HASTINGS MATH.

BINOMIAL EXPANSION

For expanding brackets with high exponents from the form: $(a+b)^n$

Introduction:

One common method of expanding brackets is using

PASCAL'S TRIANGLE

1   : Start on $(x+y)^0$

1 + 1 $1+1=2$

1   2   1

1   3 + 3   1 $3+3=6$

1   4   6   4   1

1   5 + 10   10   5   1

1   6   15   20   15   6   1

$5+10=15$

* Pascal's triangle works by adding the above two number together each time.
* the rows go with the exponents of the brackets you want to expand, starting with $(x+y)^0$.

* Pascal's triangle tells us about the coefficients of each term in the expanded form

EXAMPLE 1:

Expand using pascal's triangle:

(a) $(x+2y)^3$

ANSWER:

The exponent is 3; therefore we use row 4 of the triangle (row 1 is 0)

row 4: 1 3 3 1

STEPS:

1. coefficients = ?
2. a of each term = ?
3. b of each term = ?
4. final expantion = ?

from form $(a+b)^n$

ANSWER (using steps)

1. coefficients = ?

our coefficients are 1 3 3 1

2. a of each term = ?

the powers of a go in descending order, starting with n from the bracket $(a+b)^n$

in this example $n = 3$

a: ∴ x^3 x^2 x^1 x^0

3) b of each term = ?

the powers of b go in ascending order and end on n from the bracket $(a+b)^n$

b: $\therefore (2y)^0 \quad (2y)^1 \quad (2y)^2 \quad (2y)^3$

4) final expansion = ?

for the final expantion, each term has a constant coefficient, and an a value, and a b value.

We times these together each time.

then the terms are added in the final expansion.

Term 1 Term 2 Term 3 Term 4

$$(1)(x)^3(2y)^0 + (3)(x)^2(2y)^1 + (3)(x)^1(2y)^2 + (1)(x)^0(2y)^3$$

$$\therefore \quad (x+2y)^3$$

$$= x^3 + 3x^2(2y) + 3x(4y^2) + (2y)^3$$

$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

NOTE: this may seem complicated at first but it gets easy really quickly with practice.

A QUICKER METHOD FOR FINDING THE COEFFICIENT

^nC_r is read as 'n choose r'.

→ this is on most calculators and will give you your coefficient without Pascal's amazing triangle.

about ^nC_r:

$$^nC_r = \frac{n!}{r!(n-r)!}$$

→ this formula will work if your calculator does not have ^nC_r option

→ $n!$ is n factorial

eg $3! = 3 \times 2 \times 1$

$0! = 1$

using ^nC_r in binomial expansion:

EXAMPLE 2: what is the coefficient of the 3rd term when you expand $(a+b)^4$?

ANSWER: ^nC_r

n → 4 (given)

r → the position of the term minus 1

$\therefore$ 3-1

$\therefore r = 2$

$$\therefore \; ^4C_2 = \frac{4!}{2!(4-2)!}$$

$$= 6$$

→ or just directly stick $^4C_2$ into your calculator.

A QUICK METHOD FOR FINDING EACH TERM IN THE BINOMIAL EXPANSION

To find a term, use this: (I call this the AnyTermFormula)

Binomial expansion AnyTermFormula

$$\boxed{{}^nC_r\, a^{n-r} b^r}$$

taken from $(a+b)^n$

r is the position of the term minus 1

also known as the general term

So: if we use the AnyTermFormula and apply it to the entire expansion, we have a formula for the Binomial expansion in full, which looks like this:

$$\boxed{(a+b)^n = {}^nC_r\, a^{n-r} b^r + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_r\, a^{n-r} b^r}$$

where the number of terms will be $n+1$

n is a positive whole number

but most textbooks use this long thing:

$$(x+b)^n = a^n + {}^nC_r\, a^{n-1} b + {}^nC_r\, a^{n-2} b^2 + \ldots + b^n$$

both work.

NOTE THAT $\binom{n}{r} = {}^nC_r$

BINOMIAL EXPANSION: TYPES OF QUESTIONS

① ⓐ finding a specific term in the expansion (or the coefficient of)
(see EXAMPLE 2 above)

ⓑ Finding the term independent of x (constant term)

② expanding the binomial

③ finding the coefficient of a specific x ↳ when r is not given
(see example 3 below)

④ finding a random unknown in the original binomial, eg: $(x+qx)^3 \rightarrow q$ is an unknown.

