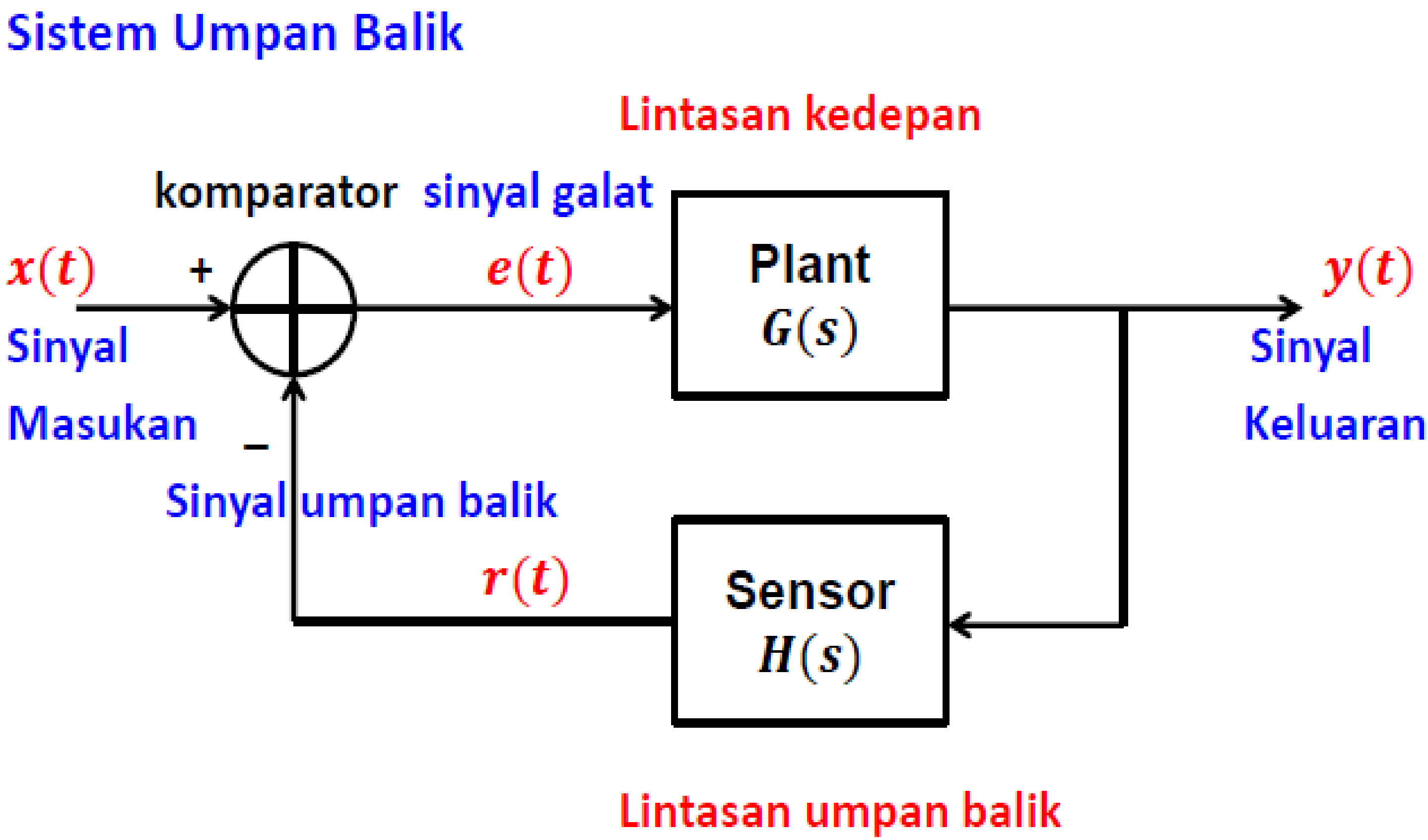
Pengolahan Sinyal Dalam Waktu Kontinyu

Bab 7. Pengantar Sistem Umpan Balik Linier



Dosen:

Dr. Suhartono Tjondronegoro

Isi Kuliah

- Bab 0. Pendahuluan.
- Bab 1. Sinyal Waktu Kontinyu.
- Bab 2. Sistem Waktu Kontinyu.
- Bab 3. Deret Fourier.
- Bab 4. Transformasi Fourier.
- Bab 5. Transformasi Laplace.
- Bab 6. Pengantar Filter Analog.
- **Bab 7. Pengantar Sistem Umpan Balik Linier.**

Pengantar Sistem Umpan Balik Linier

- Pendahuluan.
- Definisi umpan balik (feedback).
- Konsep dasar sistem umpan balik.
- Analisa sensitivitas.
- Effek umpan balik terhadap gangguan atau derau.
- Analisa distorsi.
- Keuntungan-keuntungan umpan balik dan beaya umpan balik.
- Penguat Operasional
- Sistem kendali.
- Respons transient.
- Masalah stabilitas.
- Metoda Root-locus.
- Diagram Bode.

Pendahuluan

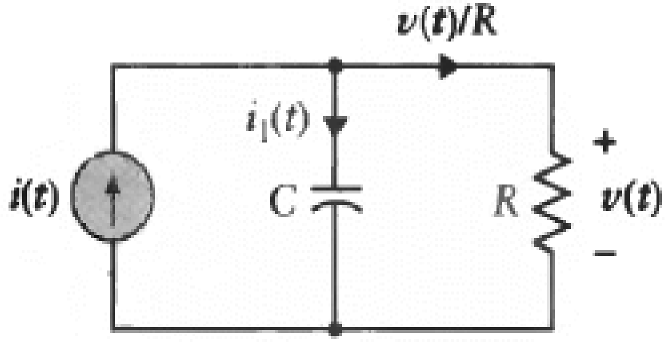
- Umpan balik (feedback) adalah sebuah konsep rekayasa (engineering) yang sangat penting.
- Adanya umpan balik (feedback) diperlukan didalam perancangan penguat daya (power amplifiers), penguat operasional (operational amplifiers), filter digital, sistem kendali (control systems), dan aplikasi lain.
- Umpan balik (Feedback) dipertimbangkan pada saat desain sebuah sistem dengan tujuan spesifik:
  - Memperbaiki sifat linier sistem,
  - Mengurangi sensitivitas penguatan (gain) sistem terhadap variasi nilai-nilai parameter-parameter sistem,
  - Mengurangi pengaruh gangguan dari eksternal terhadap operasi sistem.
- Keuntungan-keuntungan diatas dibayar dengan tingkah laku sistem yang lebih kompleks.

Definisi Umpan Balik (Feedback) (1)

- **Definisi:**
- **Umpan balik (Feedback)** adalah sebagian dari sinyal keluaran sistem yang dikembalikan ke sinyal masukan, membentuk sebuah loop ketergantungan diantara sinyal-sinyal didalam sistem.
- Perhatikan rangkaian RC:

  $i(t) = i_1(t) + i_2(t)$

  $v(t) = Ri_2(t) = \frac{1}{C}\int_{-\infty}^{t} i_1(\tau)d\tau.$



- Persamaan diatas dapat ditulis sebagai:

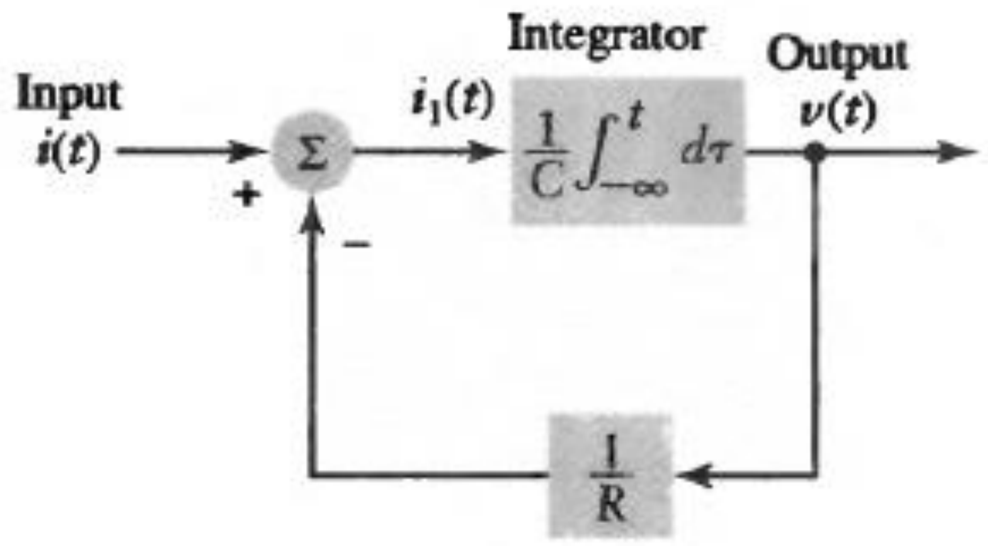
$$i_1(t) = i(t) - \frac{1}{R}v(t) \text{ dan } v(t) = \frac{1}{C}\int_{-\infty}^{t} i_1(\tau)d\tau$$

$$\text{Maka } i_1(t) = i(t) - \frac{1}{RC}\int_{-\infty}^{t} i_1(\tau)\,d\tau$$

Definisi Umpan Balik (Feedback) (2)

- Representasi diagram blok rangkaian paralel RC:

- Sistem adalah sebuah sistem umpan balik dengan Kapasitor C ada di lintasan kedepan dan konduktansi $\frac{1}{R}$ ada di-lintasan umpan balik.



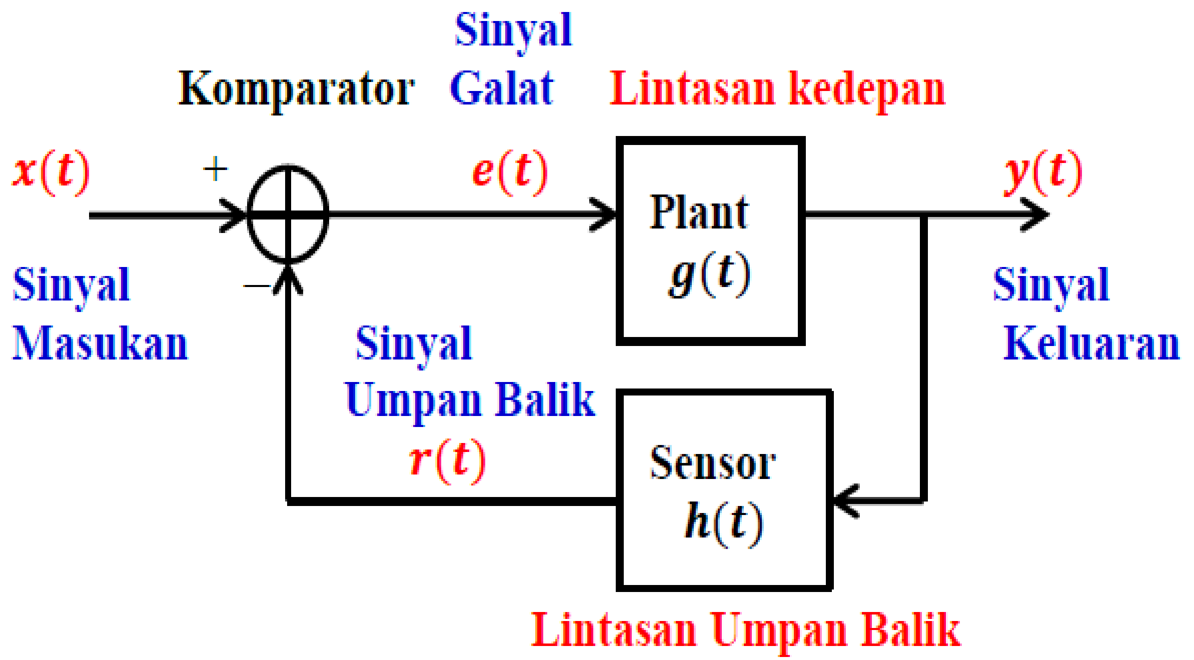
- Rangkaian paralel RC yang digambarkan disini adalah sebagai contoh sistem umpan balik, tergantung cara pandang bagaimana tingkah laku masukan-keluaran sistem diformulasikan.

Definisi Umpan Balik (Feedback) (3)

- Tujuan utama adalah mempelajari SLTBTW, dimana didalam diagram blok-nya mempunyai **loop umpan balik**, sistem tersebut disebut **sistem umpan balik linier**.
- Motivasi untuk mempelajari sistem umpan balik linier:
  - Keuntungan praktis aplikasi umpan balik dapat dicapai.
  - Pengertian tentang masalah stabilitas, menjamin bahwa sistem umpan balik (feedback system) stabil untuk semua kondisi operasi.
- Fokus bab ini dua issue penting diatas.

Konsep Dasar Sistem Umpan Balik (1)

- **Diagram blok sistem umpan balik (representasi kawasan waktu):**
- **Plant**, memproses sinyal galat $e(t)$ untuk menghasilkan sinyal keluaran $y(t)$.



- **Sensor**, mengukur sinyal keluaran $y(t)$ untuk menghasilkan sinyal umpan balik $r(t)$.
- **Komparator**, menghitung perbedaan antara sinyal masukan (referensi) $x(t)$ dengan sinyal umpan balik $r(t)$, untuk menghasilkan sinyal galat $e(t) = x(t) - r(t)$.

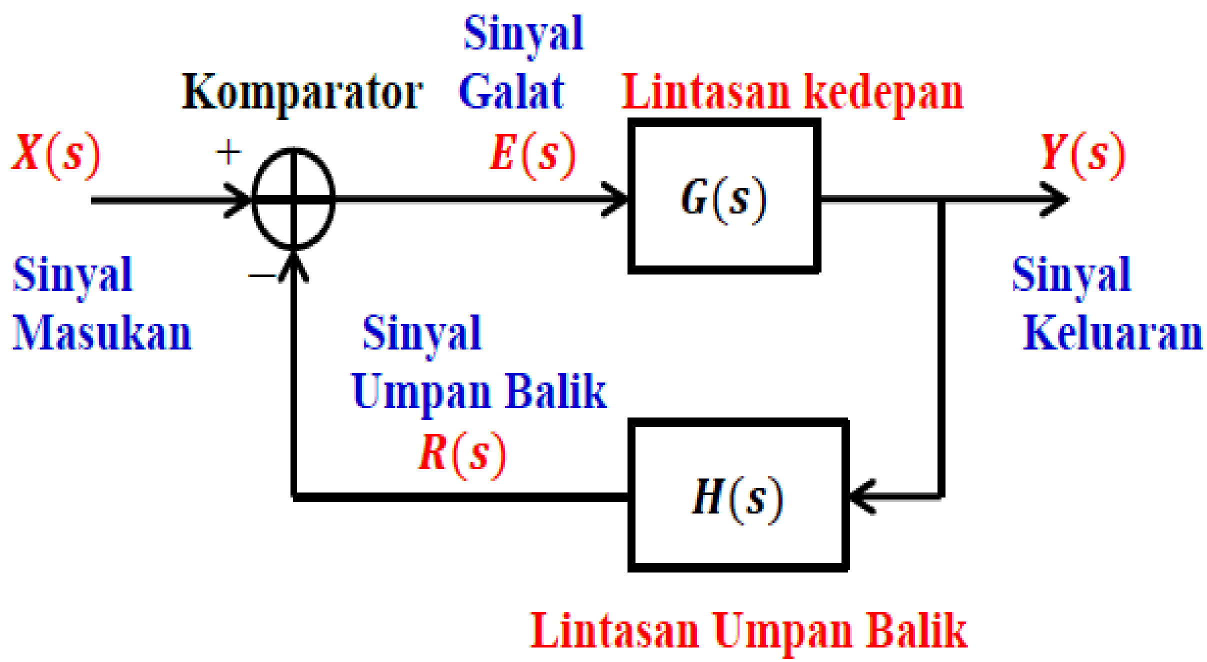
Konsep Dasar Sistem Umpan Balik (2)

- **Diagram blok sistem umpan balik (representasi kawasan-s):**
- Bila $x(t) \overset{L_u}{\Leftrightarrow} X(s)$,

  $y(t) \overset{L_u}{\Leftrightarrow} Y(s)$,

  $r(t) \overset{L_u}{\Leftrightarrow} R(s)$,

  $e(t) \overset{L_u}{\Leftrightarrow} E(s)$



- Bila $H(s)$ menyatakan fungsi transfer plant dan $H(s)$ menyatakan fungsi transfer sensor.

$$G(s) = \frac{Y(s)}{E(s)} \text{ dan } H(s) = \frac{R(s)}{Y(s)}$$

Diperoleh $\frac{Y(s)}{G(s)} = E(s) = X(s) - R(s) = X(s) - H(s)Y(s)$.

$$Y(s) = X(s)G(s) - G(s)H(s)Y(s)$$

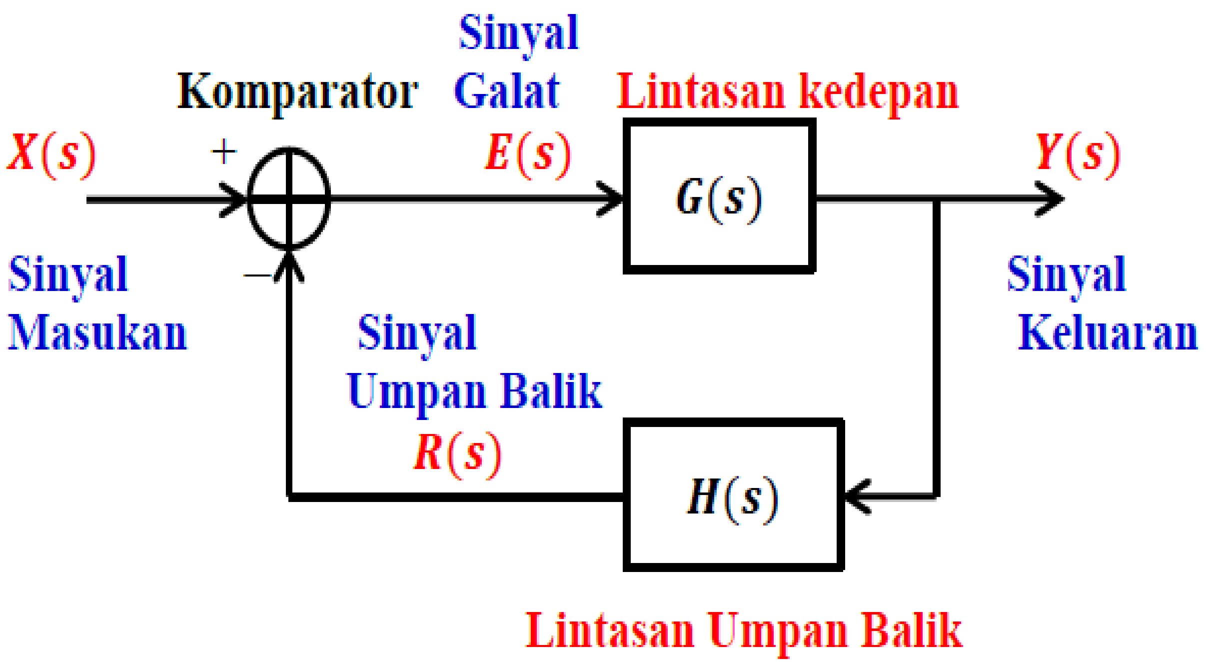
Konsep Dasar Sistem Umpan Balik (3)

- **Fungsi transfer Loop-tertutup**:

$$T(s) = \frac{Y(s)}{X(s)}$$

$$Y(s)[1 + G(s)H(s)] = G(s)X(s)$$

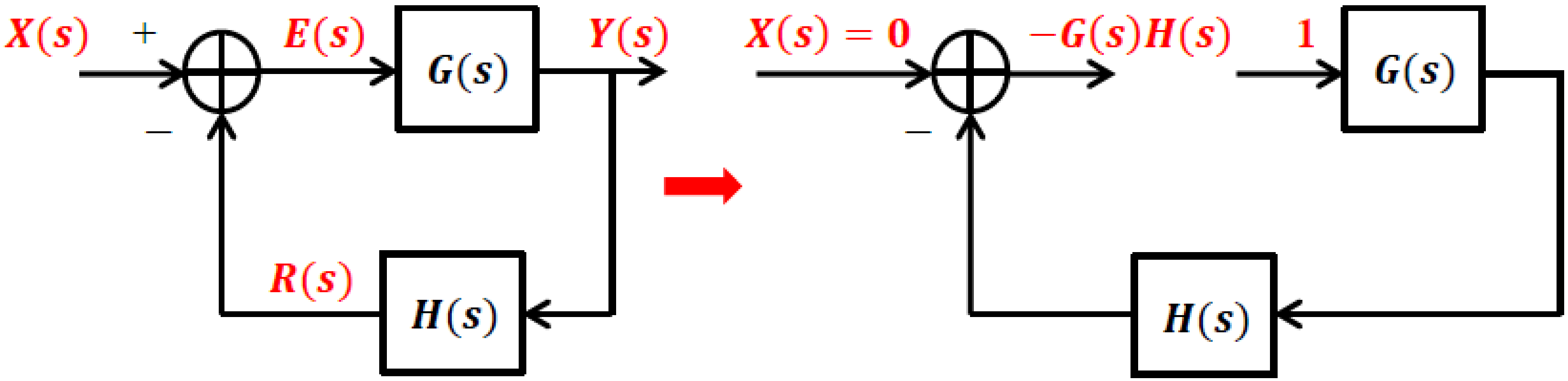
$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$



- Penyebutan **"closed-loop"** disini dipakai untuk menekankan fakta bahwa ada sebuah "**closed signal transmission loop**", dimana sinyal mengalir didalam sistem.
- Besaran $1 + G(s)H(s)$ memberikan sebuah ukuran tindakan umpan balik terhadap $G(s)$.

Konsep Dasar Sistem Umpan Balik (4)

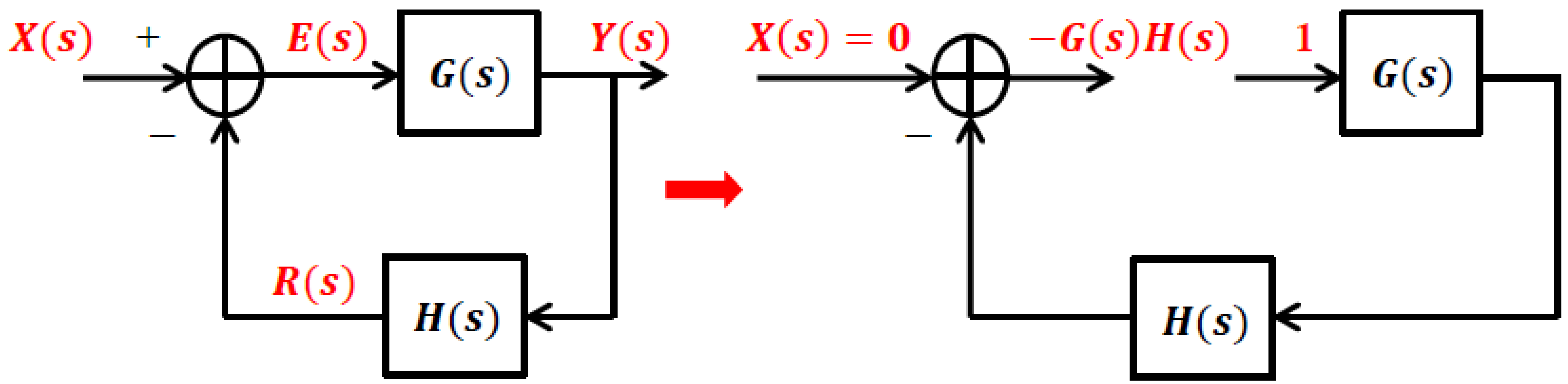
- **Sistem umpan balik:**



- Perhatikan gambar sebelah kanan:
- Sinyal $x(t) = 0$, maka $X(s) = 0$.
- Loop umpan balik menuju $G(s)$ dibuka.
- Sebuah sinyal test dengan nilai transformasi Laplace = 1 diberikan ke $G(s)$, sehingga nilai keluaran komparator adalah $-G(s)H(s)$.

Konsep Dasar Sistem Umpan Balik (4)

- **Sistem umpan balik:**



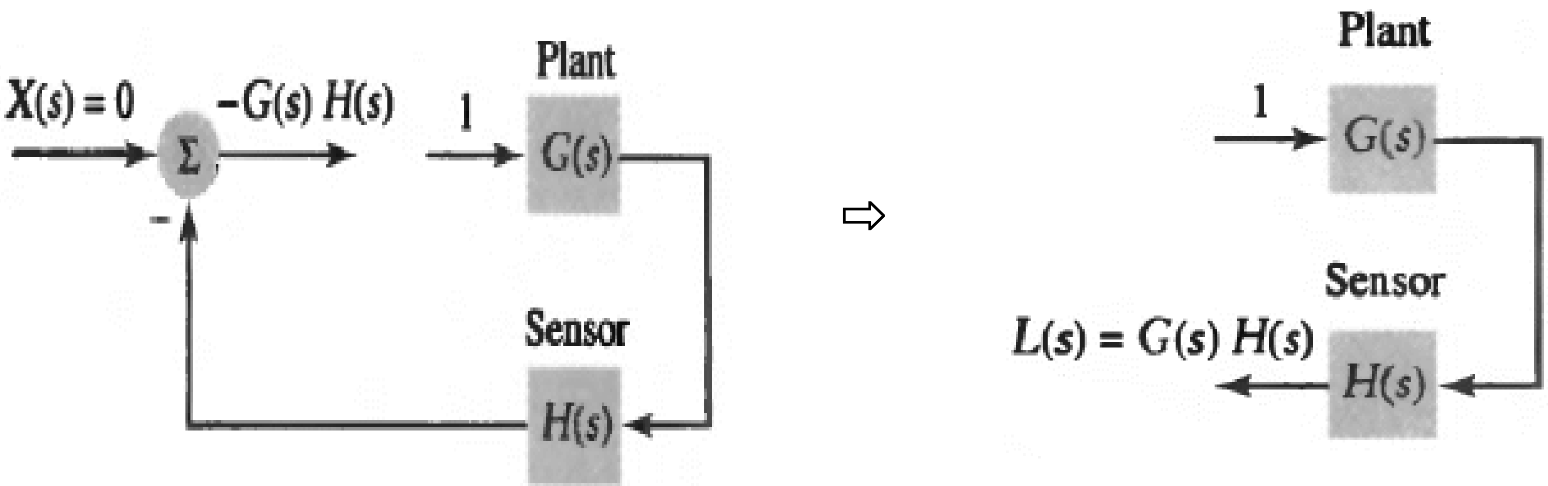
- Perbedaan antara sinyal test satuan dengan sinyal yang di-umpan balikkan adalah = $1 + G(s)H(s)$, besaran ini disebut **"return difference"**.

$$F(s) = 1 + G(s)H(s)$$

- Besaran hasil kali $G(s)H(s)$ disebut **"loop transfer function" (fungsi transfer loop)** sistem, $L(s) = G(s)H(s)$.

