10th Material Subject: Special Distribution of Discrete Random Variable

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

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WELCOME

TABLE OF CONTENTS:

1. **Bernoulli**
2. **Binomial**
3. **Hyper-geometric**
4. **Poisson**

LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Understand the assumptions for some common discrete probability distributions**
2. **Calculate discrete probability distribution to calculate probabilities in specific applications**
3. **Calculate probabilities and determine means and variances for some common discrete probability distributions**

BERNOULLI

A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli** trial.

- A Bernoulli distribution outcomes can only take two values, either **1 (success/yes)** or **0 (failure/no)**.
- It is usual to denote the probabilities by **p** and **q**, **p** as "**success**" and **q** as "**failure**", **p** and **q** must be non-negative, and their sum must be equal to one, **q = 1 - p**.
- Suppose **X** is Bernoulli distribution of probability success **p**, we can write:

$$\mathbf{X} \Rightarrow \mathbf{BER}(\mathbf{p}) \tag{1}$$

- The **Probability Mass Function** of Bernoulli Distribution:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} p & , x = 1 \\ q & , x = 0 \\ 0 & , \text{otherwise} \end{cases} \tag{2}$$

BERNOULLI

- The **Mean** of Bernoulli Distribution:

$$E(X) = \sum x \cdot f)X(x) = (1 \cdot p) + (0 \cdot 1) = p \tag{3}$$

- The **Variance** of Bernoulli Distribution:

$$E(X^2) = \sum x^2 \cdot f_X(x) = (1^2 \cdot p) + (0^2 \cdot 1) = p$$

$$Var(X) = \sigma_X^2 = E(X^2) - (E(X))^2 = p - p^2 = p(1 - p) = pq \tag{4}$$

- The **Moment Generation Function** of Bernoulli Distribution:

$$E(e^{tx}) = \sum e^{tx} \cdot f_X(x) = (e^{t \cdot 1} \cdot p) + (e^{t \cdot 0} \cdot q) = e^t p + q \tag{5}$$

BERNOULLI

Example: An experiment was made in choosing a students randomly out of 10 students (consist of 4 female and 6 male). If **X** is a random variable that is states the selection of female students, specify:

a. Probability Mass Function (PMF) of random variable **X**

b. Mean / Expected Value random variable **X**

c. Variance random variable **X**

d. Moment Generation Function random variable **X**

Answer: Probability for success **p** states if the student chosen is a female, so that $\mathbf{p} = \frac{4}{10}$, create the $\mathbf{q} = 1 - \mathbf{p} = \frac{6}{10}$.

a. Finally, the PMF can be written:

$$\mathbf{f_X(x)} = \begin{cases} \frac{4}{10} & , x = 1 \\ \frac{6}{10} & , x = 0 \\ 0 & , \text{otherwise} \end{cases} \tag{6}$$

BERNOULLI

b. Mean / Expected Value of random variable **X**

$$E(X) = p = \frac{4}{10}$$

c. Variance of random variable **X**

$$Var(X) = pq = \frac{4}{10} \cdot \frac{6}{10} = \frac{24}{100}$$

d. Moment Generation Function of random variable **X**

$$E(e^{tx}) = e^{t}p + q = \frac{4}{10}e^{t} + \frac{6}{10}$$

BINOMIAL

Binomial is a random experiment that consists of **n** Bernoulli trials.

- A Binomial distribution outcomes can only take two values, either **1 (success/yes)** or **0 (failure/no)**.
- It is usual to denote the probabilities by **p** and **q**, **p** as "**success**" and **q** as "**failure**", **p** and **q** must be non-negative, and their sum must be equal to one, **q = 1 - p**.
- Suppose **X** is Binomial distribution with **n** trial and probability success **p**, we can write:

$$\mathbf{X} \Rightarrow \mathbf{BIN}(\mathbf{n}, \mathbf{p}) \tag{7}$$

- The **Probability Mass Function** of Binomial Distribution:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} C_x^n \cdot p^x \cdot q^{n-x} & , x = 0, 1, 2, \cdots, n \\ 0 & , \textbf{otherwise} \end{cases} \tag{8}$$

BINOMIAL

- The **Mean** of Binomial Distribution:

$$E(X) = np \tag{9}$$

- The **Variance** of Binomial Distribution:

$$Var(X) = npq \tag{10}$$

- The **Moment Generation Function** of Binomial Distribution:

$$E(e^{tx}) = (e^{t}p + q)^{n} \tag{11}$$

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Example: An experiment was carried out 8 times. The experiment was made in choosing a students randomly out of 10 students (consist of 4 female and 6 male). If **X** is a random variable that is states the selection of female students, specify:

a. Probability Mass Function (PMF) of random variable **X**

b. Mean / Expected Value random variable **X**

c. Variance random variable **X**

d. Moment Generation Function random variable **X**

Answer: The experiment was carried out 8 times, then $\mathbf{n = 8}$. Probability for success **p** states if the student chosen is a female, so that $\mathbf{p = \frac{4}{10}}$, create the $\mathbf{q = 1 - p = \frac{6}{10}}$.

