VECTORS

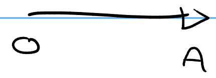
Including the following:

WHAT ARE VECTORS

1. Vectors have MAGNITUDE (length / distance) and direction

2. SCALARS Only have magnitude

3. To write: typed a vector



$\mathbf{a}$ OR $\mathbf{OA}$

handwritten: $\underline{a}$ OR OA OR $\tilde{a}$

4. The length / magnitude / modulus of a vector can be written as

a OR $|a|$

5. Vectors in the directions x, y, z on the cartesian plane are written i, j, k.

6. Magnitude-direction form: $(r; \theta)$

length (r), angle (θ)

General rule:

$$(r;\theta) \rightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = (r\cos\theta)\,i + (r\sin\theta)\,j$$

* for 2D

7 Component form: $\begin{pmatrix}x\\y\end{pmatrix} = xi + yj$

$* 2D$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = xi + yj + zk$

$* 3D$

8 POSITION VECTOR

$\vec{OP}$

origin joins point p

9 $\vec{AB}$ is $b - a$

$\vec{OB}$ $\vec{OA}$

10 Angle between vectors:

θ: $\cos\theta = \dfrac{a.b}{|a|.|b|}$

to find θ use:

where $a.b = a_1b_1 + a_2b_2$ $* 2D$

$= a_1b_1 + a_2b_2 + a_3b_3$ $* 3D$

11 $\vec{AB} \neq \vec{BA}$

direction matters

12 Magnitude of a vector: $|a| = \sqrt{x^2+y^2}$ $* 2D$

$|b| = \sqrt{x^2+y^2+z^2}$ $* 3D$

FINDING VECTORS

$\overrightarrow{AB}$ or $\begin{pmatrix} x \\ y \end{pmatrix}$

x and y show the movement on the x and y axis

Example 1

Find $\overrightarrow{AB}$, where $A(1;4)$ and $B(3;7)$

Answer:

$\overrightarrow{AB}$: $x: 3-1$
$y: 7-4$ $\therefore \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ or $2i+3j$

* 2D

Example 2

find $\overrightarrow{DE}$ where $D\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$ & $E\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$

Answer

$\overrightarrow{DE}$: $x: 2-1$
$y: -5-3$
$z: 0-7$ $= \begin{pmatrix} 1 \\ -8 \\ -7 \end{pmatrix}$

* 3D

or $\overrightarrow{DE} = 1i - 8j - 7k$

FINDING MAGNITUDE OF A VECTOR

Use $|a| = \sqrt{x^2+y^2}$

$|b| = \sqrt{x^2+y^2+z^2}$

Example 3

Find the magnitude of $\vec{AB}$ if

$$\vec{AB} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Answer

$$|\vec{AB}| = \sqrt{6^2+9^2}$$

$$= \sqrt{36+81}$$

$$= 3\sqrt{13}$$

Example 4

Find the magnitude of $\vec{AB}$, where $\vec{AB} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$

Answer:

$$|\vec{AB}| = \sqrt{6^2+2^2+(-4)^2}$$

$$= \sqrt{36+4+16}$$

$$= \sqrt{56}$$

$$= 2\sqrt{14}$$

FINDING UNIT VECTORS

* Unit vectors have a magnitude of 1

Steps ① find the magnitude of the vector
② divide the vector by the magnitude

Example 5

Find the unit vector of $\vec{AB}$ if $\vec{AB} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

Answer

① $|\vec{AB}| = \sqrt{4^2+9^2}$

$= \sqrt{16+81}$

$= \sqrt{97}$

② unit vector $= \dfrac{1}{\sqrt{97}} \times \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

divide by $\sqrt{97}$ this way

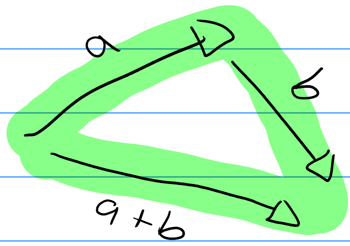
$= \begin{pmatrix} \frac{1}{\sqrt{97}} \times 4 \\ \frac{1}{\sqrt{97}} \times 9 \end{pmatrix} = \begin{pmatrix} \frac{4\sqrt{97}}{97} \\ \frac{9\sqrt{97}}{4} \end{pmatrix}$

multiplying by scalar

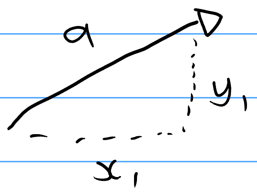
see p.

