8th Material Subject: Joint Probability Distribution (Discrete)

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

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WELCOME

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1. **Joint Probability Mass Functions**
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LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Use joint probability mass functions to calculate probabilities**
2. **Calculate marginal and conditional probability distributions from joint probability distributions**
3. **Interpret and calculate covariance and correlations between random variables**

JOINT PROBABILITY MASS FUNCTION

For simplicity, we begin by considering random experiments in which only two random variables, called **Bi-variate**.The **Joint Probability Mass Function** of the discrete random variables **X** and **Y**, denoted as $\mathbf{f_{XY}(xy)}$, satisfies:

$$\mathbf{f_{XY}(xy) \geq 0} \tag{1}$$

$$\mathbf{\sum_{X}\sum_{Y} f_{XY}(xy) = 1} \tag{2}$$

$$\mathbf{f_{XY}(xy) = P(X = x \;\; \text{and} \;\; Y = y) = P(X = x) \cap P(Y = y)} \tag{3}$$

MARGINAL PROBABILITY MASS FUNCTION

The **Marginal Probability Mass Function** of the discrete random variables $\mathbf{X}$ and $\mathbf{Y}$, denoted as $\mathbf{f_X(x)}$ or $\mathbf{f_Y(y)}$, satisfies:

$$\mathbf{f_X(x) = P(X = x) = \sum_y f_{XY}(x, y)} \tag{4}$$

$$\mathbf{f_Y(y) = P(Y = y) = \sum_x f_{XY}(x, y)} \tag{5}$$

JOINT CUMULATIVE DISTRIBUTION FUNCTION

Remember that, for a random variable **X**, we define the CDF as $\mathbf{F_X(x) = P(X \leq x)}$. Now, if we have two random variables **X** and **Y** and we would like to study them jointly, we can define the **Joint Cumulative Function** as follows:

$$\mathbf{F_{XY}(x, y) = P(X \leq x \ \ and \ \ Y \leq y) = P(X \leq x) \cap P(Y \leq y)} \tag{6}$$

INDEPENDENT BIVARIATE

The random variable **X** and **Y** become **independent**, if only:

$$\mathbf{f_{XY}(x, y) = P(X = x) \cdot P(Y = y) = f_X(x) \cdot f_Y(y)} \tag{7}$$

or:

$$\mathbf{F_{XY}(x, y) = P(X \leq x) \cdot P(Y \leq y) = F_X(x) \cdot F_Y(y)} \tag{8}$$

COVARIANCE AND CORRELATION

When two random variables **X** and **Y** are **not independent**, it is frequently of interest to assess how strongly they are related to one another. The **Covariance** between two random variables **X** and **Y** equal to:

$$\mathbf{Cov}(\mathbf{XY}) = \mathbf{E}(\mathbf{XY}) - \mathbf{E}(\mathbf{X}) \cdot \mathbf{E}(\mathbf{Y}) \tag{9}$$

Where, the joint expectation should be:

$$\mathbf{E}(\mathbf{XY}) = \sum \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{f_{XY}}(\mathbf{xy}) \tag{10}$$

The **Correlation Coefficient** of **X** and **Y**, equal to:

$$\mathbf{Cor}(\mathbf{XY}) = \rho_{\mathbf{XY}} = \frac{\mathbf{Cov}(\mathbf{XY})}{\sigma_{\mathbf{x}} \cdot \sigma_{\mathbf{y}}} \tag{11}$$

EXAMPLE

Example: Will randomly pick two balls from a box that contains of three blue, two red and three green ball. If:

X = Random variables are declared elected as a blue ball

Y = Random variables are declared elected as a red ball

a. Determine range of random variable **X**

b. Determine range of random variable **Y**

c. Determine range of joint random variable **X** and **Y**

d. Determine the joint PMF of **X** and **Y**

e. Determine the marginal PMF of **X**

f. Determine the marginal PMF of **X**

g. Are the random variables **X** and **Y** independent?

h. If your answers are not independent, specify **Cov(XY)]** and ρ_{XY}

EXAMPLE

Answer:

a. Since **X** is a random variables declared elected as a blue ball, the range of **X** will:

$$\mathbf{R_X} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$$

b. Since **Y** is a random variables declared elected as a red ball, the range of **Y** will:

$$\mathbf{R_Y} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$$

c. And range of joint random variable **X** and **Y**

$$\mathbf{R_{XY}} = \{(\mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{2}), (\mathbf{1}, \mathbf{0}), (\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{0})\}$$