⑤ Two bracket questions.

ⓐ when r is not given

ⓑ finding an unknown q/p etc

⑥ Binomial estimation questions
(applying the expansion to approximate stuff)

⑦ Questions where n is an unknown.

(this method uses

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

and expands the factorial to solve for n.)

EXAMPLES:

1) a) Finding the coefficient of a specific term in the expansion

(see EXAMPLE 2 above)

EXAMPLE 3 → (when r is not given)

Find the coefficient of x^3 in the following expansion. $(2x+4)^8$

ANSWER:

STEPS: 1) $n=?$ $r=?$ $a=?$ $b=?$
2) sub in what you have to AnyTermFormula
3) $r=?$
→ notice we don't have r yet, find r
4) sub r into AnyTermFormula
5) solve and answer the question.

1) $n=8$ $r=?$ $a=2x$ $b=4$

2) $^8C_r \times (2x)^{8-r} \times (4)^r$

3) $r=?$

from question $^8C_r \times 2^8 \times 2^{-r} \times \underbrace{x^8 \times x^{-r}}$ $\times 4^r$

→ this is what we need to find r

$x^3 = x^8 \cdot x^{-r}$ → this says 'what r gives the coefficient of x^3

$3 = 8-r$

$r = 5$

(4) $^8C_5 \times (2x)^3 \times (4)^5$

(5) $^8C_5 \times 4^5 \times 2^3 \times x^3$ → check! is this the correct x?

$458752x^3$

$\therefore$ 458752 is the coefficient

(b) Finding the term independent of x (constant term)

STEPS: (1) Put all known values into AnyTermFormula
(2) let x-values $= x^0$ to find r
(3) sub in r and solve.

EXAMPLE 4

Give the constant term for the following binomial:
$(2x+3)^3$

ANSWER:

(1) $n=3$ $r=?$ $a=2x$ $b=3$

$^3C_r \times (2x)^{3-r} \times (3)^r$

(2) $x^0 = x^3 \times x^{-r}$

$0 = 3-r$
$r = 3$

(3) $^3C_3 \times (2x)^0 \times 3^3 = 27$

2 expanding the binomial

EXAMPLE 5

Expand the following: $(x+5)^4$

ANSWER:

STEPS:
1. find coefficients with ${}^nC_r$
2. find all a's (a's exponent decreases)
3. find all b's (b's exponent increases)
4. write completed expansion and simplify

1	T_1	T_2	T_3	T_4	T_5
	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$
2	x^4	x^3	x^2	x^1	x^0
3	5^0	5^1	5^2	5^3	5^4

these 3 steps can be done at once.

4 $${}^4C_0 x^4 5^0 + {}^4C_1 x^3 5^1 + {}^4C_2 x^2 5^2 + {}^4C_3 x^1 5^3 + {}^4C_4 x^0 5^4$$

$$= 1x^4 1 + 4x^3 5 + 6x^2 25 + 4x\,125 + 1 \times 1 \times 625$$

$$= x^4 + 20x^3 + 150x^2 + 500x + 625$$

③ finding the coefficient of a Specific x ↳ when r is not given

(see EXAMPLE 3 above)

④ finding a random unknown in the original binomial, eg: $(x+qx)^3 \to q$ is an unknown.

EXAMPLE 6

The coefficient of x^3 is equal to 15 in the binomial expansion of

$(1+kx)^{10}$ where k is a constant.

Find the value of k.

ANSWER:

STEPS: ① $a=?$ $b=?$ $r=?$ $n=?$
② sub known values into AnyTermFormula
③ use x's to find r
④ $15x^3 = x^3$ term and solve.

① $a=1$ $b=kx$ $r=?$ $n=10$

② $^{10}C_r \times (1)^{10-r} \times (kx)^r$

③ $x^r = x^3$

$r=3$

④ $^{10}C_3 \times (1)^7 \times (kx)^3 = 15x^3$

$^{10}C_3 \times k^3 \times x^3 = 15x^3$

$^{10}C_3 k^3 = 15$

$k^3 = \dfrac{15}{^{10}C_3}$

$k = \sqrt[3]{\dfrac{15}{^{10}C_3}}$

$k = \dfrac{1}{2}$

5 Two bracket questions.

a) when r is not given

General note for two bracket questions:

* Expand both brackets
  (if the exponent is 1, $n=1$, then the bracket is already expanded)
* For the final answer, multiply the answers from each bracket together
* Only find terms that are needed if it does not ask you to fully expand the binomial.