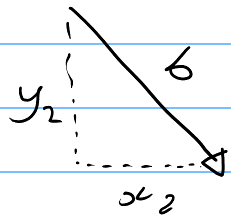
$\approx \begin{pmatrix} 22.16 \\ 0.406 \end{pmatrix}$

ADDING VECTORS



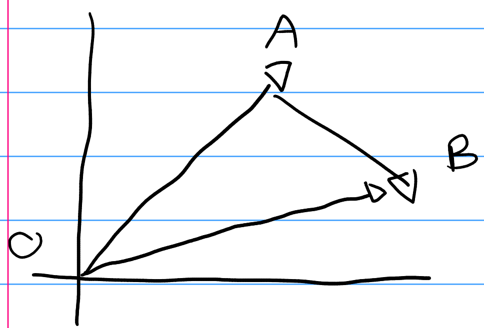
Steps: ① Add x-values and y-values





$$a + b = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Example 6



$$\overrightarrow{OA} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

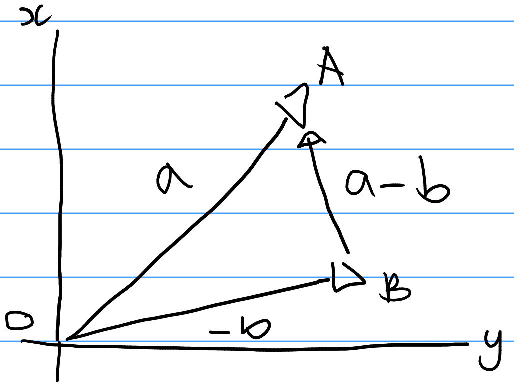
find $\overrightarrow{OB}$

Answer: $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$$= \begin{pmatrix} 8 \\ 12 \end{pmatrix} + \begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 8+4 \\ 12-6 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

SUBTRACTING VECTORS

Subtracting vectors is adding the negative of a vector, or, reversing the direction.



$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{BA}$

$\therefore$ $\overrightarrow{BA}$ is negative (reversed) $\overrightarrow{OB}$ plus $\overrightarrow{OA}$

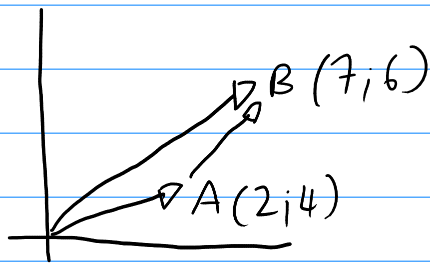
note that direction matters

$\overrightarrow{BA} \neq \overrightarrow{AB}$

Example 7

A (2;4) and B (7;6) on the cartesian plane

find $\overrightarrow{AB}$



Answer

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7-2 \\ 6-4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

MULTIPLYING VECTORS

Multiplying two vectors is known as the 'dot product' or 'scalar product'.

*2D

General rule: $\tilde{a} \times \tilde{b} = (a_x \times b_x) + (a_y \times b_y)$

*3D

$\tilde{a} \times \tilde{b} = (a_x \times b_x) + (a_y \times b_y) + (a_z \times b_z)$

Example 7

$\tilde{a} = \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix}$ anb $\tilde{b} = \begin{pmatrix} 3 \\ 2 \\ 10 \end{pmatrix}$

find the dot product.

Answer

$\tilde{a} \times \tilde{b} = (4 \times 3) + (7 \times 2) + (9 \times 10)$

$= 12 + 14 + 90$

$= 116$

PEPENDICULAR VECTORS

If two vectors are perpendicular,

the dot product $= 0$

Example 8

Prove that $\tilde{a}$ is perpendicular to $\tilde{b}$.

