QUASI-HEREDITARY ORDERINGS OF AN-TYPE NAKAYAMA ALGEBRAS

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Abstract. abstract

1. Introduction

The theory of quasi-hereditary algebra has being extensively studied since introduced by E. Cline, B. Parshall and L. Scott in the famous paper[1] in 1988. The properties of a quasi-hereditary algebra A are depended heavily on a specific ordering on the set of the iso-class of simple modules $\mathcal{S}$ such that every indecomposable projective module is filtered by the Weyl modules produced by a special process related to the ordering. This ordering is then called to be a quasi-hereditary ordering on A. So a quasi-hereditary algebra usually means not only the algebra but also the specified quasi-hereditary ordering, q-ordering for short. This yields a natural question that how many q-orderings does an arbitrary algebra A admit? Denote by $q(A)$ this number and suppose the cardinality $|\mathcal{S}| = n$, which is the rank of the Grothendieck group of A, then it is obvious that $0 \leq q(A) \leq n!$. A classical result proved by Dlab and Ringel is that $q(A) = n!$ if and only if A is hereditary [2].

Although it is difficult to give an explicit formula for $q(A)$ for arbitrarily given algebra A, some interesting approaches have being made for monomial algebras. In 1990, M. Uematsu and K. Yamagata proved that Morita equivalence preserves quasi-heredity [3], that is, if two algebras A and B are Morita equivalent, then A is quasi-hereditary if and only if so is B. Another intersting result they proved is that a Nakayama algebra A is quasi-hereditary if and only if A has a simple module of projective dimension 2. In 2000, the third author of the present paper and Y. Li proved [4] that $q(A) \leq \dfrac{2}{3}n!$ for tree-type quasi-hereditary algebras of one generator and conjectured that this inequality holds for all non-hereditary algebras, namely, they conjectured that $q(A) > \dfrac{2}{3}n!$ if and only if A is hereditary. We call this conjecture "q-ordering conjecture". In 2008, the first author and L. Wu and C. Gao gave a necessary and sufficient condition for an ordering on $\mathcal{S}$ to be a q-ordering for A_n-type algebras with exactly two generators and obtained an explicit formula for $q(A)$ in [5], and affirmatively answered q-ordering conjecture in that case.

In 2019, E.L.Green and S. Schroll [6] obtained a necessary and sufficient condition for existence of a q-ordering on $\mathcal{S}$ for all monomial algebras by introducing a monomial algebra to a given algebra, thus gave a necessary and sufficient criterion to determine if a monomial algebra is quasi-hereditary.

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A recent interesting result is due to R. Marczinzik and E. Sen [7] in 2022. They call a finite dimensional algebra A S-connected if the projective dimensions of the simple A-modules form an interval. They prove that a Nakayama algebra A is S-connected if and only if A is quasi-hereditary.

The main aim of the present paper is to classify all q-orderings and give an explicit formula for $q(A)$ for Nakayama algebras. This formula will give a positive answer for q-ordering conjecture for all Nakayama algebras. However, we will give a counter example to show that q-ordering conjecture fails in the case that A is not monomial.

Unless otherwise specified, all algebras are supposed to be finite dimensional over a fixed algebraically closed field K and the unit element of an algebra A is denoted as 1_A.

2. Preliminaries

2.1. **Quasi-hereditary Algebras.** Let A be an algebra, Λ a (finite) poset in bijective correspondence with the set of iso-classes $\mathcal{S}$ of simple (left) A-modules. The poset Λ is called the weight poset of A and the elements in Λ weights of A. For each weight $\lambda \in \Lambda$, denote by $E(\lambda)$ (or $E(A, \lambda)$) the corresponding simple module, $P(\lambda)$ the projective cover of $E(\lambda)$. Let $\Delta(\lambda)$ be the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu), \mu \preceq \lambda$. Denote by Δ the full subcategory of all $\Delta(\lambda), \lambda \in \Lambda$ in A-mod. The modules in Δ are called Weyl modules. Denote by $\mathcal{F}(\Delta)$ the class of all A-modules which have a Δ-filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Δ for $1 \leq i \leq t$. Modules in $\mathcal{F}(\Delta)$ is usually called to be good (A-)modules. For simplicity, $\mathcal{F}(\Delta)$ is simply denoted as $\mathcal{F}$ or $\mathcal{F}_A$.

Definition 2.1. *The algebra A is said to be quasi-hereditary with respective to $(\Lambda, \preceq)$ if for each $\lambda \in \Lambda$ we have*
(1) $End_A(\Delta(\lambda)) \simeq k$;
(2) $P(\lambda) \in \mathcal{F}(\Delta)$.

By (A, Λ) we denote a quasi-hereditary algebra A with weight poset Λ.

Throughout, we use $\mathscr{O}(A)$ to denote the set of all q-orderings of A.

Definition 2.2. *Let (A, Λ) be a quasi-hereditary algebra with weight poset $\Lambda = \{1 < 2 < ... < n\}$,and the set of Weyl modules $\Delta(1), ..., \Delta(n)$. Let B be a subalgebra of A(hence B and A have the same unit element and there is a bijection between their weight posets) such that the following conditions are satisfied:*

(D) the algebra $(B, \leq)$ is directed(i.e. quasi-hereditary with simple Weyl modules and injective co-Weyl modules) with respect to the partial order $\leq$ induced from Λ;
(T) tensor induction $- \otimes_B A$ is an exact functor, i.e. A is projective as a left B-module;
(W) for each i with $1 \leq i \leq n$ the following holds:

$$E(B, i) \otimes_B A \simeq \Delta(A, i).$$

Then B is called an exact Borel subalgebra of (A, Λ).

B is called a strong exact Borel subalgebra of A if and only if both the following condition (S) and the above conditions (T) and (W) are satisfied:

(S) There is a maximal semisimple subalgebra $S(A)$ of A which is also a maximal semisimple subalgebra of B,thus simple A-modules and simple B-modules can (and will) be identified.

If A is basic, then the above two notions coincide.

