4th Material Subject: Conditional Probability & Bayes Theorem

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

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WELCOME

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LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Use Bayes theorem to calculate conditional probabilities**
2. **Interpret and calculate conditional probabilities of events**

CONDITIONAL PROBABILITY

If event **A** and **B** are **Dependent**, the probability of an event **B** under the knowledge that the outcome will be in event **A** is denoted as:

$$\mathbf{P}(\mathbf{B}|\mathbf{A}) = \frac{\mathbf{P}(\mathbf{A} \cap \mathbf{B})}{\mathbf{P}(\mathbf{A})} \quad , \quad \text{with } \mathbf{P}(\mathbf{A}) > \mathbf{0} \tag{1}$$

and this is called the **Conditional Probability** of **B** given **A**. Otherwise, the probability of an event **A** under the knowledge that the outcome will be in event **B** is denoted as:

$$\mathbf{P}(\mathbf{A}|\mathbf{B}) = \frac{\mathbf{P}(\mathbf{A} \cap \mathbf{B})}{\mathbf{P}(\mathbf{B})} \quad , \quad \text{with } \mathbf{P}(\mathbf{B}) > \mathbf{0} \tag{2}$$

and this is called the **Conditional Probability** of **A** given **B**.

Example: A digital communication channel has an error rate of 1 bit per every 1000 transmitted (**BER** $= \frac{1}{1000}$). Errors are rare, but when they occur, they tend to occur in bursts that affect many consecutive bits. If a single bit is transmitted. However, if the previous bit was in error because of the bursts, we might believe that the probability that the next bit will be in error is greater than is transmitted.

MULTIPLICATION AND TOTAL PROBABILITY RULES

The conditional probability definition in equation (**1**) and (**2**) can be rewritten to provide a formula known as the **Multiplication Rule** for probabilities.

$$\mathbf{P}(\mathbf{A} \cap \mathbf{B}) = \mathbf{P}(\mathbf{B}|\mathbf{A}) \cdot \mathbf{P}(\mathbf{A}) = \mathbf{P}(\mathbf{A}|\mathbf{B}) \cdot \mathbf{P}(\mathbf{B}) \tag{3}$$

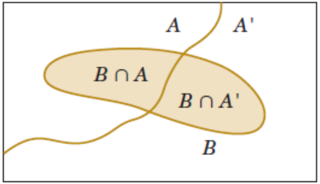


Figure 1: Partitioning an event into two mutually exclusive subsets, where **A** and **A**$'$ are partition of $\mathcal{U}$

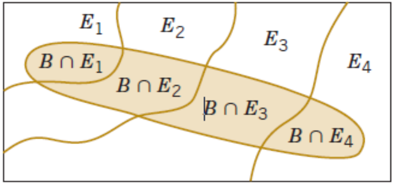


Figure 2: Partitioning an event into several mutually exclusive subsets, where $\mathbf{E_1}, \mathbf{E_2}, \cdots \mathbf{E_k}$ are partition of $\mathcal{U}$

MULTIPLICATION AND TOTAL PROBABILITY RULES

From Figure 1, shown **A** and $\mathbf{A}^{'}$ are partition of $\mathcal{U}$ and both are mutually exclusive events:

$$\mathbf{P}(\mathbf{A} \cup \mathbf{A}^{'}) = \mathbf{P}(\mathbf{A}) + \mathbf{P}(\mathbf{A}^{'}) \tag{4}$$

The following **Total Probability Rule** for event **B** is obtained:

$$\mathbf{B} = \mathbf{B} \cap \mathcal{U} = \mathbf{B} \cap (\mathbf{A} \cup \mathbf{A}^{'}) = (\mathbf{B} \cap \mathbf{A}) \cup (\mathbf{B} \cap \mathbf{A}^{'})$$