Konsep Dasar Sistem Umpan Balik (6)

- **Sistem umpan balik :**



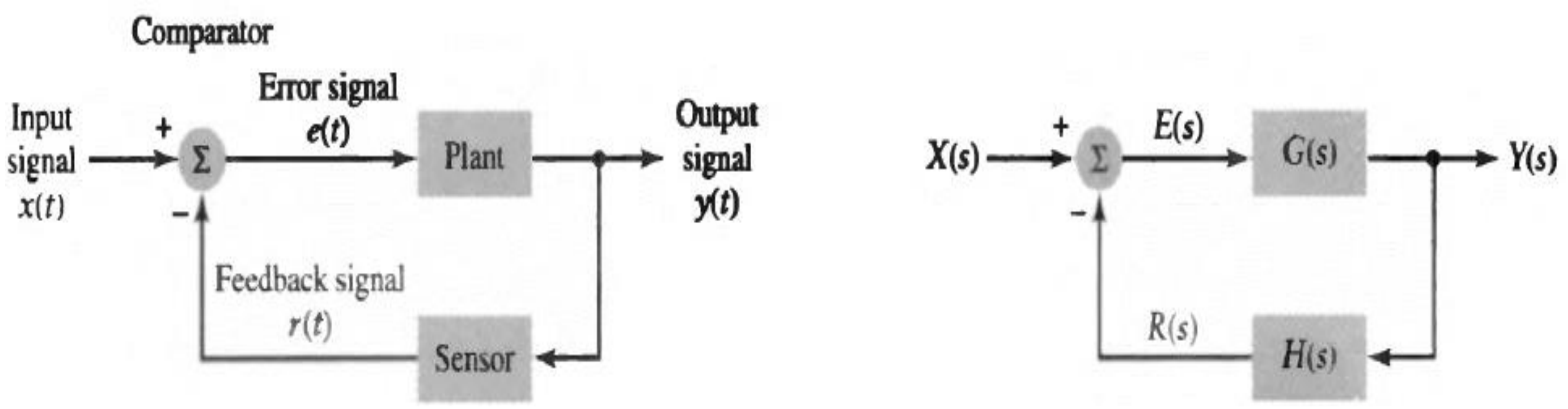
- Hubungan **"return difference"** $F(s)$ dengan **"loop transfer function"** $L(s)$:

$$F(s) = 1 + L(s).$$

- Pemakaian $G(s)H(s)$ dan $L(s)$, dapat dipertukarkan bila merujuk **"the loop transfer function"**.

Umpan Balik Negatif dan Positif

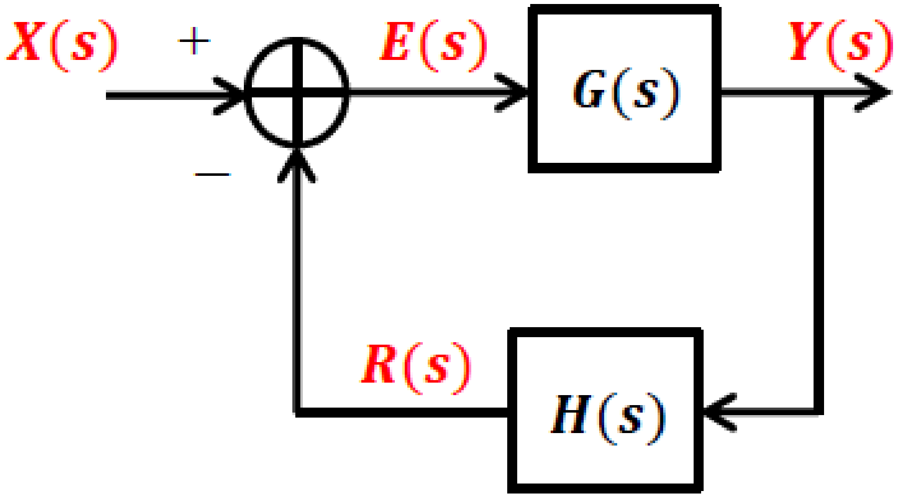
- **Diagram blok sistem umpan balik :**



- Umpan balik digambar disebut negatif, bila komparator diganti dengan adder, maka umpan balik disebut positif.
- Untuk $s = j\Omega$, fungsi transfer loop $G(j\Omega)H(j\Omega)$ mempunyai phasa yang berubah dengan frekuensi Ω, bila phasa $G(j\Omega)H(j\Omega)$ adalah nol, maka kondisi umpan balik adalah umpan balik negatif, bila phasa $G(j\Omega)H(j\Omega)$ adalah 180^0, konfigurasi yang sama bertindak seperti umpan balik positif.

Analisa Sensitivitas (1)

- Motivasi utama memakai umpan balik adalah mengurangi sensitivitas fungsi transfer loop tertutup (closed-loop transfer function) sistem di gambar:



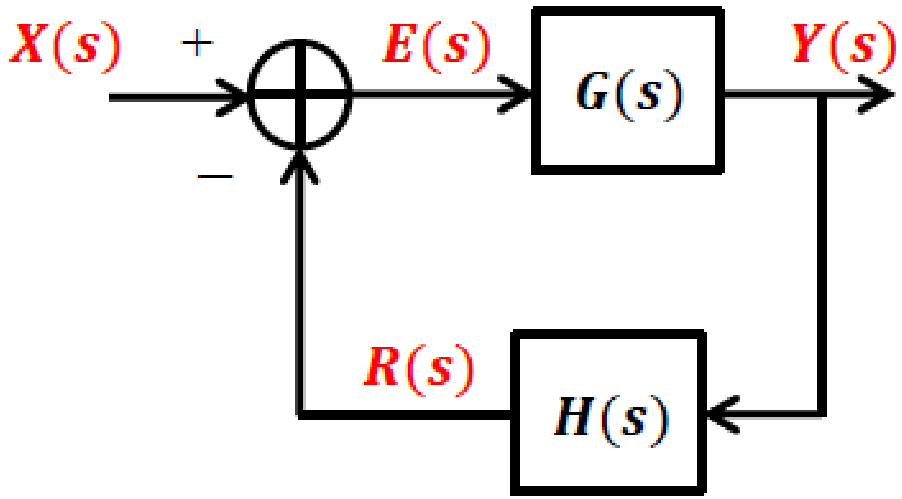
mengubah fungsi transfer "plant".

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \Rightarrow T = \frac{G}{1 + GH}$$

- G adalah penguatan (gain) "plant".
- T adalah penguatan (gain) "closed-loop" sistem "feedback".

Analisa Sensitivitas (2)

- **Sistem umpan balik:**
- Diandaikan penguatan (gain) G berubah sebesar ΔG.



- Differensiasi $T = \frac{G}{1+GH}$ terhadap G, diperoleh perubahan di T adalah

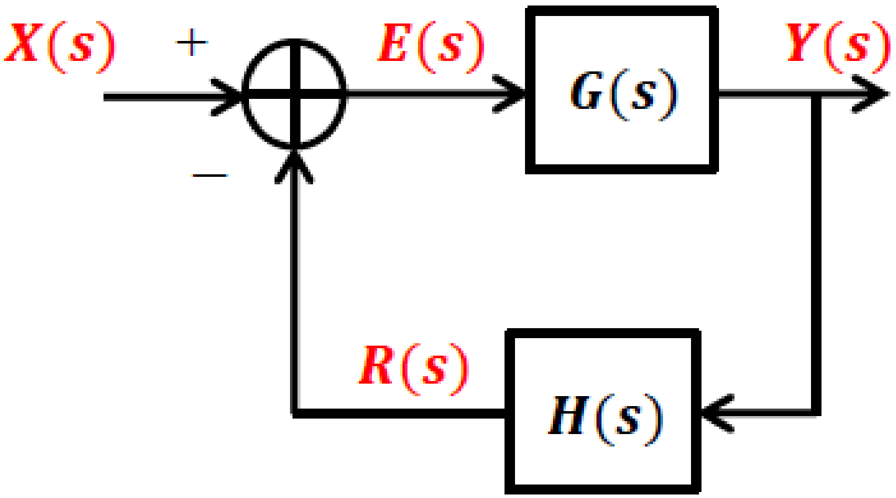
$$\Delta T = \frac{\partial T}{\partial G}\Delta G = \frac{1}{(1+GH)^2}\Delta G$$

- Sensitivitas T terhadap perubahan di G didefinisikan oleh

$$S_G{}^T = \frac{\Delta T/T}{\Delta G/G}$$

Analisa Sensitivitas (3)

- **Sistem umpan balik:**
  Sensitivitas T terhadap G adalah persentasi perubahan T dibagi dengan persentasi perubahan G.



- Memakai $T(s) = \frac{G(s)}{1+G(s)H(s)}$
- dan $\Delta T = \frac{1}{(1+GH)^2}\Delta G$ di $S_G{}^T = \frac{\Delta T/T}{\Delta G/G}$ menghasilkan

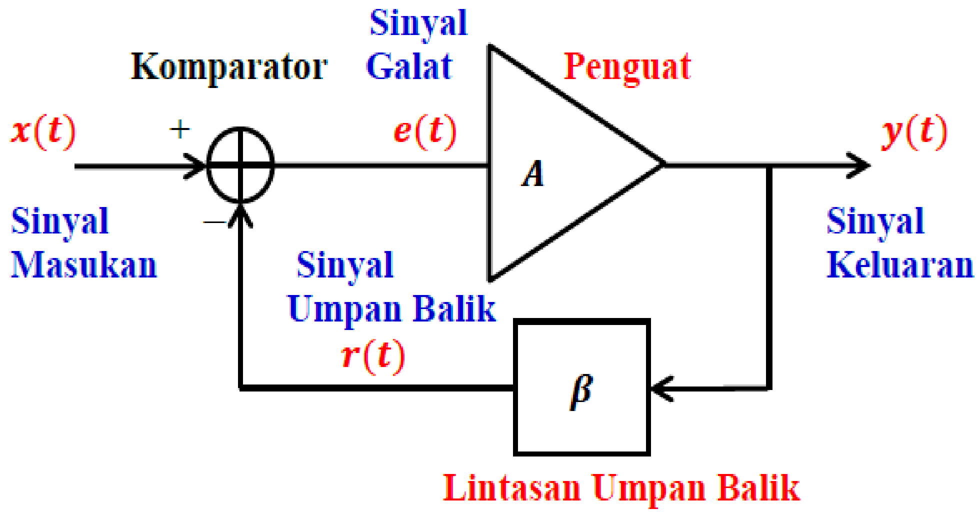
$$S_G{}^T = \frac{1}{1+GH} = \frac{1}{F}$$

- Menunjukkan bahwa sensitivitas T terhadap G adalah $= \frac{1}{F}$.
- Bila gain loop $GH \gg 1$, maka

$$T = \frac{G}{1+GH} \approx \frac{1}{H} \text{ dan } S_G{}^T = \frac{1}{1+GH} \approx \frac{1}{GH}$$

Penguat dengan umpan balik (1)

- Perhatikan sebuah penguat dengan umpan balik:
- Sistem terdiri dari sebuah penguat linier dan sebuah umpan balik yang dibuat dari resistor.



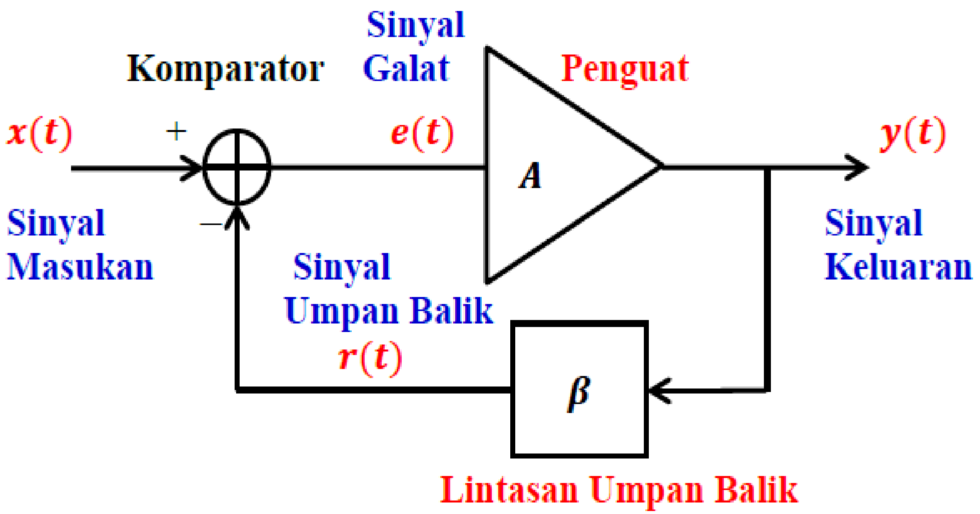
- Penguat mempunyai gain A,

  dan umpan balik mengatur nilai β ($\beta < 1$) dari keluaran yang diberikan ke masukan. Nilai gain $A = 1000$.

(a) Tentukan nilai β yang akan menghasilkan “closed-loop gain” $T = 10$.

(b) Andaikata gain A berubah sebesar 10%, berapa besar persentase perubahan yang terjadi di “closed-loop gain” T.

Penguat dengan umpan balik (2)



- Solusi:
- $G = A$ dan $H = \beta$.
- $T = \frac{1}{1+\beta A}$.

(a) $\beta = \frac{1}{A}\left(\frac{A}{T} - 1\right)$

$$\beta = \frac{1}{1000}\left(\frac{1000}{10} - 1\right) = 0{,}099\ .$$

(b) Sensitivitas "closed-loop gain" T relatif terhadap A adalah

$$T = \frac{A}{1+\beta A} \Rightarrow S_A{}^T = \frac{1}{1+\beta A} = \frac{1}{1+0.099x1000} = \frac{1}{100}$$

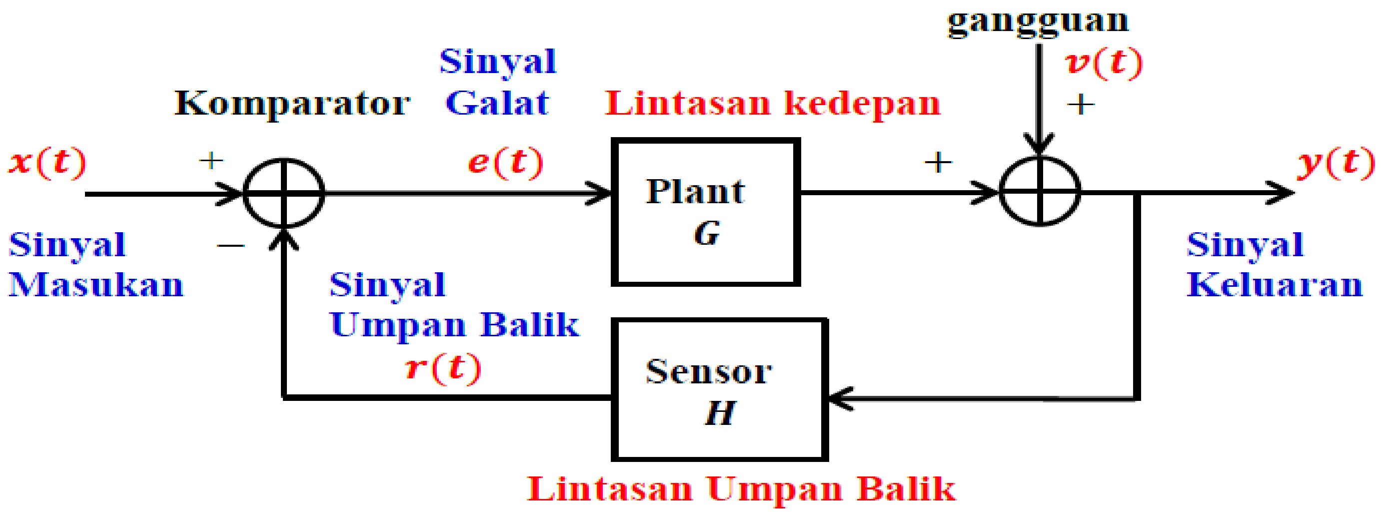
$$S_G{}^T = \frac{\Delta T/T}{\Delta G/G} \Rightarrow \frac{\Delta T}{T} = S_G{}^T \frac{\Delta A}{A}$$

Dengan perubahan 10% di A, persentasi perubahan di T adalah

$$\frac{\Delta T}{T} = S_G{}^T \frac{\Delta A}{A} = \frac{1}{100} x10\% = 0.1\%$$

Effek umpan balik terhadap gangguan (disturbance) atau derau (noise) (1)

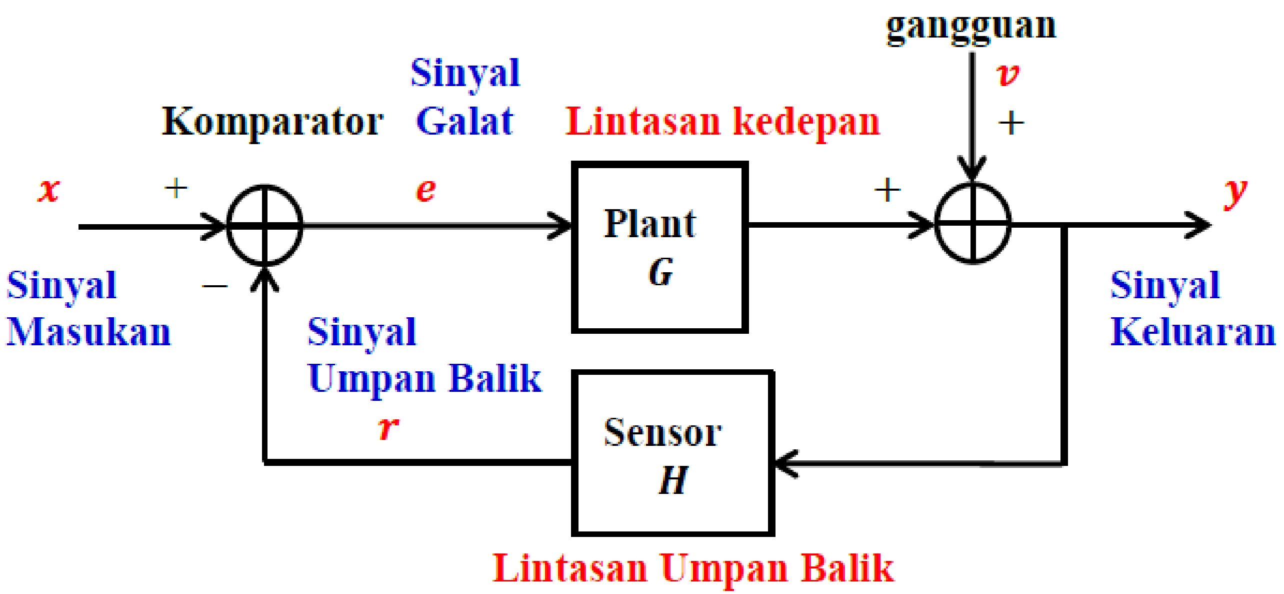
- Pemakaian umpan balik berpengaruh positif terhadap kinerja sistem: akan mengurangi effek gangguan (disturbance) atau derau (noise) yang ada didalam loop umpan balik.
- Perhatikan sistem umpan balik satu-loop:



- Kedua nilai G dan H dianggap sebagai parameter konstan.
- Sistem mengandung sinyal gangguan (disturbance), disebut $v(t)$, ada didalam loop.

Effek umpan balik terhadap gangguan (disturbance) atau derau (noise) (2)

- **Sistem umpan balik:**
- Kita buat sinyal gangguan $\nu = 0$.

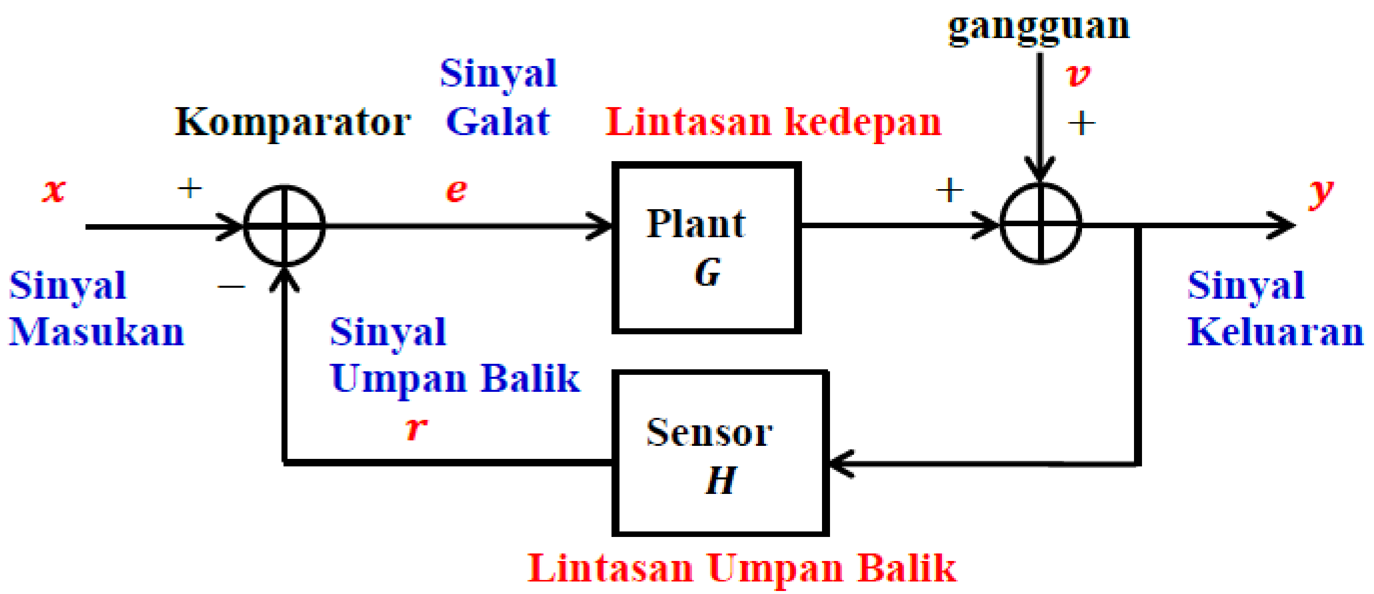


- "Closed-loop" gain sistem dengan sinyal x sebagai masukan adalah $\frac{G}{1+GH}$. Maka, sinyal keluaran akibat hanya ada x adalah:

$$y\big|_{\nu=0} = \frac{G}{1+GH}x$$

Effek umpan balik terhadap gangguan (disturbance) atau derau (noise) (3)

- **Sistem umpan balik:**

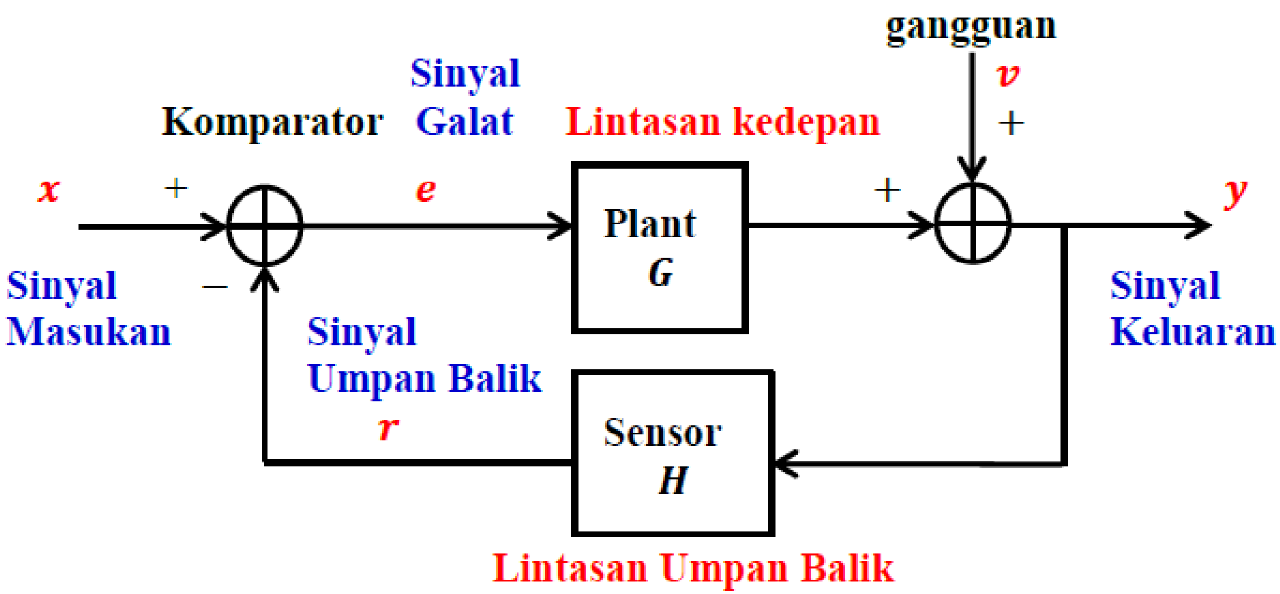


- Kita buat sinyal masukan $x = 0$. "Closed-loop" gain sistem dengan sinyal ν sebagai sinyal dari luar adalah $\frac{1}{1+GH}$. Keluaran sistem:

$$y\Big|_{x=0} = \frac{1}{1 + GH}\nu$$

Effek umpan balik terhadap gangguan (disturbance) atau derau (noise) (4)

- **Sistem umpan balik:**



- Keluaran akibat gabungan x dan v:

$$y = \frac{G}{1 + GH} x + \frac{1}{1 + GH} \text{v}$$

- Jelas bahwa penggunaan umpan balik memberikan effek pengurangan gangguan v dengan faktor $1 + GH$ (akibat "return difference" F).

Analisis Distorsi (1)

- Nonlinieritas muncul di-sistem physik apabila sistem diberi masukan diluar (melebihi) rentang operasi linier.
- We may improve the linearity of such a system by applying feedback aroud it.
- To investigate this important effect, we may proceed in one of two ways:
  - The output of the system is expressed as a nonlinear function of the input, and a pure sine wave is used as the input signal.
  - The input to the system is expressed as a nonlinear function of the output.
- The latter approach may seem strange at first sight; however, it is more general in formulation and provides a more intuitively satisfying description of how feedback affects the nonlinear behavior of a system.

