HASTINGS MATH.

INTEGRATION AND DIFFERENCIATION 2

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POWER RULE AND CHAIN RULE

POWER RULE

if $f(x)^n$

then $f'(x) = n \times f(x)^{n-1}$

REVERSE POWER RULE

if $\int f(x)\,dx$

then $\dfrac{f(x)^{n+1}}{n+1}$

CHAIN RULE

If $g(x) = (f(x))^n$

then $g'(x) = n\,(f(x))^{n-1} \times f'(x)$

REVERSE CHAIN RULE

If $\int (f(x))^n\,dx$

then $\dfrac{(f(x))^{n+1}}{(n+1) \times (f'(x))} + C$

EXAMPLES USING TANGENTS TO THE CURVE

EXAMPLE 1

Calculate the values of a and b if $f(x) = ax^2 + bx + 5$ has a tangent at $x = -1$ which is defined by the equation $y = -7x + 3$

ANSWER:

$a = ?$ $b = ?$

There are two unknowns, therefore we need two equations

$m_{tangent} = -7$

$m_{tangent} = f'(x)$ the gradient of the tangent is equal to the first derivative of the curve

$f'(x) = ?$

$f(x) = ax^2 + bx + 5$

$\therefore f'(x) = 2ax + b$

$\therefore -7 = 2ax + b$

sub $x = -1$ — x is a point on the gradient $f'(x)$

$-7 = -2a + b$

$b = 2a - 7 \quad \ldots (1)$ 1st equation

at point $x = -1$, the tangent touches the curve

$y = ?$

sub $x = -1$ into tangent

$y = -7(-1) + 3$

$= 10$

$\therefore (-1;10)$ is a point on the curve sub $(-1;10)$ into $f(x)$

$$10 = a(-1)^2 + b(-1) + 5$$
$$10 = a - b + 5$$
$$b = a - 5 \quad \ldots (2)$$

2nd equation

let (1) = (2)

$$2a - 7 = a - 5$$
$$\therefore a = 2$$

sub $a = 2$ into (1)

$$b = 2(2) - 7$$
$$= 4 - 7$$
$$= -3$$

EXAMPLE 2

Determine the gradient of the tangent of the graph of $f(x) = -3x^3 - 4x + 5$ at $x = -1$

Answer

$$f'(x) = -9x^2 - 4$$

sub $x = -1$

$$f'(-1) = -9(-1)^2 - 4$$
$$= -13$$

OTHER NOTES ON TANGENTS

NORMAL TO THE CURVE: perpendicular to tangent

$$\therefore m_1 \times m_2 = -1$$

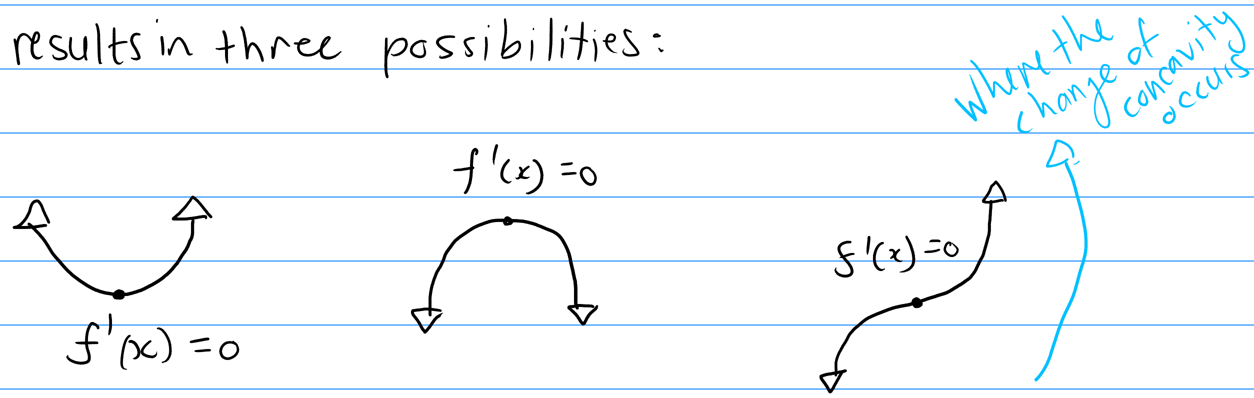
STATIONARY POINTS ON THE CURVE

Stationary points are the turning points of the curve

To determine stationary points without sketching:

