6th Material Subject: Discrete Univariate Random Variable

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

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السلام عليكم ورحمة الله وبركاته

WELCOME

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LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Determine probabilities from probability mass functions and the reverse**
2. **Determine probability mass functions from cumulative distribution functions and the reverse**
3. **Calculate probability mass function, means and variances for discrete random variable transformation**

PRELIMINARY

Many physical systems can be modeled by the same or similar random experiments and random variables. The distribution of the random variables involved in each of these common systems can be analyzed, and the results can be used in different applications and examples. In this chapter, we present the analysis of several random experiments and discrete random variables that frequently arise in applications.

Examples: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable **X** denote the number of lines in use. Then **X** can assume any of the integer values 0 through 48. When the system is observed, if 10 lines are in use, $\mathbf{x} = \mathbf{10}$.

PROBABILITY DISTRIBUTIONS AND PROBABILITY MASS FUNCTIONS

The **Probability Distribution** of a random variable **X** is a description of the probabilities associated with the possible values of **X**. The distribution is often specified by just a list of the possible values of each.

Example 1: The sample space of a sequence of three fair coin flips is all $2^3 = 8$ possible sequences of outcomes $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. If **X** is a random variable that states the number of **head**, then the range of **X** is $R_x = \{0, 1, 2, 3\}$. The probability distribution should be:

$$P(X = 0) = \frac{1}{8} \qquad P(X = 1) = \frac{3}{8}$$

$$P(X = 2) = \frac{3}{8} \qquad P(X = 3) = \frac{1}{8}$$

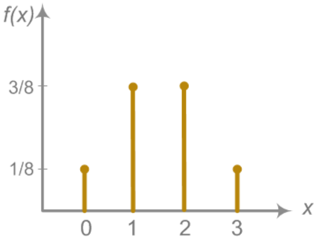

Figure 1: Probability distribution for fair a coin

PROBABILITY DISTRIBUTIONS AND PROBABILITY MASS FUNCTIONS

Example 2: There is a chance that a bit transmitted through a digital transmission is received in error. Let **X** equal the number of bits error in the next 4 bits transmitted. The possible values for **X** are $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. Suppose that the probabilities are:

$$\mathbf{P}(\mathbf{X} = \mathbf{0}) = \mathbf{0.6561}$$

$$\mathbf{P}(\mathbf{X} = \mathbf{1}) = \mathbf{0.2916}$$

$$\mathbf{P}(\mathbf{X} = \mathbf{2}) = \mathbf{0.0486}$$

$$\mathbf{P}(\mathbf{X} = \mathbf{3}) = \mathbf{0.0036}$$

$$\mathbf{P}(\mathbf{X} = \mathbf{4}) = \mathbf{0.0001}$$

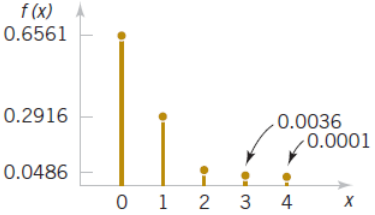

Figure 2: Probability distribution for bits in error

PROBABILITY DISTRIBUTIONS AND PROBABILITY MASS FUNCTIONS

For a discrete random variable **X** with possible values $x_1, x_2, x_3, \cdots, x_n$, a **Probability Mass Function**, denoted by **f(x)**, is a function such that:

$$f(x_i) \geq 0 \tag{1}$$

$$\sum_{i=1}^{n} f(x_i) = 1 \tag{2}$$

$$f_x(i) = P(X = x_i) \tag{3}$$

PROBABILITY DISTRIBUTIONS AND PROBABILITY MASS FUNCTIONS

The Probability mass function for Example 1 can be written as:

$$\mathbf{f(x)} = \begin{cases} \frac{1}{8} & \text{, for } x = 0, 3 \\ \frac{3}{8} & \text{, for } x = 1, 2 \\ 0 & \text{, otherwise} \end{cases}$$

The Probability mass function for Example 2 can be written as:

$$\mathbf{f(x)} = \begin{cases} 0.6561 & \text{, for } x = 0 \\ 0.2916 & \text{, for } x = 1 \\ 0.0486 & \text{, for } x = 2 \\ 0.0036 & \text{, for } x = 3 \\ 0.0001 & \text{, for } x = 4 \\ 0 & \text{, otherwise} \end{cases}$$