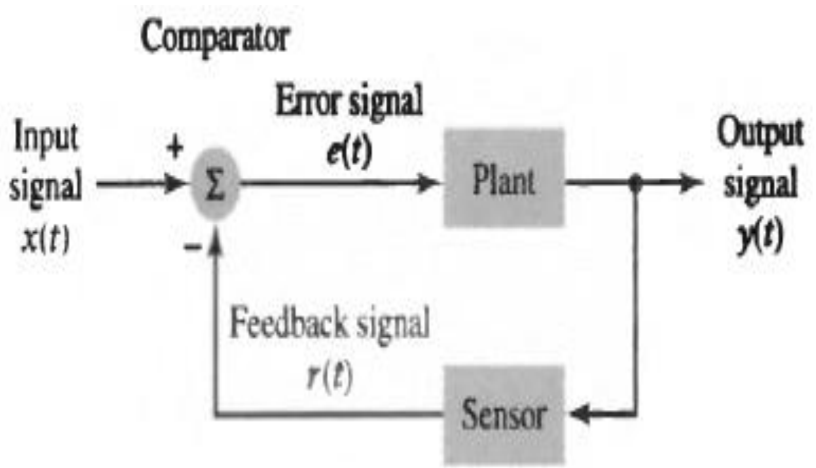
Analisis Distorsi (2)

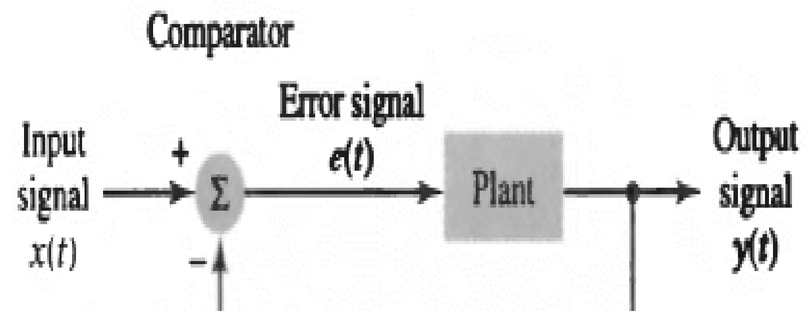
- Consider a feedback system in which the dependence of the error e on the system output y is represented by $e = a_1 y + a_2 y^2$, where a_1 and a_2 are constants.
- The linear term $a_1 y$ represents the desired behavior of the plant, and the parabolic term $a_2 y^2$ accounts for its deviation from linearity.
- Let the parameter H determine the fraction of the plant output y that is feedback to the input.
- With x denoting the input applied to the feedback system, we may write $e = x - Hy$.
- $e = a_1 y + a_2 y^2 = x - Hy$.
- We get $x = (a_1 + H)y + a_2 y^2$

Analisis Distorsi (3)

- Differentiating x with respect to y yields $\frac{dx}{dy} = a_1 + H + 2a_2 y$

  $\frac{dx}{dy} = a_1 + H\left(1 + \frac{2a_2}{a_1+H} y\right)$ which holds in the presence of feedback.
- In the absence of feedback, the plant operates by itself,





$$e = a_1 y + a_2 y^2 \Rightarrow x = a_1 y + a_2 y^2$$
$$\frac{dx}{dy} = a_1 + 2a_2 y = a_1\left(1 + \frac{2a_2}{a_1} y\right)$$

Analisis Distorsi (4)

- With feedback $\frac{dx}{dy} = a_1 + H\left(1 + \frac{2a_2}{a_1+H}y\right)$.
- Without feedback $\frac{dx}{dy} = a_1\left(1 + \frac{2a_2}{a_1}y\right)$
- The application of feedback has reduced the distortion due to deviation of the plant from linearity by the factor

$$D = \frac{\frac{2a_2 y}{a_1 + H}}{\frac{2a_2 y}{a_1}} = \frac{a_1}{a_1 + H}$$

- From $x = a_1 y + a_2 y^2$, we see that $a_1 = \frac{1}{G}$.

$$D = \frac{\frac{1}{G}}{\frac{1}{G} + H} = \frac{1}{1 + GH} = \frac{1}{F}$$

- the distortion is reduced by a factor equal to the return difference F.

Benefits of Feedback

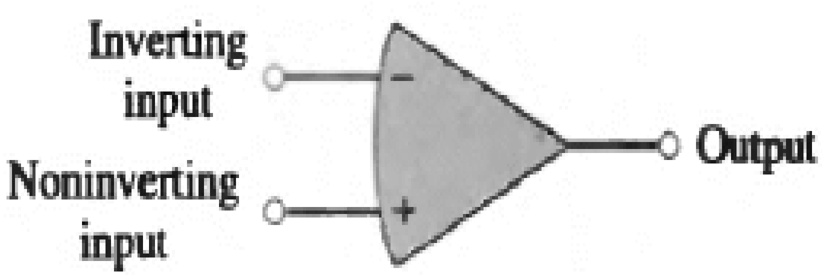
- We see that the return difference $F = 1 + GH$ plays a central role in the study of feedback systems:
- **Control of sensitivity**, the application of feedback to a plant reduces the the sensitivity of the closed-loop gain of the feedback system to parameter variations in the plant by a factor equal to F.
- **Control of the effect of an internal disturbance**, the transmission of a disturbance from some point inside the loop of the feedback system to the closed-loop output of the system is also reduced by a factor equal to F.
- **Control of distortion in a nonlinear system**, distortion due to nonlinear effects in the plant is again reduced a factor equal to F.
- These improvements in overall system performance resulting from the application of feedback are of immense engineering importance.

Biaya sistem umpan balik

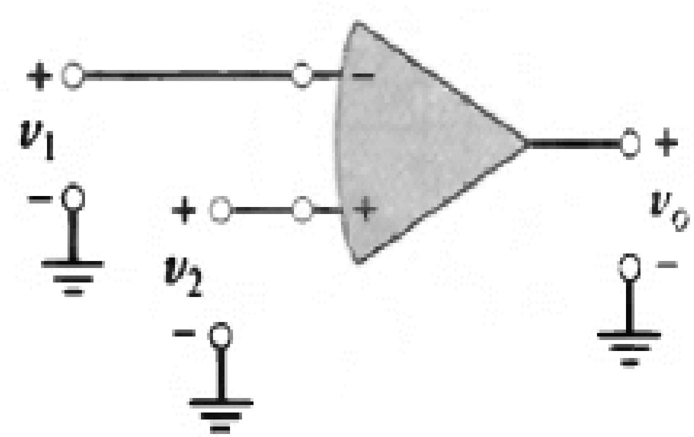
- Ada biaya terhadap keuntungan yang diperoleh dari aplikasi umpan balik di sistem kendali (kontrol, pengaturan):
- **kompleksitas bertambah**, the application of feedback to a control system requires the addition of new components. Thus, there is the cost of increased system complexity.
- **Reduce gain**, in the absence of feedback, the transfer function of a plant is $G(s)$. When feedback is applied to the plant, the transfer function of the system is $\frac{G(s)}{F(s)}$, where $F(s) = 1 + G(s)H(s)$.
- **Possible instability**, often, an open loop system is stable. When feedback is applied to the system, there is a real possibility that the closed-loop system may become unstable.
- Secara umum, keuntungan sistem umpan balik lebih besar daripada kerugiannya.

Operational Amplifiers (1)

- An operational amplifiers, or an op amp, provides the basis for realizing a transfer function with prescribed poles and zeros in a relatively straightforward manner.
- An op amp has two input terminals, one inverting and other noninverting, and an output terminal.
- Conventional symbol for operational amplifier

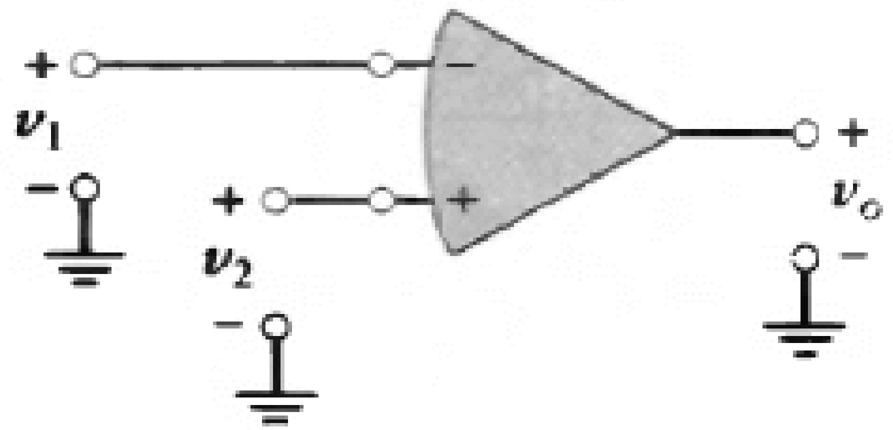


Operational amplifier with input and output voltages.



Operational Amplifiers (2)

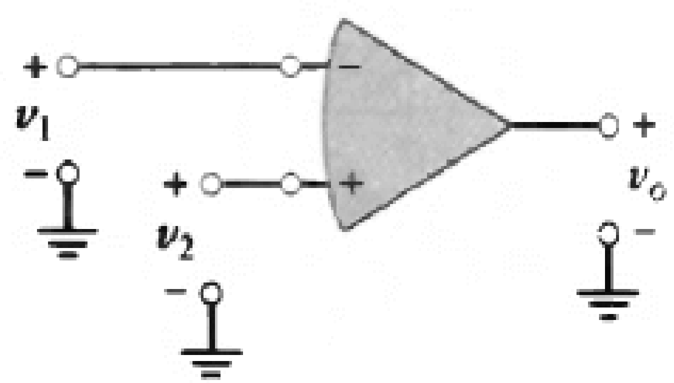
- An ideal model for an operational amplifiers encompasses four assumptions:



1. The op amp acts as a voltage-controlled voltage source described by the input-output relation $v_o = A(v_2 - v_1)$ where v_1 and v_2 are the signals applied to the inverting and noninverting input terminals, respectively, and v_o the output signal.
2. The open loop voltage gain A has a constant value that is very large compare with unity, which means that, for a finite output signal v_o, we must have $v_1 \approx v_2$. This property is referred to as virtual ground.

Operational Amplifiers (3)

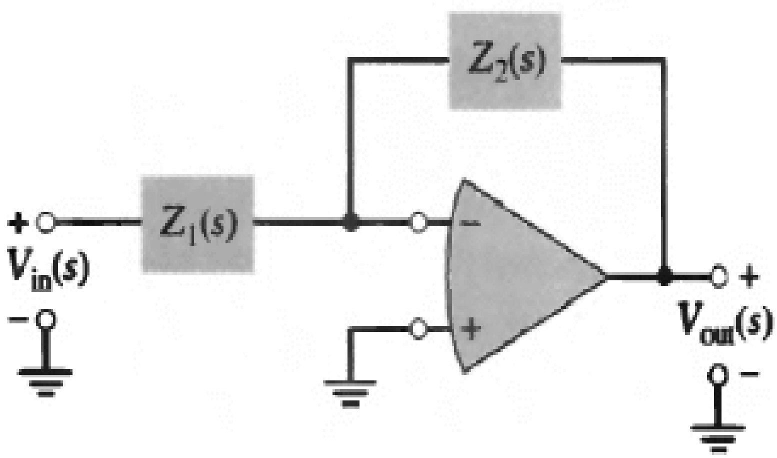
- An ideal model for an operational amplifiers encompasses four assumptions:



3. The impedance between the two input terminals is infinitely large, and so is the impedance between each one of them and the ground, which means that the input terminal currents are zero.
4. The output impedance is zero.

Operational Amplifiers (4)

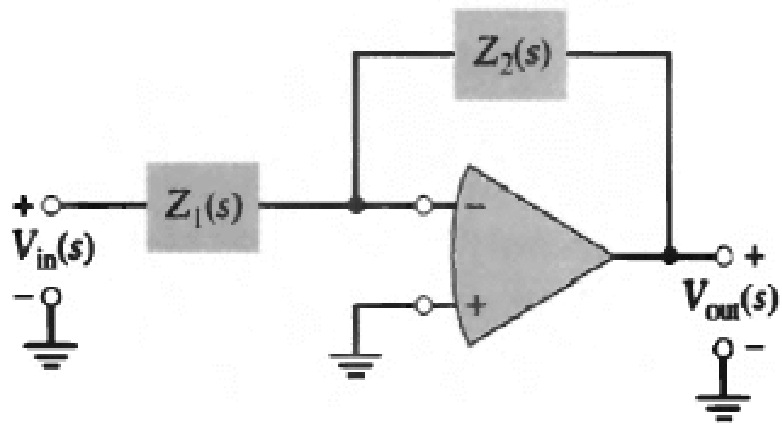
- Typically, the operational amplifiers is not used in an open loop fashion. It is normally used as the amplifier component of a feedback circuit in which the feedback controls the closed-loop transfer function of the circuit.
- Op amp embedded in a single-loop feedback circuit.
- $Z_1(s)$ and $Z_2(s)$ represent the input element and feedback Element of the circuit, respectively.
- $V_{in}(s)$ and $V_{out}(s)$ denote the Laplace transforms of the input and output voltage signals, respectively.



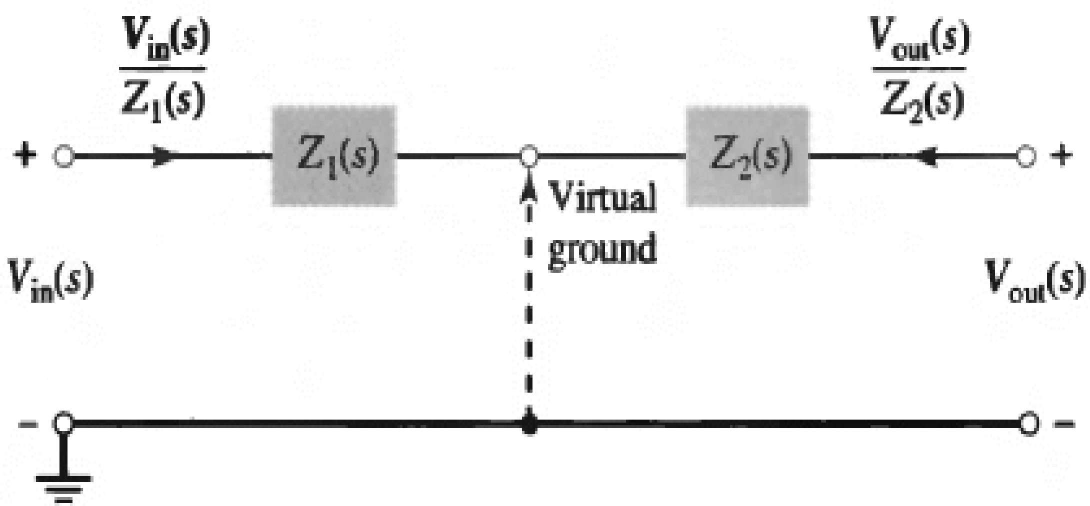
Operational Amplifiers (5)

- Using the ideal model to describe the operational amplifier, we may construct the model:

- Op amp embedded in a single-loop feedback circuit.



- Ideal model for the feedback circuit.



Operational Amplifiers (6)

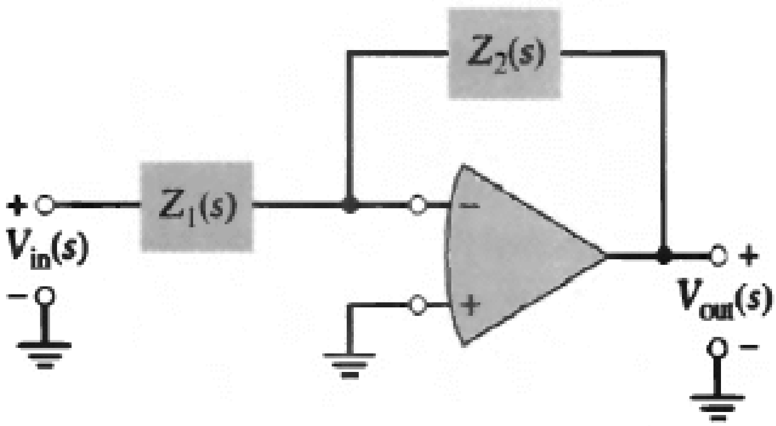
- **Properties:**

2. The open loop voltage gain A has a constant value that is very large compare with unity, which means that, for a finite output signal v_o, we must have $v_1 \approx v_2$. This property is referred to as virtual ground.
3. The impedance between the two input terminals is infinitely large, and so is the impedance between each one of them and the ground, which means that the input terminal currents are zero.

- The following condition may be derived from properties 2 and 3 of the ideal op amp: $\frac{V_{in}(s)}{Z_1(s)} \cong -\frac{V_{out}(s)}{Z_2(s)}$.
- The closed-loop transfer function of the feedback circuit is $T(s) = \frac{V_{out}(s)}{V_{in}(s)} \cong -\frac{Z_2(s)}{Z_1(s)}$.

Operational Amplifiers (7)

- Op amp embedded in a single-loop feedback circuit.



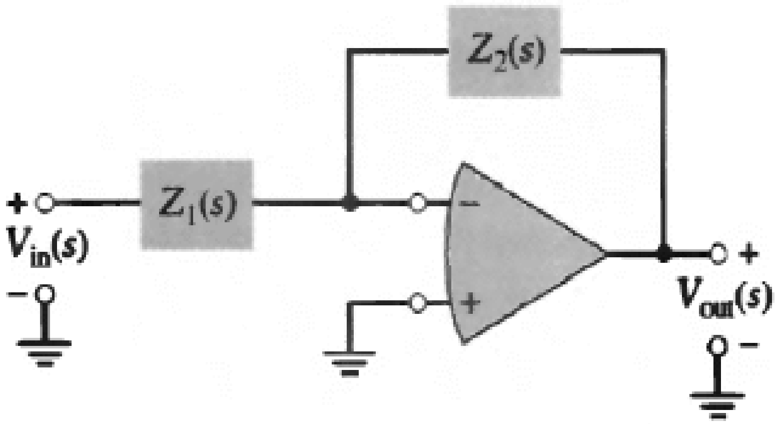
- The closed-loop transfer function of the feedback circuit is

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} \cong -\frac{Z_2(s)}{Z_1(s)}.$$

- The general feedback formula: $T(s) = \frac{G(s)}{1+G(s)H(s)}$
- The feedback element $Z_2(s)$ is connected in parallel to the amplifier at both its input and output ports. This would suggest the use of currents as the basis for representing the input signal $x(t)$ and feedback signal $r(t)$.

Operational Amplifiers (8)

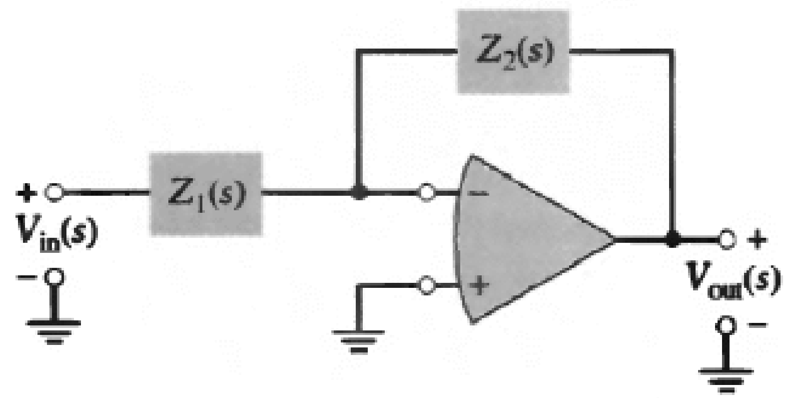
- Op amp embedded in a single-loop feedback circuit.



- The application of feedback in the above system has the effect of making the input impedance measured looking into the operational amplifier small compared with both $Z_1(s)$ and $Z_2(s)$.
- Let $Z_{in}(s)$ denote this input impedance.
- We have the current input signal $x(t) \overset{L}{\Leftrightarrow} X(s) = \frac{V_{in}(s)}{Z_1(s)}$, and the current output signal $r(t) \overset{L}{\Leftrightarrow} R(s) = -\frac{V_{out}(s)}{Z_2(s)}$

Operational Amplifiers (9)

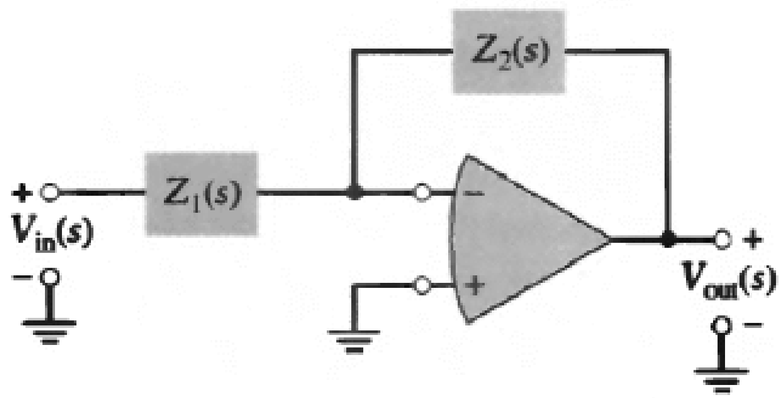
- Op amp embedded in a single-loop feedback circuit.



- The error signal $e(t)$, defined as the difference between $x(t)$ and $r(t)$, is applied across the input terminals of the operational amplifier to produce an output voltage equal to $v_{out}(t)$.
- With $e(t) \overset{L}{\Leftrightarrow} E(s)$, viewed as a current signal, we may invoke the following:
  - According to Ohm's law the voltage produced across the input terminal of the op amp is $Z_{in}(s)E(s)$, where $Z_{in}(s)$ is the input impedance.
  - A voltage gain equal to $-A$.

Operational Amplifiers (10)

- Op amp embedded in a single-loop feedback circuit.



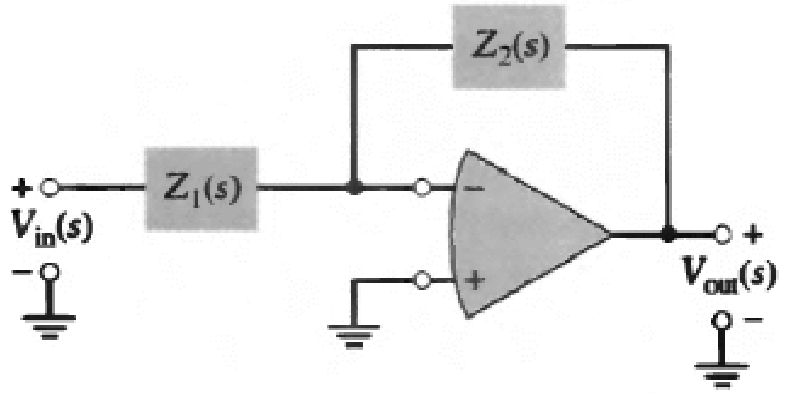
- We express the Laplace transform of the voltage $y(t)$ produced across the output terminals of the op amp as

$$Y(s) = V_{out}(s) = -AZ_{in}(s)E(s).$$

- The transfer function of the op amp, viewed as the plant is $G(s) = \frac{Y(s)}{E(s)}$, it follow that $G(s) = -AZ_{in}(s)$.
- The transfer function of the feedback path is $H(s) = \frac{R(s)}{Y(s)}$
- With $R(s) = -\frac{V_{out}(s)}{Z_2(s)}$ and $V_{out}(s) = Y(s) \Rightarrow H(s) = -\frac{1}{Z_2(s)}$

Operational Amplifiers (11)

- Op amp embedded in a single-loop feedback circuit.



