Deret Mac Laurin, $z_0 = 0$

Taylor, $z_0 \neq 0$

1. Diketahui fungsi

$$f(z) = \frac{1}{2+z}$$

a. Gambarkan daerah keanalitikan/kekonvergenan dari $f(z)$, apabila dideretkan sebagai Deret Mac Laurin ! (5 poin)

b. Tentukan Deret Mac Laurin dari $f(z)$! (7 poin)

c. Gambarkan daerah keanalitikan/kekonvergenan dari $f(z)$, apabila dideretkan sebagai Deret Taylor di $z = 1$! (5 poin)

d. Tentukan Deret Taylor dari $f(z)$ yang dideretkan di $z = 1$! (8 poin)

a. $f(z) = \dfrac{1}{2+z} \longrightarrow \boxed{\dfrac{1}{1-kz}}$

$= \dfrac{1}{2} \cdot \boxed{\dfrac{1}{1+\frac{z}{2}}}$

$= \dfrac{1}{2} \cdot \boxed{\dfrac{1}{1-\boxed{\left(-\frac{z}{2}\right)}}} \rightarrow kz$

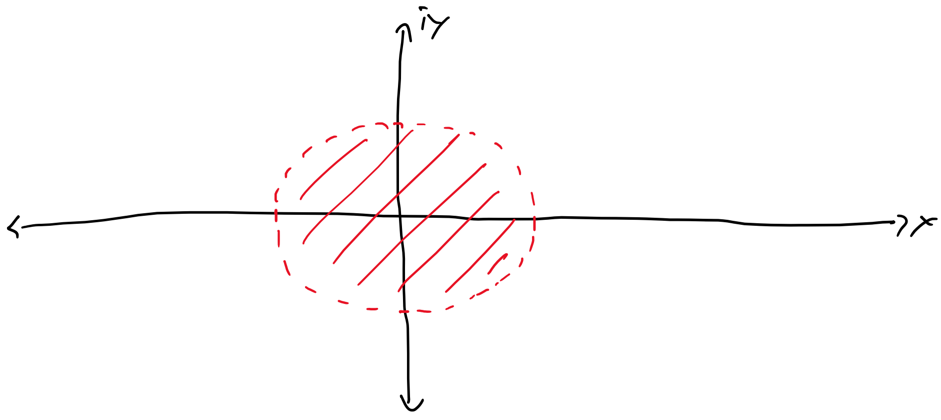
$|kz| < 1$

$\left|-\frac{z}{2}\right| < 1$

$\left|\frac{1}{2}\right| |z| < 1$

$|z| < |2|$

$|z| < 2$



b. $f(z) = \frac{1}{2+z} = \overset{a}{\frac{1}{2}} \, \frac{1}{1-(-\frac{z}{2})} \rightarrow \sum_{n=0}^{\infty} a.(kz)^n$

$= \frac{1}{2}(kz)^0 + \frac{1}{2}(kz)^1 + \frac{1}{2}(kz)^2 + \text{---}$

$= \frac{1}{2} + \frac{1}{2}\left(-\frac{z}{2}\right) + \frac{1}{2}\left(-\frac{z}{2}\right)^2 + \text{---}$

$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{z}{2}\right)^n$

$= \sum_{n=0}^{\infty} \boxed{\frac{1}{2}.(-1)^n.\frac{z^n}{2^n}} \rightarrow$ $2^n . 2 = 2^n . 2^1$
$= 2^{n+1}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^{n+1}}$

c. $f(z) = \frac{1}{2+z} \Rightarrow \boxed{\frac{1}{2+(z-z_0)}, \quad z_0 = 0 \rightarrow \text{Maclaurin} \quad z_0 \neq 0 \rightarrow \text{Taylor}}$ Bentuk umum

