXAMPLES:

1. a) finding a specific term in the expansion (the coefficient of)

(see EXAMPLE 2 above)

EXAMPLE 3 → (when r is not given)

Find the coefficient of x^3 in the following expansion. $(2x+4)^8$

ANSWER:

STEPS:
1. $n=?$ $r=?$ $a=?$ $b=?$
2. sub in what you have to AnyTermFormula
3. $r=?$
   - notice we don't have r yet, find r
4. sub r into AnyTermFormula
5. solve and answer the question.

1. $n=8$ $r=?$ $a=2x$ $b=4$
2. ${}_8C_r \times (2x)^{8-r} \times (4)^r$
3. $r=?$

from question $\leftarrow$ ${}_8C_r \times 2^8 \times 2^{-r} \times x^8 \times x^{-r} \times 4^r$

$x^3 = x^8 \cdot x^{-r}$ → this is what we need to find r

$3 = 8-r$

$r = 5$ → this says 'what r gives the coefficient of x^3'

④ $^8C_5 \times (2x)^3 \times (4)^5$

⑤ $8c_5 \times 4^5 \times 2^3 \times x^3$ → check! is this the correct x?

$458752x^3$

∴ 458752 is the coefficient

(b) Finding the term independent of x (constant term)

STEPS: ① Put all known values into AnyTermFormula
② let x-values $= x^0$ to find r
③ sub in r and solve.

EXAMPLE 4

Give the constant term for the following binomial:
$(2x+3)^3$

ANSWER:

① $n=3 \quad r=? \quad a=2x \quad b=3$

$^3C_r \times (2x)^{3-r} \times (3)^r$

② $x^0 = x^3 \times x^{-r}$

$0 = 3-r$

$r = 3$

③ $^3C_3 \times (2x)^0 \times 3^3 = 27$

(2) expanding the binomial

EXAMPLE 5

Expand the following: $(x+5)^4$

ANSWER:

STEPS:
1. find coefficients with ${}^nC_r$
2. find all a's (a's exponent decreases)
3. find all b's (b's exponent increases)
4. write completed expansion and simplify

(1) T_1 T_2 T_3 T_4 T_5

${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

(2) x^4 x^3 x^2 x^1 x^0

(3) 5^0 5^1 5^2 5^3 5^4

these 3 steps can be done at once.

(4) ${}^4C_0x^45^0 + {}^4C_1x^35^1 + {}^4C_2x^25^2 + {}^4C_3x^15^3 + {}^4C_4x^05^4$

$= 1x^41 + 4x^35 + 6x^225 + 4x125 + 1\times1\times625$

$= x^4 + 20x^3 + 150x^2 + 500x + 625$

③ finding the coefficient of a specific x (see EXAMPLE 3 above) ↳ when r is not given

④ finding a random unknown in the original binomial, eg: $(x+qx)^3$ → q is an unknown.

EXAMPLE 6

The coefficient of x^3 is equal to 15 in the binomial expansion of

$(1+kx)^{10}$ where k is a constant.

Find the value of k.

ANSWER:

STEPS: ① $a=?$ $b=?$ $r=?$ $n=?$
② Sub known values into Any Term Formula
③ use x's to find r
④ $15x^3$ = x^3 term and solve.

① $a=1$ $b=kx$ $r=?$ $n=10$

② ${}^{10}C_r \times (1)^{10-r} \times (kx)^r$

③ $x^r = x^3$
$r=3$

④ ${}^{10}C_3 \times (1)^7 \times (kx)^3 = 15x^3$
${}^{10}C_3 \times k^3 \times x^3 = 15x^3$
${}^{10}C_3 k^3 = 15$

$k^3 = \frac{15}{{}^{10}C_3}$

$k = \sqrt[3]{\frac{15}{{}^{10}C_3}}$

$k = \frac{1}{2}$

5 Two bracket questions.

9 when r is not given

General note for two bracket questions:

* Expand both brackets
  (if the exponent is 1, $n=1$, then the bracket is already expanded)
* For the final answer, multiply the answers from each bracket together
* Only find terms that are needed if it does not ask you to fully expand the binomial.