CUMULATIVE DISTRIBUTION FUNCTION

An alternate method for describing a random variables probability distribution is with **Cumulative Distribution Functions** such as $\mathbf{P}(\mathbf{X} \leq \mathbf{x})$. The cumulative distribution function of a discrete random variable **X**, denoted as **F(x)**, is:

$$\mathbf{F}(\mathbf{x}) = \mathbf{P}(\mathbf{X} \leq \mathbf{x}) = \sum_{\mathbf{x_i} \leq \mathbf{x}} \mathbf{f}(\mathbf{x_i}) \tag{4}$$

$$\mathbf{0} \leq \mathbf{F}(\mathbf{x}) \leq \mathbf{1} \tag{5}$$

$$\text{If} \quad \mathbf{x} \leq \mathbf{y} \quad \text{then} \quad \mathbf{F}(\mathbf{x}) \leq \mathbf{F}(\mathbf{y}) \tag{6}$$

CUMULATIVE DISTRIBUTION FUNCTION

The Cumulative Distribution Function for Example 1 can be written as:

$$\mathbf{F(x)} = \begin{cases} 0 & , x < 0 \\ \frac{1}{8} & , 0 \leq x < 1 \\ \frac{4}{8} & , 1 \leq x < 2 \\ \frac{7}{8} & , 2 \leq x < 3 \\ 1 & , x \geq 3 \end{cases}$$

The Cumulative Distribution Function for Example 2 can be written as:

$$\mathbf{f(x)} = \begin{cases} 0 & , x < 0 \\ 0.6561 & , 0 \leq x < 1 \\ 0.9477 & , 1 \leq x < 2 \\ 0.9963 & , 2 \leq x < 3 \\ 0.9999 & , 3 \leq x < 4 \\ 1 & , x \geq 4 \end{cases}$$

CUMULATIVE DISTRIBUTION FUNCTION

Furthermore, cumulative probabilities can be used to find the probability mass function of a discrete random variable.

Example: Consider the Example 2, (a). calculate the probability that three or fewer bits are in error and (b) Maximum two bits in error?

Answer:

$$P(X \leq 3) = f(0) + f(1) + f(2) + f(3) = 0.6561 + 0.2916 + 0.0486 + 0.0036 = 0.9999$$

$$P(X \leq 2) = f(0) + f(1) + f(2) = 0.6561 + 0.2916 + 0.0486 = 0.9963$$

This approach can also be used to determine the PMF $f(3)$

$$f(3) = P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.9999 - 0.9963 = 0.0036$$

MEAN & VARIANCE OF A DISCRETE RANDOM VARIABLE

The **MEAN** is a measure of the center or middle of the probability distribution, and the **VARIANCE** is a measure of the dispersion, or variability in the distribution.

- The **Mean** or expected value of the discrete random variable **X**, denoted as μ or **E(X)**, is:

$$\mu = \mathbf{E}(\mathbf{X}) = \sum_{\mathbf{x}} \mathbf{x} \cdot \mathbf{f}(\mathbf{x}) \tag{7}$$

- The **Variance** of the discrete random variable **X**, denoted as σ^2 or **Var(X)**, is:

$$\sigma^2 = \mathbf{Var}(\mathbf{X}) = \mathbf{E}(\mathbf{x} - \mu)^2 = \sum_{\mathbf{x}} (\mathbf{x} - \mu)^2 \cdot \mathbf{f}(\mathbf{x}) = \mathbf{E}(\mathbf{X}^2) - (\mathbf{E}(\mathbf{X})^2) \tag{8}$$

- The **Standard Deviation** of the discrete random variable **X**, denoted as σ or, is:

$$\sigma = \sqrt{\sigma^2} \tag{9}$$

MEAN & VARIANCE OF A DISCRETE RANDOM VARIABLE

The **MEAN, VARIANCE** and **STANDARD DEVIATION** for Example-1, with he Probability mass function:

$$\mathbf{f(x)} = \begin{cases} \frac{1}{8} & \text{, for } x = 0, 3 \\ \frac{3}{8} & \text{, for } x = 1, 2 \\ 0 & \text{, otherwise} \end{cases}$$