a. Finally, the PMF can be written:

$$\mathbf{f_X(x)} = \begin{cases} C_x^8 \cdot \left(\frac{4}{10}\right)^x \cdot \left(\frac{6}{10}\right)^{8-x} & , x = 0, 1, 2, 3, 4, 5, 6, 7, 8 \\ 0 & , \textbf{otherwise} \end{cases} \tag{12}$$

BINOMIAL

b. Mean / Expected Value of random variable **X**

$$E(X) = n \cdot p = 8 \cdot \frac{4}{10} = \frac{32}{10}$$

c. Variance of random variable **X**

$$Var(X) = n \cdot p \cdot q = 8 \cdot \frac{4}{10} \cdot \frac{6}{10} = \frac{192}{100}$$

d. Moment Generation Function of random variable **X**

$$E(e^{tx}) = (e^t p + q)^n = \left(\frac{4}{10}e^t + \frac{6}{10}\right)^8$$

HYPERGEOMETRIC

Hypergeometric experiments have the same characteristics with binomial. The difference is hypergeometric states the number of successful events in samples taken as WOR **(Without Replacement)** samples.

- A set of **N** objects contains **K** objects classified as successes **N** $-$ **K** objects classified as failures. A sample of size **n** objects is selected randomly (without replacement) from the **N** objects where **K** $\leq$ **N** and **n** $\leq$ **N**. The random variable **X** that equals the number of successes in the sample is a hypergeometric random.

$$\mathbf{X} \Rightarrow \mathbf{HYP}(\mathbf{N}, \mathbf{n}, \mathbf{K}) \tag{13}$$

- The **Probability Mass Function** of Hypergeometric Distribution:

$$\mathbf{f_X(x)} = \begin{cases} \frac{C_x^K \cdot C_{n-x}^{N-K}}{C_n^N} & , x = 0, 1, 2, \cdots, n \\ 0 & , \textbf{otherwise} \end{cases} \tag{14}$$

(15)

HYPERGEOMTERIC

- The **Mean** of Hypergeometric Distribution:

$$E(X) = \frac{n \cdot K}{N} = n \cdot p \tag{16}$$

- The **Variance** of Hypergeometric Distribution:

$$Var(X) = n \cdot p \cdot q \cdot \left(\frac{N - n}{N - 1}\right) \tag{17}$$

Example: In an observation tube there are 16 bacteria, 6 among them are good bacteria that are able to change substances food becomes nutrition and vitamins. A laboratory assistant will take 5 bacteria randomly and Without Replacement. If **X** states that bad bacteria have been picked, determine:

a. Probability Mass Function (PMF) random variable **X**

b. Mean / Expected Value random variable **X**

c. Variance random variable **X**

d. $f_X(0)$, $f_X(1)$, $f_X(2)$, $f_X(3)$, $f_X(4)$ and $f_X(5)$

Answer: **X** is a random variable that states the selection of bad bacteria, there are 16 total objects consisting of $K = 10$ and $N - K = 6$, then $p = \frac{K}{N} = \frac{10}{16} = \frac{5}{8}$ and $q = 1 - p = \frac{3}{8}$, so that:

$$X \Rightarrow HYP(16, 5, 10) \tag{18}$$

HYPERGEOMTERIC

a. Probability Mass Function (PMF) random variable **X**

$$\mathbf{f_X(x)} = \begin{cases} \frac{C_x^{10} \cdot C_{5-x}^{6}}{C_5^{16}} & , x = 0, 1, 2, 3, 4, 5 \\ 0 & , \text{otherwise} \end{cases}$$

b. Mean / Expected Value random variable **X**

$$\mathbf{E(X) = \frac{n \cdot K}{N} = n \cdot p = 5 \cdot \frac{5}{8} = \frac{25}{8}}$$

c. Variance random variable **X**

$$\mathbf{Var(X) = n \cdot p \cdot q \cdot \left(\frac{N - n}{N - 1}\right) = 5 \cdot \frac{5}{8} \cdot \frac{3}{8} \cdot \frac{11}{15} = \frac{55}{64}}$$