- Using:

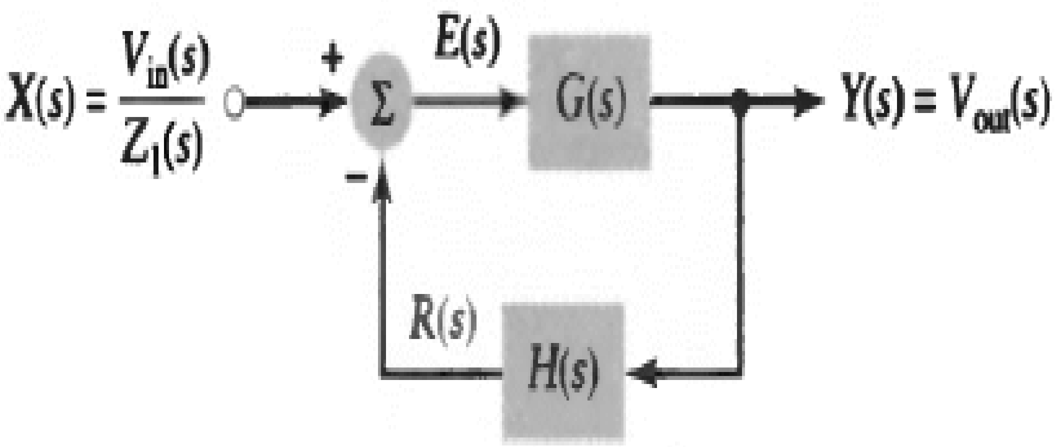
$$X(s) = \frac{V_{in}(s)}{Z_1(s)}$$

$$Y(s) = V_{out}(s) = -AZ_{in}(s)E(s).$$

$$G(s) = -AZ_{in}(s)$$

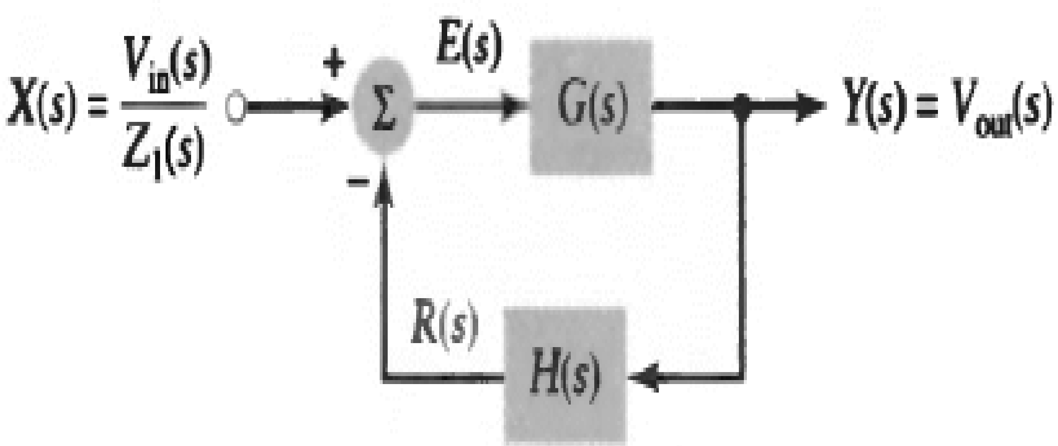
$$H(s) = -\frac{1}{Z_2(s)}$$

- We have the equivalent feedback circuit

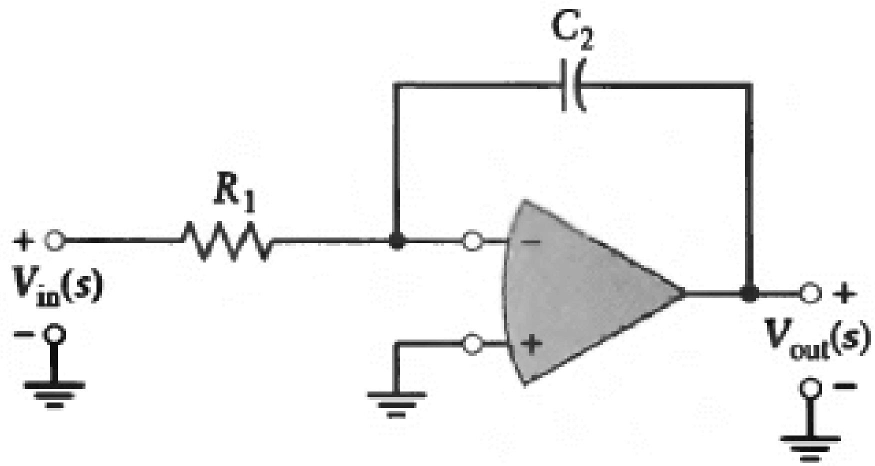


Operational Amplifiers (12)

- We find:
- $\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)}$
- $\frac{Y(s)}{X(s)} = \frac{-AZ_{in}(s)}{1+(-AZ_{in}(s))\left(-\frac{1}{Z_2(s)}\right)}$
- $\frac{Y(s)}{X(s)} = \frac{-AZ_{in}(s)}{1+\frac{AZ_{in}(s)}{Z_2(s)}}$
- $\frac{Y(s)}{X(s)} = \frac{V_{out}(s)}{V_{in}(s)/Z_1(s)} = \frac{-AZ_{in}(s)}{1+\frac{AZ_{in}(s)}{Z_2(s)}} \Rightarrow \frac{V_{out}(s)}{V_{in}(s)} = \frac{-AZ_{in}(s)}{Z_1(s)\left[1+\frac{AZ_{in}(s)}{Z_2(s)}\right]}.$
- Since $A \gg 1$, we may approximate $\frac{V_{out}(s)}{V_{in}(s)} \cong -\frac{Z_2(s)}{Z_1(s)}$



Integrator

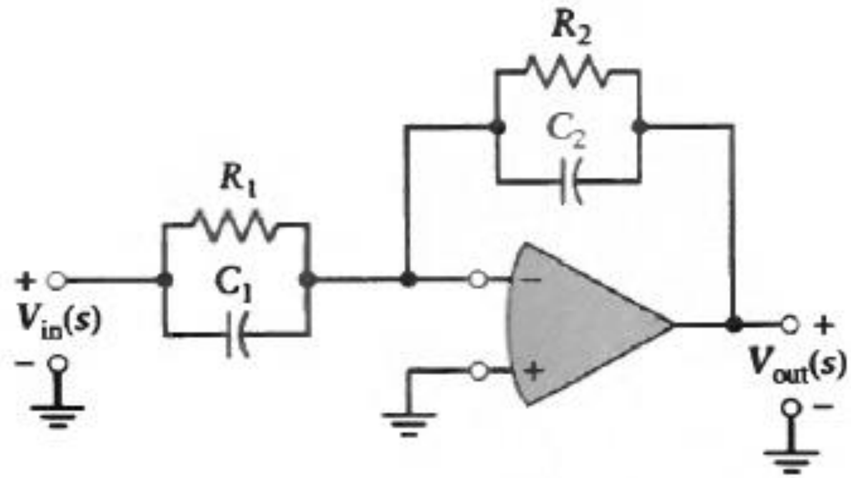


- The impedance
- $Z_1(s) = R_1$
- $Z_2(s) = \frac{1}{sC_2}$
- The closed-loop

  transfer function $T(s) \cong -\frac{1}{sC_2R_1}$
- Which shows that the closed-loop transfer function has a pole at the origin.
- Since division by the complex variable s corresponds to integration in time, we conclude that this circuit performs integration on yhe input signal.

Op Amp Circuit with RC elements

- $Z_1(s) = \frac{R_1}{1+sC_1R_1}$
- $Z_2(s) = \frac{R_2}{1+sC_2R_2}$

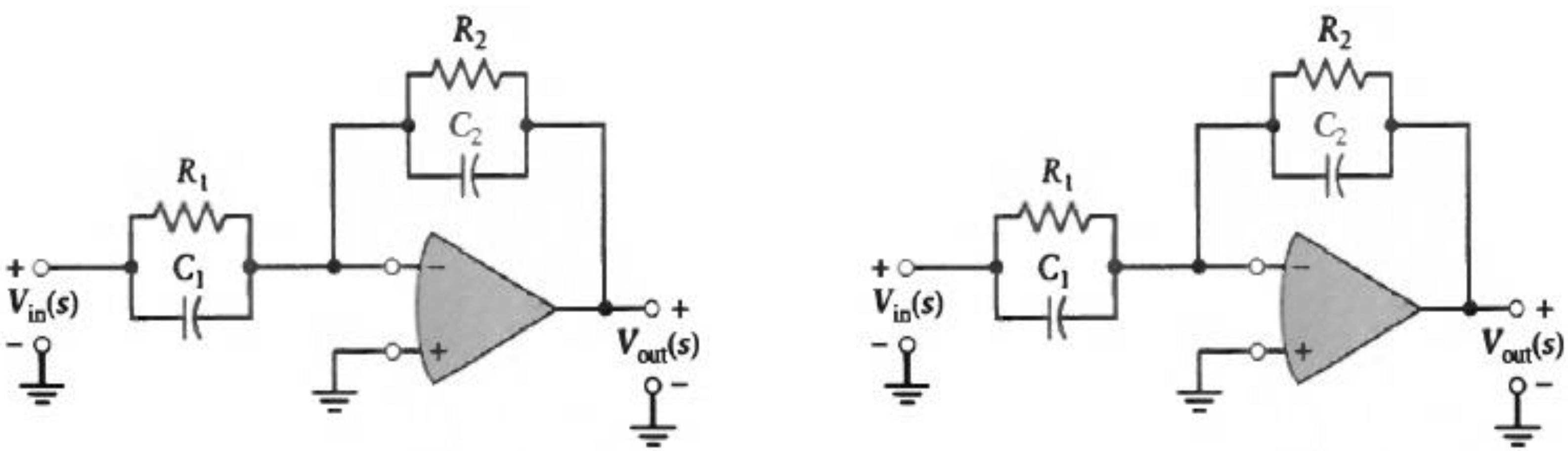


- The closed-loop transfer function: $T(s) \cong -\frac{R_2}{R_1}\frac{1+sC_1R_1}{1+sC_2R_2}$

which has a zero at $s = -\frac{1}{C_1R_1}$ and a pole at $s = -\frac{1}{C_2R_2}$.

Active Filters (1)

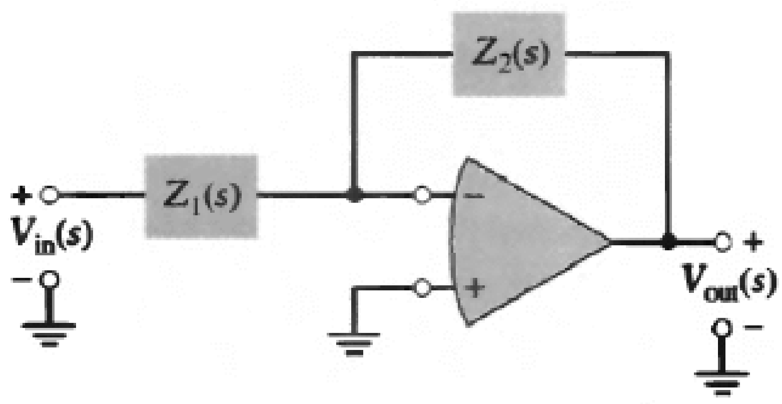
- We may design filters by using op amp; filters synthesized in this way are referred to as active filters.
- By cascading different versions of the basic circuit:



- It is possible to synthesize an overall transfer function with arbitrary real poles and arbitrary real zeros.

Active Filters (2)

- Elaborate forms of the impedances $Z_1(s)$ and $Z_2(s)$ in



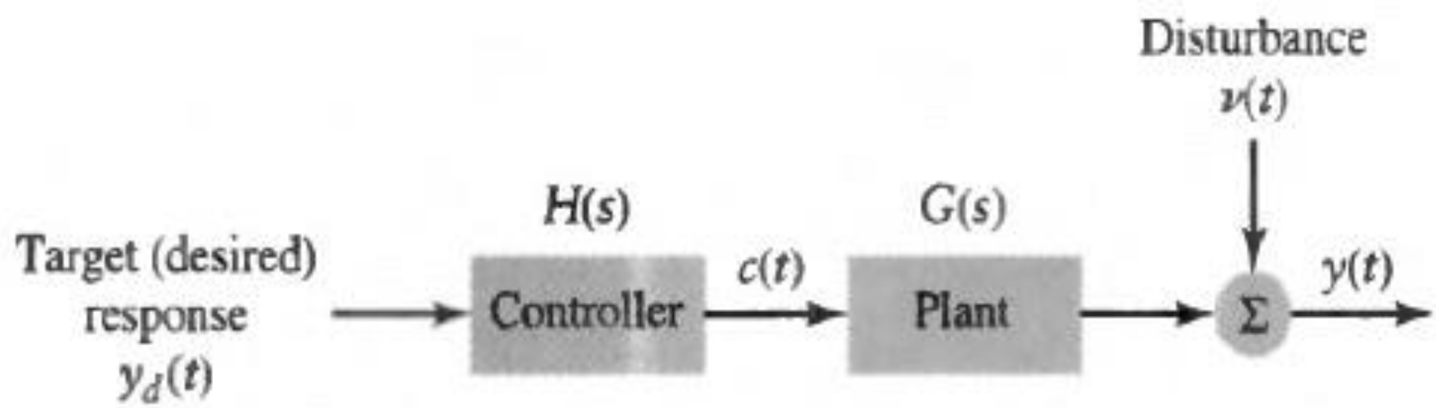
- We can realize a transfer function with arbitrary complex poles and zeros.
- Active filters offer an advantage by eliminating the need for using inductors, and offer the advantages of continuous-time operation.

Sistem Kendali (Kontrol / Pengaturan)

- Perhatikan sebuah "plant" yang dapat dikontrol.
- Fungsi sistem kendali adalah untuk mendapatkan kendali yang teliti terhadap "plant" agar keluaran "plant" dekat dengan respons yang diinginkan.
- Hal ini dicapai melalui pengaturan (perubahan) yang benar terhadap masukan "plant".
- Dua jenis dasar sistem kendali (kontrol / pengaturan):
  - **Kendali loop terbuka**, dimana pengaturan (perubahan) terhadap masukan "plant" diperoleh langsung dari respons target.
  - **Kendali loop tertutup**, dimana umpan balik dipakai untuk mengatur "plant".
- Untuk kedua jenis sistem kendali, respons target adalah masukan sistem kendali.

Sistem Kendali Loop Terbuka (1)

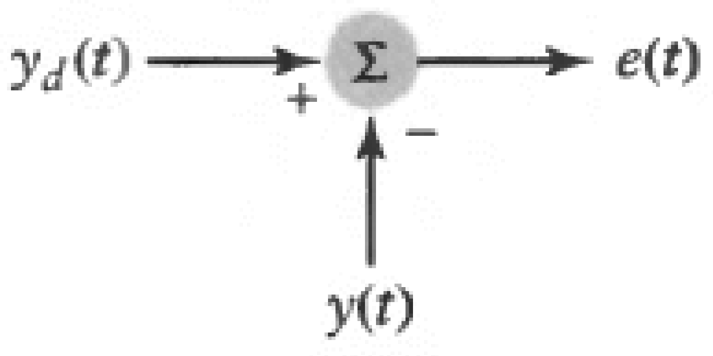
- Diagram blok sistem kendali loop terbuka.



- Dinamika "plant" dinyatakan oleh fungsi transfer $G(s)$.
- Kontroller (pengatur), dinyatakan oleh fungsi transfer $H(s)$, bereaksi terhadap respons target $y_d(t)$ untuk menghasilkan sinyal pengatur yang diinginkan $c(t)$.
- Besaran gangguan (disturbance) $v(t)$ dimasukkan kedalam sistem untuk memperhitungkan adanya derau (noise) dan distorsi (distortion) yang dihasilkan oleh keluaran "plant".

Sistem Kendali Loop Terbuka (2)

- Konfigurasi untuk menghitung sinyal error (galat):

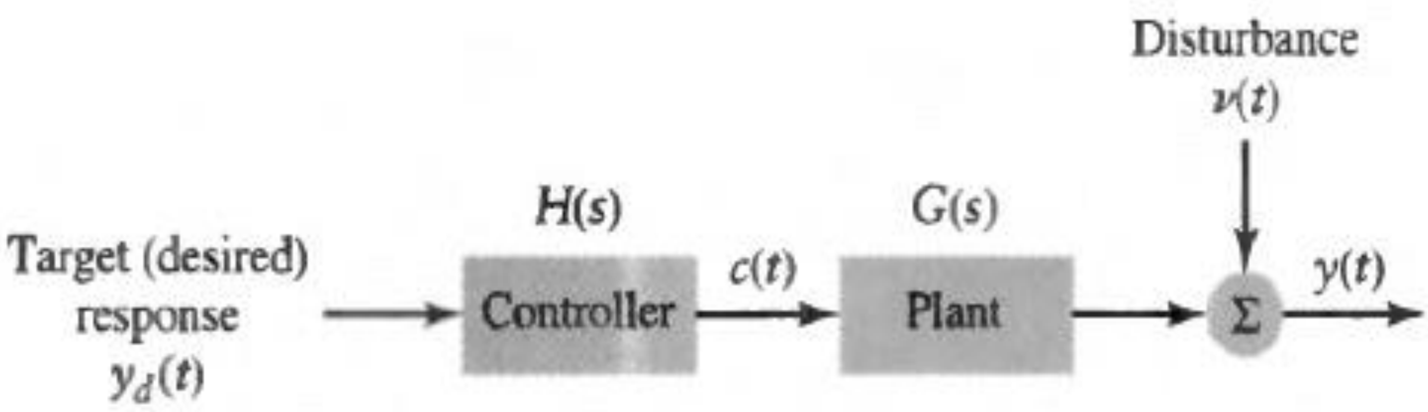


sinyal error $e(t)$ sebagai perbedaan antar respons target $y_d(t)$ dengan keluaran sistem aktual $y(t)$; artinya $e(t) = y_d(t) - y(t)$.

- Let $y_d(t) \overset{L}{\Leftrightarrow} Y_d(s)$, $y(t) \overset{L}{\Leftrightarrow} Y(s)$, and $e(t) \overset{L}{\Leftrightarrow} E(s)$.
- Then $e(t) = y_d(t) - y(t) \overset{L}{\Leftrightarrow} E(s) = Y_d(s) - Y(s)$

Open-Loop Control (3)

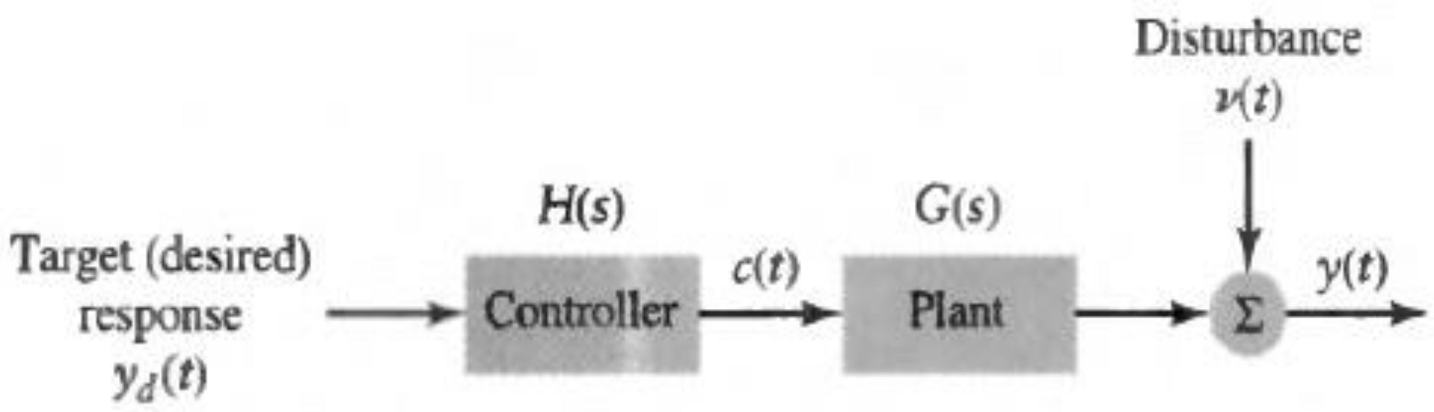
- The block diagram of an open-loop control system.



- Let $v(t) \overset{L}{\Leftrightarrow} N(s)$
- We have $Y(s) = G(s)H(s)Y_d(s) + N(s)$.
- And $E(s) = Y_d(s) - Y(s)$.
- Then $E(s) = [1 - G(s)H(s)]Y_d(s) - N(s)$.
- The error $e(t)$ is minimized by setting $1 - G(s)H(s) = 0$.
- For this condition: $H(s) = \frac{1}{G(s)}$.

Open-Loop Control (4)

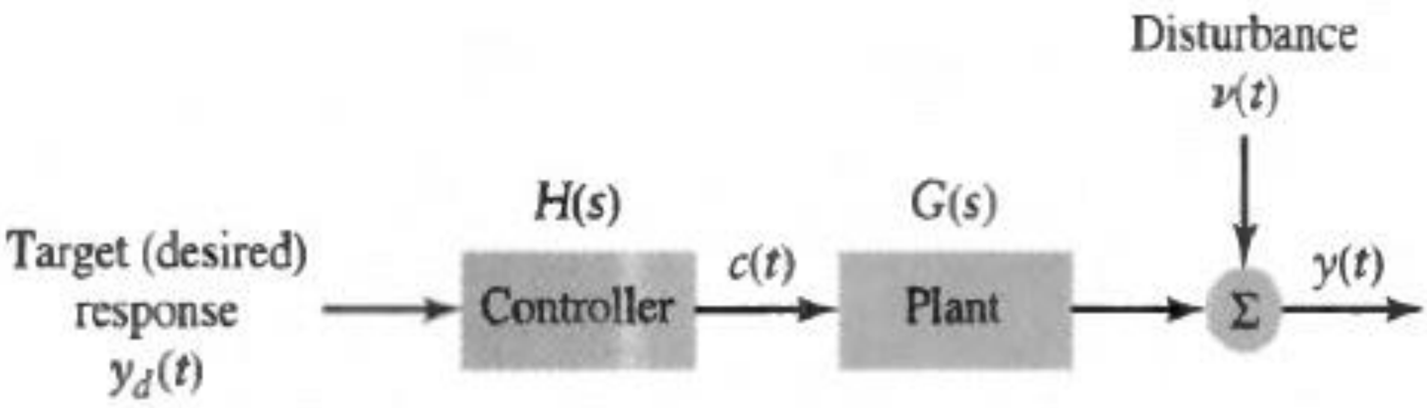
- The block diagram of an open-loop control system.



- If $y_d(t) = 0 \Rightarrow y(t) = v(t)$.
- The best that an open-loop control system can do is to leave the disturbance $v(t)$ unchanged.
- The overall transfer function of the system(in the absence of the disturbance $v(t)$ is simply $T(s) = \frac{Y(s)}{Y_d(s)} = G(s)H(s)$.
- Ignoring the dependence on s, and assumming that H does not change

Open-Loop Control (5)

- The block diagram of an open-loop control system.



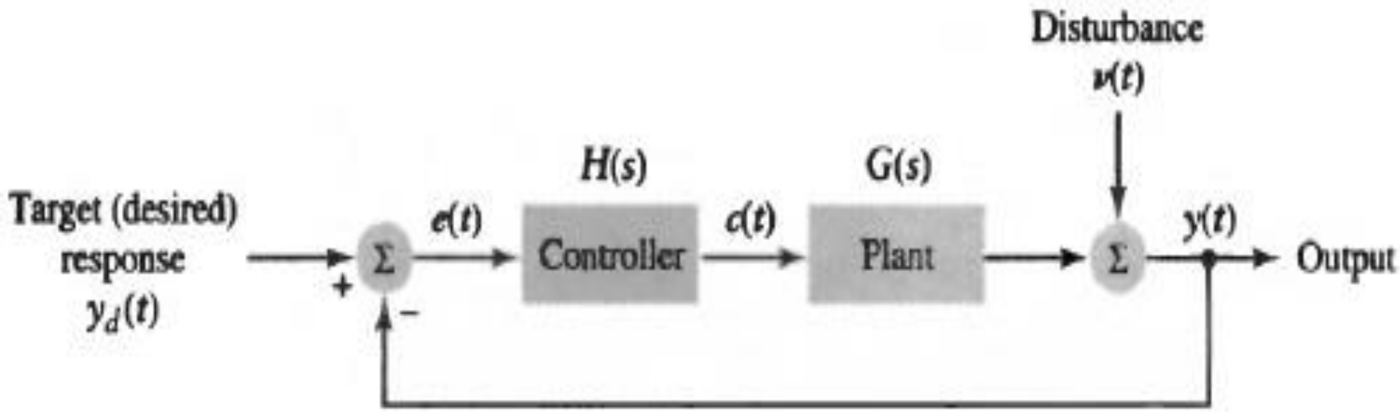
- The sensitivity of T with respect to changes in G is

$$S_G{}^T = \frac{\Delta T/T}{\Delta G/G} = \frac{H\Delta G/(GH)}{\Delta G/G} = 1.$$

- A percentage change in G is translated into equal percentage change in T.

Closed-Loop Control (1)

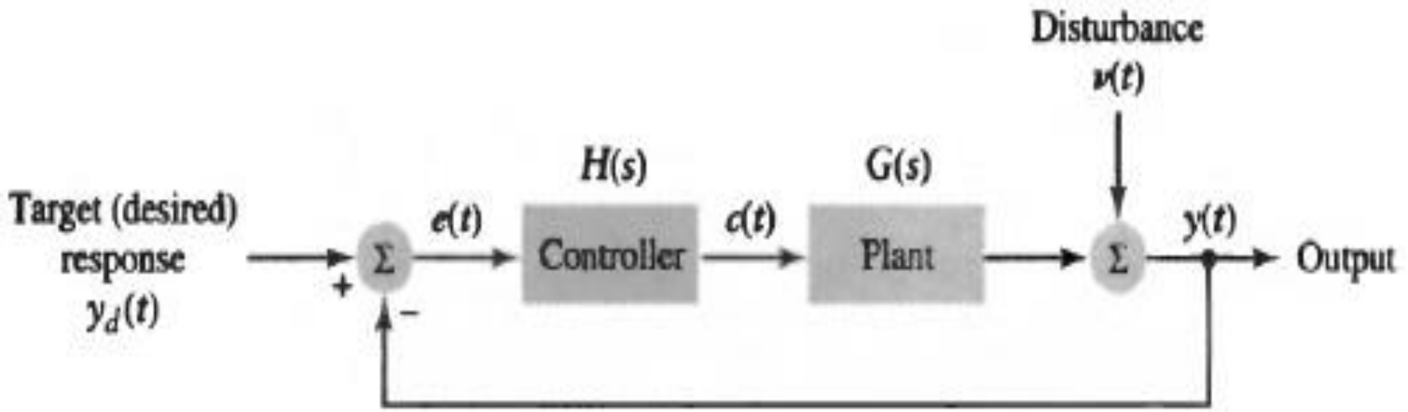
- The block diagram of an closed-loop control system (control system with unity feedback).



- The plant and controller are represented by the transfer functions $G(s)$ and $H(s)$, respectively.
- The controller or compensator in the forward path preceding the plant is the only 'free' part of the system that is available for adjustment by the system designer.

Closed-Loop Control (2)

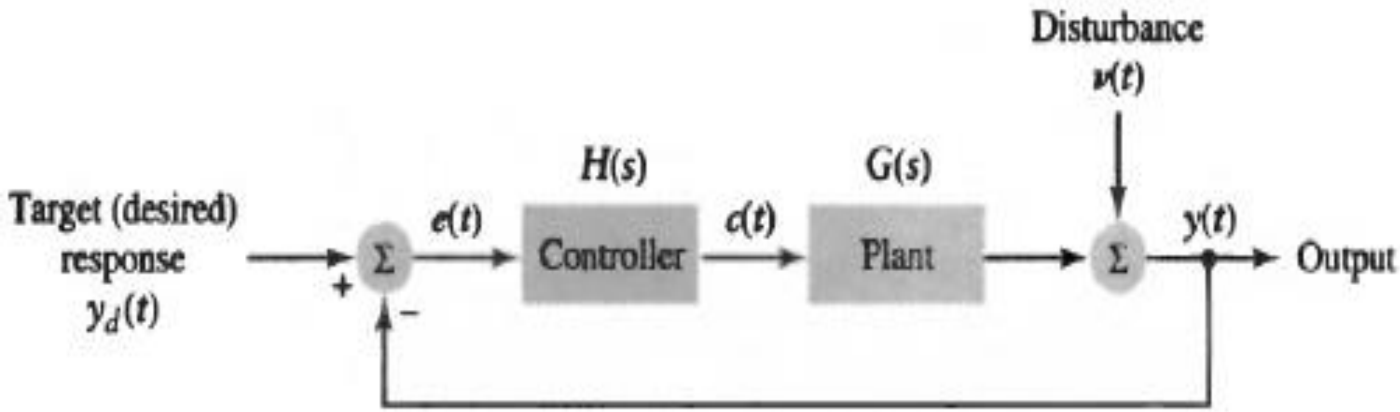
- The block diagram of an closed-loop control system (control system with unity feedback).



- This closed-loop control system is referred to as a single-degree-of-freedom (1-DOF) structure.
- The sensor (measuring the output signal to produce a feedback signal) is perfect, the transfer function of the sensor is unity, and noise produced by the sensor is zero.