2.2. **Nakayama Algebras.** For any $n \geq 2$, an algebra A is called to be a *Nakayama algebra of A_n type* provided A is a factor algebra KQ_n/I of the following quiver algebra

$$Q_n: \quad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \cdots \longrightarrow \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$$

with I an admissible (two-sided) ideal of KQ_n. The oriented quiver Q_n is usually called to be linearly ordered Dynkin diagram A_n.

Note that any admissible ideal I of KQ_n is generated by a set $\{g_1, \ldots, g_k\}$ of generators, where each g_i is a path with starting vertex $s(g_i)$ and terminal vertex $t(g_i)$ of at least length 2 and $1 \leq s(g_i) < s(g_j) \leq n$ whenever $i < j$. If the number of generators is required to be as small as possible, then every admissible ideal I of KQ_n can be written as $I =< g_1, \ldots, g_k >$ in a unique way and we will adopt this expression.

For simplicity, the simple A-module S_x corresponding to the vertex x (namely, $S_x = A/AeA$ with $e + x = 1_A$) is simply denoted as x. This notation will make no confuse when we denote by $\preceq$ the partial order of the weight poset Λ. Sometimes we also use $i \succeq j$ to mean $j \preceq i$.

A Nakayama algebra of A_n type is obviously quasi-hereditary with respect to the trivial q-ordering $n \preceq n-1 \preceq \cdots \preceq 1$. What about the other simple orderings? Let us check the following easy example.

Example 2.3. *Let $A = KQ_3/ < \alpha_2\alpha_1 >$ with*

$$Q_3: \quad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3$$

Choose the weight poset Λ with ordering the simple modules as $1 \succeq 2 \succeq 3$, then $P(k) = \Delta(k)$, for $k = 1, 2, 3$, so $1 \succeq 2 \succeq 3$ is a q-ordering. Similarly, one checks easily that $1 \succeq 3 \succeq 2$ is also a q-ordering. However, if one chooses the weight poset Λ with ordering the simple modules as $2 \succeq 1 \succeq 3$, then $P(k) = \Delta(k)$, for $k = 2, 3$, but $\Delta(k) = 1$ and $P(1) \notin \mathscr{F}$, so $2 \succeq 1 \succeq 3$ is not a q-ordering. In fact, none of the other 3 orderings is a q-ordering. So $q(A) = 2$.

One of our purpose is to show that $q(A) \leq \dfrac{2}{3}n!$ for Nakayama algebras A of A_n type, so proving that q-ordering conjecture is true in this case. Our main tool is the Green-Schroll theory.

3. Green-Schroll theory and application to Nakayama Algebra

We recall briefly the theory developed by Green and Schroll in [6].

Let A be an algebra. An (two-sided) ideal I in A is called heredity if it is idempotent (namely $I^2 = I$), annihilating by the Jacobson radical $J(A)$ (namely $IJ(A)I = 0$ and projective as a left (or right, equivalently) A-module.

Let Q be a finite quiver with vertex set Q_0 and arrow set Q_1. An algebra Λ is called to be a monomial algebra of type Q if Λ is isomorphic to a quotient of the quiver algebra KQ modulo an ideal I generated by paths. Let T be a set of paths in Q. A vertex $v \in Q_0$

is called to be not properly internal to T if, for each $t \in T$, v can only occur in t as either the origin or end vertex of t.

Green and Schroll prove the following Green-Schroll theorem (Proposition 3.2 of [6]) .

Theorem 3.1. *Let $A = KQ/I$ be a finite dimensional monomial algebra and let T be a minimal set of generators of paths of I. For $v \in Q_0$, the ideal AvA is heredity if and only if v is not properly internal to T.*

The relationship between heredity ideals and quasi-hereditary algebras is as follows. An algebra A is quasi-hereditary if there exists a chain of ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{i-1} \subset I_i \subset \cdots \subset I_m = A \tag{1}$$

such that I_i/I_{i-1} is a heredity ideal in $A/I_{i-1}, \forall i$. The sequence (1) is called to be *a heredity chain* for A.

We will apply Green-Schroll theorem to Nakayama algebras of A_n type. We first fix some notations.

Since Nakayama algebras are serial, so any indecomposable module has exactly one composition series (from top to socle) and is completely determined with its top and socle, we will use the interval symbol $[t; s]$ to represent the unique indecomposable module with top t and socle s. Thus, the indecomposable projective module $P(j)$ with $top(P(j)) = j$ is exactly the module $[j; j + d - 1]$, where $d = dim_K P(j)$. From this representation, one also sees easily that $rad(P(j)) = [j + 1, j - 1 + d]$. Notice that a simple module x has simplicity 0 in an indecomposable module $[t; s]$ if and only if the positive integers x, s, t satisfying $x < t$ or $x > s$, namely $x \notin [t, s]$ as numbers. So in this case we simply write $x \notin [t; s]$.

To deal with the q-orderings, we make use of that fact that the weight poset Λ is a reordering of the first n positive numbers. Note first that the condition (1) in Definition 2.1 is automatically satisfied for all directed algebras. Another easy fact of a directed Nakayama algebra is that it has a unique simple projective module $P(n)$ and obviously $\Delta(n) = P(n)$ under all simple orderings. So, $P(n)$ always satisfies the condition (2) in the Definition2.1.

The restriction of Green-Schroll theorem 3.1 to Nakayama algebra of A_n type is exactly the following

Theorem 3.2. *Let $A = KQ/I$ be a Nakayama algebra of A_n type with $I =< g_1, \ldots, g_k >$, where each g_i is a path from the vertex $s(g_i)$ of length at least 2. Then a simple orderings $v_1 \preceq v_2 \preceq \cdots \preceq v_n$ is a q-ordering if and only if $v_j \preceq s(g_i)$ or $v_j \preceq t(g_i), \forall v_j \in [s(g_i), t(g_i)], \forall i \in [1, k]$.*

Proof. We prove by induction on the number $n \leq 3$ of vertices.