$\tilde{a}\begin{pmatrix}-6\\12\end{pmatrix}$ $\tilde{b}\begin{pmatrix}6\\3\end{pmatrix}$

Answer

$$\tilde{a} \times \tilde{b} = (-6\times 6) + (12\times 3)$$

$$= -36 + 36$$

$$= 0$$

$$\therefore \tilde{a} \perp \tilde{b}$$

PARALLEL VECTORS

For parallel vectors, there is a constant scalar

If $\tilde{a} \times k = \tilde{b}$

then $\tilde{a} \parallel \tilde{b}$

Steps: ① attempt to find a potential constant scalar by dividing $\tilde{b}$ x-value by $\tilde{a}$ x-value

② Test the scalar on the y-value and z-value.

If the scalar gets all be values of $\tilde{b}$ then $\tilde{a} \parallel \tilde{b}$.

Example 8

Prove that $\tilde{a} \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ is parallel to $\tilde{b} \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}$

Answer:

$4 \div 2 = 2$
$8 \div 4 = 2$
$6 \div 3 = 2$

$\therefore$ 2 is a constant scalar

$\therefore \tilde{a} \parallel \tilde{b}$ because $2\tilde{a} = \tilde{b}$

MAGNITUDE - DIRECTION FORM $(r;\theta)$

General rule: $(r;\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = (r\cos\theta)i + (r\sin\theta)j$ → component form.

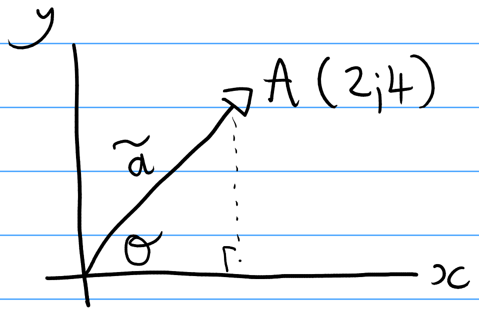
Steps:

1. The magnitude / modulus / length of a vector is the absolute value of the length. $|\bar{a}|$
   This is done using Pythagorus

2. The direction of a vector can be deduced using the angle between the vector and the x-axis
   This is done using $\tan\theta = m$ SOH CAH TOA

Example 9

Write the vector $\tilde{a}$ in magnitude direction form



Answer

$|\tilde{a}| = \left|\pm\sqrt{2^2+4^2}\right|$ → length of the vector $\tilde{a}$ (using Pyth.)

$= +\sqrt{20}$

θ: $\tan\theta = \frac{4}{2}$

$\theta = \tan^{-1}(2) = 63{,}435$

$\therefore (r;\theta) \rightarrow (\sqrt{20}; 63{,}435)$

FROM MAGNITUDE-DIRECTION TO COMPONENT FORM

Steps ① sketch

② apply rule $(r;\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

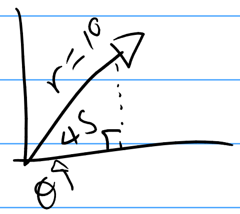
$= r\cos\theta\, i + r\sin\theta\, j$

Example 10

Write the vector $(5;45°)$ in component form.

Answer:

① $r = 10$
$\theta = 45°$



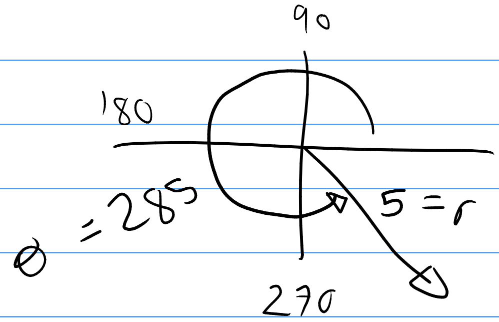
② $(10;45) = \begin{pmatrix} 10\cos 45 \\ 10\sin 45 \end{pmatrix} = 5\sqrt{2}\, i + 5\sqrt{2}\, j$

Example 11

Write the vector $(5;285°)$ in component form

Answer

① $r = 5$
$\theta = 285°$



② $(5;285°) = \begin{pmatrix} 5\cos 285° \\ 5\sin 285° \end{pmatrix} = 1.294i - 4.83j$