let $f'(x) = 0$

this results in three possibilities:



local minimum　　local maximum　　point of inflection

To determine which of the three it is, one method is to construct a table

EXAMPLE 3

Determine the local min, max and point of inflection of $f(x) = x^3 + 3x^2 - 9x - 27$ using the first derivative only.

Answer: **Steps:**
1. find $f'(x) = 0$
2. construct a table to determine the nature of the stationary points.
3. midpoint stationary points is PoI

1. $f(x) = x^3 + 3x^2 - 9x - 27$

$f'(x) = 3x^2 + 6x - 9$

$3x^2 + 6x - 9 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

any value below and above $x = -3$ (choose)

2.

x-value$_1$:	$x = -4$	$x = -3$	$x = -2$
Gradient :	$f'(-4) = 15$	$f'(-3) = 0$	$f'(-2) = -9$
	positive	S.P	negative
	/	.	\

$\therefore$ local max

x-value$_2$:	$x = 0$	$x = 1$	$x = 2$
Gradient:	$f'(0) = -9$	$f'(1) = 0$	$f'(2) = 15$
	negative		positive
	\	.	/

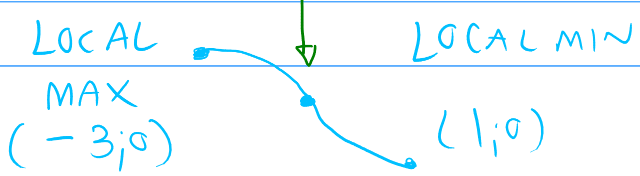
$\therefore$ local min

3. POI: point of inflection :

$\frac{-3+1}{2} = -1$

$f(-1) = -16$

POI $(-1;16)$



THE SECOND DERIVATIVE

Another way to determine the nature of stationary points on the curve is making use of the properties of the second derivative

EXAMPLE 4

Use the first and second derivative to find
a) the local max and min
b) the point of inflection
c) where the curve is concave up or concave down.

$f(x) = x^3 + 3x^2 - 9x - 27$ → same curve as above

Answer:

$f(x) = x^3 + 3x^2 - 9x - 27$

$f'(x) = 2x^2 + 6x - 9$ → 1st derivative

$f''(x) = 6x + 6$ → 2nd derivative

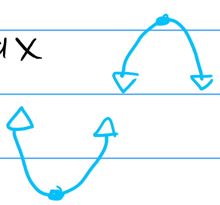
a) local max / min $f'(x) = 0$

$\therefore x = -3$ or $x = 1$ → as in example 3

for local min	$f''(x) > 0$	$\therefore$ positive
for local max	$f''(x) < 0$	$\therefore$ negative
for POI	$f''(a) = 0$	