Let $n = 3$. Then $k = 1$ and the condition $v_j \preceq s(g_1)$ or $v_j \preceq t(g_1), \forall v_j \in [s(g_1), t(g_i 1]$ is sufficient and necessary is a direct consequence of Theorem 2 in [4] or Lemma 3.3 in [5].

Suppose the theorem is true when $n = p$. Consider the case $n = p + 1$. Denote by e_n the primitive idempotent corresponding to the vertex n and take $L = Ae_nA$. Now, Green-Schroll Theorem assures that L is a heredity ideal of A. (It can also be proven directly as follows. Since L is generated by idempotent so it is obviously idempotent; L is also annihilated by $J(A)$ since the indecomposable projective module e_nA is exactly the simple module S_n, so $LJ(A)L = LJ(A)S_n = L0 = 0$ by Nakayama Lemma; finally, $L = Ae_nA = Ae_n$ is a left projective A-module. Therefore, L is a heredity ideal of A.)

Thus a simple ordering

$$j_1 \preceq j_2 \preceq \cdots ... \preceq j_n \tag{2}$$

is a q-ordering on A if and only if the corresponding simple ordering (that is, deleting n from (2) and keeping the ordering of all other simples) is a q-ordering on A/L. By induction hypothesis, every simple modules $S_{v_j} \not\simeq S_n, S_{v_j} \in [s(g_i), t(g_i)]$ must obey the requirements $v_j \preceq s(g_i)$ or $v_j \preceq t(g_i), \forall i \in [1, k]$.

There are following two cases relating the simple module S_n.

First, $t(g_k) \neq n$, then S_n is innocent to the quasi-hereditary property of any simple ordering. Thus the simple ordering (2) is a q-ordering on A if and only if the corresponding simple ordering is a q-ordering on A/L.

Second, $t(g_k) = n$, then according to Green-Schroll Theorem, Ae_nA is heredity if and only if S_n satisfies the conditions $j \preceq s(g_i)$ or $j \preceq n$ for ll $j \in [s(g_k), n]$. So the simple ordering (2) is a q-ordering on A if and only if the corresponding simple ordering is a q-ordering on A/L and $j \preceq s(g_i)$ or $j \preceq n$ for ll $j \in [s(g_k), n]$.

Therefore, the conditions are sufficient and necessary for A to be quasi-hereditary. □

4. Formula for q-orderings of Nakayama algebras of A_n type

Lemma 4.1. *Suppose $rad(P(k+1)) = P(k)$. If the weight poset Λ satisfying $P(k+1) \in \mathscr{F}$, then $P(k) \in \mathscr{F}$.*

Proof. By the assumption, the socles of $P(k+1)$ and $P(k)$ coincide, say $Soc(P(k)) = s$, then $P(k+1) = [k+1; s], P(k) = [k; s]$. Suppose $\Delta(k) = [k; s-l]$, $0 \leq l \leq s-k$. There are 3 cases.

(1) $l = 0$.

Then $\Delta(k) = [k; s] = P(k)$, thus $\mathcal{P}(k) \in \mathscr{F}$.

(2) $l = s-k$.

Then $\Delta(k) = k$. It is obvious that $P(k) \in \mathscr{F}$ when $P(k+1) \in \mathscr{F}$.

(3) $0 < l < s-k$.

Then $\Delta(k) = [k; s-l]$. Thus k is the largest element in the subset $\{k, \ldots, s-l\}$ of Λ. so, $s-l+1 \notin \Delta(i), \forall\ k \leq i \leq s-l$. Since $P(+1) \in \mathscr{F}$, there exists a module $M = [s-l+1; s] \in \mathscr{F}$. Thus $P(k) \in \mathscr{F}$ as $P(k)$ is filtered by good modules M and $\Delta(k)$. □

4.1. **Induction Process.** We proceed to get an explicit formula for $q(A)$ by an induction process. Namely, we connect an algebra A of A_n type with another algebra B of A_{n+1} type such that $q(B)$ can be computed directly from $q(A)$. This can be done since there is a canonical embedding of algebras

$$\iota_n : \quad KQ_n \hookrightarrow KQ_{n+1}. \tag{3}$$

Namely, ι_n is defined by $\iota_n(i) = i, \iota_n(\alpha_j) = \alpha_j, 1 \leq i \leq n, 1 \leq j \leq n-1$. Thus every algebra of A_n type is closely connected with another algebra $\hat{A}$ of A_{n+1} type by the very embedding ι_n as follows

$$\iota_n : \quad \hat{A} = KQ_{n+1}/ < \iota(I) > . \tag{4}$$

Lemma 4.2. *Let $A = KQ_n/I$ and $\hat{A} = KQ_{n+1}/ < \iota_n(I) >$. Then $q(\hat{A}) = (n+1)q(A)$.*

Proof. To avoid confusion, $P(i)$ respectively $Q(i)$ will stand for indecomposable projective $\hat{A}$-module respectively A-module.

First suppose the last generator of I is the path from s to n, i.e. $k = s, g_s = \prod_{i=1}^{n-1} \alpha_i$.

Then $P(n+1) \in \mathscr{F}_{\hat{A}}$ since $P(n+1)$ is a simple module.

By the definition of g_s, $radP(k) = P(k+1), k = s+1, \ldots, n$. By 4.1, $P(n), \ldots, P(s+1) \in \mathscr{F}_{\hat{A}}$.

Though $radP(s) = P(s+1)$ might fail, we claim that $P(s) \in \mathscr{F}_{\hat{A}}$. To see this, note that $P(s) = [s; n-1]$ and $P(s+1) = [s+1; n+1]$. There are following cases.

(1) $\Delta(s) = [s; n-1]$. Then $P(s) = \Delta(s) \in \mathscr{F}_{\hat{A}}$.

(2) $\Delta(s) = [s; s+i]$, $0 \leq i < n-s-1$. Then, since $[s+i+1; n-1] \in \mathscr{F}(A)$, $n = max\{s+i+1, s+i+2, \ldots, n\}$. $n+1$ doesn't influence the conditions.