EXAMPLE 7

Find the independent term of the expansion of the following: $(2x+7)^8(4x+3)^3$

STEPS: for both brackets:

1. write what you know into AnyTermFormula
2. For independent term $x^0 = x$ term to solve for r
3. sub r's

4. times the two answers together for final answer.

ANSWER:

(1) $(2x+7)^8$

$n=8 \quad a=2x \quad b=7$

$^8C_r \times (2x)^{8-r} \times 7^r$

(2) $x^{8-r} = x^0$

$x^8 \times x^{-r} = x^0$

$8-r=0$

$r=8$

(3) $^8C_8 \times (2x)^{8-8} \times 7^8$

$= 1 \times 7^8$

$= 7^8$

$(4x+3)^3$

$n=3 \quad a=4x \quad b=7$

$^3C_r \times (4x)^{3-r} \times 3^r$

$x^{3-r} = x^0$

$r=3$

$^3C_3 \times (4x)^{3-3} \times 3^3$

$= 1 \times 3^3$

$= 3^3$

(4) $7^8 \times 3^3 = 155649627$

EXAMPLE 8

Find the term in x and x^2 in the following expansion:

$(4-x)(2-4x)^6$

STEPS: (1) Check which bracket(s) need to be expanded

(2) For two bracket questions, the final answer will be multiplied, therefore to be safe, check for all terms in x up to the highest power asked for. eg. x^0, x^1, x^2 for x^2.

3 Write what you know into AnyTermFormula
4 $x^0 = x$ terms
$x^1 = x$ terms
$x^2 = x$ terms
etc
to solve for r in each case
(or by inspection if possible)
5 sub r's to get terms
6 multiply for final answer.

ANSWER:

1 $(4-x)$ already expanded

$(2-4x)^6$ needs to be expanded

2 for x^2 need x^0, x^1, x^2

3 $^6C_r \times (2)^{6-r} \times (-4x)^r$

4 x term: x^r

$\therefore x^0 = x^r$
$0 = r$

$\therefore x^1 = x^r$
$1 = r$

$\therefore x^2 = x^r$
$2 = r$

5 x^0: $^6C_0 \times (2)^{6-0} \times (-4x)^0$
$= 1 \times 2^6 \times 1$
$= 2^6$

x^1: $^6C_1 \times (2)^{6-1} \times (-4x)^1$
$= 6 \times 2^5 \times -4x$
$= -768x$

x^2: ${}^6C_2 \times (2)^{6-2} \times (-4x)^2$
$= 15 \times 2^4 \times 16x^2$

$= 3840x^2$

(6) $(3840x^2 - 768x + 2^6)(4 - x)$

all possible x^1 and x^2

$3840x^2 \times 4 + (-768x \times 4) + (-768x) \times (-x) + (2^6) \times (-x)$

$= 15360x^2 - 3072x + 768x^2 - 64x$

$= 16128x^2 - 3136x$

EXAMPLE 9

(a) Find the first 3 terms in the binomial expansion

$\left(x - \frac{2}{x}\right)^5$

these types of questions follow on themselves. The answer for (a) is likely to be used in (b).

(b) Find the coefficient of x in the following binomial expansion: $\left(4 + \frac{1}{x^2}\right)\left(x - \frac{2}{x}\right)^5$

ANSWER:

(a) STEPS: (1) for the first three terms, sub all info into Any Term Formula and solve

(1) ${}^5C_0 \times (x)^{5-0} \times \left(-\frac{2}{x}\right)^0$ $\quad T_1: r = 0$

$= 1 \times x^5 \times 1$
$= x^5$

$^5C_1 \times (x)^{5-1} \times \left(-\frac{2}{x}\right)^1$ T_2 : $r=1$

$= 5 \times x^4 \times -\frac{2}{x}$

$= -10x^3$

$^5C_2 \times (x)^{5-2} \times \left(-\frac{2}{x}\right)^2$ T_3 $r=2$

$= 10 \times x^3 \times \frac{4}{x^2}$

$= 10 \times \frac{4x^3}{x^2}$

$= 40x$

$\therefore x^5 - 10x^3 + 40x$

(b) STEPS: (1) use the previous expansion (check it works)
(2) multiply the brackets to find terms in x