$$\mathbf{E(X)} = \sum_{\mathbf{x}} \mathbf{x} \cdot \mathbf{f(x)} = \left(\mathbf{0} \cdot \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(\mathbf{1} \cdot \frac{\mathbf{3}}{\mathbf{8}}\right) + \left(\mathbf{2} \cdot \frac{\mathbf{3}}{\mathbf{8}}\right) + \left(\mathbf{3} \cdot \frac{\mathbf{1}}{\mathbf{8}}\right) = \frac{\mathbf{3}}{\mathbf{2}}$$

$$\mathbf{E(X^2)} = \sum_{\mathbf{x}} \mathbf{x^2} \cdot \mathbf{f(x)} = \left(\mathbf{0^2} \cdot \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(\mathbf{1^2} \cdot \frac{\mathbf{3}}{\mathbf{8}}\right) + \left(\mathbf{2^2} \cdot \frac{\mathbf{3}}{\mathbf{8}}\right) + \left(\mathbf{3^2} \cdot \frac{\mathbf{1}}{\mathbf{8}}\right) = \frac{\mathbf{24}}{\mathbf{8}} = \mathbf{3}$$

$$\sigma^2 = \mathbf{Var(X)} = \mathbf{E(X^2)} - (\mathbf{E(X)})^2 = \mathbf{3} - \left(\frac{\mathbf{3}}{\mathbf{2}}\right)^2 = \frac{\mathbf{3}}{\mathbf{4}} \quad \textbf{and} \quad \sigma = \sqrt{\sigma^2} = \sqrt{\frac{\mathbf{3}}{\mathbf{4}}} = \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}$$

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

Suppose **X** is a discrete random variable with mean **E(X)** and variance **Var(X)**, if there is a new random variable denoted by $\mathbf{Y} = \mathbf{aX} + \mathbf{b}$ (for any constant of **a** and **b**), then:

$$\mathbf{E}(\mathbf{Y}) = \mathbf{a} \cdot \mathbf{E}(\mathbf{X}) + \mathbf{b} \tag{10}$$

$$\mathbf{Var}(\mathbf{Y}) = \mathbf{a}^2 \cdot \mathbf{Var}(\mathbf{X}) \tag{11}$$

Equation 10 is obtained from:

$$\mathbf{E}(\mathbf{Y}) = \mathbf{E}(\mathbf{aX} + \mathbf{b}) = \sum (\mathbf{aX} + \mathbf{b}) \cdot \mathbf{f}(\mathbf{x}) = \sum \mathbf{aX} \cdot \mathbf{f}(\mathbf{x}) + \sum \mathbf{b} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{a} \sum \mathbf{X} \cdot \mathbf{f}(\mathbf{x}) + \mathbf{b} \sum \mathbf{f}(\mathbf{x})$$

Since $\sum \mathbf{x} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{E}(\mathbf{X})$ and $\sum \mathbf{f}(\mathbf{x}) = \mathbf{1}$, then:

$$\mathbf{E}(\mathbf{Y}) = \mathbf{a} \cdot \mathbf{E}(\mathbf{X}) + \mathbf{b}$$

In the same way, try to describe the Equation 11.

DISCRETE RANDOM VARIABLE TRANSFORMATION

Suppose $\mathbf{X}$ is a discrete random variable with mean probability mass function $\mathbf{f_X(x)}$, if there is a new random variable denoted by $\mathbf{Y} = \mathbf{g(X)}$, a one-to-one transformation between the values of Y and X, then the probability mass function of $\mathbf{Y}$:

$$\mathbf{f_Y(y) = f_X(g^{-1}(x))} \tag{12}$$

Where $\mathbf{g^{-1}(x)}$ is an inverse function of $\mathbf{Y} = \mathbf{g(X)}$.

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

Suppose **X** is a discrete random variable with mean **E(X)** and variance **Var(X)**, if there is a new random variable denoted by $\mathbf{Y} = \mathbf{aX} + \mathbf{b}$ (for any constant of **a** and **b**), then:

$$\mathbf{E(Y) = a \cdot E(X) + b} \tag{13}$$

$$\mathbf{Var(Y) = a^2 \cdot Var(X)} \tag{14}$$

Equation 10 is obtained from:

$$\mathbf{E(Y) = E(aX + b) = \sum (aX + b) \cdot f(x) = \sum aX \cdot f(x) + \sum b \cdot f(x) = a \sum X \cdot f(x) + b \sum f(x)}$$

Since $\mathbf{\sum x \cdot f(x) = E(X)}$ and $\mathbf{\sum f(x) = 1}$, then:

$$\mathbf{E(Y) = a \cdot E(X) + b}$$

In the same way, try to describe the Equation 11.