$f''(-3) = -12$ $\therefore$ negative $\therefore$ local max

$f''(1) = 12$ $\therefore$ positive $\therefore$ local min



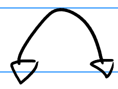
(b) POI: $f''(x) = 0$

$6x + 6 = 0$

$x = -1$

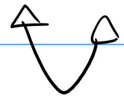
(c) CONCAVITY

local max is concave down



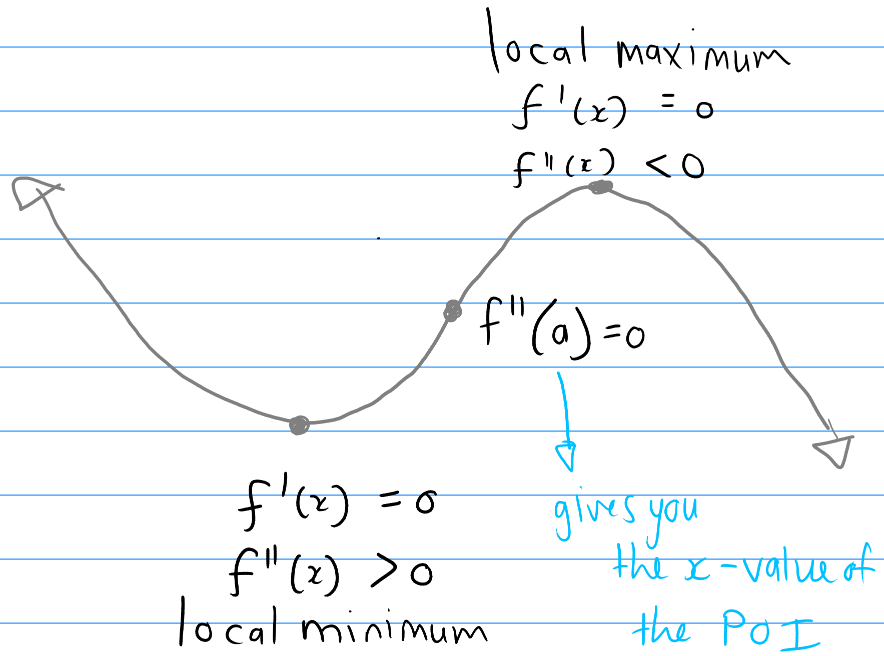
$\therefore$ at $x = -3$

local min is concave up



$\therefore$ at $x = 1$

SUMMARY OF FIRST AND SECOND DERIVATIVE:

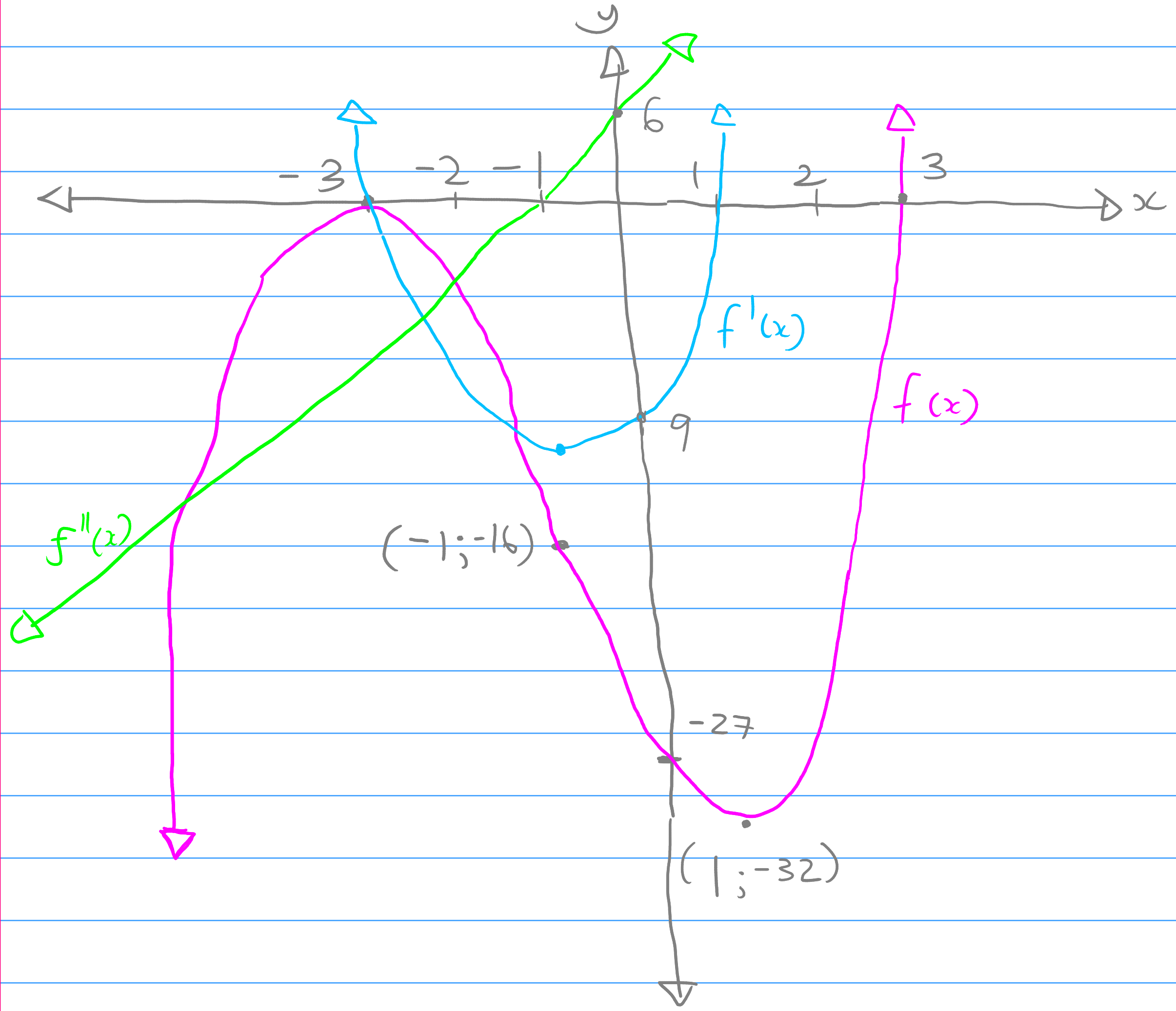


HOW THE CURVE, 1st and 2nd DERIVATIVES RELATE TO EACH OTHER

$f(x) = x^3 + 3x^2 - 9x - 27$

$f'(x) = 2x^2 + 6x - 9$

$f''(x) = 6x + 6$



$f'(x)$: x-intercepts are the same as the x's of the stationary points

: x of turning point is x of $f(x)$ POI

$f''(x)$: x-intercept at $f(x)$ POI and $f'(x)$ turning point

: where it is negative (under x-axis), $f(x)$ is concave down and $f'(x)$ has a negative gradient

: where it is positive (above x-axis), $f(x)$ is concave up and $f'(x)$ has a positive gradient

SKETCHING CUBIC GRAPHS

STEPS: 1 y-int, let $x=0$

2 x-int(s), let $y=0$ and use either

→ factor theorum

→ two bracket method

$(\quad)(\quad)$

↳ use calculator tables to find first factor

3 Stationary points

let $f'(x)=0$

sub x's into $f(x)$ to find corresponding y's

4 use a table / 2nd derivative test / by inspection to test for max / min

Max: $f''(x)<0$

min $f''(x)>0$

5 POI: let $f''(a)=0$

solve for a

sub a into $f(x)$ to find corresponding y.

EXAMPLE 5

Sketch the following

$f(x) = x^3 + 3x^2 - 9x - 27$ → same curve as above

Answer: → working only. See sketch above

1 y-int, let $x = 0$

$y = -27$

$\therefore (0; -27)$

2 x-int(s), let $y = 0$

$x^3 + 3x^2 - 9x - 27$

$(x+3)$ is factor → from tables mode on calculator

↳ can get all x-ints from here, but required to prove for exams.