Closed-Loop Control (3)

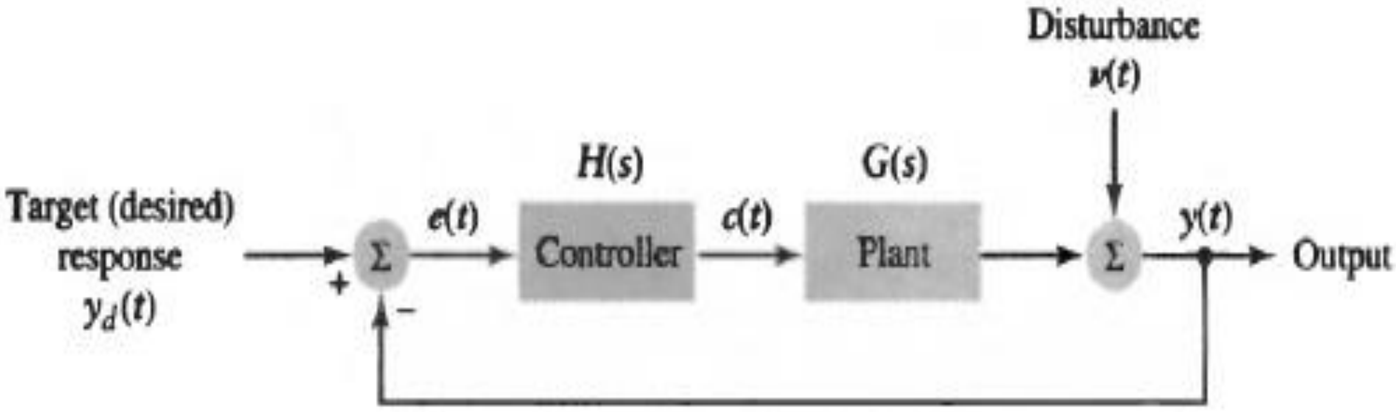
- The block diagram of an closed-loop control system (control system with unity feedback).



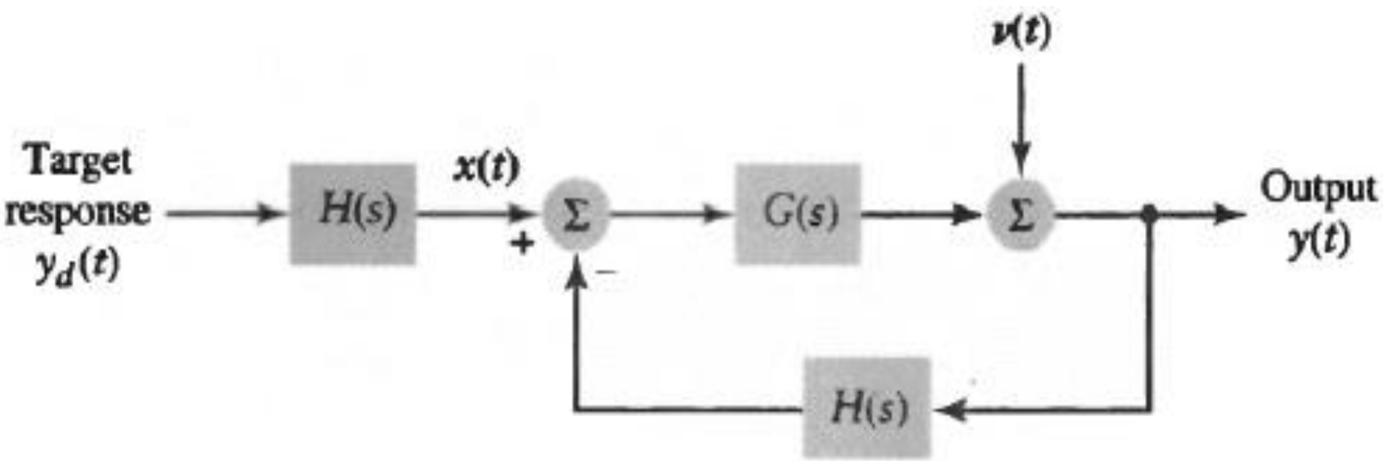
- The actual output $y(t)$ of the plant is fed back directly to the input of the system. The system is said to be a unity-feedback system.
- The controller is actuated by the “measured” error $e(t)$, defined as $e(t) = y_d(t) - y(t)$.

Closed-Loop Control (4)

- **For the purpose of analysis**

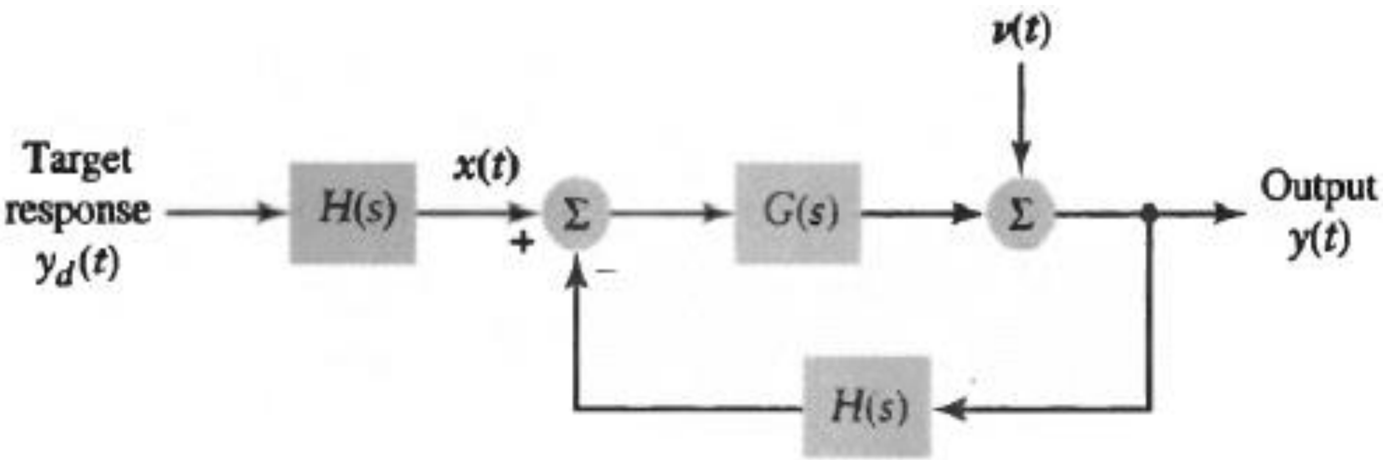


⇩

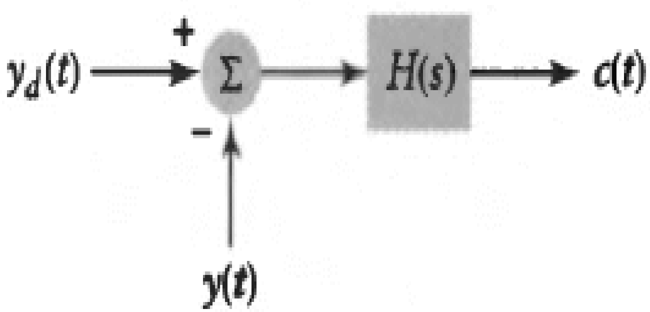


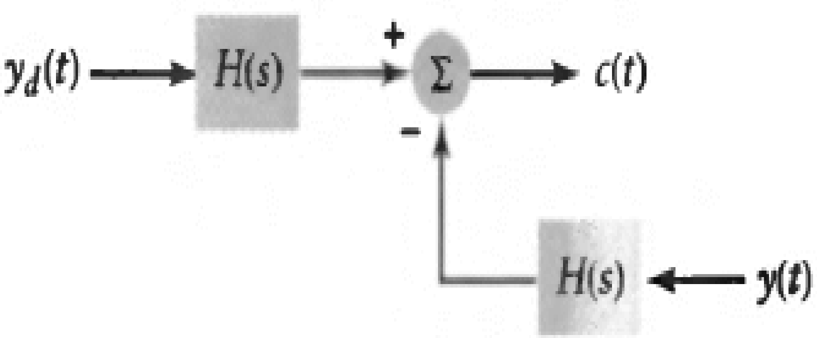
Closed-Loop Control (5)

- For the purpose of analysis



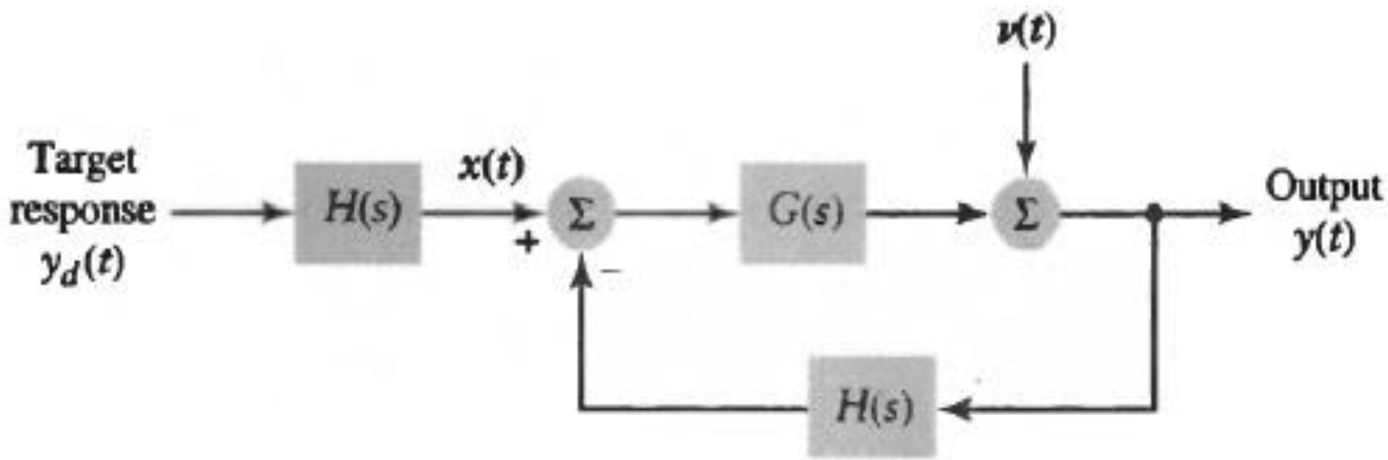
- Use of the equivalence between the two block diagrams:



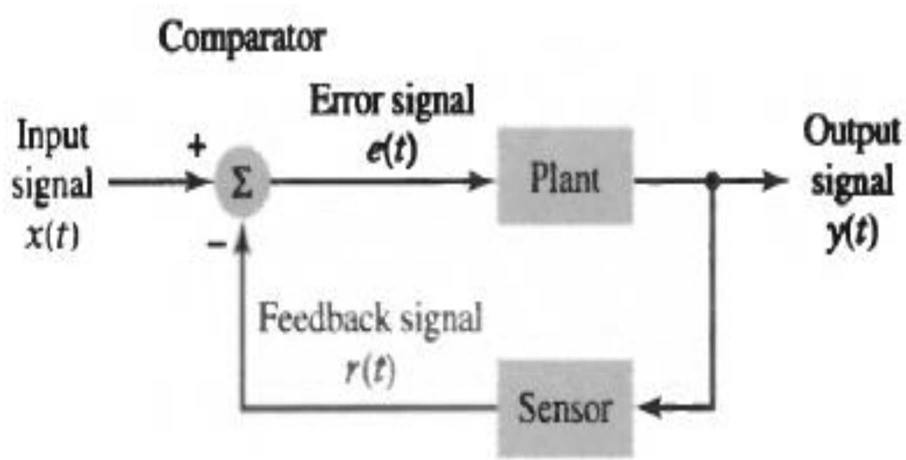


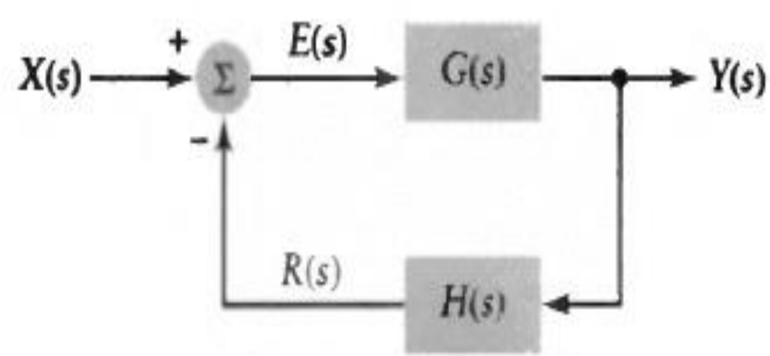
Closed-Loop Control (6)

- The feedback control system



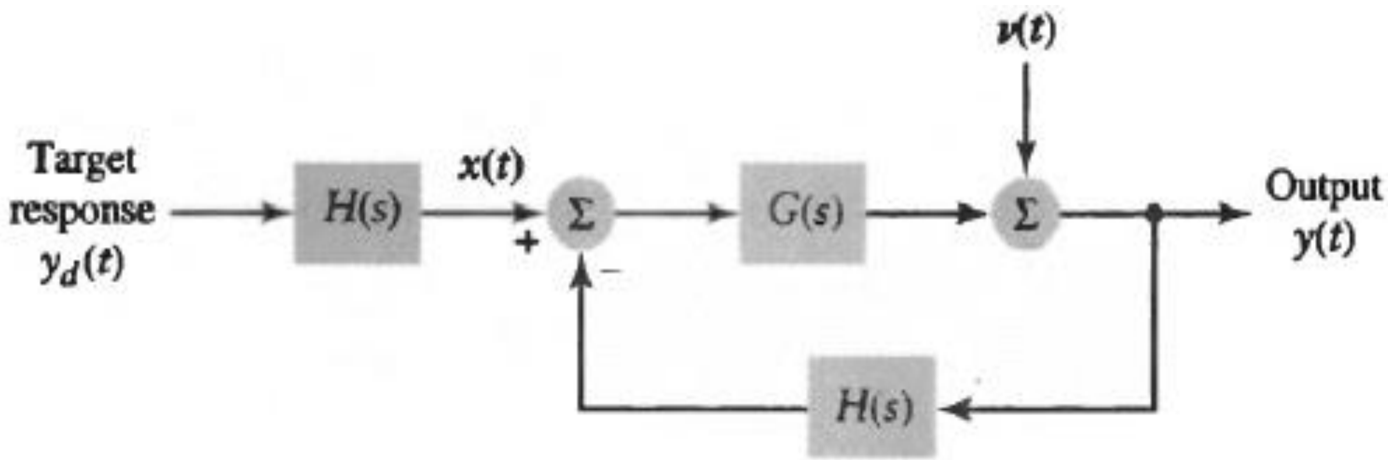
- Compare with:





Closed-Loop Control (7)

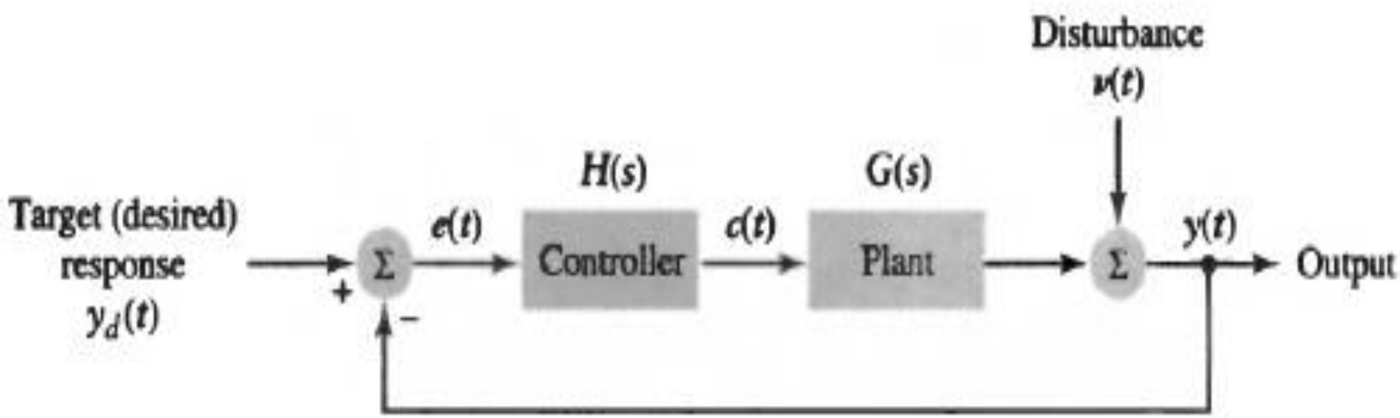
- The feedback control system



- $X(s) = H(s)Y_d(s)$, using $\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)}$,
- The closed-loop transfer function of the 1-DOF system:
- $T(s) = \frac{Y(s)}{Y_d(s)} = \frac{Y(s)}{X(s)}\frac{X(s)}{Y_d(s)} = \frac{G(s)H(s)}{1+G(s)H(s)}$.
- Assuming that $G(s)H(s) \gg 1$ for all values of s that are of interest, we have $T(s) \cong 1$.

Closed-Loop Control (8)

- With the disturbance $v(t) = 0$, we have $y(t) = y_d(t)$.
- It is desirable to have a large loop gain $G(s)H(s)$.
- Under this condition, the system



has the potential to achieve the desired goal of accurate control, the actual output $y(t)$ of the system closely approximating the target response $y_d(t)$.

Closed-Loop Control (9)

- **Without controller**

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

With controller and unity feedback

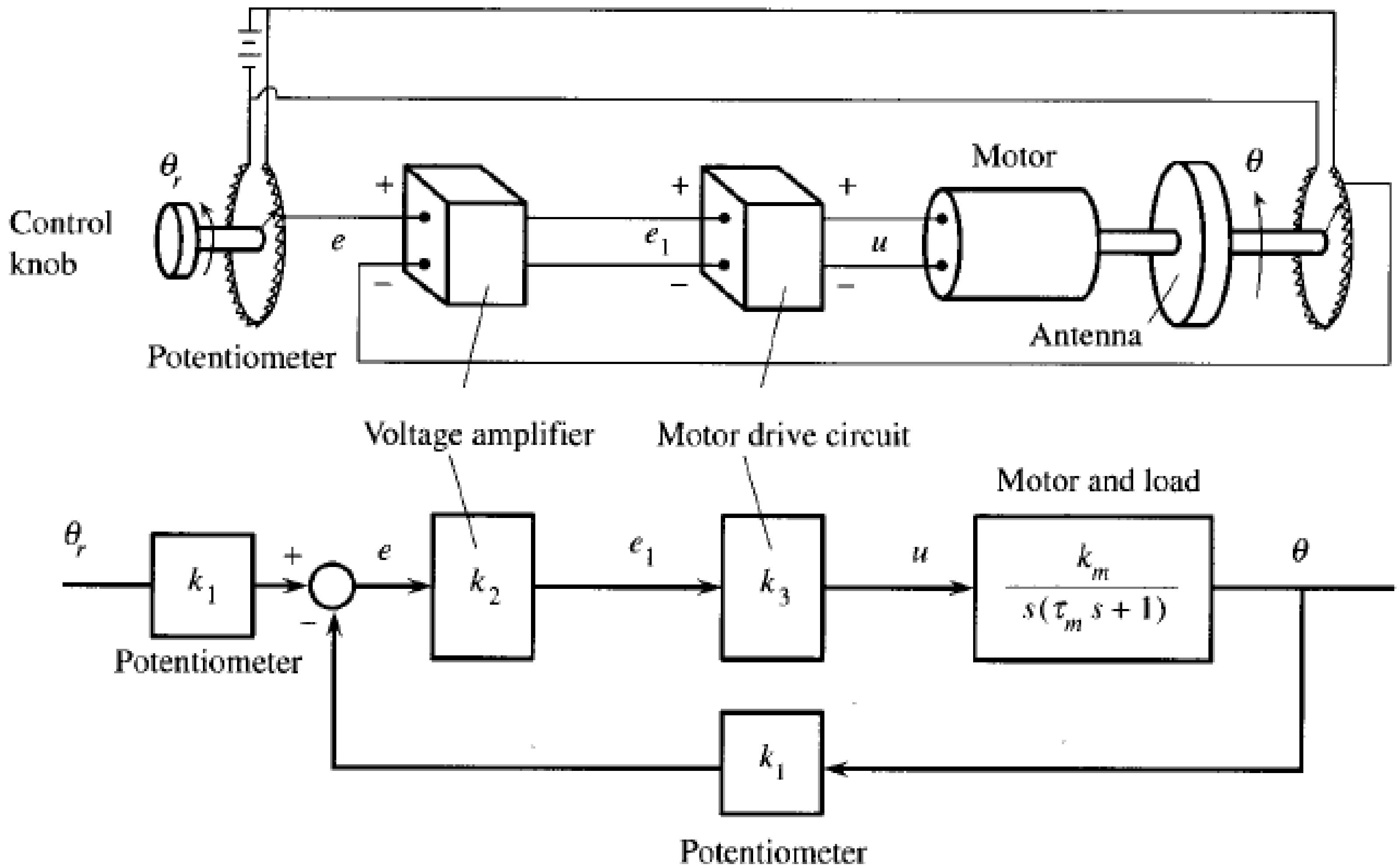
$$T(s) = \frac{G(s)H(s)}{1+G(s)H(s)}$$

- **There are good reasons for using a large loop gain**:
  - The sensitivity of the closed-loop control system $T(s)$ is reduced by a factor equal to the return difference $F(s) = 1 + G(s)H(s)$.
  - The disturbance $v(t)$ inside the feedback loop is reduced by the same factor $F(s)$.
  - The effect of distortion due to nonlinear behavior of the plant is also reduced by $F(s)$.

Examples of Control System

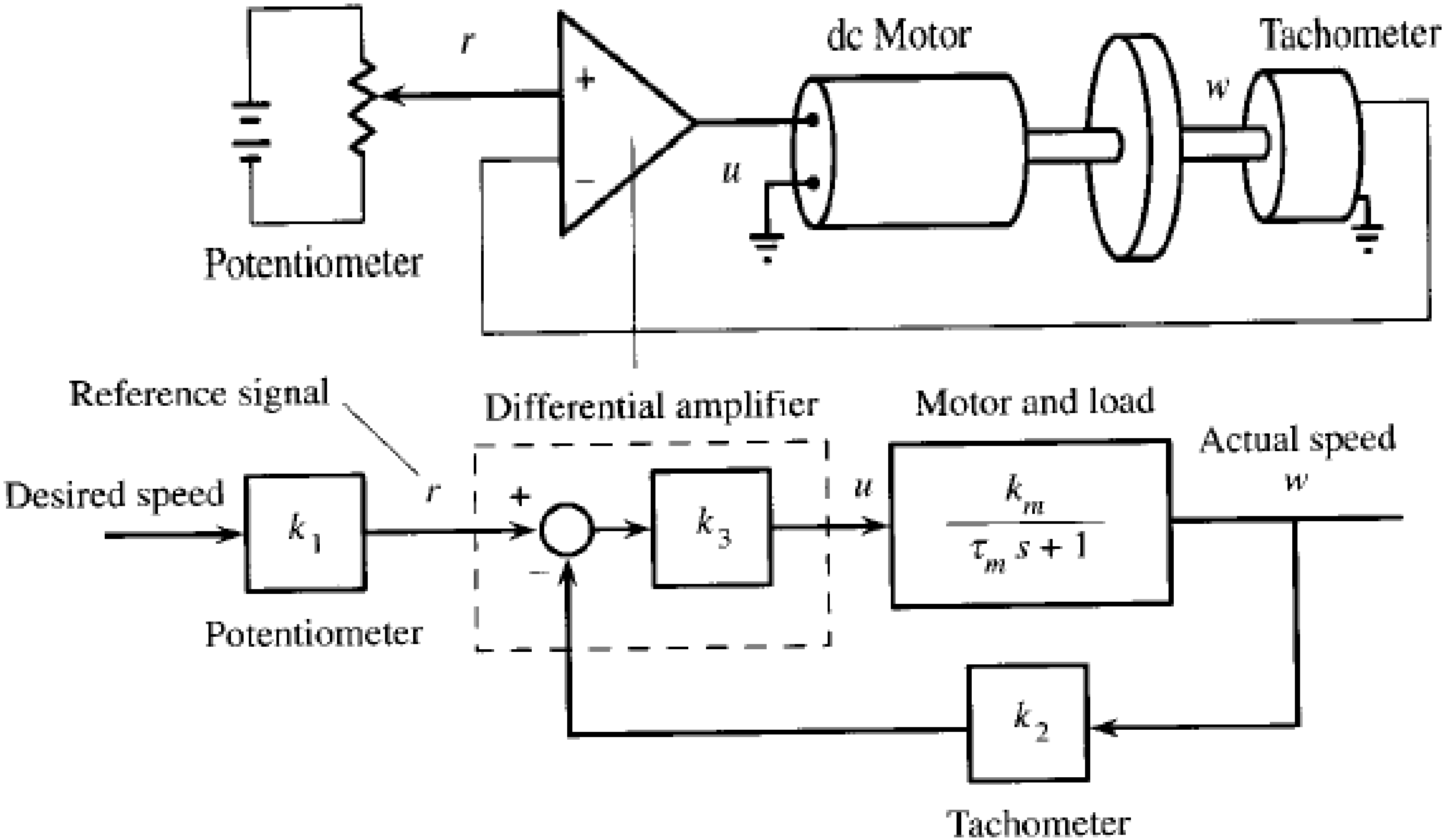
- Position Control Systems
- Velocity Control Systems
- Speed Control System
- Temperature Control Systems

Position Control Systems



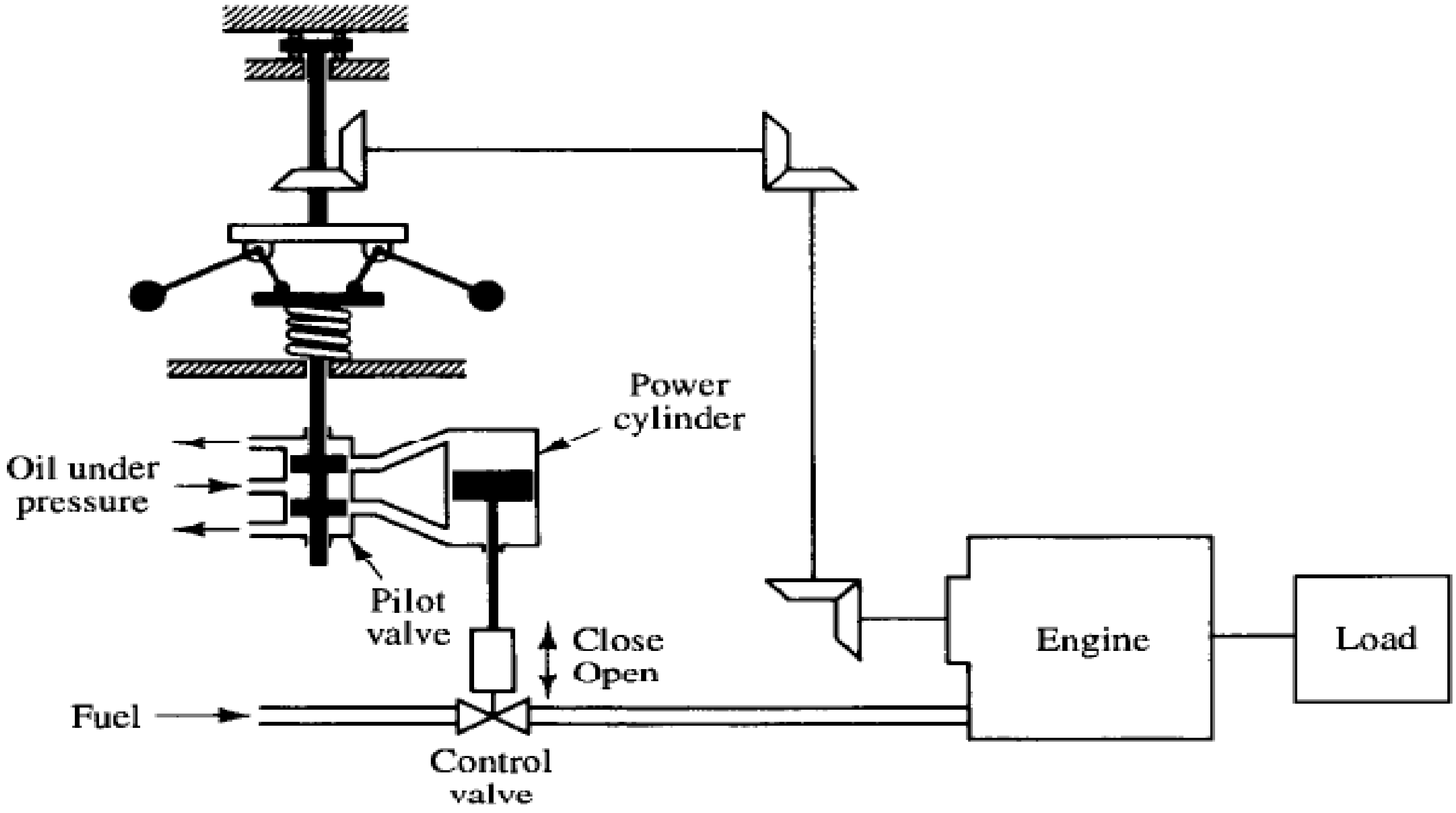
- Analog and Digital Control System Design; Chi-Tsong Chen; Sounders Publishing.