HYPERGEOMTERIC

d. $f_X(0)$, $f_X(1)$, $f_X(2)$, $f_X(3)$, $f_X(4)$ and $f_X(5)$

$$f_X(0) = \frac{C_0^{10} \cdot C_{5-0}^{6}}{C_5^{16}} = \frac{1 \cdot 6}{4368} = \frac{6}{4368}$$

$$f_X(1) = \frac{C_1^{10} \cdot C_{5-1}^{6}}{C_5^{16}} = \frac{10 \cdot 15}{4368} = \frac{150}{4368}$$

$$f_X(2) = \frac{C_2^{10} \cdot C_{5-2}^{6}}{C_5^{16}} = \frac{45 \cdot 20}{4368} = \frac{900}{4368}$$

$$f_X(3) = \frac{C_3^{10} \cdot C_{5-3}^{6}}{C_5^{16}} = \frac{120 \cdot 15}{4368} = \frac{1800}{4368}$$

$$f_X(4) = \frac{C_4^{10} \cdot C_{5-4}^{6}}{C_5^{16}} = \frac{210 \cdot 6}{4368} = \frac{1260}{4368}$$

$$f_X(5) = \frac{C_5^{10} \cdot C_{5-5}^{6}}{C_5^{16}} = \frac{252 \cdot 1}{4368} = \frac{252}{4368}$$

POISSON

Poisson first introduced by Sim eon-Denis Poisson(1781 - 1840). The Poisson distribution states the number of discrete events (sometimes also called "**arrivals**") that occur during certain time intervals.

- The random variable **X** that equals the number of events in a Poisson process is a poisson random variable with parameter $\lambda > \mathbf{0}$:

$$\mathbf{X} \Rightarrow \mathbf{POI}(\lambda) \tag{19}$$

- The **Probability Mass Function** of Poisson Distribution:

$$\mathbf{f_X(x)} = \begin{cases} \frac{e^{-\lambda}.\lambda^x}{x!} & , x = 0, 1, 2, \cdots \\ 0 & , \textbf{otherwise} \end{cases} \tag{20}$$

POISSON

- The **Mean** of Poisson Distribution:

$$E(X) = \lambda \tag{21}$$

- The **Variance** of Poisson Distribution:

$$Var(X) = \lambda \tag{22}$$

- The **Moment Generation Function** of Poisson Distribution:

$$E(e^{tx}) = e^{\lambda(e^{t}-1)} \tag{23}$$

POISSON

Example: Phone calls that enter the complaint center meet the Poisson process with an entrance average of 2.5 call / minute. Calculate:

a. Probability Mass Function (PMF) of random variable **X**

b. Mean / Expected Value of random variable **X**

c. Variance of random variable **X**

d. Moment Generation Function of random variable **X**

e. The probability of no more than 3 incoming calls in one minutes

f. The probability of less than 5 incoming calls in one minutes

POISSON

Answer:

a. Probability Mass Function (PMF) random variable **X**

$$\mathbf{f_X(x)} = \begin{cases} \frac{e^{-2.5} \cdot 2.5^{x}}{x!} & , x = 0, 1, 2, 3, \cdots \\ 0 & , \text{otherwise} \end{cases}$$

b. Mean / Expected Value random variable **X**

$$\mathbf{E(X)} = \lambda = \mathbf{2.5}$$

c. Variance random variable **X**

$$\mathbf{Var(X)} = \lambda = \mathbf{2.5}$$

POISSON

d. Moment Generation Function of random variable **X**

$$E(e^{tx}) = e^{2.5(e^t - 1)} \tag{24}$$

e. The probability of no more than 3 incoming calls in one minutes

$$P(X \leq 3 = f_X(0) + f_X(1) + f_X(2) + f_X(3)$$

$$P(X \leq 3 = \frac{e^{-2.5} \cdot 2.5^0}{0!} + \frac{e^{-2.5} \cdot 2.5^1}{1!} + \frac{e^{-2.5} \cdot 2.5^2}{2!} + \frac{e^{-2.5} \cdot 2.5^3}{3!} = 0.7576$$

POISSON

f. The probability of less than 5 incoming calls in one minutes

$$P(X \leq 4 = f_X(0) + f_X(1) + f_X(2) + f_X(3) + f_X(4)$$

$$P(X \leq 3 = \frac{e^{-2.5} \cdot 2.5^0}{0!} + \frac{e^{-2.5} \cdot 2.5^1}{1!} + \frac{e^{-2.5} \cdot 2.5^2}{2!} + \frac{e^{-2.5} \cdot 2.5^3}{3!} + \frac{e^{-2.5} \cdot 2.5^4}{4!} = 0.8912$$

Thank You