9th Material Subject: Joint Probability Distribution (Continuous)

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

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WELCOME

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LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Use joint probability density functions to calculate probabilities**
2. **Calculate marginal and conditional probability distributions from joint probability distributions**
3. **Interpret and calculate covariance and correlations between random variables**

JOINT PROBABILITY DENSITY FUNCTION

For simplicity, we begin by considering random experiments in which only two random variables, called **Bi-variate**. The **Joint Probability Density Function** of the continuous random variables **X** and **Y**, denoted as $\mathbf{f_{XY}(xy)}$, satisfies:

$$\mathbf{f_{XY}(xy) \geq 0} \tag{1}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f_{XY}(xy)\,dx\,dy = 1} \tag{2}$$

$$\mathbf{f_{XY}(xy) = P(X = x \quad and \quad Y = y) = P(X = x) \cap P(Y = y)} \tag{3}$$

MARGINAL PROBABILITY DENSITY FUNCTION

The **Marginal Probability Density Function** of the continuous random variables $\mathbf{X}$ and $\mathbf{Y}$, denoted as $\mathbf{f_X(x)}$ or $\mathbf{f_Y(y)}$, satisfies:

$$\mathbf{f_X(x) = P(X = x) = \int f_{XY}(x, y)\, dy} \tag{4}$$

$$\mathbf{f_Y(y) = P(Y = y) = \int f_{XY}(x, y)\, dx} \tag{5}$$

JOINT CUMULATIVE DISTRIBUTION FUNCTION

Remember that, for a random variable $\mathbf{X}$, we define the CDF as $\mathbf{F_X(x) = P(X \leq x)}$. Now, if we have two random variables **X** and **Y** and we would like to study them jointly, we can define the **Joint Cumulative Function** as follows:

$$\mathbf{F_{XY}(x, y) = P(X \leq x \ \text{ and } \ Y \leq y) = P(X \leq x) \cap P(Y \leq y)} \tag{6}$$

INDEPENDENT BIVARIATE

The random variable **X** and **Y** become **independent**, if only:

$$f_{XY}(x, y) = P(X = x) \cdot P(Y = y) = f_X(x) \cdot f_Y(y) \tag{7}$$

or:

$$F_{XY}(x, y) = P(X \leq x) \cdot P(Y \leq y) = F_X(x) \cdot F_Y(y) \tag{8}$$

COVARIANCE AND CORRELATION

When two random variables **X** and **Y** are **not independent**, it is frequently of interest to assess how strongly they are related to one another. The **Covariance** between two random variables **X** and **Y** equal to:

$$\mathbf{Cov(XY) = E(XY) - E(X) \cdot E(Y)} \tag{9}$$

Where, the joint expectation should be:

$$\mathbf{E(XY) = \int\int x \cdot y \cdot f_{XY}(xy)\, dx\, dy} \tag{10}$$

The **Correlation Coefficient** of **X** and **Y**, equal to:

$$\mathbf{Cor(XY)} = \rho_{\mathbf{XY}} = \frac{\mathbf{Cov(XY)}}{\sigma_{\mathbf{x}} \cdot \sigma_{\mathbf{y}}} \tag{11}$$

EXAMPLE

Example: Suppose that **X** and **Y** are two continuous random variable with joint PDF:

$$\mathbf{f_{XY}(xy)} = \begin{cases} c(x+y) & \text{, for } \mathbf{0 < x < 3} \text{ and } \mathbf{0 < y < 3} \\ 0 & \text{, for x and y otherwise} \end{cases}$$

a. Determine the value of **c**

b. Determine the marginal PDF of **X**

c. Determine the marginal PDF of **Y**

d. Determinan the $\mathbf{P(1 < x < 2)}$

e. Determinan the $\mathbf{P(x \geq 1)}$

f. Determinan the $\mathbf{P(y \leq 2.5)}$

Answer:

a. The value of **c** must qualify the joint pdf that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f_{XY}(xy)\ dx\ dy = 1}$

$$\int_0^3\int_0^3 \mathbf{cxy\ dx\ dy = 1} \ \rightarrow \int_0^3 \left(\frac{\mathbf{cx^2y}}{\mathbf{2}}\Big|_0^3\right) \mathbf{dy = 1} \ \rightarrow \int_0^3 \left(\frac{\mathbf{9cy}}{\mathbf{2}}\right) \mathbf{dy = 1} \ \rightarrow \left(\frac{\mathbf{9cy^2}}{\mathbf{4}}\Big|_0^3\right) \mathbf{dy = 1}$$

$$\frac{\mathbf{81c}}{\mathbf{4}} = \mathbf{1} \quad \rightarrow \mathbf{c} = \frac{\mathbf{4}}{\mathbf{81}}$$

b. The marginal PDF of **X**

$$\mathbf{f_X(x)} = \int_0^3 \frac{\mathbf{4}}{\mathbf{81}}\mathbf{xy\ dy} = \frac{\mathbf{2}}{\mathbf{81}}\mathbf{xy^2}\Big|_0^3 = \frac{\mathbf{2}}{\mathbf{9}}\mathbf{x}$$

So, the marginal PDF for **X** is:

$$\mathbf{f_X(x)} = \begin{cases} \frac{2}{9}x & , 0 < x < 3 \\ 0 & , \textit{otherwise} \end{cases}$$

c. The marginal PDF of **Y**

$$f_Y(y) = \int_0^3 \frac{4}{81} xy\,dx = \frac{2}{81} x^2 y \Big|_0^3 = \frac{2}{9} y$$

So, the marginal PDF for **Y** is:

$$f_Y(y) = \begin{cases} \frac{2}{9}y & , 0 < y < 3 \\ 0 & , \textit{otherwise} \end{cases}$$

d. The $P(1 < x < 2)$

$$P(1 < x < 2) = \int_1^2 \frac{2}{9} x\,dx = \frac{1}{9} x^2 \Big|_1^2 = \frac{3}{9}$$

e. The $P(x \geq 1)$

$$P(x \geq 1) = P(1 < x < 3) = \int_1^3 \frac{2}{9} x\,dx = \frac{1}{9} x^2 \Big|_1^3 = \frac{8}{9}$$

f. The $\mathbf{P}\left(\mathbf{y} \le \mathbf{2.5}\right)$

$$\mathbf{P}\left(\mathbf{y} \le \mathbf{2.5}\right) = \mathbf{P}\left(\mathbf{0} < \mathbf{y} < \mathbf{2.5}\right) = \int_{0}^{2.5} \frac{2}{9}\mathbf{y}\,\mathbf{dy} = \frac{1}{9}\mathbf{y}^2\Big|_0^{2.5} = \frac{\mathbf{6.25}}{\mathbf{9}}$$

Thank You