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

Example: If $\mathbf{Y} = \mathbf{3X} + \mathbf{2}$ is the transformation of random variable **X** with probability mass function:

$$\mathbf{f_X(x)} = \begin{cases} \frac{2x+1}{16} & \text{, for } x = 0, 1, 2, 3 \\ 0 & \text{, otherwise} \end{cases}$$

Determine the $\mathbf{f_Y(y)}$, $\mathbf{E(Y)}$ and $\mathbf{Var(Y)}$!
Answer: First, we can write down the probability distribution for **X**:

x	**0**	**1**	**2**	**3**
$\mathbf{f_X(x)}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

Now, calculate the mean and variance of **X**

$$E(X) = \sum_x x \cdot f(x) = \left(0 \cdot \frac{1}{16}\right) + \left(1 \cdot \frac{3}{16}\right) + \left(2 \cdot \frac{5}{16}\right) + \left(3 \cdot \frac{7}{16}\right) = \frac{34}{16}$$

$$E(X^2) = \sum_x x^2 \cdot f(x) = \left(0^2 \cdot \frac{1}{16}\right) + \left(1^2 \cdot \frac{3}{16}\right) + \left(2^2 \cdot \frac{5}{16}\right) + \left(3^2 \cdot \frac{7}{16}\right) = \frac{86}{16}$$

$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = \frac{86}{16} - \left(\frac{34}{16}\right)^2 = \frac{55}{64}$$

So that, the mean and variance of **Y**:

$$E(Y) = 3 \cdot E(X) + 2 = \frac{67}{8} \quad \text{and} \quad Var(Y) = 3^2 \cdot Var(X) = \frac{495}{64}$$

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

To find the probability mass function of **Y**:

$$\mathbf{Y} = \mathbf{3X} + \mathbf{2} \quad \text{then the inverse become} \quad \mathbf{g}^{-1}(\mathbf{x}) = \frac{\mathbf{Y} - \mathbf{2}}{\mathbf{3}}$$

$$\mathbf{f_Y}(\mathbf{y}) = \mathbf{f_X}\left(\mathbf{g}^{-1}(\mathbf{x})\right)$$

$$\mathbf{f_Y}(\mathbf{y}) = \begin{cases} \frac{2\cdot\left(\frac{Y-2}{3}\right)+1}{16} & \text{, for } \mathbf{Y} = \mathbf{3X} + \mathbf{2} \text{ with } x = 0, 1, 2, 3 \\ 0 & \text{, otherwise} \end{cases}$$

$$\mathbf{f_Y}(\mathbf{y}) = \begin{cases} \frac{2Y-1}{48} & \text{, for } y = 2, 5, 8, 11 \\ 0 & \text{, otherwise} \end{cases}$$

MEAN AND VARIANCE IN RANDOM VARIABLE TRANSFORMATION

We can also calculate the **mean** and **variance** of Y from $\mathbf{f_Y(y)}$ and compare with the results we have before.

The probability distribution of **Y**:

y	2	5	8	11
$f_Y(y)$	$\frac{3}{48}$	$\frac{9}{48}$	$\frac{15}{48}$	$\frac{21}{48}$

$$E(Y) = \sum_y y \cdot f_Y(y) = \left(2 \cdot \frac{3}{48}\right) + \left(5 \cdot \frac{9}{48}\right) + \left(8 \cdot \frac{15}{48}\right) + \left(11 \cdot \frac{21}{48}\right) = \frac{402}{48} = \frac{67}{8}$$

$$E(Y^2) = \sum_y y^2 \cdot f + Y(y) = \left(2^2 \cdot \frac{3}{48}\right) + \left(5^2 \cdot \frac{9}{48}\right) + \left(8^2 \cdot \frac{15}{48}\right) + \left(11^2 \cdot \frac{21}{48}\right) = \frac{3738}{48} = \frac{623}{8}$$

$$\sigma^2 = \text{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{623}{8} - \left(\frac{67}{8}\right)^2 = \frac{495}{64}$$

Thank You