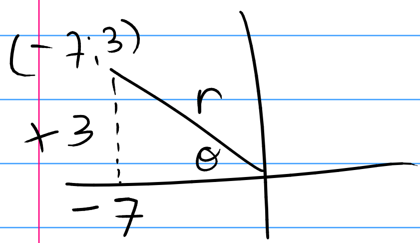
FROM COMPONENT FORM TO MAGNITUDE-DIRECTION FORM

Steps
1. sketch
2. find r using distance formula $\sqrt{i^2+j^2}$
3. find θ using $\tan\theta = \frac{O}{A}$
4. Check quadrant for final θ

Example 12

Write the vector $-7i + 3j$ in magnitude-direction form.

Answer

1. 

2. $r = \sqrt{(-7)^2+(3)^2}$

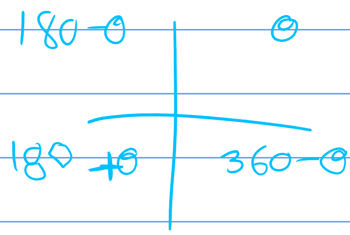
$= \sqrt{58}$

3. $\tan\theta = \frac{O}{A} = \frac{3}{7}$ → use positive tan

$\theta = \tan^{-1}\left(\frac{3}{7}\right)$

$= 23,2$

4. 2nd quadrant $\therefore$ $180 - \theta$



$\therefore 180 - 23.2$

$= 156.80$

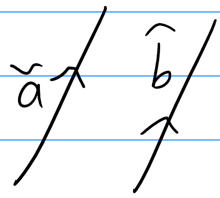
$\therefore (r;\theta) = (\sqrt{58}; 156.8^\circ)$

EQUAL VECTORS

Equal vectors have equal length and direction.

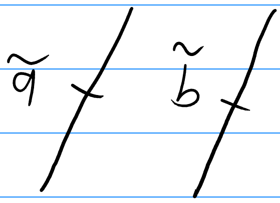
$\therefore$ if $\tilde{a} = \tilde{b}$

then



(same direction)

and



(same length)

POSITION VECTORS

Position vectors join a point to the origin eg. $\overrightarrow{OA}$

Component form does not give the position of the vector.

Example 13

Given

$L(4;4)$; $M(-2;-1)$ and $N(2;3)$

(I) Write the position vector of L in component form

Answer

Position vector of L is $\overrightarrow{OL}$

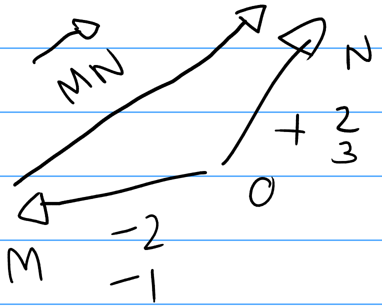
$$\therefore \overrightarrow{OL} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4i + 4j$$

(II) Write $\overrightarrow{MN}$ in component form

Answer

$\overrightarrow{MN} = ?$

sketch:



backwards

$$\therefore \overrightarrow{MN} = -\begin{pmatrix} -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{matrix} 4 \\ 4 \end{matrix} = 4i + 4j$$

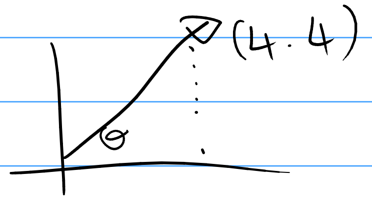
(iii) What can be deduced about $\overrightarrow{OL}$ and $\overrightarrow{MN}$?

