7th Material Subject: Continuous Univariate Random Variable

Undergraduate of Telecommunication Engineering

MUH1F3 - PROBABILITY AND STATISTICS

Telkom University
Center of eLearning & Open Education Telkom University
Jl. Telekomunikasi No.1, Bandung - Indonesia
http://www.telkomuniversity.ac.id

Lecturer: Nor Kumalasari Caecar Pratiwi, S.T., M.T. (caecarnkcp@telkomuniversity.ac.id)

Telkom University

السلام عليكم ورحمة الله وبركاته

WELCOME

TABLE OF CONTENTS:

LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Determine probabilities from probability density functions and the reverse**
2. **Determine probabilities and probability density functions from cumulative distribution functions and the reverse**
3. **Calculate means and variances for continuous random variables**

PROBABILITY DISTRIBUTIONS AND PROBABILITY DENSITY FUNCTIONS

A **Probability Density Function**, denoted by $\mathbf{f(x)}$ can be used to describe the probability distribution of a continuous random variable **X**. The probability that **X** is between **a** and **b** is determined as the integral of $\mathbf{f(x)}$ from **a** to **b**. For continuous random variable **X**, a **Probability Density Function** is a function such that:

$$\mathbf{f(x) \geq 0} \tag{1}$$

$$\int_{-\infty}^{\infty} \mathbf{f(x)\,dx = 1} \tag{2}$$

$$\mathbf{P(a \leq X \leq b)} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)\,dx} = \text{area under f(x) from a to b} \tag{3}$$

If **X** is a continuous random variable, for any $\mathbf{x_1}$ and $\mathbf{x_1}$

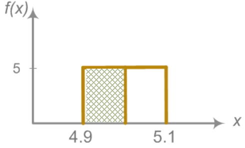
$$\mathbf{P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)} = \int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{f(x)\,dx} \tag{4}$$

Example: Let the continuous random variable **X** denote the current measured in a thin copper wire in milliamperes (mA). Assume that the range of **X** is $\mathbf{4.9 - 5.1mA}$, and the probability density function of **X** is:

$$\mathbf{f(x)} = \begin{cases} 5 & \text{, for } 4.9 \leq x \leq 5.1 \\ 0 & \text{, otherwise} \end{cases} \quad (5)$$

What is the probability that a current measurement is less than 5 mA?

Answer: The probability density function is shown in Figure below. The shaded area in indicates the probability $\mathbf{P(X < 5)}$.



$$\mathbf{P(X < 5) = \int_{4.9}^{5} f(x)dx = \int_{4.9}^{5} 5\ dx}$$

$$\mathbf{P(X < 5) = 5x \Big|_{4.9}^{5} = 5(5 - 4.9) = 0.5}$$

CUMULATIVE DISTRIBUTION FUNCTION

An alternate method for describing a random variables probability distribution is with **Cumulative Distribution Functions** such as $\mathbf{P(X \leq x)}$. The cumulative distribution function of a continuous random variable $\mathbf{X}$, denoted as **F(x)**, is:

$$\mathbf{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du} \quad \text{, for} \quad -\infty < \mathbf{x} < \infty \tag{6}$$

$$\mathbf{0 \leq F(x) \leq 1} \tag{7}$$

$$\text{If} \quad \mathbf{x \leq y} \quad \text{then} \quad \mathbf{F(x) \leq F(y)} \tag{8}$$

If the Cumulative Distribution Functions $\mathbf{F(x)}$ given, the probability density function $\mathbf{f(x)}$ will bi simply calculate:

$$\mathbf{f(x) = \frac{d}{dx} F(x)} \tag{9}$$

CUMULATIVE DISTRIBUTION FUNCTION

Example: Calculate the Cumulative distribution function for continuous random variable **X** denote the current measured in a thin copper wire in milliamperes (mA). Assume that the range of **X** is $\mathbf{4.9 - 5.1mA}$, and the probability density function of **X** is:

$$\mathbf{f(x)} = \begin{cases} 5 & \text{, for } 4.9 \leq x \leq 5.1 \\ 0 & \text{, otherwise} \end{cases}$$

Answer: The CDF $\mathbf{F(x) = 0}$ for $\mathbf{x < 4.9}$ and:

$$\mathbf{F(x) = P(X < x) = \int_{4.9}^{x} f(u)\,du = \int_{4.9}^{x} 5\,du = 5u \Big|_{4.9}^{x} = 5(x - 4.9) = 5x - 24.5}$$