EXAMPLE

d. The joint PMF of **X** and **Y**

Suppose **X = 0** and **Y = 0**, meaning that no blue or red balls are drawn. The two balls are taken from green balls. So that:

$$\mathbf{f_{XY}(0,0) = \frac{3C0 \cdot 2C0 \cdot 3C2}{8C2} = \frac{3}{28}}$$

While **X = 0** and **Y = 1**, meaning that no blue drawn. The two balls are taken from 1 red and 1 green ball. So that:

$$\mathbf{f_{XY}(0,1) = \frac{3C0 \cdot 2C1 \cdot 3C1}{8C2} = \frac{6}{28}}$$

And **X = 0** and **Y = 2**, meaning that the two balls are taken from red. So that:

$$\mathbf{f_{XY}(0,2) = \frac{3C0 \cdot 2C2 \cdot 3C0}{8C2} = \frac{1}{28}}$$

In the same way, it can be calculated for $\mathbf{f_{XY}(1,0)}$, $\mathbf{f_{XY}(1,1)}$ and $\mathbf{f_{XY}(2,0)}$.

EXAMPLE

The joint PMF for **X** and **Y** shown below:

		Y		
		0	**1**	**2**
X	**0**	3/28	6/28	1/28
	1	9/28	6/28	0
	2	3/28		0

e. The marginal PMF of **X**

$$f_X(0) = \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28}$$

$$f_X(1) = \frac{9}{28} + \frac{6}{28} = \frac{15}{28} \quad \text{and} \quad f_X(2) = \frac{3}{28} = \frac{3}{28}$$

EXAMPLE

f. Determine the marginal PMF of **Y**

$$f_Y(0) = \frac{3}{28} + \frac{9}{28} + \frac{3}{28} = \frac{15}{28} \quad , \quad f_Y(1) = \frac{6}{28} + \frac{6}{28} = \frac{12}{28} \quad \text{and} \quad f_Y(2) = \frac{1}{28}$$

g. Random variables **X** and **Y** independent if $f_{XY}(xy) = f_X(x) \cdot f_Y(y)$

$$f_{XY}(0,0) = f_X(0) \cdot f_Y(0)$$

$$\frac{3}{28} \neq \frac{10}{28} \cdot \frac{15}{28}$$

So, Random variables **X** and **Y** are not independent

EXAMPLE

i. The **Cov(XY)]** and ρ_{XY} are

$$E(X) = \sum x \cdot f_X(x) = \left(0 \cdot \frac{10}{28}\right) + \left(1 \cdot \frac{15}{28}\right) + \left(2 \cdot \frac{3}{28}\right) = \frac{21}{28}$$

$$E(Y) = \sum y \cdot f_Y(y) = \left(0 \cdot \frac{15}{28}\right) + \left(1 \cdot \frac{12}{28}\right) + \left(2 \cdot \frac{1}{28}\right) = \frac{14}{28}$$

$$E(XY) = \sum x \cdot y \cdot f_{XY}(xy) = \left(0 \cdot 0 \cdot \frac{3}{28}\right) + \left(0 \cdot 1 \cdot \frac{6}{28}\right) + \left(0 \cdot 2 \cdot \frac{1}{28}\right) + \left(1 \cdot 0 \cdot \frac{9}{28}\right)$$

$$+ \left(1 \cdot 1 \cdot \frac{6}{28}\right) + (1 \cdot 2 \cdot 0) + \left(2 \cdot 0 \cdot \frac{3}{28}\right) + (2 \cdot 1 \cdot 0) + (2 \cdot 2 \cdot 0) = \frac{6}{28}$$

$$Cov(XY) = E(XY) - (E(X) \cdot E(Y)) = \frac{6}{28} - \left(\frac{21}{28} \cdot \frac{14}{18}\right) = -\frac{9}{56}$$

EXAMPLE

$$E(X^2) = \sum x^2 \cdot f_X(x) = \left(0^2 \cdot \frac{10}{28}\right) + \left(1^2 \cdot \frac{15}{28}\right) + \left(2^2 \cdot \frac{3}{28}\right) = \frac{27}{28}$$

$$Var(X) = \sigma_x^2 = E(X^2) - (E(X))^2 = \frac{27}{28} - \left(\frac{21}{28}\right)^2 = \frac{45}{112} \quad \text{then} \quad \sigma_x = \sqrt{\frac{45}{112}}$$

$$E(Y^2) = \sum y^2 \cdot f_Y(y) = \left(0^2 \cdot \frac{15}{28}\right) + \left(1^2 \cdot \frac{12}{28}\right) + \left(2^2 \cdot \frac{1}{28}\right) = \frac{16}{28}$$

$$Var(Y) = \sigma_y^2 = E(Y^2) - (E(Y))^2 = \frac{14}{28} - \left(\frac{21}{28}\right)^2 = \frac{9}{112} \quad \text{then} \quad \sigma_x = \sqrt{\frac{9}{112}}$$

$$\rho_{XY} = \frac{-\frac{9}{56}}{\sqrt{\frac{45}{112}} \cdot \sqrt{\frac{9}{112}}} = -0.894$$

Thank You