(1) $(x^5 - 10x^3 + 40x)\left(4 + \frac{1}{x^2}\right)$ ✓

$(x^5 - 10x^3 + 40x)(4 + x^{-2})$

2 $-10x^3 \times x^{-2} + 40x \times 4$

$= -10x + 160x$
$= 150x$

$\therefore$ 150 is the coefficient of x

5 Two bracket questions.
b finding an unknown q/p etc

EXAMPLE 10

Find the value of k for which there is no term in x^2 in the expansion of

$$(1+kx)(2-x)^6$$

ANSWER:

STEPS: for 'no term in x^2', find the x^2, then equal to zero

1. Expand first 3 terms of second bracket
2. times brackets to find x^2 terms
3. let x^2 terms $= 0x^2$ and solve for k

① $(2-x)^6$

T_1: ${}^6C_0 \times (2)^{6-0} \times (-x)^0 = 1 \times 2^6 \times 1 = 64$

T_2: ${}^6C_1 \times (2)^{6-1} \times (-x)^1 = 6 \times 2^5 \times -x = -192x$

T_3: ${}^6C_2 \times (2)^{6-2} \times (-x)^2 = 15 \times 2^4 \times x^2 = 240x^2$

$\therefore (64 - 192x + 240x^2)$

② $(64 - 192x + 240x^2)(1+kx)$

$$240x^2 \times 1 + (-192x \times kx) = 0x^2$$
$$240x^2 - 192kx^2 = 0x^2$$
$$240 - 192k = 0$$
$$240 = 192k$$
$$k = 1.25$$

6 Binomial estimation questions

(applying the expansion to approximate stuff)

EXAMPLE 11

a) Find the first four terms of the binomial expansion

$$\left(1-\frac{x}{4}\right)^{10}$$

b) Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

ANSWER:

a)
T_1: ${}^{10}C_0 \times 1 \times \left(-\frac{x}{4}\right)^0 = 1$

T_2: ${}^{10}C_1 \times 1 \times \left(-\frac{x}{4}\right)^1 = -\frac{x}{4} \times 10 = -\frac{10}{4}x$

T_3: ${}^{10}C_2 \times 1 \times \left(-\frac{x}{4}\right)^2 = 45 \times 1 \times \frac{x^2}{16} = \frac{45}{16}x^2$

T_4: ${}^{10}C_3 \times 1 \times \left(-\frac{x}{4}\right)^3 = 120 \times 1 \times -\frac{x^3}{64} = -\frac{120}{64}x^3$

$\therefore\ 1 - 2.5x + 2.8125x^2 - 1.875x^3$

b) **STEPS:**

1. for estimation, let the original given bracket equal the given value and solve for x
2. sub the answer of x into the expanded binomial from question a

① let $(1-\frac{x}{4})^{10} = 0.975^{10}$

$$1-\frac{x}{4} = 0.975$$

$$x = -(0.975-1)\times 4$$

$$x = \frac{1}{10} = 0.1$$

② sub $x=0.1$ into

$$1-2.5x+2.8125x^2-1.875x^3$$

$$\therefore 1-2.5(0.1)+2.8125(0.1)^2-1.875(0.1)^3$$

$$= 0.77625$$

$= 0.7763$ to four decimal places.

7 Questions where n is an unknown.

EXAMPLE 12

When $(1-2x)^p$ is expanded, the coefficient of x^2 is 40 $p>0$

a $p=?$

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

STEPS:
1. find the correct r for x^2 term
2. use ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ to sub coefficient 40 and correct r.
3. expand the factorial to solve for n.

1. $(1-2x)^p$

$${}^nC_r \times (1)^{n-r} \times (-2x)^r$$

$x^r = x^2 \longrightarrow$ use x^2 because coefficient of 40 is given.

$r=2$

2. $40 = {}^nC_2$

NOTE: to expand the factorial follow this proof:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

sub $r=2$

$$^{n}C_{2} = \frac{n!}{2!(n-2)!}$$

sub $^{n}C_{2} = 40$

$$40 = \frac{n!}{2!(n-2)!}$$

3 $$40 \times 2! = \frac{n!}{(n-2)!}$$

$$80 = \frac{n \times (n-1) \times \cancel{(n-2)!}}{\cancel{(n-2)!}}$$

n! expanded until you reach the denominator to cancel the !

$$80 = n(n-1)$$
$$80 = n^2 - n$$

$$n^2 - n - 80 = 0$$
$$(n+4)(n-5) = 0$$

$n = 5$ OR $n = -4$ invalid

$\therefore n = 5 = p$