Therefor:

$$\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B} \cap \mathbf{A}) + \mathbf{P}(\mathbf{B} \cap \mathbf{A}^{'}) = \mathbf{P}(\mathbf{B}|\mathbf{A}) \cdot \mathbf{P}(\mathbf{A}) + \mathbf{P}(\mathbf{B}|\mathbf{A}^{'}) \cdot \mathbf{P}(\mathbf{A}^{'}) \tag{5}$$

MULTIPLICATION AND TOTAL PROBABILITY RULES

In general, see Figure 2, shown $\mathbf{E_1}, \mathbf{E_2}, \cdots \mathbf{E_k}$ are partition of $\mathcal{U}$ and they are mutually exclusive events:

$$\mathbf{P}(\mathbf{E_1} \cup \mathbf{E_2} \cup \cdots \cup \mathbf{E_k}) = \mathbf{P}(\mathbf{E_1}) + \mathbf{P}(\mathbf{E_2}) + \cdots + \mathbf{P}(\mathbf{E_k}) \tag{6}$$

The following **Total Probability Rule** for event **B** is obtained:

$$\mathbf{B} = \mathbf{B} \cap \mathcal{U} = \mathbf{B} \cap (\mathbf{E_1} \cup \mathbf{E_2} \cup \cdots \cup \mathbf{E_k}) = (\mathbf{B} \cap \mathbf{E_1}) \cup (\mathbf{B} \cap \mathbf{E_2}) \cup \cdots \cup (\mathbf{B} \cap \mathbf{E_k})$$

Therefor:

$$\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B} \cap \mathbf{E_1}) + \mathbf{P}(\mathbf{B} \cap \mathbf{E_2}) + \cdots + \mathbf{P}(\mathbf{B} \cap \mathbf{E_k})$$

$$\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B}|\mathbf{E_1}) \cdot \mathbf{P}(\mathbf{E_1}) + \mathbf{P}(\mathbf{B}|\mathbf{E_2}) \cdot \mathbf{P}(\mathbf{E_2}) + \cdots + \mathbf{P}(\mathbf{B}|\mathbf{E_k}) \cdot \mathbf{P}(\mathbf{E_k}) \tag{7}$$

MULTIPLICATION AND TOTAL PROBABILITY RULES

In general, a collection of event $\mathbf{E_1}, \mathbf{E_2}, \cdots \mathbf{E_k}$ such that $\mathbf{E_1} \cup \mathbf{E_2} \cup \cdot \cup \mathbf{E_k} = \mathcal{U}$ said to be exhaustive.

$$\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B} \cap \mathbf{E_1}) + \mathbf{P}(\mathbf{B} \cap \mathbf{E_2}) + \cdots + \mathbf{P}(\mathbf{B} \cap \mathbf{E_k})$$

$$\mathbf{P}(\mathbf{B}) = \mathbf{P}(\mathbf{B}|\mathbf{E_1}) \cdot \mathbf{P}(\mathbf{E_1}) + \mathbf{P}(\mathbf{B}|\mathbf{E_2}) \cdot \mathbf{P}(\mathbf{E_2}) + \cdots + \mathbf{P}(\mathbf{P}|\mathbf{E_k}) \cdot \mathbf{P}(\mathbf{E_k}) \tag{8}$$

The Equation **(5)** and **(8)** called the **Total Probability Rule**

EXAMPLE

A photocopying company using 3 machines with a percentage of each use 55 %, 30 % and 15 %. Each machine produces its own defects 8 %, 5 %, and 4 %. From the overall results of photocopying from all machines, one sheet is chosen randomly.

a. Draw a tree diagram for the above situation!

b. Calculate the probability selected result is **good**!

c. Calculate the probability selected result is **defect**!

d. If the selected result is **defect**, calculate the probability of those results coming from the second machine!

e. If the selected result is **good**,calculate the probability that the result will come from the first machine!