$= \frac{1}{2+(z-1)}$

$= \frac{1}{1+z}$

$= \frac{1}{1-(-z)}$

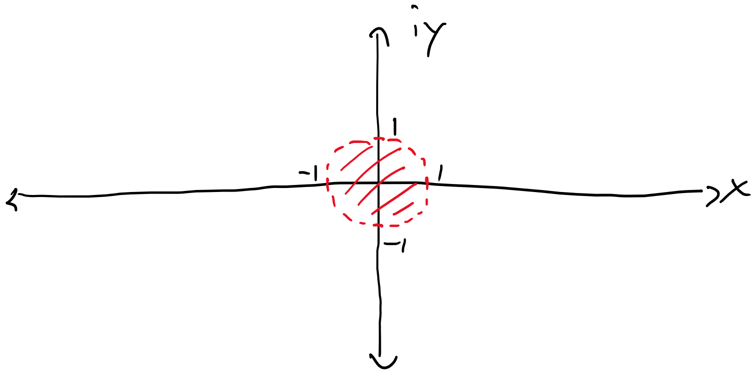
$|-z| < 1$

$|z| < 1$

$\left|\frac{2}{z}\right| < 1$

$|2| < |z|$

$|z| > 2$



d. $$f(z) = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n$$

$$f(z) = \frac{1}{2+z} \quad ; \quad z_0 = 2i \qquad (a-ib)(a+ib) = a^2 + b^2$$

$$= \frac{1}{2+(z-2i)}$$

$$= \frac{1}{(2+z)-2i} \times \frac{(2+z)+2i}{(2+z)+2i}$$

$$= \frac{(2+z)-2i}{(2+z)^2 + 2^2}$$

$$= \frac{z+2-2i}{4+4z+z^2+4}$$

$$= \frac{z+2-2i}{z^2+4z+8} \qquad \boxed{\frac{1}{1-kz}} \qquad z_{1,2} = \frac{-4 \pm \sqrt{-16}}{2}$$

$$z_{1,2} = -2 \pm -2i$$

$$= \frac{z+2-2i}{(z-(-2-2i))(z-(-2+2i))}$$

$$= \frac{A}{z-(-2-2i)} + \frac{B}{(z-(-2+2i))}$$

2. Diketahui

$$f(z) = \frac{2z-1}{(z^2+16)(z-i)^2}$$

a. Tentukan semua titik singular dari $f(z)$ dan jenis kutub/ordenya ! (6 poin)

b. Hitunglah residu $f(z)$ pada masing-masing titik singularnya ! (12 poin)

c. Misalkan lintasan C : $|z| = 1,5$ dengan arah positif atau berlawanan jarum jam. Berdasarkan hasil perhitungan 2.a dan 2.b hitunglah integral berikut ! (7 poin)

$$\oint_C \frac{2z-1}{(z^2+16)(z-i)^2}\,dz$$

$$\boxed{\begin{aligned}(a-b)(a+b) &= a^2-b^2\\ (a-bi)(a+bi) &= a^2+b^2\end{aligned}}$$

a. $f(z) = \dfrac{2z-1}{(z^2+16)(z-i)^2}$

$= \dfrac{2z-1}{(z+4i)(z-4i)(z-i)^2}$

$z = -4i$, orde 1

$z = 4i$, orde 1

$z = i$, orde 2

b. $z = -4i \rightarrow (z+4i)$

$g(z) = \dfrac{2z-1}{(z-4i)(z-i)^2}$ → Pers. Residu ←

$g(-4i) = \dfrac{2(-4i)-1}{(-4i-4i)(-4i-i)^2}$

$\text{Res}_{z=-4i} = \dfrac{-8i-1}{-8i(-5i)^2}$

$i^2 = -1$

$(-5)^2 i^2$

$25.(-1)$

$= \dfrac{-8i-1}{-8i(-25)}$

$= \dfrac{-8i-1}{200i} \times \dfrac{-200i}{-200i}$

$= \dfrac{-\cancel{1600}^{8} + \cancel{200}^{1} i}{\cancel{40.000}_{200}}$

$\text{Res}_{z=-4i} = \dfrac{-8+i}{200}$ → Nilai Residu ←

$z = 4i \rightarrow (z-4i)$

$g(z) = \dfrac{2z-1}{(z+4i)(z-i)^2}$

$g(4i) = \dfrac{2(4i)-1}{(4i+4i)(4i-i)^2}$

$= \dfrac{8i-1}{8i(3i)^2}$

$= \dfrac{8i-1}{8i(-9)}$

$= \dfrac{8i-1}{-72i} \cdot \dfrac{72i}{72i}$

$\text{Res}_{z=4i} = \dfrac{\cancel{72}(-8-i)}{72^{\cancel{2}}}$

$\text{Res}_{z=4i} = \dfrac{-8-i}{72}$

$z = i \rightarrow (z-i)^2$

$$\boxed{\mathrm{Res}_{z=z_0} = z_0 \cdot \frac{1}{(n-1)!} \cdot g^{n-1}(z) \Big|_{z=z_0}}$$

n = orde

$$= i \cdot \frac{1}{(2-1)!} \cdot g^{2-1}(z) \Big|_{z=i}$$

$$= i \cdot \frac{1}{1!} \cdot \frac{d}{dz}\left(\frac{2z-1}{z^2+16}\right) \Big|_{z=i}$$