Finally, for $\mathbf{x \geq 5.1}$ the CDF $\mathbf{F(x) = 1}$

CUMULATIVE DISTRIBUTION FUNCTION

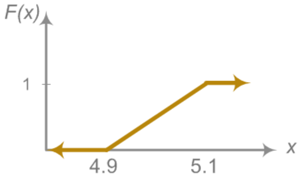


Figure 1: The Graph of Cumulative Distribution Function

MEAN & VARIANCE OF A CONTINUOUS RANDOM VARIABLE

The **MEAN** and **VARIANCE** can also be defined for a continuous random variable. Integration replaces summation in the discrete definitions.

- The **Mean** or expected value of the continuous random variable **X**, denoted as μ or **E(X)**, is:

$$\mu = \mathbf{E}(\mathbf{X}) = \int_{-\infty}^{\infty} \mathbf{x} \cdot \mathbf{f}(\mathbf{x})\, \mathbf{dx} \tag{10}$$

- The **Variance** of the discrete random variable **X**, denoted as σ^2 or **Var(X)**, is:

$$\sigma^2 = \mathbf{Var}(\mathbf{X}) = \mathbf{E}(\mathbf{x} - \mu)^2 = \int_{-\infty}^{\infty} (\mathbf{x} - \mu)^2 \cdot \mathbf{f}(\mathbf{x}) = \mathbf{E}(\mathbf{X}^2) - (\mathbf{E}(\mathbf{X})^2) \tag{11}$$

- The **Standard Deviation** of the discrete random variable **X**, denoted as σ or, is:

$$\sigma = \sqrt{\sigma^2} \tag{12}$$

MEAN & VARIANCE OF A DISCRETE RANDOM VARIABLE

The **MEAN, VARIANCE** and **STANDARD DEVIATION** for Example 1 before are:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{4.9}^{5.1} x \cdot 5\,dx = \frac{5x^2}{2}\Big|_{4.9}^{5.1} = \frac{5(5.1^2 - 4.9^2)}{2} = 5$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{4.9}^{5.1} x^2 \cdot 5\,dx = \frac{5x^3}{3}\Big|_{4.9}^{5.1} = \frac{5(5.1^3 - 4.9^3)}{3} = 25.003$$

$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = 25.0033 - (5)^2 = 0.0033 \quad \text{and} \quad \sigma = \sqrt{\sigma^2} = \sqrt{0.0033} = 0.057$$

CONTINUOUS RANDOM VARIABLE TRANSFORMATION

Suppose that $\mathbf{X}$ is a continuous random variable with probability density function $\mathbf{f_X(x)}$, the function $\mathbf{Y = g(x)}$ is a one-to-one transformation between the values of $\mathbf{Y}$ and $\mathbf{X}$, so that the equation $\mathbf{Y = g(x)}$ can be uniquely solved for $\mathbf{x}$ in terms of $\mathbf{y}$. The probability distribution of $\mathbf{Y}$ is:

$$\mathbf{f_Y(y) = f_X(g^{-1}(y)) \cdot |J|} \tag{13}$$

where $\mathbf{J = \frac{d}{dy} g^{-1}(y)}$ is called the **Jacobian Transform**.

CONTINUOUS RANDOM VARIABLE TRANSFORMATION

Example: Let **X** be a continuous random variable with probability distribution:

$$\mathbf{f_X(x)} = \begin{cases} \frac{x}{8} & , 0 \leq x \leq 4 \\ 0 & , \text{otherwise} \end{cases} \tag{14}$$

Find the probability distribution of $\mathbf{Y = 2X + 4}$.

Example: The inverse solution is $\mathbf{g^{-1}(y) = \frac{Y-4}{2}}$ and from this, we find the Jacobian to be $\mathbf{J = \frac{d}{dy}\left(\frac{Y-4}{2}\right) = \frac{1}{2}}$. Therefore, from Equation 31, the probability distribution of **Y** is:

$$\mathbf{f_Y(y)} = \begin{cases} f_X(g^{-1}(y)) \cdot |J| = \frac{\frac{Y-4}{2}}{8} \cdot \frac{1}{2} = \frac{Y-4}{32} & , \text{for } 4 \leq y \leq 12 \\ 0 & , \text{otherwise} \end{cases}$$

Thank You