In A, $P(s) = [s; n-l]$, $P(s+1) = [s+1; n]$.

Therefore, what we claim is right. We can also claim that $P(s-1), \ldots, P(1) \in \mathscr{F}(\hat{A})$ with the same reason or by 4.1 .

Above all, $m+1$ does not affect whether an order is quasi-hereditary, which means $\dfrac{q(\hat{A})}{q(A)} = n+1$. □

Definition 4.3. $\mathscr{O}_{A/I}[\alpha, \beta] :=$ *the set of orders of* $\{\alpha, \alpha+1, \ldots, \beta\}$ *satisfying the quasi-hereditary orders of* A/I.

4.2. **Preparation.**

Lemma 4.4. *If A is the simple reverse path extension of A_m, then* $\dfrac{q(A)}{q(A_m)} = m+1$.

Proof. Suppose the simple module added is S_0. By 4.1, $P(0) \in \mathscr{F}(A)$ for all orders in $\mathscr{O}(A_m)$. So $\dfrac{q(A)}{q(A_m)} = m+1$.

□

5. Mobility Theorem

Definition 5.1. *Suppose $n \leq m$. $q_A(n,m)$ means orderings of $[n,m]$ make $P(i) \in \mathscr{F}(A)$, $i = n, \ldots, m$*

Proposition 5.2. $q_{A_n}(1,n) = q(A_n)$

Proof. Verify the definitions of them can get it obviously. □

Proposition 5.3. $(m-n+i+j+1)!q_A(n,m) = (m-n+1)!q_A(n-i, m+j)$ *if the addition or the reduction is simple.(It means every extension is a simple path extension or a simple reverse path extension.)*

Proof. Transforming $q_A(n,m)$ into the condition that **??** and 4.4 can be used through 5.2. It's natural by the iteration of the above operations.

The first operation: $q_A(n,m) = q_{B_{m-n+1}}(1, m-n+1) = q(B_{m-n+1})$, $q_A(n-1,m) = q_{B_{m-n+2}}(1, m-n+2) = q(B_{m-n+2})$. By **??**, $\dfrac{q(B_{m-n+2})}{q(B_{m-n+1})} = m-n+2$, so $(m-n+2)q_A(n,m) = q_A(n-1,m)$. □

Proposition 5.4. *Ideal* $I_1 =< g_1, \ldots, g_{k_1} >$, $I_2 =< g'_1, \ldots, g'_{k_2} >$, $g_i =< \alpha_{m_i} \ldots \alpha_{t_i} >$, $g'_i =< \alpha_{m'_i} \ldots \alpha_{t'_i} >$, $\alpha_{m_{k_1}}$ *is in the front of* $\alpha_{t'_1 - 1}$, *which means the generators of* I_1 *and* I_2 *are irrelevant. Suppose* $s_1 = m_{k_1} + 1 - t_1$, $s_2 = m'_{k_2} + 1 - t'_1$,*which are the lengths of* I_1 *and* I_2. *Then*

$$q(A_n/(I_1 + I_2)) = q_{A_n/(I_1+I_2)}(t_1, t_1 + s_1) q_{A_n/(I_1+I_2)}(t'_1, t'_1 + s_2) C^{s_1+1}_{s_1+s_2+2} \frac{n!}{(s_1 + s_2 + 2)!}$$

Proof. Because of the irrelevance of I_1 and I_2, we know that the quasi-hereditary orders in $\mathscr{O}_{A_n/(I_1+I_2)}(t_1, t_1 + s_1)$and $\mathscr{O}_{A_n/(I_1+I_2)}(t'_1, t'_1 + s_2)$ are irrelevant .

To compute the $q(A_n/(I_1 + I_2))$, the quasi-hereditary orderings must preserve both orderings in $\mathscr{O}_{A_n/(I_1+I_2)}(t_1, t_1 + s_1)$ and $\mathscr{O}_{A_n/(I_1+I_2)}(t'_1, t'_1 + s_2)$ whose lengths are s_1 and s_2.

Considering the orderings of $\{t_1, \ldots, t_1 + s_1, t'_1, \ldots, t'_1 + s_2\}$ satisfying $\mathscr{O}(A_n)$. There are $C^{s_1+1}_{s_1+s_2+2}$ types of permutations with two certain orderings $\sigma_1 \in \mathscr{O}_{A_n/(I_1+I_2)}(t_1, t_1 + s_1)$ and $\sigma_2 \in \mathscr{O}_{A_n/(I_1+I_2)}(t'_1, t'_1 + s_2)$.

What's more $\{1, \ldots, t_1 - 1, t'_1 + s_2 + 1, \ldots, n\}$ can be inserted into everywhere in any certain ordering of $\{t_1, \ldots, t_1 + s_1, t'_1, \ldots, t'_1 + s_2\}$ satisfying the quasi-hereditary orderings. There are $\frac{n!}{(s_1+s_2+2)!}$ types of permutations.

By combinatorics, we have that:

$$q(A_n/(I_1 + I_2)) = q_{A_n/(I_1+I_2)}(t_1, t_1 + s_1) q_{A_n/(I_1+I_2)}(t'_1, t'_1 + s_2) C^{s_1+1}_{s_1+s_2+2} \frac{n!}{(s_1 + s_2 + 2)!}$$

□

Definition 5.5. $I_1 \cong I_2$ *means* I_1 *and* I_2 *have the same configuration. For example,*

$$I_1 =< g_1, g_2, \ldots, g_s >, \;\; I_2 =< g'_1, g'_2, \ldots, g'_s >$$

$$g_i =< \alpha_{m_i - 1} \ldots \alpha_{n_i} >, \;\; g'_i =< \alpha_{m_i - 1 + k} \ldots \alpha_{n_i + k} >$$

Theorem 5.6. *If* $I_1 \cong I_2$, *then for a hereditary algebra* A, $q(A/I_1) = q(A/I_2)$.