Velocity Control Systems



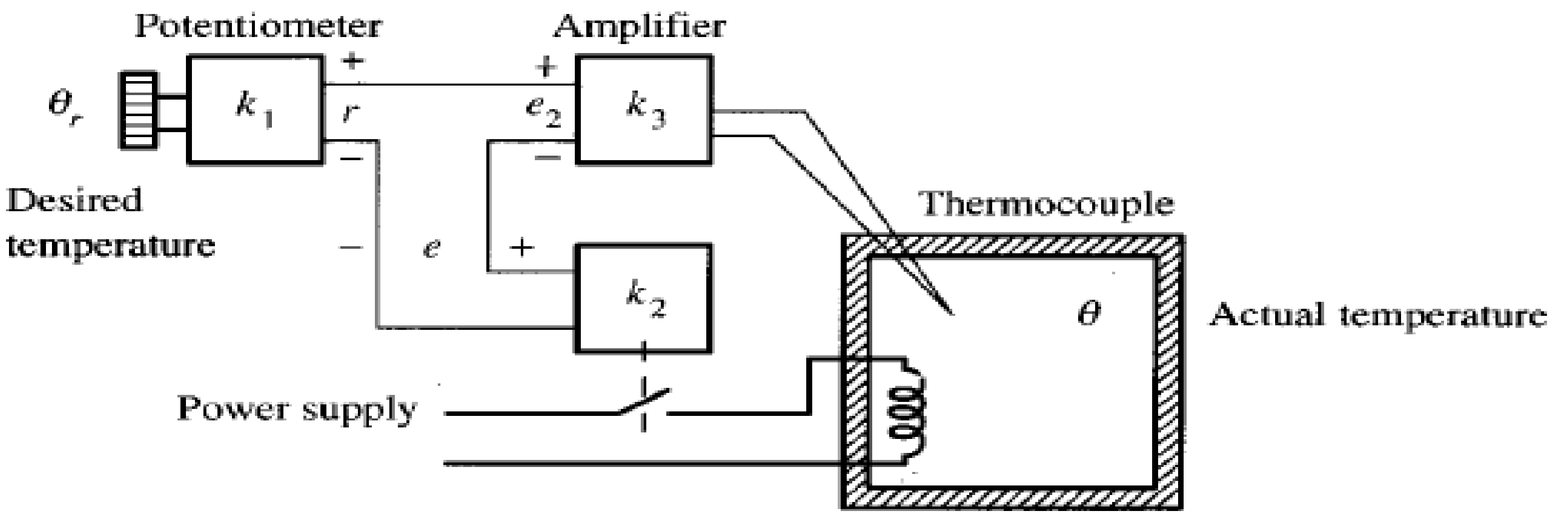
- Analog and Digital Control System Design; Chi-Tsong Chen; Sounders Publishing.

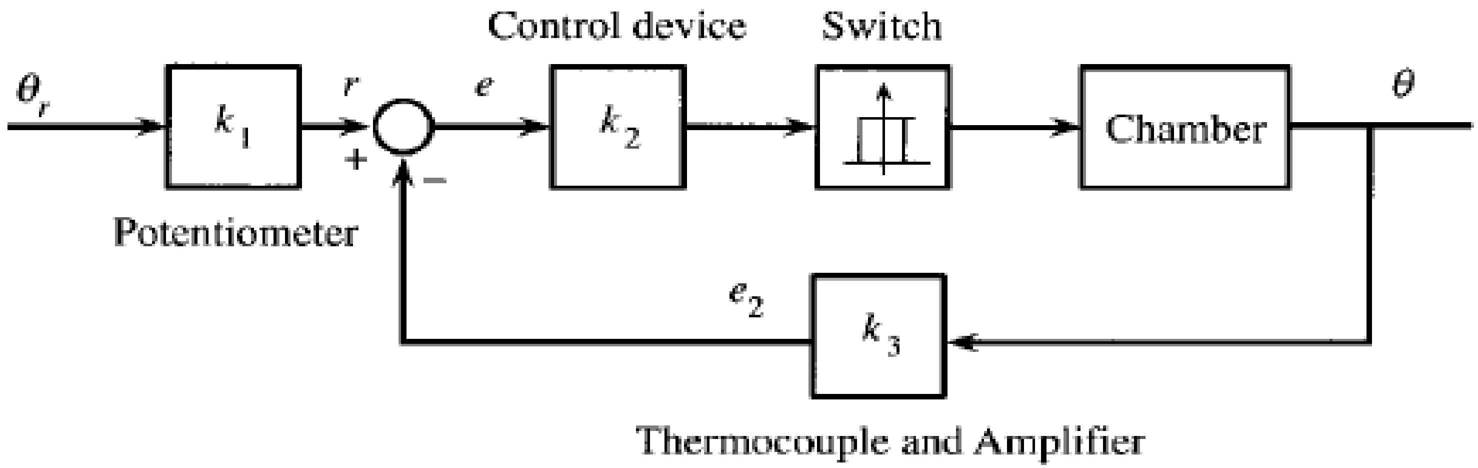
Speed Control System



- Modern Control Engineering; K. Ogata; Prentice-Hall, 1997.

Temperature Control Systems





- Analog and Digital Control System Design; Chi-Tsong Chen; Sounders Publishing.

Transient Response of Low-Order Systems

- For the material on **the stability analysis of feedback control systems**, we find it informative to examine the transient response of first-order and second-order systems.
- Their transient analysis forms the basis for a better understanding of higher order systems.
- First order system:

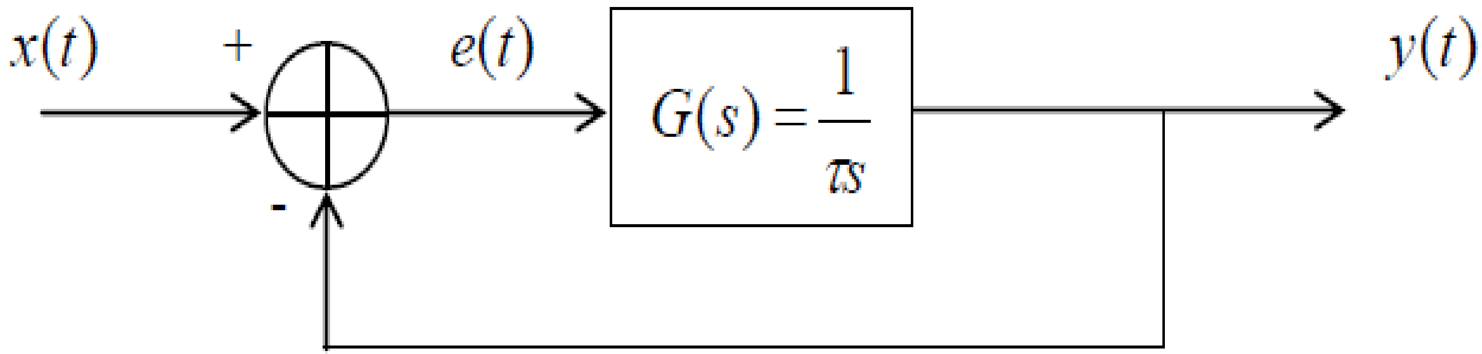
$$T(s) = \frac{T(0)}{\tau s + 1}$$

- Second order system:

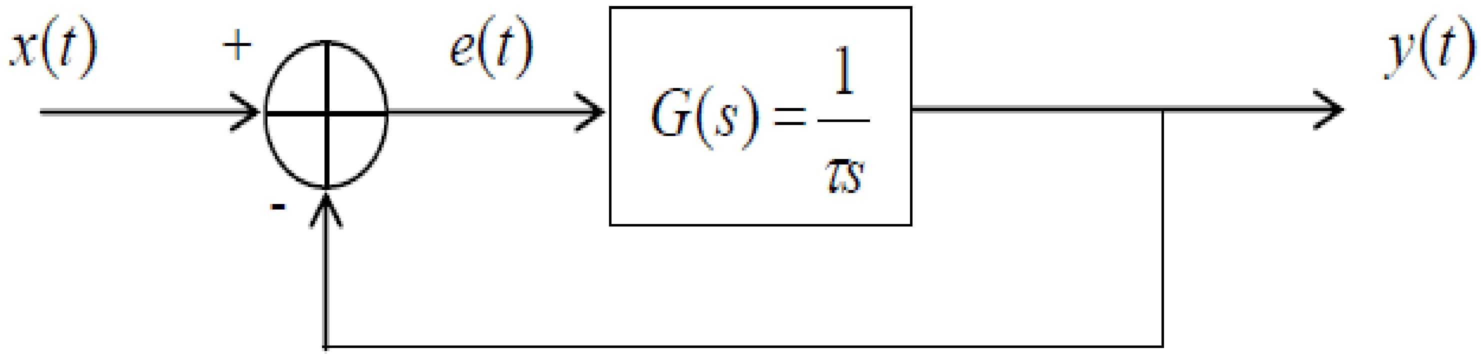
$$T(s) = \frac{T(0){\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Transient Response of First-Order System (1)

- Standard form: $T(s) = \frac{T(0)}{\tau s+1}$, where $T(0)$ is the **gain** of the system at $s = 0$, and τ is measured in units of time and is referred to as the **time constant** of the system.
- Assuming $T(0) = 1 \Rightarrow T(s) = \frac{1}{\tau s+1}$



Transient Response of First-Order System (2)



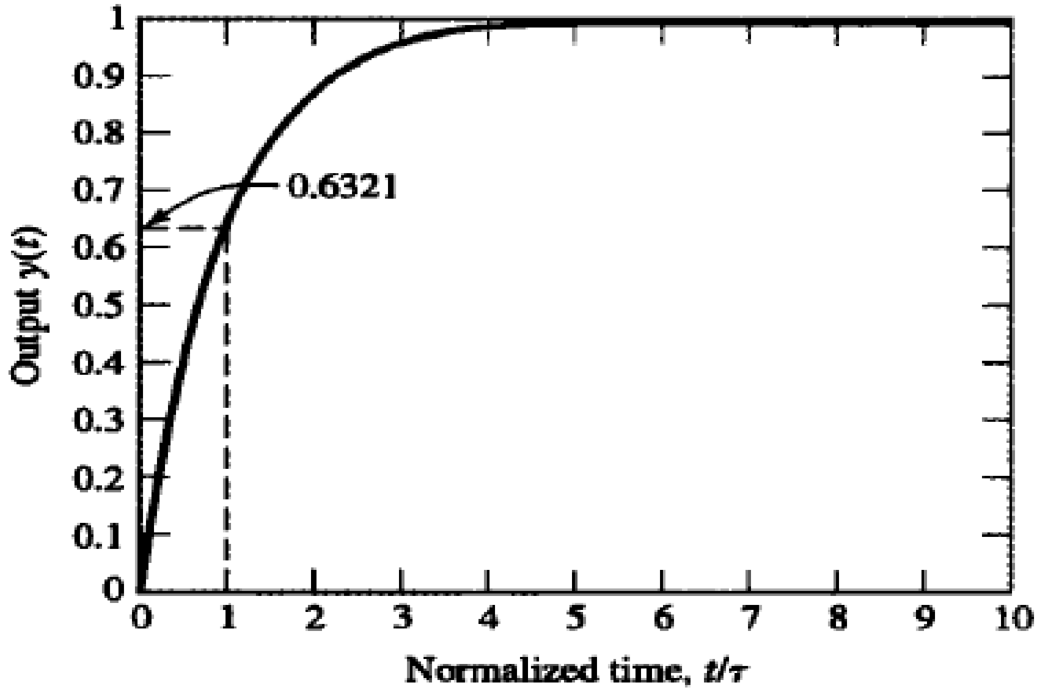
- The single pole of $T(s) = \frac{1}{\tau s+1}$ is located at $s = -\frac{1}{\tau}$
- For a step input: $Y_d(s) = \frac{1}{s}$, the response of the system has the LT

$Y(s) = \frac{1}{s(\tau s+1)} \Rightarrow Y(s) = \frac{1}{s(\tau s+1)} = \frac{1}{s} - \frac{\tau}{\tau s+1}$

- $Y(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} \overset{L^{-1}}{\Longrightarrow} y(t) = \left(1 - e^{-t/\tau}\right)u(t).$

Transient Response of First-Order System (3)

- The step response of the system $y(t) = \left(1 - e^{-t/\tau}\right)u(t)$.



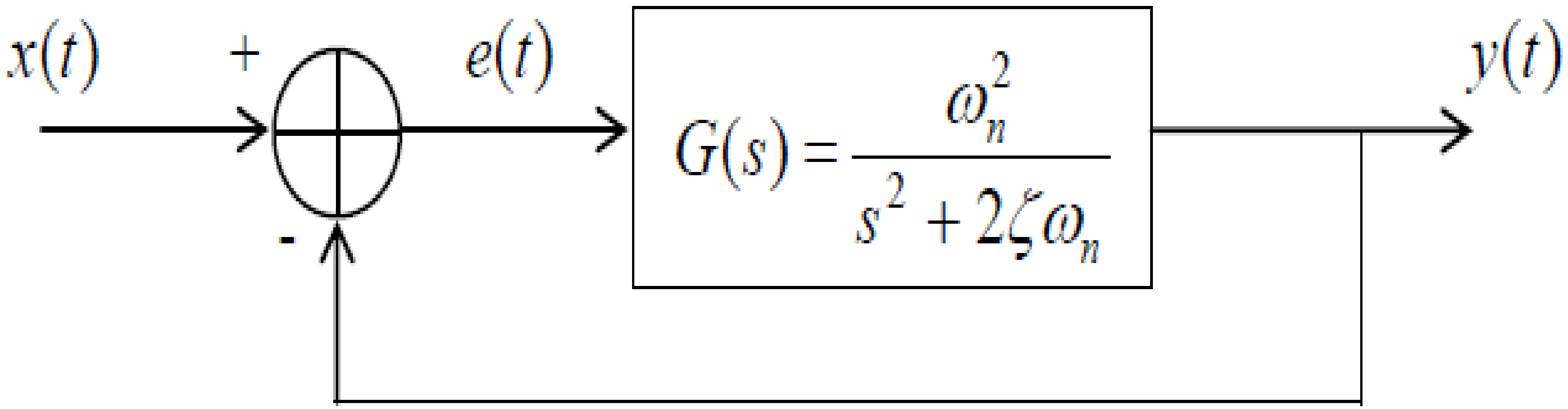
- At $t = \tau$, the response $y(t)$ reaches 63.21% of its final value, τ is the "time constant".

Transient Response of Second-Order System (1)

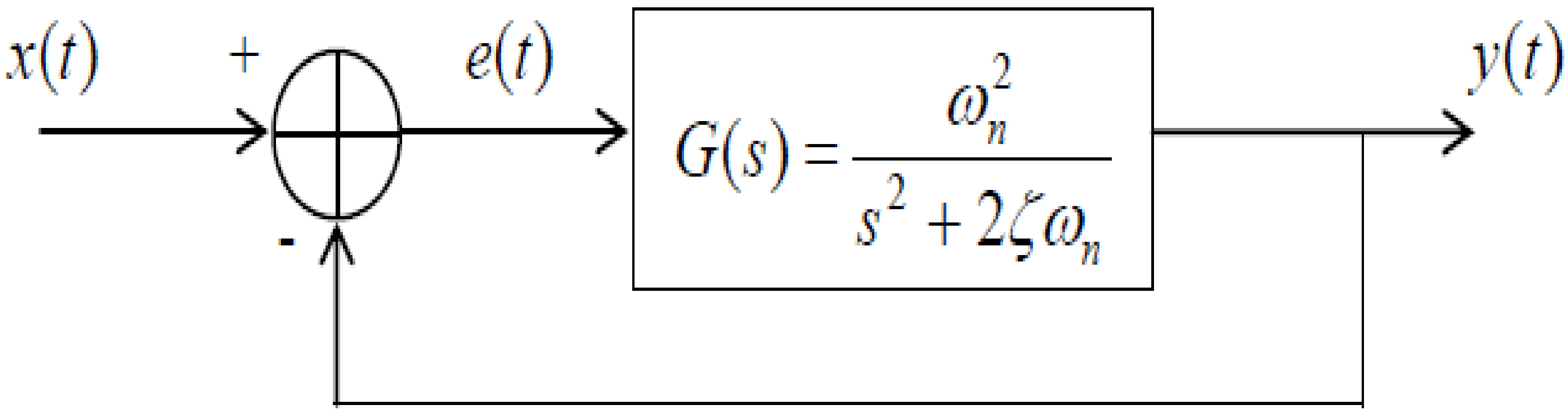
- Standard form: $T(s) = \frac{T(0){\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$,

  where $T(0)$ is the gain of the system at $s = 0$, ω_n is called the **undamped frequency** of the system, and ζ is called the **damping ratio**.

- Assuming $T(0) = 1 \Rightarrow T(s) = \frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$



Transient Response of Second-Order System (2)



- The poles of the system are located at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.
- For a step input: $Y_d(s) = \frac{1}{s}$, the response of the system has the LT

$$Y(s) = \frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)}$$

Transient Response of Second-Order System (3)

- For a step input: $Y_d(s) = \frac{1}{s}$, the response of the system has the LT
  $Y(s) = \frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)}$
- We assume that the poles of $T(s)$: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
  are complex with real parts < 0, $0 < \zeta < 1$,
- The step response of the system as the exponentially damped sinusoidal signal

$$y(t) = \left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)\right]u(t)$$

The time constant of the exponentially damped sinusoid is defined by $\tau = 1/\zeta\omega_n$.

The frequency of the exponentially damped sinusoid is $\omega_n\sqrt{1-\zeta^2}$, $\omega_n > 0$.

Transient Response of Second-Order System (4)

1. $0 < \zeta < 1$.

   The two poles of $T(s)$ constitute a complex-conjugate pair.

   $$y(t) = \left[1 \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2} + tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)\right] u(t).$$

   The system is said to be **underdamped**.

2. $\zeta > 1$.

   The two poles of $T(s)$ are real.

   $$y(t) = \left(1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2}\right) u(t).$$

   $\tau_1 = \frac{1}{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$ and $\tau_2 = \frac{1}{\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}$.

   $k_1 = \frac{1}{2}\left(1 + \frac{\zeta}{\sqrt{\zeta^2-1}}\right)$ and $k_2 = \frac{1}{2}\left(1 - \frac{\zeta}{\sqrt{\zeta^2-1}}\right)$.

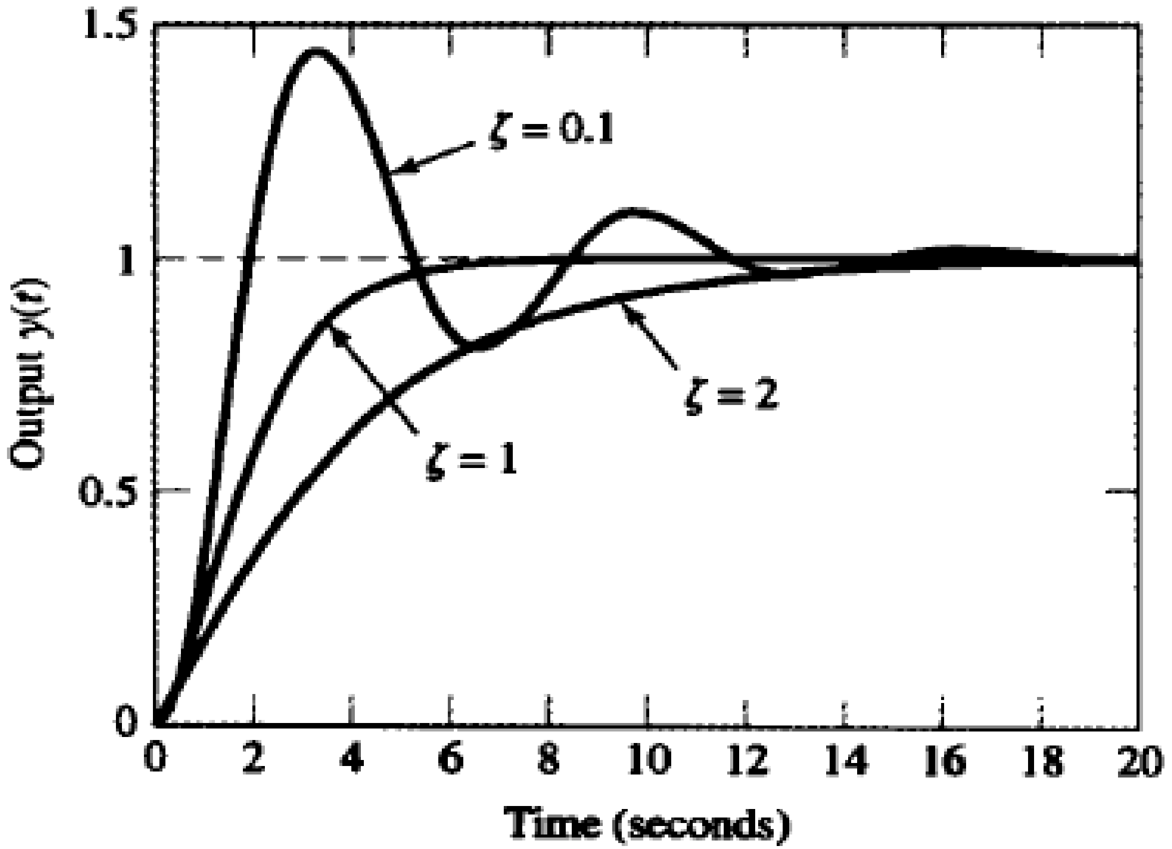
   The system is said to be **overdamped**.

Transient Response of Second-Order System (5)

3. $\zeta = 1$. The two poles of $T(s)$ are coincident at $s = -\omega_n$.

$y(t) = \left(1 - e^{-t/\tau} - te^{-t/\tau}\right)u(t)$. $\tau = 1/\omega_n$.

The system is said to be **critically damped**.



The Stability (1)

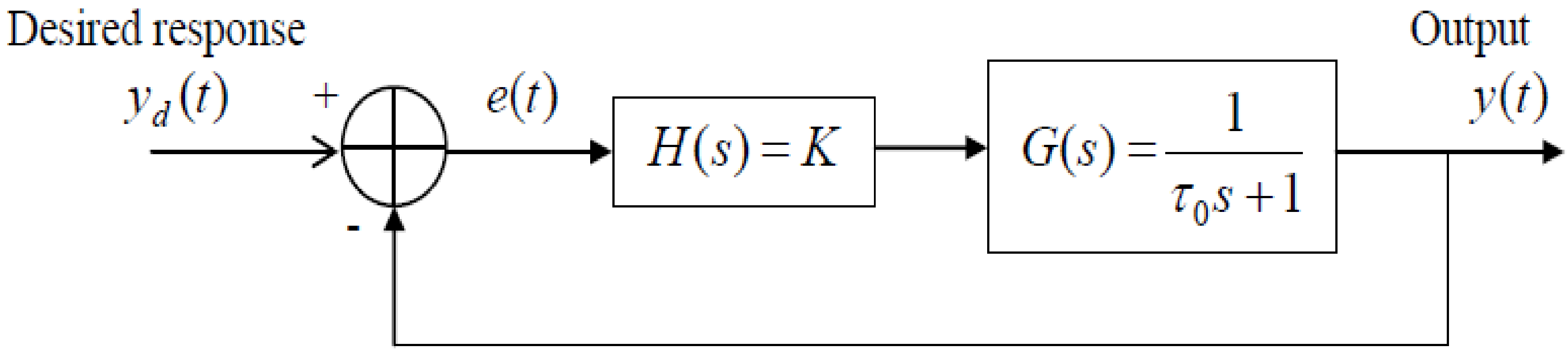
- A large loop gain $G(s)H(s)$ is required to make the closed-loop transfer function $T(s)$ of a feedback system:
  - less sensitive to variations in the values of parameters,
  - mitigate the effects of disturbance or noise,
  - and reduce nonlinear distortion.
- It would be tempting to propose the following recipe for improving the performance of a feedback system: make the loop gain $G(s)H(s)$ of the system as large as possible in the passband of the system.
- Unfortunately, the utility of this simple recipe is limited by a **stability problem** that is known to arise in feedback systems under certain conditions: if the number of poles contained in $G(s)H(s)$ is three or higher, then the system becomes more prone to **instability** and therefore more difficult to control as the loop gain is increased.