EXAMPLE 7

Find the independent term of the expansion of the following: $(2x+7)^8(4x+3)^3$

STEPS: for both brackets:

1. write what you know into $AnyTermFormula$
2. for independent term $x^0 = x$ term to solve for r
3. sub r's

4. times the two answers together for final answer.

ANSWER:

① $(2x+7)^8$

$n=8 \quad a=2x \quad b=7$

$^8C_r \times (2x)^{8-r} \times 7^r$

$(4x+3)^3$

$n=3 \quad a=4x \quad b=7$

$^3C_r \times (4x)^{3-r} \times 3^r$

② $x^{8-r} = x^0$

$x^8 \times x^{-r} = x^0$

$8-r=0$

$r=8$

$x^{3-r} = x^0$

$r=3$

③ $^8C_8 \times (2x)^{8-8} \times 7^8$

$= 1 \times 7^8$

$= 7^8$

$^3C_3 \times (4x)^{3-3} \times 3^3$

$= 1 \times 3^3$

$= 3^3$

④ $7^8 \times 3^3 = 155649627$

EXAMPLE 8

Find the term in x and x^2 in the following expansion:

$(4-x)(2-4x)^6$

STEPS: ① Check which bracket(s) need to be expanded

② For two bracket questions, the final answer will be multiplied, therefore to be safe, check for all terms in x up to the highest power asked for. eg. x^0, x^1, x^2 for x^2.

(3) Write what you know into AnyTermFormula

(4) $x^0 = x$ terms
$x^1 = x$ terms
$x^2 = x$ terms
etc
to solve for r in each case
(or by inspection if possible)

(5) sub r's to get terms

(6) multiply for final answer.

ANSWER:

(1) $(4-x)$ already expanded

$(2-4x)^6$ needs to be expanded

(2) for x^2 need x^0, x^1, x^2

(3) ${}^6C_r \times (2)^{6-r} \times (-4x)^r$

(4) x term: x^r

$\therefore x^0 = x^r$

$0 = r$

$\therefore x^1 = x^r$

$1 = r$

$\therefore x^2 = x^r$

$2 = r$

(5) x^0: ${}^6C_0 \times (2)^{6-0} \times (-4x)^0$

$= 1 \times 2^6 \times 1$

$= 2^6$

x^1: ${}^6C_1 \times (2)^{6-1} \times (-4x)^1$

$= 6 \times 2^5 \times -4x$

$= -768x$

x^2: ${}^6C_2 \times (2)^{6-2} \times (-4x)^2$

$= 15 \times 2^4 \times 16x^2$

$= 3840x^2$

⑥ $(3840x^2 - 768x + 2^6)(4 - x)$ → all possible x^1 and x^2

$3840x^2 \times 4 + (-768x \times 4) + (-768x) \times (-x) + (2^6) \times (-x)$

$= 15360x^2 - 3072x + 768x^2 - 64x$

$= 16128x^2 - 3136x$

EXAMPLE 9

(a) Find the first 3 terms in the binomial expansion

$\left(x - \frac{2}{x}\right)^5$ → these types of questions follow on themselves. The answer for (a) is likely to be used in (b).