Proof. Without loss of generality, suppose

$$I_1 =< g_1, g_2, \ldots, g_s >, I_2 =< g'_1, g'_2, \ldots, g'_s >$$

$$g_i =< \alpha_{m_i - 1} \ldots \alpha_{n_i} >, g'_i =< \alpha_{m_i - 1 + k} \ldots \alpha_{n_i + k} >$$

By 5.3, $q(A/I_1) = \frac{n!}{(m_s - n_1 + 1)!} q_{A/I_1}(n_1, m_s)$, $q(A/I_2) = \frac{n!}{(m_s - n_1 + 1)!} q_{A/I_2}(n_1 + k, m_s + k)$. Since $I_1 \cong I_2$, $q_{A/I_1}(n_1, m_s) = q_{A/I_2}(n_1 + k, m_s + k)$. So the proof is over.

□

Corollary 5.7. *If* $I_i \cong J_i$, I_i *and* I_j *are irrelevant,* J_i *and* J_j *are irrelevant,* $\forall i = 1, 2, 3, \ldots, m$ *and* $i \neq j$, *then*

$$q(A/\sum_{i=1}^{m} I_i) = q(A/\sum_{i=1}^{m} J_i)$$

Proof. By 5.4, suppose the width of I_i and J_i is s_i. The start point of I_i is a_i. The start point of J_i is b_i. (width means the number of arrows that are enclosed by an ideal)

(if $I_1 =< g_1, g_2, \ldots, g_s >$, $g_1 =< \alpha_{m_1 - 1} \ldots \alpha_{a_i} >$, a_i is called the start point of I_i)

By $I_i \cong J_i$, we have:

$$q_{A/\cup_{i=1}^m I_i}(a_i, a_i + s_i) = q_{A/\cup_{i=1}^m J_i}(b_i, b_i + s_i)$$

Then we have:

$$\begin{aligned} q(A/\sum_{i=1}^m I_i) &= \Pi_{i=1}^m q_{A/\sum_{i=1}^m I_i}(a_i, a_i + s_i)\frac{n!}{\Pi_{i=1}^m (s_i+1)!} \\ &= \Pi_{i=1}^m q_{A/\sum_{i=1}^m J_i}(b_i, b_i + s_i)\frac{n!}{\Pi_{i=1}^m (s_i+1)!} \\ &= q(A/\sum_{i=1}^m J_i) \end{aligned}$$

□

6. T_1 GENERATORS

In this essay, we'd like to deal with this type of algebra with m generators T_1 like:

$$I_{T_1} =< g_1, g_2, \ldots, g_m >, g_i =< \alpha_{n-m+i} \ldots \alpha_i >, \ \forall i = 1, 2, \ldots, m$$

We can find that:

$$P(k) = [k; k+n-m] \ \forall k = 1, 2, \ldots, m$$
$$P(s) = [s; n], \ \forall s = m+1, \ldots, n-1$$

By 4.1 and 6.3, we can tell that the Quasi-Hereditary orders of this type of algebra are equivalent with the following conditions:

$$\max\{k, k+n-m\} = \max\{k, k+1, \ldots, \ k+n-m\}, \ \forall k = 1, 2, \ldots, m$$

Before we start, we introduce an important lemma.

Lemma 6.1. *Let* $y_k = \max\{k, k+n-m\} = \max\{k, k+1, \ldots, \ k+n-m\}, \ \forall k = 1, 2, \ldots, m$*,* $\{y_1, y_2, \ldots, y_m\}$ *are the largest* m *elements of* $\{1, 2, \ldots, n\}$*.*

Proof. We assume M is the set of the largest m elements of $\{1, 2, \ldots, n\}$.

$y_k = \max\{k, k+n-m\} = \max\{k, k+1, \ldots, \ k+n-m\}, \ \forall k = 1, 2, \ldots, m,$

y_k must be larger than $n-m$ elements.

So $k \in M$.

□

Lemma 6.2. *The following propositions are equivalent:*

(1)$y_k = \max\{k, k+n-m\} = \max\{k, k+1, \ldots, \ k+n-m\}, \ \forall k = 1, 2, \ldots, m$

(2)$\forall \ j \in \{1, 2, \ldots, m\}$*, s.t.* $1 \succeq 2 \succeq \cdots > j, \ n \succeq n-1 \succeq \cdots \succeq n-m+j+1$ *and* $\{1, 2, \ldots, j\} \cup \{n, n-1, \ldots, n-m+j+1\}$ *are the largest* m *elements of* $\{1, 2, \ldots, n\}$*.*

Proof. We assume M is the set of the largest m elements of $\{1, 2, \ldots, n\}$.

(1)⇒ (2):

(i) If $\{i | y_i = i\} \neq \emptyset$, let k is the largest natural number of $\{i | y_i = i\}$.

① Suppose $y_{k-1} = k+n-m-1$, then $k = y_k \succeq k+n-m-1 = y_{k-1} \succeq k$, impossible.

② Suppose $y_{k-1} = k-1$, then $k-1 \succeq k, \ k-1, k \in M$, no problem.

Repeating the method above, we can tell that:

$$y_i = i, \ \ \forall i = 1, 2, \ldots, k$$

And $1 \succeq 2 \succeq \cdots \succeq k$. By 6.1, $1, 2, \ldots, k \in M$.

By the assumption above, if $k = m$, (1)$\Rightarrow$ (2) holds naturally.

If $k < m$, we can tell $y_{k+1} = k+1+n-m$. If not, $y_{k+1} = k+1$ is contradictory to $k = \max\{i|y_i = i\}$.

By $y_{k+1} = k+1+n-m$, we can tell that:

$$k+1+n-m = \max\{k+1, k+2, \ldots, \ k+n-m, k+1+n-m\}$$

$$y_{k+2} = k+n-m+2 = \max\{k+2, k+3, \ldots, \ k+1+n-m, k+2+n-m\}$$

So we have $y_{k+2} \succeq y_{k+1}$, which implies $n > n-1 > \cdots > k+1+n-m$.