The Stability (2)

- In the design of a feedback system, the task is therefore not only to **meet the various performance requirements** imposed on the system for satisfactory operation inside a prescribed passband, but also to ensure that the **system is stable and remains stable** under all possible operating conditions.
- The stability of a feedback system, is completely determined by the location of the system's poles or **natural frequencies** in the $s-$ plane.
- The natural frequencies of a linear feedback system with closed-loop transfer function $T(s)$ are defined as the roots of the characteristic equation $A(s) = 0$, where $A(s)$ is the denominator polynomial of $T(s)$.
- The feedback system is stable if the roots of this characteristic equation are all confined to the left half of the $s-$ plane.

The Stability of 1st Order Feedback System (1)

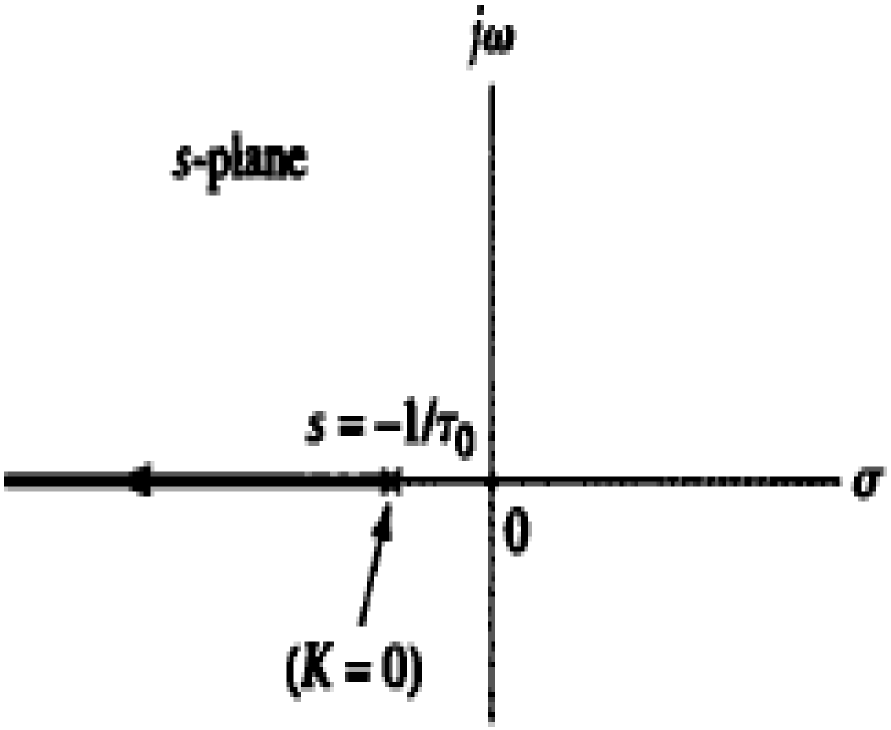
- The 1st order feedback system with unity feedback.



- The loop transfer function: $G(s)H(s) = \frac{K}{\tau_0 s+1}$, where τ_0 is the open-loop time constant of the system and K is an adjustable loop gain. $G(s)H(s)$ has a single pole at $s = -1/\tau_0$.
- The closed-loop transfer function: $T(s) = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K}{\tau_0 s+K+1}$.
- The characteristic equation: $\tau_0 s + K + 1 = 0$

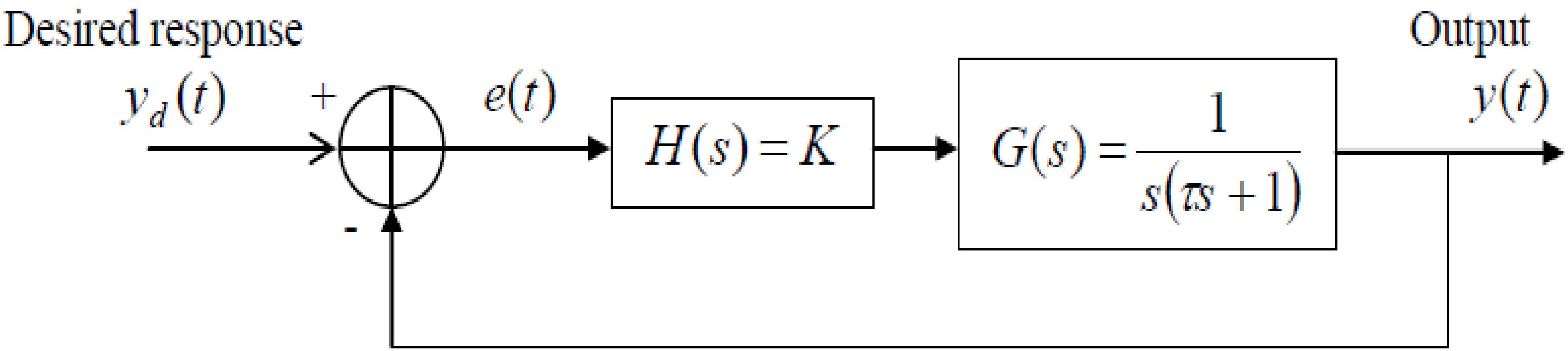
The Stability of 1st Order Feedback System (2)

- The characteristic equation: $\tau_0 s + K + 1 = 0$, which has a single pole at $s = -(K+1)/\tau_0$.
- As K is increased, this root moves long the real axis of the $s-$plane.
- It remains confined to the left half of the $s-$plane for $K > -1$.
- The 1st order feedback system with a loop transfer function $G(s)H(s) = \frac{K}{\tau_0 s+1}$ is stable for all $K > -1$.



The Stability of 2nd Order Feedback System (1)

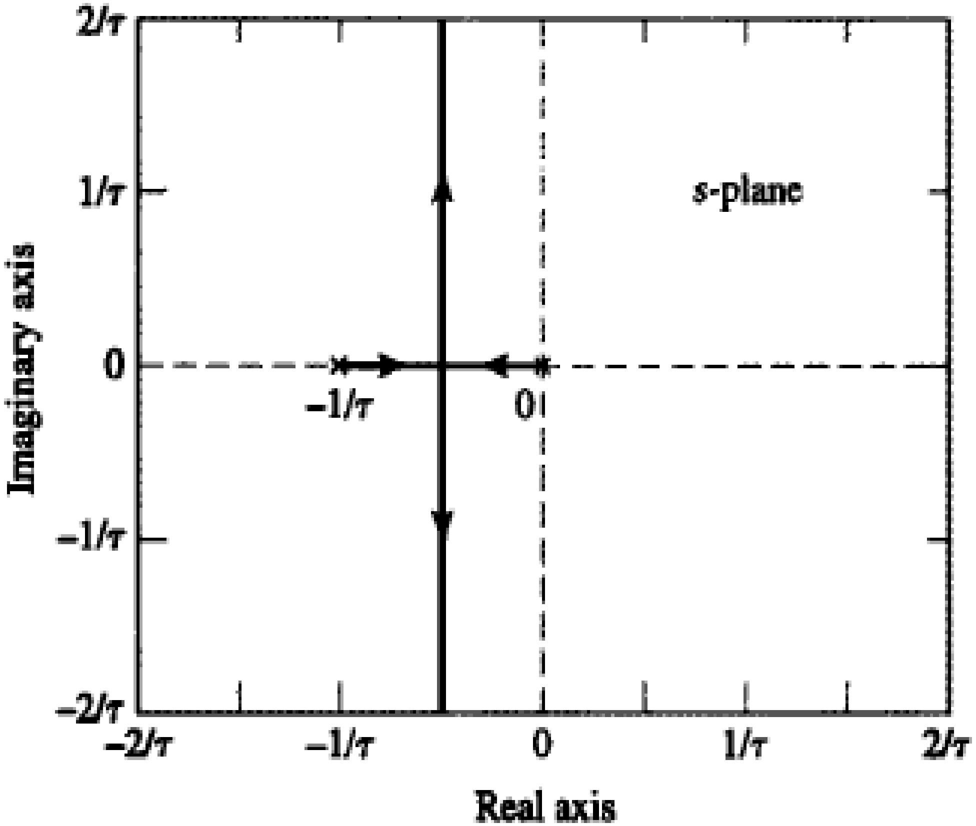
- The 2nd order feedback system with unity feedback.



- The loop transfer function: $G(s)H(s) = \frac{K}{s(\tau s+1)}$, where K is an adjustable loop gain measured in rad/s.
- $G(s)H(s)$ has a simple poles at $s = 0$ and $s = -1/\tau$.
- The closed-loop transfer function: $T(s) = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K}{\tau s^2+s+K}$.
- The characteristic equation: $\tau s^2 + s + K = 0$

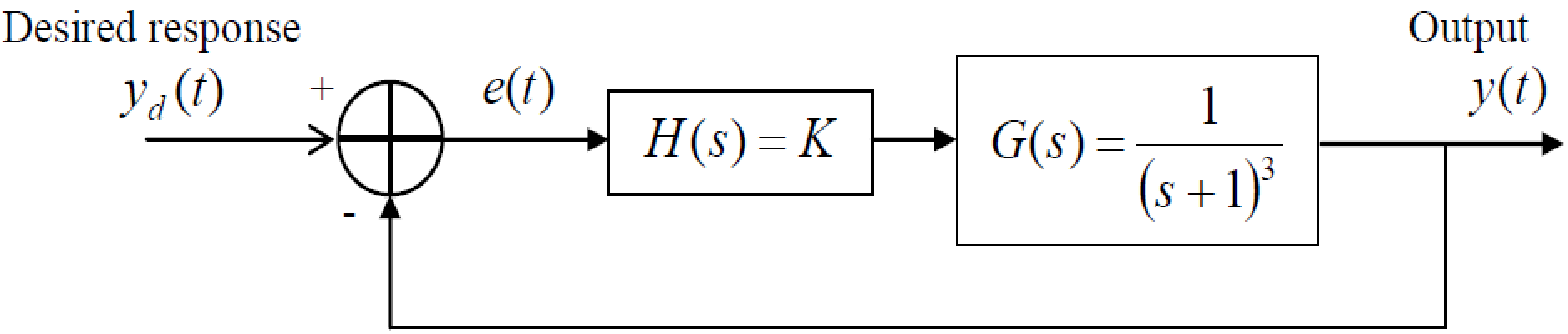
The Stability of 2nd Order Feedback System (2)

- The characteristic equation: $\tau s^2 + s + K = 0$, which has a pair of roots: $s = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \frac{K}{\tau}}$.
- For $K = 0$, the CE has a root at $s = 0$ and at $s = -1/\tau$.
- As K is increased, the 2 roots move toward each other.
- For $K = 1/4\tau \Rightarrow s = -1/2\tau$.
- For $K > 1/4\tau$, roots are complex.
- The 2nd order feedback system with a loop transfer function $G(s)H(s) = \frac{K}{s(\tau s+1)}$ is stable for all $K \geq 0$.



The Stability of 3rd Order Feedback System (1)

- The 3rd order feedback system with unity feedback.



- The loop transfer function: $G(s)H(s) = \frac{K}{(s+1)^3}$.
- The closed-loop transfer function:

$$T(s) = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K}{s^3+3s^2+3s+K+1}.$$

- The characteristic equation: $s^3 + 3s^2 + 3s + K + 1 = 0$

The Stability of 3rd Order Feedback System (2)

- How variations in K affect the stability of the system.

Roots of the Characteristic Equation $\boldsymbol{s^3+3s^2+3s+K+1=0}$		
K	Roots	Comment
0	Third-order root at $s=-1$	
5	$s=-2.71$ $s=-0.1450\pm j1.4809$	Stable
10	$s=-3.1544$ $s=0.0772\pm j1.8658$	Unstable

- The loop gain K has a profound influence on the stability of the system.
- The majority of feedback systems used in practice are of order 3 or higher. The stability of such system is a problem of importance.

Routh-Hurwitz Criterion (1)

- The Routh-Hurwitz criterion provides a simple procedure for ascertaining whether all the roots of a polynomial $A(s)$ have negative real parts (i.e., lie in the left-half of the $s-$plane), without having to compute the roots of $A(s)$.
- Let $A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$, where $a_n \neq 0$.
- The procedure begins by arranging all the coefficients of $A(s)$ in the form of two rows as follows:

Row n:	a_n	a_{n-2}	a_{n-4}	…
Row $n-1$:	a_{n-1}	a_{n-3}	a_{n-5}	…

- If the order n of $A(s)$ is even, coefficient a_0 belong to row n, then a zero is placed under a_0 in row $n-1$.
- Construct row $n-2$ by using the entries of rows n and $n-1$ in accordance with the following formula:

Routh-Hurwitz Criterion (2)

- Construct row $n-2$ by using the entries of rows n and $n-1$ in accordance with the following formula:

Row $n-2$: $\frac{a_{n-1}a_{n-2}-a_n a_{n-3}}{a_{n-1}}$ $\quad\quad$ $\frac{a_{n-1}a_{n-4}-a_n a_{n-5}}{a_{n-1}}$ $\quad\quad$ …

- Next, the entries of rows $n-1$ and $n-2$ are used to construct row $n-3$, following a procedure similar to row $n-2$, and the process is continued until we reach row 0.
- The resulting array of $(n+1)$ rows is called the **Routh array**.
- **The Routh-Hurwitz criterion**: All the roots of $A(s)$ lie in the LH of the $s-$ plane if all the entries in the leftmost column of the Routh array are nonzero and have the same the same sign. If sign changes are encountered in scanning the leftmost column, the number of such changes is the number of roots of $A(s)$ in the RH of the $s-$ plane.

Example: 4th-Order Feedback System

- $A(s) = s^4 + 3s^3 + 7s^2 + 3s + 10.$
- Construct the Routh array of the system, and determine whether the system is stable.
- Row 4: 1 7 10
- Row 3: 3 3 0
- Row 2: $\frac{3x7-1x3}{3} = 6$ $\frac{3x10-1x0}{3} = 10$ 0
- Row 1: $\frac{6x3-3x10}{6} = -2$ 0 0
- Row 0: $\frac{-2x10-6x0}{-2} = 10$ 0 0
- There are 2 sign changes, we conclude that the system is unstable, and $A(s)$ has 2 roots in the RH of $s-$ plane.

Routh-Hurwitz Criterion (3)

- **The Routh-Hurwitz criterion** may be used to determine **the critical value of the loop gain K** for which $A(s)$ has a pair of roots on the $j\omega$ −axis, the Routh-Hurwitz test terminates prematurely in that an entire (always odd numbered) row of zeros is encountered in constructing the Routh array.
- When this happens, the feedback system is said to be **on the verge of instability**.
- **The critical value of K** is deduced from the entries of the particular row in question.
- The corresponding pair of roots on the $j\omega$ −axis is found in the auxiliary polynomial formed from the entris of the preceding row.

Example: 3rd-Order Feedback System (1)

- $L(s) = \frac{K}{(s+1)^3}$
- $A(s) = (s+1)^3 + K = s^3 + 3s^2 + 3s + 1 + K.$
- Find the value of K for which the system is on the verge of instability and find the corresponding pair of roots on the $j\omega$ −axis on the s −plane.
- Construct the Routh array of the system, we obtain:
- Row 3: 1 3
- Row 2: 3 $1 + K$
- Row 1: $\frac{3x3-1x(1+K)}{3} = \frac{8-K}{3}$ 0
- Row 0: $\frac{(8-K)x(1+K)-3x0}{3\frac{(8-K)}{3}} = 1 + K$ 0

Example: 3rd-Order Feedback System (2)

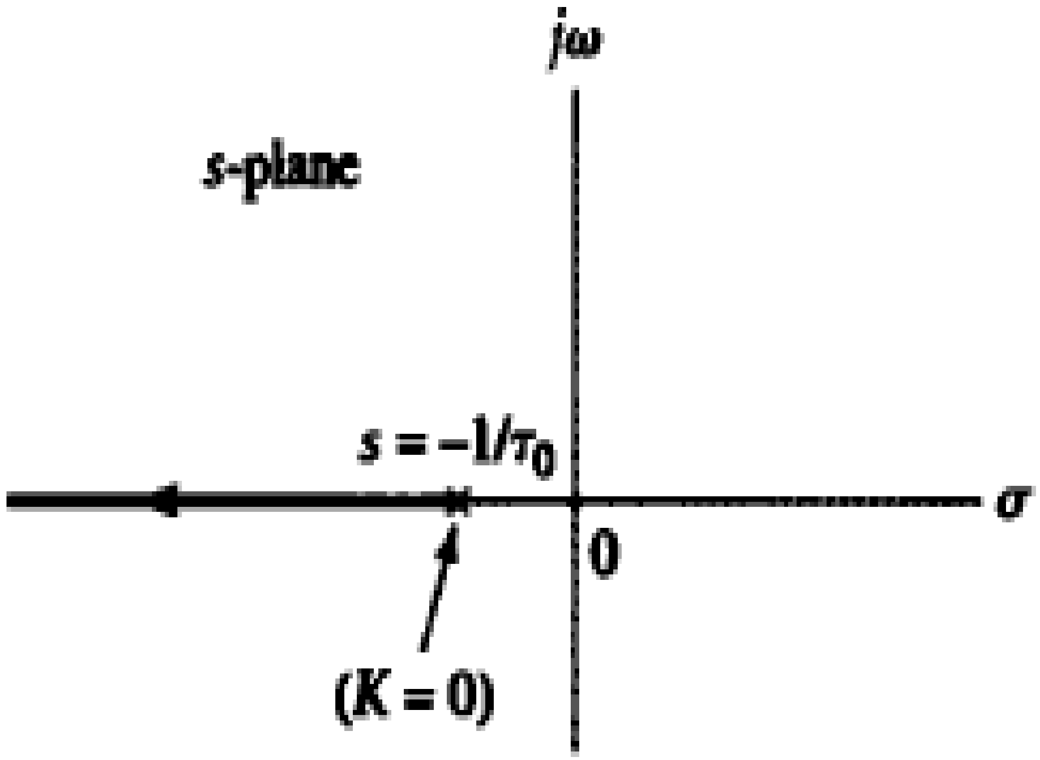
- For the only nonzero entry of row 1 to become zero, we require that $8 - K = 0 \Rightarrow K = 8$.
- The corresponding pair of roots on the $j\omega$ −axis is found in the auxiliary polynomial formed from the entris of the preceding row.
- For this value of K, the auxiliary polynomial is obtained from row 2. we have $3s^2 + 9 = 0 \Rightarrow s^2 + 3 = 0$, which has a pair of roots at $s = \pm j\sqrt{3}$. **The system is on the verge of instability**.
- For $K = 8 \Rightarrow A(s) = s^3 + 3s^2 + 3s + 9 = (s^2 + 3)(s + 3)$

Root Locus Method (1)

- The **root locus method** is **an analytical tool** for the design of a linear feedback system, with emphasis on the locations of the poles of the system's closed-loop transfer function.
- Recall that the poles of a system's transfer function determine its **transient response**.
- By knowing the locations of the closed-loop poles, we can deduce information about the **transient response** of the feedback system.
- The method derives its name from the fact that a "root locus" is the geometric path or locus traced out by the roots of the system's characteristic equation in the s −plane as some parameter (usually, but not necessarily, the loop gain) is varied from zero to infinity.

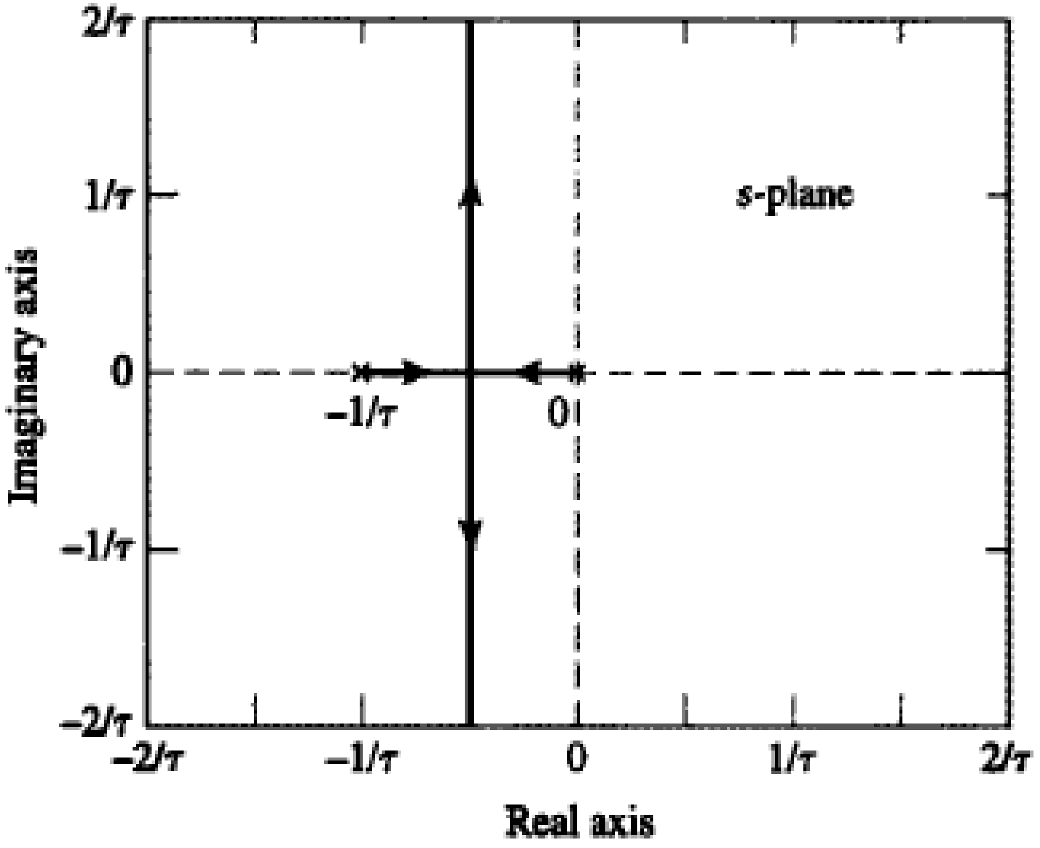
Root Locus Method (2)

- The root locus for a first-order feedback system:



Root Locus Method (3)

- The root locus for a second-order feedback system:



Root Locus Method (4)

- Construction of the root locus begins with the loop transfer function of the system:

$$L(s) = G(s)H(s) = K\,\frac{\prod_{i=1}^{M}(1 - s/c_i)}{\prod_{i=1}^{N}(1 - s/d_i)}$$

where K is the loop gain, d_i is the poles, and c_i is the zeros of $L(s)$.

- These poles and zeros are fixed numbers, independent of K.
- In a linear feedback system, they may be determined directly from the block diagram of the system, since the system is usually made up of a cascade connection of 1st and 2nd order components.
- The term “root locus” refers to a situation in which the loop gain is nonnegative-that is, $0 \leq K \leq \infty$.

Root Locus Criteria (1)

- Let $L(s) = G(s)H(s) = K\frac{\prod_{i=1}^{M}(1-s/c_i)}{\prod_{j=1}^{N}(1-s/d_j)} = K\frac{P(s)}{Q(s)}$.
- $P(s) = \prod_{i=1}^{M}\left(1 - \frac{s}{c_i}\right)$ and $Q(s) = \prod_{j=1}^{N}\left(1 - \frac{s}{d_j}\right)$
- $T(s) = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K\frac{P(s)}{Q(s)}}{1+K\frac{P(s)}{Q(s)}} = \frac{KP(s)}{Q(s)+KP(s)}$
- The characteristic equation $A(s) = Q(s) + KP(s) = 0$, equivalently: $L(s) = K\frac{P(s)}{Q(s)} = -1.$
- Since $s = \sigma + j\omega$, we may express $P(s) = |P(s)|e^{j\arg\{P(s)\}}$, where $|P(s)| = \prod_{i=1}^{M}\left|1 - \frac{s}{c_i}\right|$ and $\arg\{P(s)\} = \sum_{i=1}^{M}\arg\left\{1 - \frac{s}{c_i}\right\}$.

Root Locus Criteria (2)

- Similarly, we may express $Q(s) = |Q(s)|e^{j\arg\{Q(s)\}}$, where $|Q(s)| = \prod_{j=1}^{N}\left|1 - \frac{s}{d_j}\right|$ and $\arg\{Q(s)\} = \sum_{j=1}^{N}\arg\left\{1 - \frac{s}{d_j}\right\}$.
- Substituting:
- $$\left.\begin{aligned} |P(s)| &= \prod_{i=1}^{M}\left|1 - \frac{s}{c_i}\right| \\ \arg\{P(s)\} &= \sum_{i=1}^{M}\arg\left\{1 - \frac{s}{c_i}\right\} \\ |Q(s)| &= \prod_{j=1}^{N}\left|1 - \frac{s}{d_j}\right| \\ \arg\{Q(s)\} &= \sum_{j=1}^{N}\arg\left\{1 - \frac{s}{d_j}\right\} \end{aligned}\right\} \to K\frac{P(s)}{Q(s)} = -1.$$
- Two basic criteria for a root locus (assuming that $K \geq 0$):
  – Angle criterion.
  – Magnitude criterion.

Root Locus Criteria (3)

- **Angle criterion.**
- For a point s_l to lie on a root locus, the angle criterion $\arg\{P(s)\} - \arg\{Q(s)\} = (2k+1)\pi,\ \ k = 0, \pm 1, \pm 2, \ldots,$ must be satisfied for $s = s_l$.
- The angles $\arg[Q(s)]$ and $\arg\{P(s)\}$ are themselves determined by the angles of the pole and zero factors of $L(s)$.
- **Magnitude criterion.**
- Once a root locus is constructed, the value of the loop gain K corresponding to the point s_l is determined from the magnitude criterion $K = \frac{|Q(s)|}{|P(s)|}$, evaluated at $s = s_l$.
- The magnitudes $|Q(s)|$ and $|P(s)|$ are themselves determined by the magnitudes of the pole and zero factors of $L(s)$.