(b) Find the coefficient of x in the following binomial expansion: $\left(4 + \frac{1}{x^2}\right)\left(x - \frac{2}{x}\right)^5$

ANSWER:

(a) STEPS: (1) for the first three terms, sub all info into Any Term Formula and solve

(1) ${}^5C_0 \times (x)^{5-0} \times \left(-\frac{2}{x}\right)^0$ $T_1: r = 0$

$= 1 \times x^5 \times 1$

$= x^5$

$${}^5C_1 \times (x)^{5-1} \times \left(-\frac{2}{x}\right)^1 \qquad T_2 : r = 1$$

$$= 5 \times x^4 \times -\frac{2}{x}$$

$$= -10x^3$$

$${}^5C_2 \times (x)^{5-2} \times \left(-\frac{2}{x}\right)^2 \qquad T_3 \quad r = 2$$

$$= 10 \times x^3 \times \frac{4}{x^2}$$

$$= 10 \times \frac{4x^3}{x^2}$$

$$= 40x$$

$$\therefore x^5 - 10x^3 + 40x$$

(b) STEPS: (1) use the previous expansion (check it works)
(2) multiply the brackets to find terms in x

(1) $\left(x^5 - 10x^3 + 40x\right)\left(4 + \frac{1}{x^2}\right)$ ✓

$$\left(x^5 - 10x^3 + 40x\right)\left(4 + x^{-2}\right)$$

2 $$-10x^3 \times x^{-2} + 40x \times 4$$

$$= -10x + 160x$$

$$= 150x$$

$\therefore$ 150 is the coefficient of x

5 Two bracket questions.
b finding an unknown q/p etc

EXAMPLE 10

Find the value of k for which there is no term in x^2 in the expansion of

$$(1+kx)(2-x)^6$$

ANSWER:

STEPS: for 'no term in x^2', find the x^2, then equal to zero

1. Expand first 3 terms of second bracket
2. times brackets to find x^2 terms
3. let x^2 terms $= 0x^2$ and solve for k

(1) $(2-x)^6$

T_1: ${}^6C_0 \times (2)^{6-0} \times (-x)^0 = 1 \times 2^6 \times 1 = 64$

T_2: ${}^6C_1 \times (2)^{6-1} \times (-x)^1 = 6 \times 2^5 \times -x = -192x$

T_3: ${}^6C_2 \times (2)^{6-2} \times (-x)^2 = 15 \times 2^4 \times x^2 = 240x^2$

$\therefore (64 - 192x + 240x^2)$

(2) $(64 - 192x + 240x^2)(1+kx)$

$240x^2 \times 1 + (-192x \times kx) = 0x^2$

$240x^2 - 192kx^2 = 0x^2$

$240 - 192k = 0$

$240 = 192k$

$k = 1.25$

(6) Binomial estimation questions (applying the expansion to approximate stuff)

EXAMPLE 11

(a) Find the first four terms of the binomial expansion

$$\left(1-\frac{x}{4}\right)^{10}$$

(b) Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

ANSWER:

(a) $T_1: {}^{10}C_0 \times 1 \times \left(-\frac{x}{4}\right)^0 = 1$

$T_2: {}^{10}C_1 \times 1 \times \left(-\frac{x}{4}\right)^1 = -\frac{x}{4} \times 10 = -\frac{10}{4}x$

$T_3: {}^{10}C_2 \times 1 \times \left(-\frac{x}{4}\right)^2 = 45 \times 1 \times \frac{x^2}{16} = \frac{45}{16}x^2$

$T_4: {}^{10}C_3 \times 1 \times \left(-\frac{x}{4}\right)^3 = 120 \times 1 \times -\frac{x^3}{64} = -\frac{120}{64}x^3$

$\therefore 1 - 2.5x + 2.8125x^2 - 1.875x^3$

(b) **STEPS:**

1. for estimation, let the original given bracket equal the given value and solve for x
2. sub the answer of x into the expanded binomial from question (a)

① let $(1-\frac{x}{4})^{10} = 0.975^{10}$

$1-\frac{x}{4} = 0.975$

$x = -(0.975-1)\times 4$

$x = \frac{1}{10} = 0.1$

② sub $x=0.1$ into

$1-2.5x+2.8125x^2-1.875x^3$

$\therefore\ 1-2.5(0.1)+2.8125(0.1)^2-1.875(0.1)^3$

$= 0.77625$

$= 0.7763$ to four decimal places.