SEQUENCES AND SERIES

ARITHMETIC (LINEAR)

General formula for the nth term

of an arithmetic sequence

$$u_n = a + (n-1)d$$

$d = u_2 - u_1$
$d = u_3 - u_2$

u_1

common difference.

General formula for the sum to n

of an arithmetic series

PROOF:

$S_n = a + (a+d) + (a+2d) + \dots + (a+(n-2)d) + (a+(n-1)d)$ ... ① all terms in the series

$S_n = (a+(n-1)d) + (a+(n-2)d) + \dots + (a+2d) + (a+d) + a$ ... ② all terms in the series reversed

① + ②

$$2S_n = n(2a+(n-1)d)$$

$$S_n = \frac{n}{2}(2a+(n-1)d)$$

Another formula for S_n

$$S_n = \frac{n}{2}(a+l)$$

l is the last term

SIGMA NOTATION

→ not just used for arithmetic. Any S_n

This is used for showing a series
A series is a pattern where you add each consecutive term together

Example 1: $U_1 ; U_2 ; U_3 ; \ldots$ this is a sequence

$U_1 + U_2 + U_3 ; \ldots$ this is a series

Sigma notation: means find S_n

$$\sum_{k=1}^{n} U_k = U_1 + U_2 + U_3 + \ldots + U_n$$

finish → n

start → $k=1$

U_1 is where $k = 1$
U_2 is where $k = 2$
U_n is the last term

STEPS:
1. Determine if working with ARITHMETIC OR GEOMETRIC series (by working out the first three terms in the sequence and examining their properties)
2. Use appropriate S_n formula.

EXAMPLE 2

Calculate $\sum_{k=1}^{30} (3k+5)$

Answer:

Step ① :

$u_1 : \quad k=1 \quad : 3(1)+5 = 8$
$u_2 : \quad k=2 \quad : 3(2)+5 = 11$
$u_3 : \quad k=3 \quad : 3(3)+5 = 14$

$+3$
$+3$

ce

Δ has a common differen of $+3$
∴ arithmetic

Step ② : $S_n = \frac{n}{2}(2a+(n-1)d)$

$a = u_1 = 8$
$d = 3$
$n = 30$

$\therefore \quad S_{30} = \frac{30}{2}(2(8)+(30-1)3)$

$= 1545$

EXAMPLE 3 → classic 'trick' arithmetic question

Given: $0; -\frac{1}{2}; 0; \frac{1}{2}; 0; \frac{3}{2}; 0; \frac{5}{2}; 0; \frac{7}{2}; 0; \ldots$

(a) Write the value of the 191st term

Answer:
For this example we can split the sequence into

Even numbers: $-\frac{1}{2}; \frac{1}{2}; \frac{3}{2}; \frac{5}{2}; \ldots$
Odd numbers: $0; 0; 0; \ldots$

$\therefore$ 191st $= 0$ because it is an odd number.

(b) Find S_{500}

Answer: Split into two S_n formulas
One for even and one for odd

Remember to halve the number of terms

$$S_{250} = \frac{250}{2}\left(2\left(-\frac{1}{2}\right) + (250-1)(1)\right)$$
$$= 31\,000$$

even

$$S_{250} = \frac{250}{2}\left(2(0) + (250-1)(0)\right)$$
$$= 0$$

odd

$\therefore$ S_{250} (even) + S_{250} (odd)
$= 31000 + 0 = 31000 = S_{500}$

EXAMPLE 4

Determine the number of terms in the following series

$18 + 24 + 30 + \dots + 300$

$\underbrace{18+24}_{+6}\underbrace{+30}_{+6}$

$d = +6$

∴ common difference

∴ arithmetic

Answer:

NOTE: a common error here is to try and use the S_n formula

The question is asking for the n (position) of U_n, the last term in the series.

Use the U_n formula $U_n = a + (n-1)d$

$$\begin{aligned} 300 &= 18 + (n-1)6 \\ &= 18 + 6n - 6 \\ &= 6n + 12 \\ 288 &= 6n \\ n &= 48 \end{aligned}$$

∴ 48th position

∴ there are 48 terms in this series

EXAMPLE 5

Write the following series in Sigma notation

$(1\times2)+(5\times6)+(9\times10)+(13\times14)+\ldots+(81\times82)$

Answer

Steps: 1 Split into two sequences to write in Sigma notation (two U_n formulas)

2 find n of last term

3 write out sigma with both T_n formulas an n.

1 first U_n:

$1;\ 5;\ 9;\ 13;\ \ldots;\ 81$

$+4\ +4$

$U_n = 1+(n-1)4$
$= 1+4n-4$
$= 4n-3$

2 $81 = 4n-3$ → sub last term $U_n = 81$

$n = 21$ → n should be the same in second U_n.