Hence $\exists k \in \{1, 2, \ldots, m\}$, $1 \succeq 2 \succeq \cdots \succeq k$, $\ n \succeq n-1 \succeq \cdots \succeq n-m+k+1$ and $\{y_1, y_2, \ldots, y_m\}$ are the largest m elements of $\{1, 2, \ldots, n\}$.

$\{1, 2, \ldots, j\} \cup \{n, n-1, \ldots, n-m+j+1\}$ are the largest m elements of $\{1, 2, \ldots, n\}$.

(ii) If $\{i|y_i = i, i \in M\} = \emptyset$, $y_i = i+n-m, \ \ \forall i = 1, 2, \ldots, m$.

In the same way, $y_{i+1} \succeq y_i$ .we can also tell that $n \succeq n-1 > \cdots \succeq n-m+1$. (1)$\Rightarrow$ (2) also holds.

(2)$\Rightarrow$ (1):

$\exists j \in \{1, 2, \ldots, m\}$, $1 \succeq 2 \succeq \cdots \succeq j$, $\ n \succeq n-1 \succeq \cdots \succeq n-m+j+1$.

$\forall k = 1, 2, \ldots m$, if $k \preceq j$, $k \in \{1, 2, \ldots, j\}$ and $k = \max\{k, k+1, \ldots, j\}$. By $\{j, j+1, \ldots, k+n-m\} \cap \{n, n-1, \ldots, n-m+j+1\} = \emptyset$, we can know

$$k = \max\{k, k+n-m\} = \max\{k, k+1, \ldots, \ t, k+n-m\}$$

If $k > j$, $k+n-m \in \{n, n-1, \ldots, n-m+j+1\}$. Repeating the same proof above, we can tell that:

$$k+n-m = \max\{k, k+n-m\} = \max\{k, k+1, \ldots, k+n-m\}$$

In conclusion,

$$\max\{k, k+n-m\} = \max\{k, k+1, \ldots, k+n-m\}, \ \ \forall k = 1, 2, \ldots, m$$

□

Lemma 6.3. *Suppose* $P(k+1) = [k+1; s+1]$, $P(k) = [k; s]$.

If $P(k+1) \in \mathscr{F}$, *then* $P(k) \in \mathscr{F}$ *if and only if*

$$\max\{k, k+1, \ldots, \ s, s+1\} = \max\{k, s+1\}$$

Proof. (1) If $k = \max\{k, k+1, \ldots, \ t, t+1\}$,
then we have

$$\Delta(k) = [k; s]$$

so

$$P(k) \in \mathscr{F}.$$

(2)If $s+1 = \max\{k, \ldots, s+1\}$, $s+1 \notin \Delta(k), \ldots, \Delta(s)$.
Since $P(k+1) \in \mathscr{F}$,
we have

$$P^* = [k+1; s] \in \mathscr{F}$$

By 4.1, we can tell that

$$P(k) \in \mathscr{F}$$

(3)If there is a c s.t. $k < c < s+1$ and $c = \max\{k, \dots, s+1\}$,

$$\Delta(c) = [c; t], \quad t \geq s+1.$$

Obviously, $c \notin \Delta(i)$, $\forall i \in \{k, k+1, \dots,\ s, s+1\} \setminus \{c\}$.
As we know that $P(k) = [k; c; s]$ and $[c; s] \notin \mathscr{F}$,
so it is necessary that $P(k) \notin \mathscr{F}$

$\square$

Theorem 6.4. $q(A_n/I_{T_1}) = 2^m(n-m)!$

Proof. By 6.2,we take any j in $\{1, 2, \dots, m\}$. Now we only need to fill $\{1, 2, \dots, j\}$ into the largest m elements of $\{1, 2, \dots, n\}$, which exists C_n^j cases. The rest of the largest m elements is filled automatically with $\{n, n-1, \dots, n-m+j+1\}$. The rest $n-m$ elements is the random permutations of $\{j+1, j+2, \dots, n-m+j\}$, which exists $(n-m)!$ cases.

Then we sum up all the cases:

$$q(A_n/I_{T_1}) = \sum_{j=1}^{m} C_m^j (n-m)! = 2^m(n-m)!$$

$\square$

7. T_2 generators

In this essay,we'd like to deal with another type of algebra with m generators T_2 like:

$$I_{T_2} = < g_1, g_2, \dots, g_m >, g_i = < \alpha_{2i+1} \dots \alpha_{2i-1} >, \ \forall i = 1, 2, \dots, m$$

Consider T_2 generators, in which we have:
when k is odd and k < 2m+1:

$$P(k) = [k; k+1]$$

when k is even and k < 2m:

$$P(k) = [k; k+2]$$

when $k = 2m$:

$$P(2m) = [2m; 2m+1]$$

when $k = 2m+1$:

$$P(2m+1) = 2m+1$$

By 4.1, we can simply get that the Quasi-Hereditary orderings need to satisfy following conditions:

$$2k-1 \succeq 2k \ \ or \ \ 2k+1 \succeq 2k \succeq 2k-1, \ \ \forall k = 1, 2, ..., m$$

Lemma 7.1. *The orderings above equals to the permutations of $2m+1$ elements with no local maxima in even positions.*

Proof. $\Rightarrow$

For $2k-1 \succeq 2k$ *or* $2k+1 \succeq 2k \succeq 2k-1$, we know that $2k$ is not the maxima in $\{2k-1, 2k, 2k+1\}$, which means the orderings have no local maxima in even positions.

$\Leftarrow$

Consider permutations of length $2m+1$ with no local maxima in even positions. In these permutations, we have that $\forall k = 1, 2, ..., m$, the $2k$ must small than $2k-1$ or $2k+1$.

If $2k-1 \succeq 2k$, it's done.

If $2k-1 \preceq 2k$, $2k$ must larger than $2k+1$, which is $2k+1 \succeq 2k \succeq 2k-1$.This also satisfies the conditions.