Answer

$\overrightarrow{OL}$ and $\overrightarrow{MN}$ are equal in length and direction.
↳ (parellel)

PROOF of ∥ (CHECK)

$\overrightarrow{OL} = 4i + 4j$



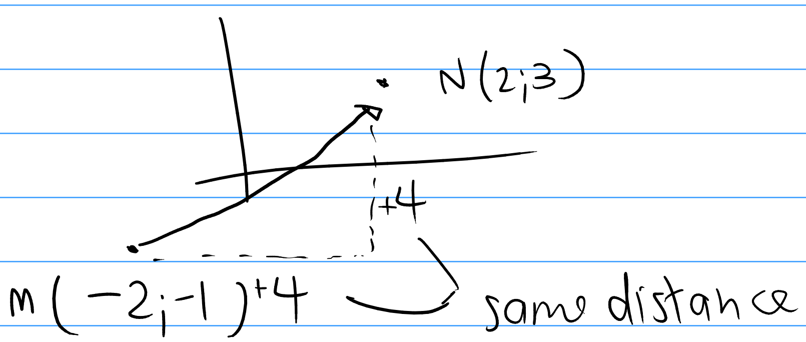
$|\overrightarrow{OL}| = \sqrt{4^2+4^2} = 4\sqrt{2}$

$\tan\theta = \frac{4}{4}$

$\theta = \tan^{-1}(1) = 45°$

$|\overrightarrow{MN}| = \sqrt{4^2+4^2} = 4\sqrt{2}$

$\tan\theta = ?$



$\therefore \tan\theta = \frac{4}{4}$
$= 45°$

EXTRA NOTE ON POSITION VECTORS

A vector that is the line segment between two points can be found by subtracting the position vectors of those points.

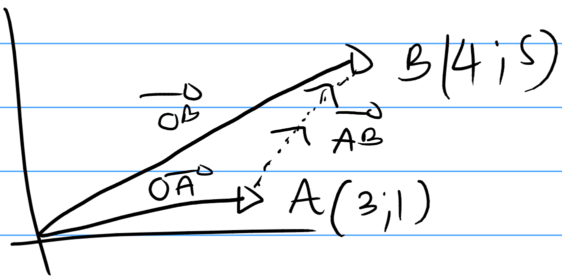
Example 14

Given $A(3;1)$ and $B(4;5)$

find $\overrightarrow{AB}$

Answer

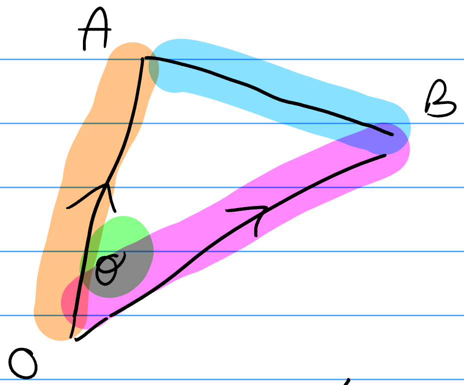
$$-\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{AB}$$



$$\overrightarrow{AB} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} = 1i + 4j$$

THE ANGLE BETWEEN TWO VECTORS (SCALAR PRODUCT)



Use cosine rule: (proof)

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2\,OA \times OB}$$

$$\therefore \cos\theta = \frac{|\vec{OA}|^2 + |\vec{OB}|^2 - |\vec{AB}|^2}{2\,|\vec{OA}| \times |\vec{OB}|}$$

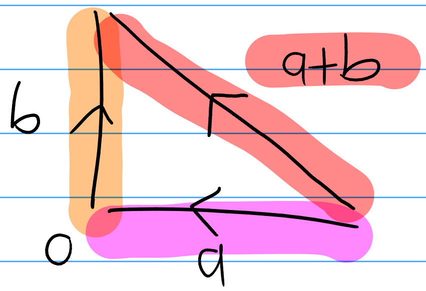
$$\therefore \cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|a| \cdot |b|}$$

and

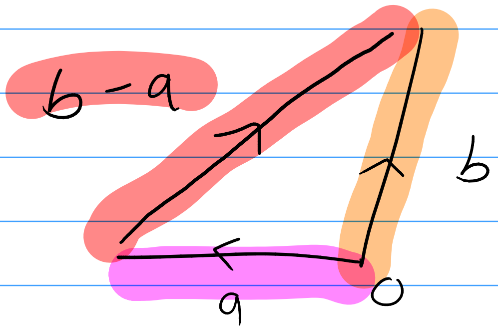
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a| \cdot |b| \cdot \cos\theta$$

POSITIVE AND NEGATIVE VECTORS – QUICK NOTE

①



②



③