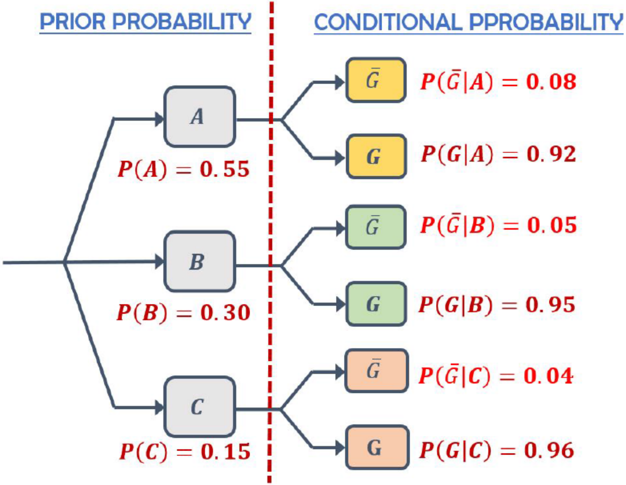
EXAMPLE

ANSWER: We first determine the possibility of any event will happen:

- **A** is event that states the selected paper from **first machine**
- **B** is event that states the selected paper from **second machine**
- **C** is event that states the selected paper from **third machine**
- **G** is event that states the selected paper in **good** condition
- $\overline{\mathbf{G}}$ is event that states the selected paper in **defect** condition

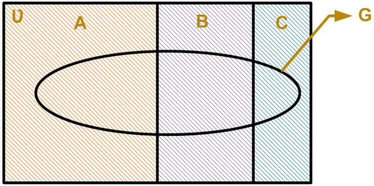
EXAMPLE

a. The tree diagram for the above situation



EXAMPLE

b. The probability selected result is **good** $\longrightarrow$ **Total Probability Rules**



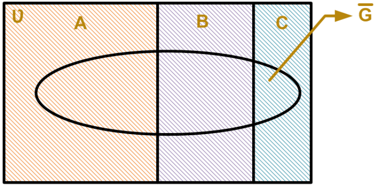
$$P(G) = P(G \cap A) + P(G \cap B) + P(G \cap C)$$

$$P(G) = P(G|A) \cdot P(A) + P(G|B) \cdot P(B) + P(G|C) \cdot P(C)$$

$$P(G) = 0.92 \cdot 0.55 + 0.95 \cdot 0.3 + 0.96 \cdot 0.15 = \frac{187}{200} = 0.935$$

EXAMPLE

c. The probability selected result is **defect** $\longrightarrow$ **Total Probability Rules**



$$P(\overline{G}) = P(\overline{G} \cap A) + P(\overline{G} \cap B) + P(\overline{G} \cap C)$$

$$P(\overline{G}) = P(\overline{G}|A) \cdot P(A) + P(\overline{G}|B) \cdot P(B) + P(\overline{G}|C) \cdot P(C)$$

$$P(G) = 0.08 \cdot 0.55 + 0.05 \cdot 0.3 + 0.05 \cdot 0.15 = \frac{13}{200} = 0.065$$

Or, simply we can calculate:

$$P(\overline{G}) = 1 - P(G) = 1 - \frac{187}{200} = \frac{13}{200} = 0.065$$

EXAMPLE

d. If the selected results are **defect**, the probability of those results coming from the second machine is $\longrightarrow$ **Conditional Probability**

$$P(B|\overline{G}) = \frac{P(C \cap \overline{G})}{P(\overline{G})} = \frac{P(B) \cdot P(\overline{G}|B)}{P(\overline{G})} = \frac{0.3 \cdot 0.05}{0.065} = \frac{3}{13} = 0.23$$

e. If the selected result is **good**, the probability that the result will come from the first machine is

$$P(A|G) = \frac{P(A \cap G)}{P(G)} = \frac{P(A) \cdot P(G|A)}{P(G)} = \frac{0.55 \cdot 0.92}{0.935} = \frac{46}{85} = 0.54$$

Thank You