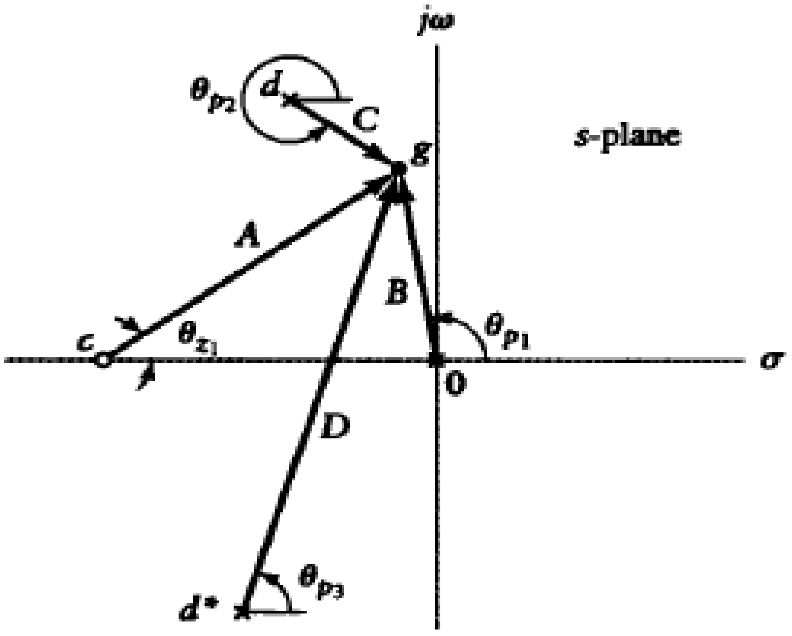
Root Locus Criteria (4)

- Consider the loop transfer function: $L(s) = \frac{K\left(1-\frac{s}{c}\right)}{s\left(1-\frac{s}{d}\right)\left(s-\frac{s}{d^*}\right)}$, which has a zero at $s = c$, a simple pole at $s = 0$, and a pair of complex-conjugate poles at $s = d, d^*$.
- Select an arbitrary trial point g in the s −plane, and construct vectors from the poles and zeros of $L(s)$ to that point.
- From the **angle criterion**:

$\arg\{P(s)\} - \arg\{Q(s)\}$
$= (2k+1)\pi,, \; k = 0, \pm 1, \pm 2, \ldots,$

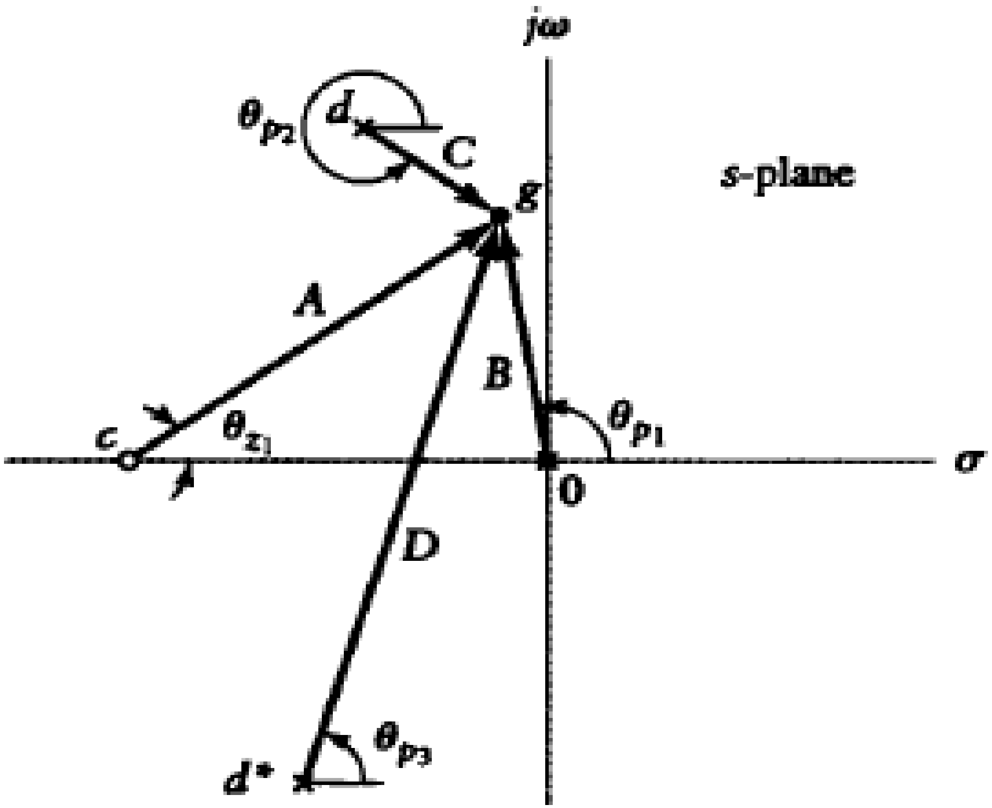
- And the **magnitude criterion**:

$K = \frac{|Q(s)|}{|P(s)|}$ to be both satisfied by the choice of point g.



Root Locus Criteria (5)

- We should find that $\theta_{z_1} - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} = (2k+1)\pi,\ k = 0, \pm 1, \dots,$ and $K = \frac{BCD}{A}$, where
- $\theta_{z_1} = \arg\left\{1 - \frac{g}{c}\right\},$
- $\theta_{p_1} = \arg\{g\},$
- $\theta_{p_2} = \arg\left\{1 - \frac{g}{d}\right\},$
- $\theta_{p_3} = \arg\left\{1 - \frac{g}{d^*}\right\},$
- $A = \left|1 - \frac{g}{c}\right|,$
- $B = |g|,$
- $C = \left|1 - \frac{g}{d}\right|, D = \left|1 - \frac{g}{d^*}\right|$



Properties of The Root Locus (1)

- Given the poles and zeros of the loop transfer function
- $L(s) = G(s)H(s) = K\frac{\prod_{i=1}^{M}\left(1-\frac{s}{c_i}\right)}{\prod_{j=1}^{N}\left(1-\frac{s}{d_j}\right)}$, we may construct an approximate form of the root locus of a linear feedback system by exploiting some basic properties of the root locus:
- **Property1**. The root locus has a number of branches equal N or M, whichever greater.
- A branch of the root locus refers to the locus of one of the roots of the characteristic equation $A(s) = 0$ as K varies from zero to infinity.
- Property 1 follows from $A(s) = Q(s) + KP(s)$, $P(s) = \prod_{i=1}^{M}\left(1 - \frac{s}{c_i}\right)$ and $Q(s) = \prod_{j=1}^{N}\left(1 - \frac{s}{d_j}\right)$

Properties of The Root Locus (2)

- **Property 2**. The root locus starts at the poles of the loop transfer function.
- For $K = 0$, the characteristic equation, $A(s) = Q(s) + KP(s)$, reduces to $Q(s) = 0$.
- The root of $Q(s) = 0$ are the same as the poles of $L(s)$, which proves that property 2 holds.
- **Property 3**. The root locus terminates on the zeros of the loop transfer function, including those zeros zeros which lie at infinity.
- As $K \to \infty$, the characteristic equation $A(s) = Q(s) + KP(s)$, reduces to $P(s) = 0$.
- The root of $P(s) = 0$ are the same as the zeros of $L(s)$, which proves that property 3 holds.

Properties of The Root Locus (3)

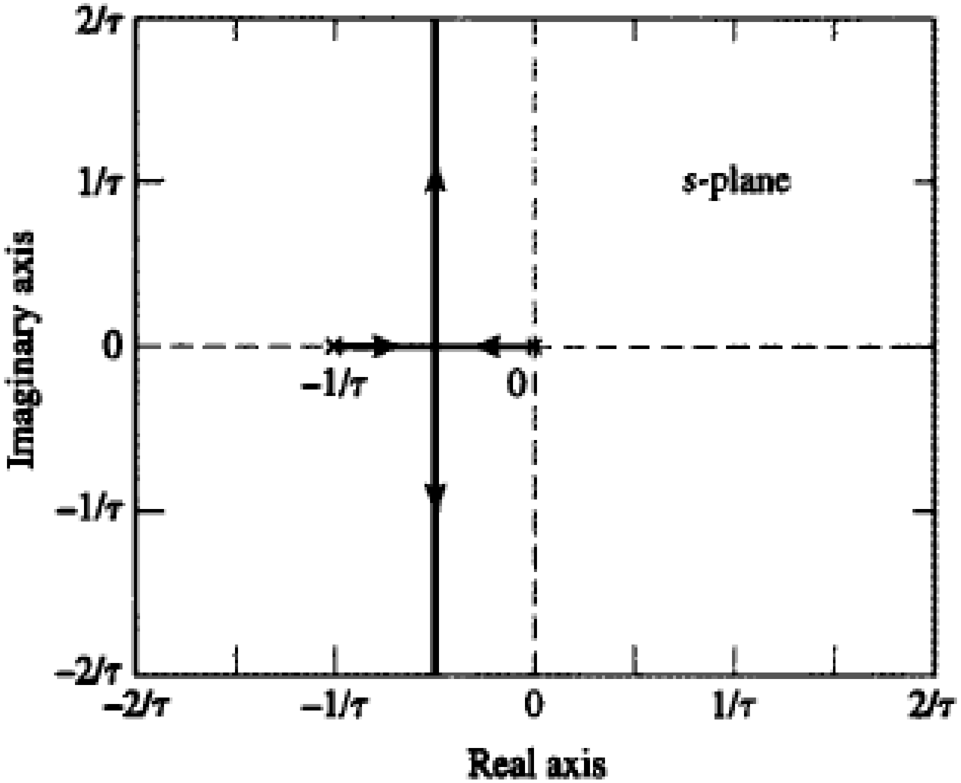
- **Property 4**. The root locus is symmetrical about the real axis of the s −plane.
- Either the poles and zeros of the $L(s)$ are real, or else they occur in complex-conjugate pairs.
- The roots of the $A(s) = Q(s) + KP(s)$, must be real or complex-conjugate pairs, from which property 4 holds.
- **Property 5**. As the loop gain K approaches infinity, the branches of the root locus tend to straight-line asymptotes with angle given by
  $\theta_k = \frac{(2k+1)\pi}{N-M}, \qquad k = 0,1,2,\ldots,|N-M|-1.$
- The asymptotes intersect at a common point on the real axis of the s −plane, the location which is defined by $\sigma_0 = \frac{\sum_{j=1}^{N} d_j - \sum_{i=1}^{M} c_i}{N-M}$.
- The intersection point $s = \sigma_0$ is called the **centroid** of the root locus.

Properties of The Root Locus (4)

- **Property 6**. The intersection points of the root locus with the imaginary axis of the s −plane, and the corresponding values of loop gain K, may be determined from the Routh-Hurwitz criterion.
- **Property 7**. The **breakaway** points, where the branches of the root locus intersect, must satisfy the condition $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$, where $L(s)$ is the loop transfer function.
- Equation $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$, but not sufficient condition for a **breakaway** point. In other words, all **breakaway** points satisfy $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$, but not all solutions of this equation are **breakaway** points.

Second-Order Feedback System

- Consider the 2nd-order feedback system, the loop transfer function of the system is $L(s) = \frac{K}{s(\tau s+1)}$.
- Find the breakaway point of the root locus of the system.
- Solution:
- $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$
- $\Rightarrow \frac{d}{ds}[s(\tau s + 1)] = 0$
- $\Rightarrow 2\tau s + 1 = 0.$
- $s = -\frac{1}{2\tau}.$

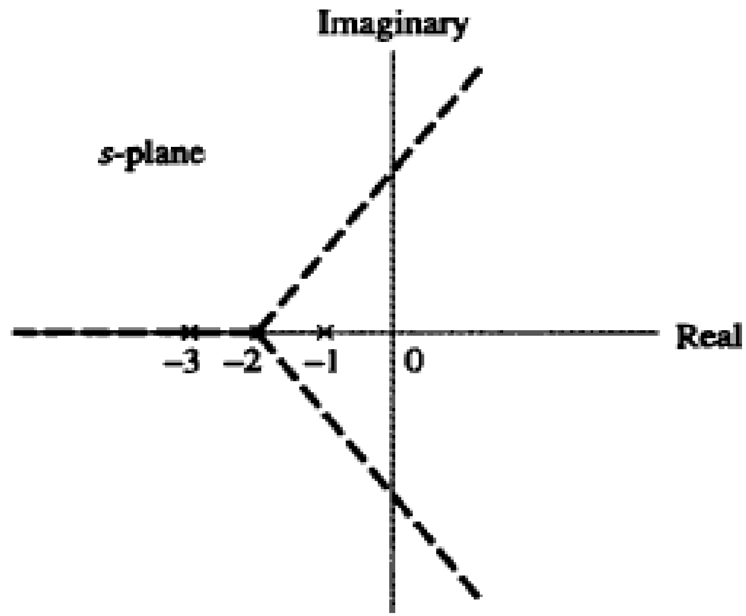


Linear Feedback Amplifier (1)

- Consider a linear feedback amplifier, the loop transfer function of the system is $L(s) = \frac{6K}{(s+1)(s+2)(s+3)}$.
- Sketch the root locus of this feedback amplifier.
- Solution:
- $L(s)$ has poles at $s = -1$, $s = -2$, and $s = -3$. All three zeros of $L(s)$ occur at infinity.
- The root locus has three branches that start from the poles and terminate at infinity.
- $\theta_k = \frac{(2k+1)\pi}{N-M}$, $k = 0,1,\ldots,|N - M| - 1$. ⇨ $\theta_k = \frac{(2k+1)\pi}{3}, k = 0,1,2$
- $\theta_0 = 60^0$, $\theta_1 = 180^0$, and $\theta_2 = 300^0$.
- The centroid: $\sigma_0 = \frac{-1-2-3}{3} = -2$.

Linear Feedback Amplifier (2)

- The asymptotes



- The characteristic equation:

$A(s) = Q(s) + KP(s)$

$A(s) = (s + 1)(s + 2)(s + 3) + 6K$

$A(s) = s^3 + 6s^2 + 11s + 6(K + 1)$

- The Routh array:

Row 3:	1	11
Row 2:	6	$6(K+1)$
Row 1:	$\frac{66-6(K+1)}{6} = 10 - K$	0
Row 0:	$6(K+1)$	0

- The critical value of K: from row 1 → $K = 10.$

Row 2, the auxiliary polynomial with $K = 10$: $6s^2 + 66 = 0.$

Linear Feedback Amplifier (3)

- The intersection points of the root locus with the imaginary axis are at $s = \pm j\sqrt{11}$.
- Using $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$

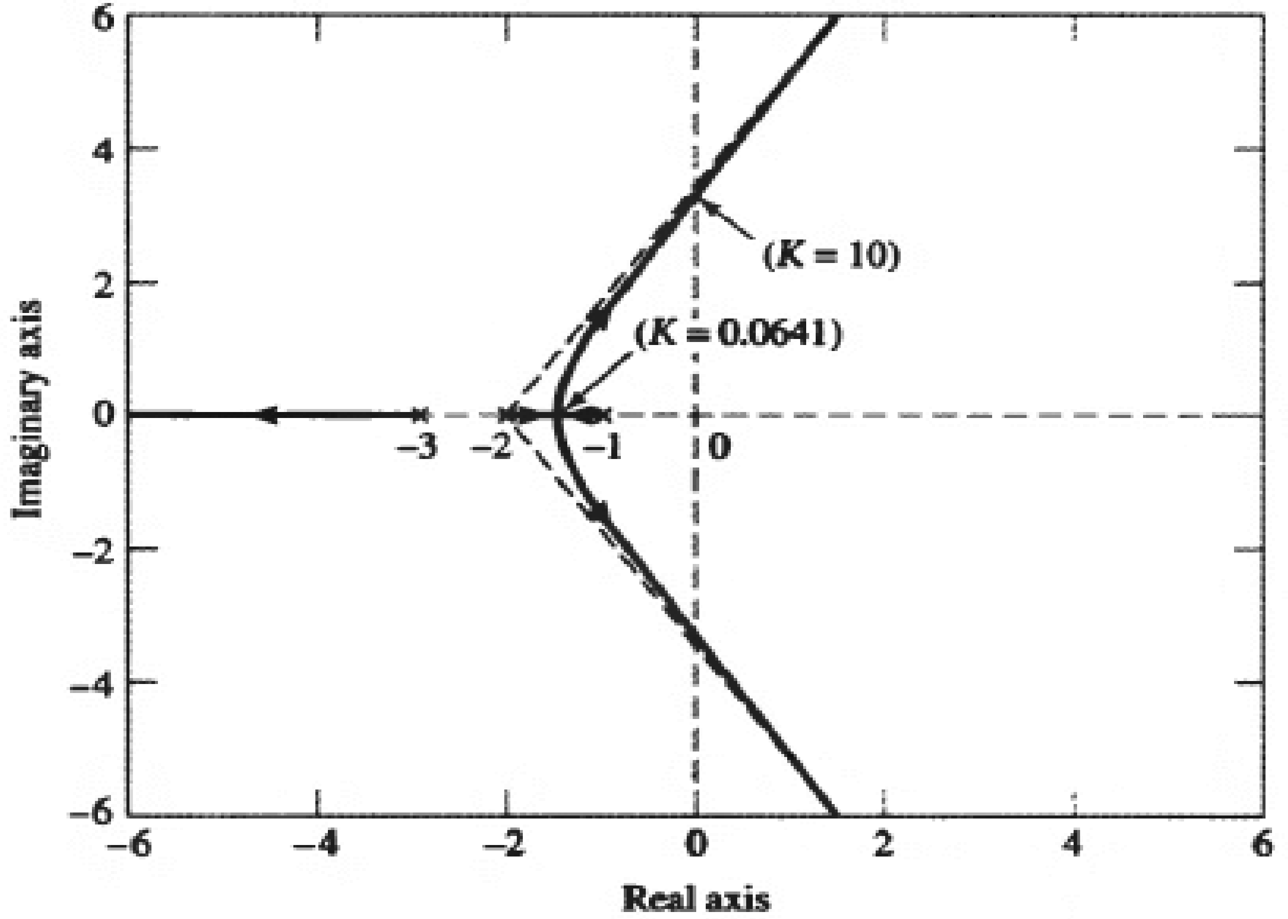
  $\Rightarrow \frac{d}{ds}(s^3 + 6s^2 + 11s + 6) = 3s^2 + 12s + 11 = 0$

  The root of this equation: $s = -1.423$ is on the root locus and is a breakaway point, and $s = -2.577$ is not on the root locus.
- For $s = -1.423$,

  $$K = \frac{|Q(s)|}{|P(s)|} = \frac{|-1.423+1|x|-1.423+2|x|-1.423+3|}{6} = 0.0641$$

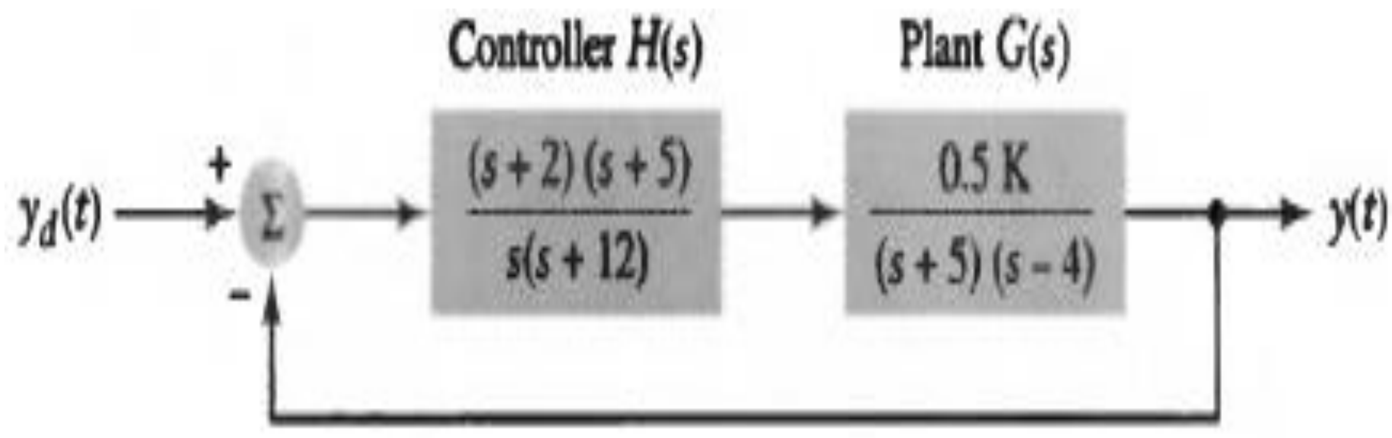
Linear Feedback Amplifier (4)

- Root locus of $L(s) = \frac{6K}{(s+1)(s+2)(s+3)}$



Unity Feedback System (1)

- Consider the unity-feedback control:



- The plant is unstable, with $G(s) = \frac{0.5K}{(s+5)(s-4)}$
- The controller has $H(s) = \frac{(s+2)(s+5)}{s(s+12)}$
- Sketch the root locus of the system, and determine the values of K for which the system is stable.
- Solution:
- The plant has 2 poles, at $s = -5$ and at $s = 4$.

Unity Feedback System (2)

- The controller has 2 zeros, at $s = -2$ and $s = -5$, and 2 poles, at $s = 0$ and at $s = -12$.
- The loop transfer function: $L(s) = G(s)H(s) = \frac{0.5K(s+2)}{s(s+12)(s-4)}$.
- The RL has 3 branches, 1 branch starts at the pole $s = -12$ and terminate at the zero $s = -2$. 2 branches start at the poles $s = 0$ and $s = 4$ and terminate at infinity.
- $\theta_k = \frac{(2k+1)\pi}{N-M}, \; k = 0,1, \dots, |N-M| - 1. \Rightarrow \theta_k = \frac{(2k+1)\pi}{2}, k = 0,1$
- $\theta_0 = 90^0, \theta_1 = 270^0$.
- The centroid: $\sigma_0 = \frac{(-12+0+4)-(-2)}{2} = -3$.
- The characteristic equation: $A(s) = Q(s) + KP(s)$

$A(s) = s^3 + 8s^2 + (0.5K - 48)s + K$

Unity Feedback System (3)

- The Routh array:

Row 3:	1	$0.5K - 48$
Row 2:	8	K
Row 1:	$\frac{8(0.5K-48)-K}{8} = \frac{3K-384}{8}$	0
Row 0:	K	0

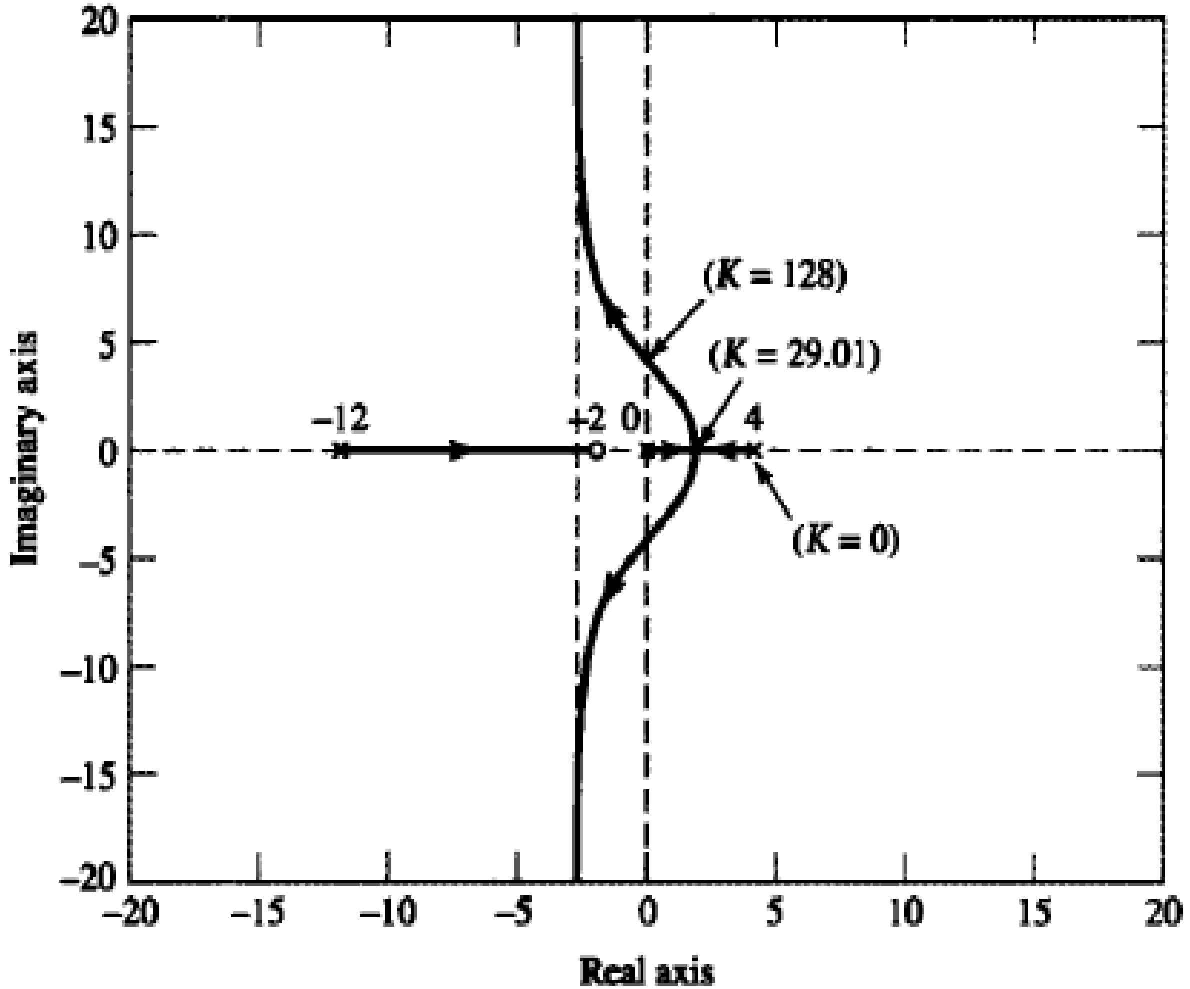
- The critical value of K: from row 1 → $K = 128$.

  Row 2, the auxiliary polynomial with $K = 128$: $8s^2 + 128 = 0$.
- The intersection points of the root locus with the imaginary axis are at $s = \pm j4$.
- Using $\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0$

  $\Rightarrow \frac{d}{ds}\left(\frac{s(s+12)(s-4)}{0.5K(s+2)}\right) = s^3 + 7s^2 + 16s - 48 = 0.$
- The breakaway point at $s = 1.6083$. The corresponding value of $K = 29.01$.

Unity Feedback System (4)

- Root locus of $L(s) = \frac{0.5K(s+2)}{s(s-4)(s+12)}$
- Unstable:
  $0 \leq K \leq 128.$
- Stable:
  $K > 128$



Reading Assignment

1. Signals and Systems; Simon Haykin, Barry Van Veen; 2nd edition, John Wiley & Sons, Inc. 2004. Chapter 9.
2. Signals and Systems; Alan V. Oppenheim, Alan S. Willsky, S. Hamid Nawab; 2nd edition, Prentice-Hall, 1997. Chapter 11.

- **End of Chapter 7. Introduction to Linear Feedback Systems.**