$(x+3)(x^2 + 0x - 9)$

$(x+3)(x^2 - 9)$

$(x+3)(x-3)(x-3)$

$\therefore x = \pm 3$

$\therefore (3;0)$ & $(-3;0)$

3 S.P's $f'(x) = 0$

$f'(x) = 3x^2 + 6$

$3x^2 + 6x - 9 = 0$

$3(x^2 + 2x - 3) = 0$

$(x - 1)(x + 3) = 0$

$x = 1$ or $x = -3$

$f(1) = -32$

$f(-3) = 0$

$\therefore$ s.p's : $(1; -32)$ & $(-3; 0)$

4 max/min :

$f''(x) = 6x+6$

$f''(1) = 6(1)+6 = 12$ $\therefore$ pos $\therefore$ min

$f''(-3) = 6(-3)+6 = -12$ $\therefore$ neg $\therefore$ max

5 POI :

$f''(x) = 0$

$6x+6 = 0$

$x = -1$

sub $x = -1$

$f(-1) = -16$

$\therefore$ POI $(-1; -16)$

NOTE

If: $f(x) = y$

then: $f'(x) = \dfrac{dy}{dx}$

$f''(x) = \dfrac{d^2y}{dx^2}$

OPTIMISATION / MODELLING

this for volume

STEPS
1. Draw a sketch of the problem
2. Find a formula to use eg. $V = 2x^3 - 6x$
3. for max/min $f'(x) = 0$

eg: if $V = 2x^3 - 6x$

$$V' = 6x^2 - 6$$

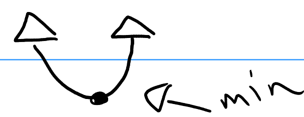
$$6x^2 - 6 = 0$$
$$x^2 = 1$$

$$x = \pm\sqrt{1}$$
$$x = \pm 1$$

4. to determine which is max/min sub x's into $f''(x)$

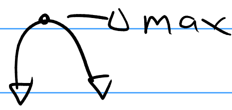
eg $V' = 6x^2 - 6$
$V'' = 12x$

sub $x = 1$
$V'' = 12$ $\therefore$ positive



$\therefore$ $x = 1$ will find the minimum volume

sub $x = -1$
$V'' = -12$ $\therefore$ negative



$\therefore$ $x = -1$ will find the maximum volume

5 sub correct x back into original formula to find max/min

eg: for maximum volume : $x = -1$

$$\therefore V = 2(-1)^3 - 6(-1)$$
$$= -2 + 6$$
$$= 4 \text{ units}^3$$

for minimum volume : $x = 1$

$$\therefore V = 2(1)^3 - 6(1)$$
$$= 2 - 6$$
$$= -4$$

invalid

→ this won't happen in most questions. only the max volume would be asked for

TYPES OF OPTIMISATION PROBLEMS

1. Maximise volume / area / profit
2. Minimise area / volume / cost / distance / time

NOTES ON DISTANCE / SPEED / TIME QUESTIONS

$s(t)$ = displacement over time
$s'(t)$ = velocity over time
$s''(t)$ = acceleration

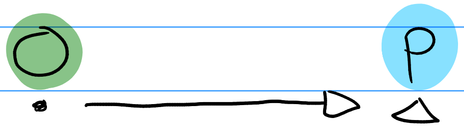
average velocity : $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$

total distance travelled (change in displacement)

average velocity is the rate of change of displacement

total time taken

Considering an object P, in motion along a straight line from a fixed point of origin O



- The distance P travels accumulates over time, no matter what direction it travels in
- The speed S of P is how fast it is travelling
- The displacement s of P is the position relative to O
- The velocity v is the rate of change of the displacement
- The acceleration a is the rate of change of velocity.

average acceleration:

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

this time using velocity functions

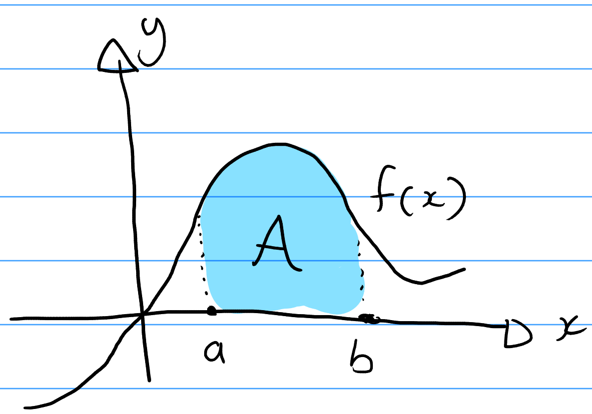
acceleration $= a(t) = v'(t) = s''(t)$

AREA UNDER THE CURVE

DEFINITE INTEGRALS

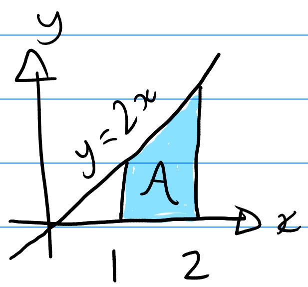
These are integrals that specify definite points to work with

General rule:



Area $= \int_a^b f(x)\,dx = F(b) - F(a)$

Example 6



→ the area between a straight line and the x-axis

Find the area of the shaded region.

Answer

Integral at $x=2$ minus integral at $x=1$

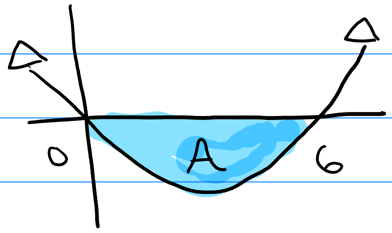
$$\int_1^2 (2x)\,dx = F(2) - F(1) \quad ; \quad F(x) = x^2$$

$$= \left[x^2\right]_1^2 = (2)^2 - (1)^2$$

→ this is how it is set out in your working

$$= 3 \text{ units}^2$$

EXAMPLE 7



area is below the x-axis here, therefore the integral is negative

Find the area of the shaded region

$f(x) = x^2 - 6x$

Answer

$\text{Area} = -\int_0^6 (x^2 - 6x)\,dx$

$F(x) = -\left(\frac{1}{3}x^3 - 3x^2\right) = -\frac{1}{3}x^3 + 3x^2$

$= \left[-\frac{1}{3}x^3 + 3x^2\right]_0^6$

$= \left(-\frac{1}{3}(6)^3 + 3(6)^2\right) - \left(-\frac{1}{3}(0)^3 + 3(0)^2\right)$

$= 36 - 0$

$= 36 \text{ units}^2$

EXAMPLE 8

$f(x) = x^3 - x^2 - 2x$

Find the total area between the x-axis and the curve

Answer:

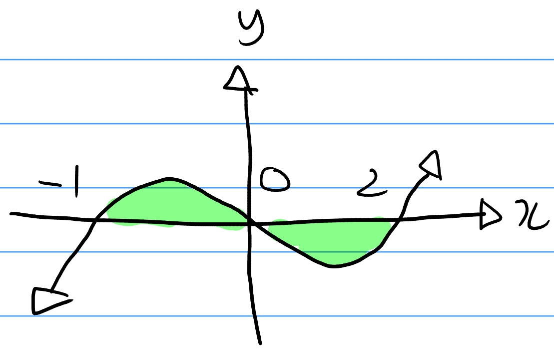
Steps
1. Sketch and shade
2. find the positive area
3. find the negative area
4. add the areas to get total area

① x-int, let $y = 0$

$x^3 - x^2 - 2x = 0$

$x = -1$ or 0 or 2

→ using tables mode on calculator



y-int let $x = 0$

$\therefore (0;0)$

② positive area $= \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_{-1}^{0}$ → $\left[F(x) \right]_a^b$

$= \left(\frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - (0)^2 \right) - \left(\frac{1}{4}(-1)^4 - \frac{1}{3}(-1)^3 - (-1)^2 \right)$