$\square$

a_m means the number of permutations of $2m-1$ elements with no local maxima in even positions.(cite https://oeis.org/A113583)

So, when it comes to A_{2m+1}, $q(A_{2m+1}/I_{T_2}) = a_{m+1}$.

Theorem 7.2. $q(A_n/I_{T_2}) = \frac{a_{m+1}n!}{(2m+1)!}$

Proof. The Quasi-Hereditary orderings need to satisfy following conditions:

$$2k-1 \succeq 2k \ \ or \ \ 2k+1 \succeq 2k \succeq 2k-1, \ \ \forall k = 1,2,...,m$$

. By **Simple path Extension Lemma**,

$$q(A_n/I_{T_2}) = \frac{n!}{(2m+1)!} q_{A_n/I_{T_2}}(1, 2m+1) = \frac{n!}{(2m+1)!} q(A_{2m+1}/I_{T_2}) = \frac{a_{m+1}n!}{(2m+1)!}$$

$\square$

When we have $q(A_n/I_{T_1})$, we can compute any single generator situation quickly.

Corollary 7.3. *(Single generator situation) If we have $I = i \to i+m$, then $q(A_n/I) = \frac{2}{m+1}n!$*

Proof. Consider $T1$ generators in A_{m+1} with only one generator which is $I = 1 \to m+1$.

So it comes that $q(A_{m+1}/I) = 2m!$

Now we consider $A_n, (n > m)$ with generator $I = i \to i+m$.

$$q_{A_n/I}(i, i+m) = q(A_{m+1}/I) = 2m!$$

By **Simple path Extension Lemma**,

$$q(A_n/I) = \frac{n!}{(m+1)!} q_{A_n/I}(i, i+m) = \frac{2}{m+1} n!$$

$\square$

Corollary 7.4. *If we have $I_i =< \alpha_{m_i} \ldots \alpha_{m_i+k_i-1} >$ and $m_i + k_i < m_{i+1}$,$\forall i = 1, 2, \ldots, t$, then*

$$q(A_n/\sum_{i=1}^{t} I_i) = \frac{2^t}{\Pi_{i=1}^t (k_i+1)} n!$$

Proof. WLOG, $m_1 = 1$,

When t = 2,

$$\begin{aligned} q(A_{m_2+k_2}/I_1+I_2) &= q_{A_{m_2+k_2}/(I_1+I_2)}(m_1, m_1+k_1) q_{A_{m_2+k_2}/(I_1+I_2)}(m_2, m_2+k_2) C_{k_1+k_2+2}^{k_1+1} \frac{(m_2+k_2)!}{(k_1+k_2+2)!} \\ &= q(A_{k_1+1}/J_1) q(A_{k_2+1}/J_2) C_{k_1+k_2+2}^{k_1+1} \frac{(m_2+k_2)!}{(k_1+k_2+2)!} \\ &= \frac{2}{k_1+1} \frac{2}{k_2+1} (k_1+1)!(k_2+1)! C_{k_1+k_2+2}^{k_1+1} \frac{(m_2+k_2)!}{(k_1+k_2+2)!} \\ &= \frac{2^2}{(k_1+1)(k_2+1)} (m_2+k_2)! \end{aligned}$$

When t=j, suppose

$$q(A_{m_j+k_j}/\sum_{i=1}^{j} I_i) = \frac{2^j}{\Pi_{i=1}^{j}(k_i+1)}(m_j+k_j)!$$

$$q(A_{m_{j+1}+k_{j+1}}/\sum_{i=1}^{j+1} I_i)$$

$$= q_{A_{m_{j+1}+k_{j+1}}/\sum_{i=1}^{j+1} I_i}(m_1, m_j+k_j)q_{A_{m_{j+1}+k_{j+1}}/\sum_{i=1}^{j+1} I_i}(m_{j+1}, m_{j+1}+k_{j+1})C_{m_j+k_j+k_{j+1}+1}^{k_{j+1}+1}\frac{(m_{j+1}+k_{j+1})!}{(m_j+k_j+k_{j+1}+1}$$

$$= q(A_{m_j+k_j}/\sum_{i=1}^{j} I_i)q(A_{k_{j+1}+1}/J_{j+1})C_{m_j+k_j+k_{j+1}+1}^{k_{j+1}+1}\frac{(m_{j+1}+k_{j+1})!}{(m_j+k_j+k_{j+1}+1)!}$$

$$= \frac{2^{j+1}}{\Pi_{i=1}^{j+1}(k_i+1)}(m_j+k_j)!(k_{j+1}+1)!C_{m_j+k_j+k_{j+1}+1}^{k_{j+1}+1}\frac{(m_{j+1}+k_{j+1})!}{(m_j+k_j+k_{j+1}+1)!}$$

$$= \frac{2^{j+1}}{\Pi_{i=1}^{j+1}(k_i+1)}(m_{j+1}+k_{j+1})!$$

So consider general t,

$$q(A_{m_t+k_t}/\sum_{i=1}^{t} I_i) = \frac{2^t}{\Pi_{i=1}^{t}(k_i+1)}(m_t+k_t)!$$

By **??** and 4.4,

$$q(A_n/\sum_{i=1}^{t} I_i) = \frac{2^t}{\Pi_{i=1}^{t}(k_i+1)}n!$$

□

8. A Proof of Conjecture in A_n-type Nakayama Algebra

Lemma 8.1. *A is a hereditary A_n type Nakayama Algebra, suppose$I_0 \lhd A$,$I_1 \lhd A$, $g \notin I_0$, $I_1 =< I_0, g >$, then $\mathscr{O}(A/I_0) \supseteq \mathscr{O}(A/I_1)$.*

Proof. (1)If $I_0 = 0$, then $\mathscr{O}(A/I_0) = \mathscr{O}(A) \supseteq \mathscr{O}(A/I_1)$

(2)If $I_0 = 0$, suppose $I_0 =< g_1, \ldots, g_k >$, g_i begins at s_i and ends ate_i, then $s_1 < \cdots < s_k$, $e_1 < \cdots < e_k$.
When $g = 0$, we see $\mathscr{O}(A/I_0) = \mathscr{O}(A/I_1)$.
When $g \neq 0$, we prove this matter in three cases. Suppose $P^0(M)$ is the projective module of A/I_0, $P^1(M)$ is the projective module of A/I_1.