1 Second U_n

$2;\ 6;\ 10;\ 14;\ \ldots;\ 82$

$+4\ +4$

$U_n = 4n-2$

2 $82 = 4n-2$ → sub last term $U_n = 82$

$n = 21$ → n should be the same as first U_n

$\therefore$ for first U_n: $\displaystyle\sum_{n=1}^{21} (4n-3)$

second U_n: $\displaystyle\sum_{n=1}^{21} (4n-2)$

3 the final answer in sigma notation is

$$\sum_{n=1}^{21} (4n-3)(4n-2)$$

GEOMETRIC

General formula for the nth term

$$u_n = a \cdot r^{n-1}$$

$$r = \frac{u_2}{u_1} \qquad r = \frac{u_3}{u_2}$$

→ geometric sequences have a common ratio r

$$a = u_1$$

Sum to n formula

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

→ use this one where $r > 1$

or

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

→ use this one where $r < 1$

if: r is negative
then: the terms of the sequence will ALTERNATE between positive and negative

if: r is positive
then: the terms of the sequence will all have the same sign as u_1.

Geometric S_n PROOF

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \quad \ldots ①$$

$$r\,S_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \quad \ldots ②$$

① − ②

$$S_n - r\,S_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a(r^n-1)}{r-1} \qquad r \neq 1$$

CONVERGENT GEOMETRIC SERIES

A convergent series is a series where S_∞ is a finite number that can be worked out

Convergent series are finite because if r is a fraction, eventually the size of the next term you are adding will be so small it does not make a mathematically significant change to the size of the sum of all the terms.

The term following that will be even smaller, so even if you add an infinite amount of terms, the answer for S_n will be the same, and therefore finite.

This happens when

$$-1 < r < 1 \qquad r \neq 0$$

To work out S_∞ use

$$S_\infty = \frac{a}{1-r}$$

NOTE: in contrast to a finite geometric series, an INFINITE geometric series will have no last term.
In other words, an infinite geometric series does not CONVERGE to a finite number.

EXAMPLE 6

It is given that $S_\infty = 450$; $a = 45$ and $r = \frac{9}{10}$

What is the smallest value for n for which

$$S_\infty - S_n < 1 ?$$

Answer

This question means what is the smallest position where the difference between S_∞ and S_n is less than 1?

STEPS: (1) Sub in known values and solve for S_n

(2) sub in S_n and solve for n

↳ this uses log law

if: $a^b = c$

then: $\log_a c = b$

(1) $S_\infty - S_n < 1$

$450 - S_n < 1$

$-S_n < 1 - 450$

$S_n > 499$ → divide by -1, inequality flips around

(2) Solve $S_n = 449$

$a = 45$

$r = \frac{9}{10}$ $\quad -1 < r < 1 \quad \therefore \quad S_n = \frac{a(1-r^n)}{1-r}$

$r \neq 0$

$$449 = \frac{45\left(1-\left(\frac{9}{10}\right)^n\right)}{1-\frac{9}{10}}$$

$$\frac{449\left(1-\frac{9}{10}\right)}{45} = 1 - \frac{9}{10}^n$$

$$-\left[\left(\frac{449\left(1-\frac{9}{10}\right)}{45}\right)-1\right] = \frac{9}{10}^n$$

$$\frac{1}{450} = \frac{9}{10}^n$$

$9^b = c$

$\log_a c = b$

$$n = \log_{\frac{9}{10}} \frac{1}{450}$$

$n = 57,984$ where $S_n = 449$

n is a whole number

$S_n > 449$

$\therefore$ Smallest $n = 58$

QUADRATIC SEQUENCES

$U_n = an^2 + bn + c$ → general form

Quadratic sequences have a second common difference

To find a, b & c:

$2a$ = Second difference
$3a + b$ = First difference
$a + b + c = U_1$

EXAMPLE 7

Find a formula for nth term for the following sequence

1, 4, 9, 16, 25, 36

Answer:

1, 4, 9, 16, 25, 36

3 5 7 9 11 → First difference

+2 +2 +2 +2 → second common difference
∴ quadratic

$2a$ = 2nd diff.	$3a+b$ = 1st diff.	$a+b+c = U_1$
$2a = 2$	$3(1) \cdot = 3$	$1 + 0 + c = 1$
$a = 1$	$b = 0$	$c = 0$

$$\therefore U_n = 1n^2 + 0n + 0 = n^2$$

SUMMARY OF FORMULAS:

Arithmetic:

$$U_n = a + (n-1)d$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_n = \frac{n}{2}(a + l)$$

$$U_2 - U_1 = d$$

$$U_3 - U_2 = d$$

$$a = U_1$$

Geometric:

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

$$S_n = \frac{a(r^n-1)}{r-1}, \quad r \neq 1$$

$$S_\infty = \frac{a}{1-r}$$

$$a = U_1$$

$$r = \frac{U_2}{U_1}$$

$$r = \frac{U_3}{U_2}$$

Sigma notation

$$\sum_{n=1}^{n} U_n$$

General

$$U_n = S_n - S_{n-1}$$

Quadratic

$$an^2 + bn + c$$

$2a = $ 2nd diff | $3a + b = $ 1st diff | $a + b + c = U_1$

TYPES OF QUESTIONS

1. given the position n, what is the value at position n?
   → see example 8
   → see example 9
2. given the value, what is the position n?
   → see example 8
   → see example 9
3. find a formula for the nth term
   → see example 7/8/9
4. Questions involving sigma notation
   → see Example 2
   → see Example 5
5. Questions using simultaneous equations
   → see example 9
6. listing certain terms in a sequence or series
   → see example 8
7. Questions involving S_n or S_∞
   → see example 10
8. Problem solving questions
   → see example 3
   → see example 4
   → see example 6
9. Examples involving extra unknowns
   → see example 8
10. Proving recurring decimals = certain value using S_∞
    → see example 11

EXAMPLE 8

Consider the following arithmetic sequence

$$3x-1;\ 5x-2;\ 4x+2$$

1. Determine the value of x
2. Determine the value of the first three terms
3. Determine a formula for the n^{th} term
4. Determine a value for the 15th term
5. Which term has a value of 302
6. Is 150 a term in the sequence. Justify your answer

Answer

① $d = U_2 - U_1 = U_3 - U_2$ → because arithmetic

$$(5x-2)-(3x-1) = (4x+3)-(5x-2)$$
$$5x-2-3x+1 = 4x+3-5x+2$$
$$x = 2$$

② $U_1 : 3(2)-1 = 5$

$U_2 : 5(2)-2 = 8$

$U_3 : 4(2)+3 = 11$

③ $U_n = a+(n-1)d$

$= 5+(n-1)3$

$= 5+3n-3$

$U_n = 3n+2$

$\underbrace{5;8}_{+3}\underbrace{;11}_{+3}$

$d=3$

$a=5$

(4) $u_{15} = 3(15)+2$
$= 47$

(5) $302 = 3n+2$

$\frac{300}{3} = n$

$n = 100$ $\therefore$ 100th term

(6) $150 = 3n+2$

$\frac{148}{3} = n$

$n = \frac{148}{3}$ n is not a whole number

$\therefore u_n \neq 150$

EXAMPLE 9

The second term in a geometric sequence is -4 and the fifth term is 32

1. Determine a formula for the nth term
2. Which term has a value of -1024?
3. Determine the eighth term in the sequence

Answer

1

$u_2 = -4 = ar$

$u_5 = 32 = ar^4$

$r = \frac{-4}{a}$ ... 1 $\qquad r = \sqrt[4]{\frac{32}{a}}$ .. 2

let 1 = 2

$\frac{-4}{a} = \sqrt[4]{\frac{32}{a}}$

$\frac{4^4}{a^4} = \frac{32}{a}$

$4^4 a = 32a^4$

$\frac{4^4}{32} = a^3$

$\sqrt[3]{\frac{4^4}{32}} = a$

$a = 2$

sub $a = 2$ into 1

$r = \frac{-4}{2}$

$r = -2$

(2) $U_n = 2(-2)^{n-1}$
$U_n = -1024$

$-1024 = 2(-2)^{n-1}$

$\frac{-1024}{2} = (-2)^{n-1}$

$-512 = (-2)^{n-1}$ → can't use negative logs / $\log_2(512)$ works

$(-2)^9 = (-2)^{n-1}$ → prime fact and drop bases

$9 = n-1$
$n = 10$
$\therefore$ the 10th term

(3) $U_8 = ar^7$
$= 2(-2)^7$
$= -256$

EXAMPLE 10

Evaluate the following and express in sigma notation

$100 + 40 + 16 + \cdots$

Answer

$\frac{40}{100} = \frac{2}{5}$ $\qquad \frac{16}{40} = \frac{2}{5}$

$\therefore$ common ratio

$\therefore$ geometric

$u_n = ar^{n-1}$

$a = 100 \quad r = \frac{2}{5}$

$u_n = 100\left(\frac{2}{5}\right)^{n-1}$

Sigma notation $\therefore$ does it have a last term?

$-1 < r < 1 \qquad r \neq 0$

has infinite terms but S_∞ exists

$\therefore \quad S_n = S_\infty$

$\sum_{n=1}^{\infty} 100\left(\frac{2}{5}\right)^{n-1} = S_\infty = \frac{a}{1-r}$

$= \frac{100}{1-\frac{2}{5}}$

$= \frac{500}{3}$

EXAMPLE 11

Show that $1,\dot{9} = 2$

ANSWER

$$1,9999\ldots = 1 + (0,9 + 0.09 + 0.009 + 0.0009 + \ldots)$$
$$= 1 + \frac{0,9}{1-0.1}$$

$\rightarrow S_\infty = \frac{a}{1-r}$

$a = 0.9$

$r = 0.1$

$$= 1 + 1$$
$$= 2$$