$= 0 - \left(-\frac{5}{12} \right)$

$= \frac{5}{12}$ units^2

③ negative area $= -\int_0^2 (x^3 - x^2 - 2x)\,dx$

$= \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2\right]_0^2$

$= \left(-\frac{1}{4}(2)^4 + \frac{1}{3}(2)^3 + (2)^2\right) - (0)$

$= \frac{8}{3} - 0$

$= \frac{8}{3}$ units^2

④ $\frac{5}{12} + \frac{8}{3} = 3\frac{1}{12}$ units^2

Example 9

$f(x) = x^2$
$g(x) = x$

area between two functions

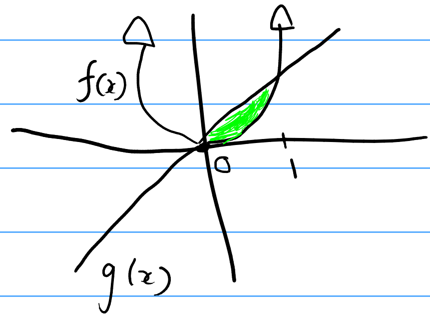
Find the area between $g(x)$ and $f(x)$

Answer: **Steps:**
1. find x-int(s) and y-int(s)
2. find limits by letting $f(x) = g(x)$ and solving for x-coordinates
3. write integral and solve

good idea to sketch here

NOTE: Always Top function minus Bottom function.

1. $f(x)$: y-int & x-int $(0;0)$
   $g(x)$: y-int & x-int $(0;0)$



2. Let $f(x) = g(x)$

$x^2 = x$
$x^2 - x = 0$
$x(x-1) = 0$
$x = 0$ or $x = 1$

top - bottom

3. $\int_0^1 (g(x) - f(x))\,dx = \text{Area}$

$\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$= \left(\frac{1}{6}\right) - (0) = \frac{1}{6}$ units^2

EXAMPLE 10

NOTE: method for area between the curve and the y-axis

$f(x) = x - 1$

find the area between the line and the y-axis from $y = 0$ to $y = 4$

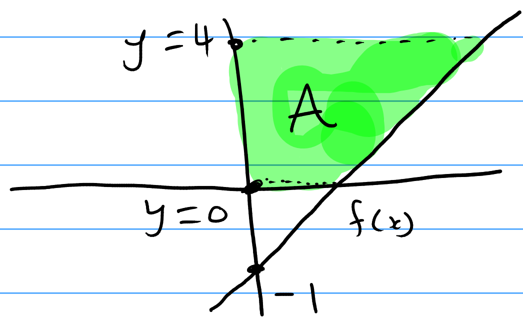
Answer

Steps:

1. isolate x (make x the subject)
2. find Area in usual way but now the integral will be with respect to y

ie: $\int_a^b f(y)\,dy$

Sketching is useful



① $f(x) = x - 1$

$y = x - 1$

$x = y + 1$

② $\int_0^4 (y+1)\,dy$

$= \left[\frac{1}{2}y^2 + y\right]_0^4$

$= 0 + \left(\frac{1}{2}(4)^2 + (4)\right)$

$= 12$ units^2

EXAMPLE 11

Determine the volume of the solid obtained by rotating the region bound by $f(x) = x^2 - 4x + 5$, $x = 1$ and $x = 4$ → see sketch on p. 25

NOTE: For revolution around x-axis:

$$\text{Volume around } x\text{-axis} = \int_a^b \pi (y)^2 \, dx$$

Answer

Steps
1. expand brackets
2. integrate with x-values

1. expand $f(x)$:

$(x^2 - 4x + 5)^2 = (x^2 - 4x + 5)(x^2 - 4x + 5)$

$x^4 - 8x^3 + 26x^2 - 40x + 25$

2. Integrate:

$$\int_1^4 (x^2 - 4x + 5)^2 \pi \, dx$$

$$= \pi \times \int_1^4 (x^2 - 4x + 5)^2 dx$$

can take π to the side

$\pi \times \int = \int \pi$

$$= \pi \times \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25) \, dx$$

$$= \pi \times \left[\frac{x^5}{5} - 2x^4 - \frac{26}{3}x^3 - 20x^2 + 25x \right]_1^4$$

$$= \pi \times (f(4) - f(1))$$ → calculator work.