① $I_1 =< g_1, \ldots, g_m, g, g_{m+1}, \ldots, g_k >$, it means $s_m < s < s_{m+1}$, $e_m < e < e_{m+1}$.
In this case, for all simple module,

$$P^0(i) = P^1(i), i = 1, \ldots, s_m, \ldots, s+1, \ldots, n; \quad P^0(j) \neq P^1(j), j = s_m+1, \ldots, s$$

This will only lead to $\Delta^0(j) \neq \Delta^1(j)$ being different, $j = s_m+1, \ldots, s$.
Therefore, for any order of $\mathscr{O}(A/I_1)$,
$P^0(i) \in \mathscr{F}(A/I_0), i = s+1, \ldots, n$.
By **Ladder Lemma**, $P^0(j) \in \mathscr{F}(A/I_0), j = s_m+1, \ldots, s$.
As $P^0(s_m) = P^1(s_m)$, we can suppose$P^0(s_m) = \Delta^1(s_m) + \sum_i \Delta^1(x_i)$.

If $\sum_i \Delta^1(x_i) = 0$, then $P^0(s_m) \in \mathscr{F}(A/I_0)$.
If $\sum_i \Delta^1(x_i) \neq 0$, then $e_m \succeq x_i$ for any i. This shows $\sum_i \Delta^0(x_i) = \sum_i \Delta^1(x_i)$, so$P^0(s_m) \in \mathscr{F}(A/I_0)$.
Similarly, we can show $P^0(i) \in \mathscr{F}(A/I_0), i = 1, \ldots, s_m$.
Overall, in this case, for any order of $\mathscr{O}(A/I_1)$, it's a quasi-hereditary order of A/I_0.

② $I_1 =< g_1, \ldots, g_k, g >$it means $s_k < s$, $e_k < e$.
In this case, for all simple module,

$$P^0(i) = P^1(i), i = 1, \ldots, s_k, \ldots, s+1, \ldots, n; \quad P^0(j) \neq P^1(j), j = s_k + 1, \ldots, s$$

This will only lead to $\Delta^0(j) \neq \Delta^1(j)$ being different, $j = s_k + 1, \ldots, s$.
Therefore, for any order of $\mathscr{O}(A/I_1)$,
$P^0(i) \in \mathscr{F}(A/I_0), i = s+1, \ldots, n$.
By 4.1, $P^0(j) \in \mathscr{F}(A/I_0), j = s_k + 1, \ldots, s$.
As $P^0(s_k) = P^1(s_k)$, we can suppose$P^0(s_k) = \Delta^1(s_m) + \sum_i \Delta^1(x_i)$.
If $\sum_i \Delta^1(x_i) = 0$, then $P^0(s_k) \in \mathscr{F}(A/I_0)$.
If $\sum_i \Delta^1(x_i) \neq 0$, then $e_k \succeq x_i$ for any i. This shows $\sum_i \Delta^0(x_i) = \sum_i \Delta^1(x_i)$, so$P^0(s_k) \in \mathscr{F}(A/I_0)$.
Similarly, we can show $P^0(i) \in \mathscr{F}(A/I_0), i = 1, \ldots, s_k$.
Overall, in this case, for any order of $\mathscr{O}(A/I_1)$, it's a quasi-hereditary order of A/I_0.

③ $I_1 =< g, g_1, \ldots, g_k >$, it means $s < s_1, < e < e_1$.
In this case, for all simple module,

$$P^0(i) = P^1(i), i = s+1, \ldots, n; \quad P^0(j) \neq P^1(j), j = 1, \ldots, s$$

This will only lead to $\Delta^0(j) \neq \Delta^1(j)$ being different, $j = 1, \ldots, s$.
Therefore, for any order of $\mathscr{O}(A/I_1)$,
$P^0(i) \in \mathscr{F}(A/I_0), i = s+1, \ldots, n$.
By 4.1, $P^0(j) \in \mathscr{F}(A/I_0), j = 1, \ldots, s$.
Overall, in this case, for any order of $\mathscr{O}(A/I_1)$, it's a quasi-hereditary order of A/I_0.

In all, $\mathscr{O}(A/I_0) \supseteq \mathscr{O}(A/I_1)$.

□

Corollary 8.2. *A is a hereditary A_n type Nakayama Algebra, suppose$I_0 \lhd A$,$I_1 \lhd A$, $g \notin I_0$, $I_1 =< I_0, g >$, then $q(A/I_0) \geq q(A/I_1)$.*

Corollary 8.3. *If A is a non-hereditary A_n type Nakayama Algebra, $q(A) \geq \frac{2}{3}n!$.*

References

[1] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[2] Vlastimil Dlab and Claus Michael Ringel. Quasi-hereditary algebras. *Illinois J. Math.*, 33(2):280–291, 1989.

[3] Morio Uematsu and Kunio Yamagata. On serial quasi-hereditary rings. *Hokkaido Math. J.*, 19(1):165–174, 1990.

[4] Yue Hui Zhang and Li Yu. Counting quasi-hereditary orderings of finite dimensional algebras. *J. MATh. TECH.*, 16(3):9–11, 2000.

[5] Yue Hui Zhang, Liu San Wu, and Chun Yan Gao. Quasi-hereditary orderings of A_n-type algebras with two generators. *J. Math. Res. Exposition*, 28(4):975–980, 2008.

[6] Edward L. Green and Sibylle Schroll. On quasi-hereditary algebras. *Bull. Sci. Math.*, 157:102797, 14, 2019.

[7] René Marczinzik and Emre Sen. A new characterization of quasi-hereditary Nakayama algebras and applications. *Comm. Algebra*, 50(10):4481–4493, 2022.