$$= 15\tfrac{9}{15} \times \pi = 15\tfrac{9}{15}\pi \text{ units}^3$$

EXAMPLE 12

$f(x) = x^3$

find the volume obtained by rotating around the y-axis within the limits $y = 0$ and $y = 4$

see sketch on p 25

ANSWER

Step ① make x the subject

② find

$$\int_a^b (f(y))^2 \cdot \pi \, dy$$

① $y = x^3$

$x = \sqrt[3]{y}$

② expand:

$$(\sqrt[3]{y})^2 = (y^{\frac{1}{3}})^2 = y^{\frac{2}{3}}$$

integrate:

$$\int_0^4 y^{\frac{2}{3}} \pi \, dy$$

$$= \pi \times \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^4$$

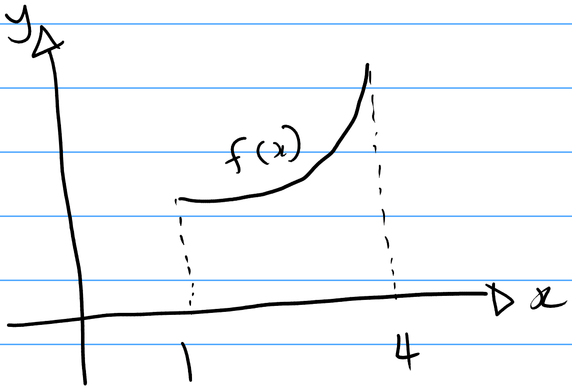
$$= \pi \times \left(\frac{3}{5} (4)^{\frac{5}{4}} \right) - 0)$$

$$= 6.047\pi \text{ units}^3$$

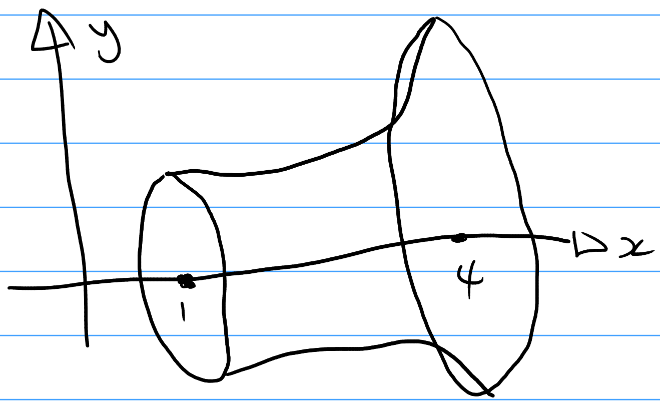
EXAMPLES 11 & 12 Sketched:

Example 11: $f(x) = x^2 - 4x + 5 \qquad x = 1 \;;\; x = 4$

Sketch with limits 1 & 4

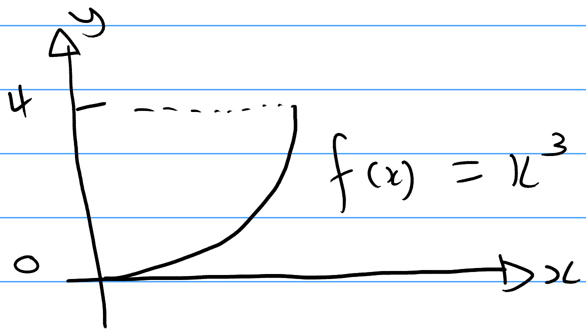


Sketch when rotated 360°



Example 12

Sketch with limits $y = 4$
$y = 0$



$f(x) = x^3$

3D sketch of revolution