$$= i \cdot \frac{2.(z^2+16)-(2z-1).2z}{(z^2+16)^2} \Big|_{z=i}$$

$$= i \cdot \frac{2z^2+32-4z^2+2z}{(z^2+16)^2} \Big|_{z=i}$$

$$= \frac{-2z^2+2z+32}{(z^2+16)^2} \, i \Big|_{z=i}$$

$$= \frac{-2i^2+2i+32}{(i^2+16)^2} \, i$$

$$= \frac{2+2i+32}{15^2} \, i$$

$$= \frac{34+2i}{225} \, i$$

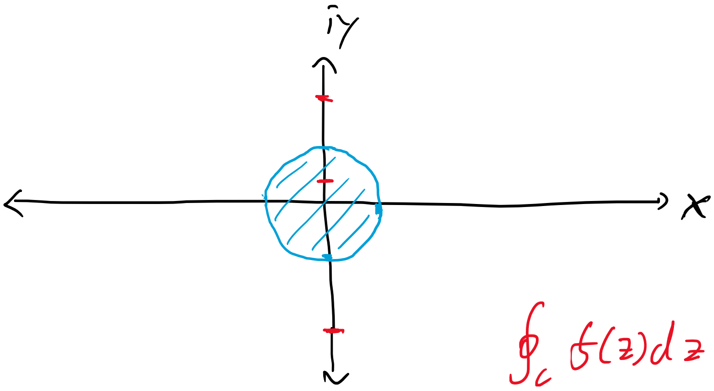
$$= \frac{-2+34i}{225}$$

$C : |z| = 1{,}5$

$z_1 = -4i$

$z_2 = 4i$

$z_3 = i$



$$\oint_C f(z)\,dz = 2\pi i\left(\operatorname{Res}_{z=z_1} + \dots + \operatorname{Res}_{z=z_n}\right)$$

$$\oint_C \frac{2z-1}{(z^2+16)(z-i)^2}\,dz = 2\pi i\left(\operatorname{Res}_{z=i}\right)$$

$$= 2\pi i\left(\frac{-2+34i}{225}\right)$$

$$= \frac{-68\pi - 4\pi i}{225}$$

3. Diberikan fungsi $f(t) = 2t$ dengan $-2 \le t \le 2$, selanjutnya fungsi $f(t)$ dipandang periodik dengan periode 4

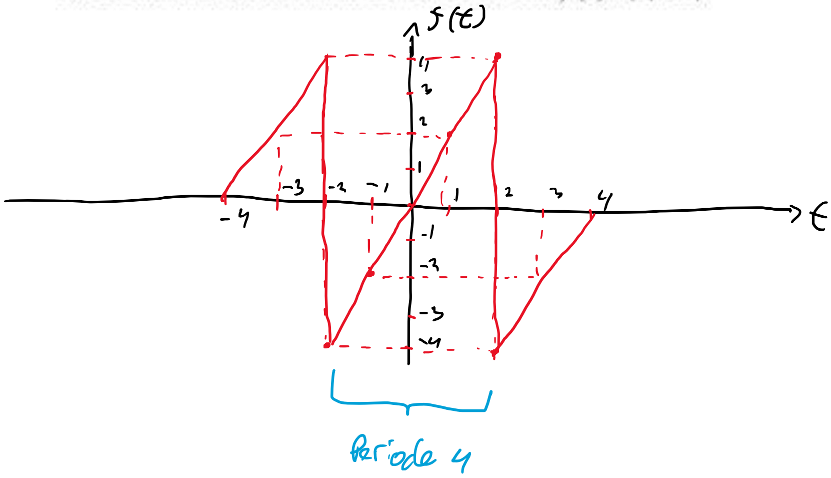
a. Gambarkan **sinyal** fungsi $f(t)$ pada interval $-4 \le t \le 4$! (5 poin)

b. Berdasarkan 3.a. apakah jenis fungsi $f(t)$ adalah fungsi ganjil, fungsi genap atau bukan keduanya ? (5 poin)

c. Berdasarkan 3.b. hitunglah 3 Koeffisien Fourier dari $f(t)$! (Gunakan sifat integral dari fungsi ganjil dan fungsi genap) (10 poin)

d. Berdasarkan 3.c.tentukan Deret Fourier dari $f(t)$! (5poin)

a.



$f(-t) = f(t)$ genap

$f(-t) = -f(t)$ ganjil

b. $f(t) = 2t$

$f(-t) = 2(-t)$

$f(-t) = -2t$

$f(-t) = -f(t) \rightarrow$ fungsi ganjil

c. a_0, a_n, b_n

Karena $f(t)$ fungsi ganjil, maka $a_0 = a_n = 0$

$$a_0 = \frac{1}{P}\int_P f(t)\,dt$$

$$a_n = \frac{2}{P}\int_P f(t) \cdot \cos\frac{2\pi n t}{P}\,dt$$

$$b_n = \frac{2}{P}\int_P f(t) \cdot \sin\frac{2\pi n t}{P}\,dt$$

P = Periode

$$P = 2-(-2) = 4$$

$$b_n = \frac{1}{P}\int_P f(t).\ \sin\frac{2\pi n t}{P}\, dt$$

$$= \frac{1}{4}\int_{-2}^{2} 2t.\ \sin\frac{2\pi n t}{4}\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{4}\int_{-2}^{2} \overset{u}{2t}.\ \overset{dv}{\sin\frac{\pi n t}{2}}\, dt$$

$$dv = \sin\frac{\pi n t}{2}\, dt$$

$$v = -\frac{2}{\pi n}\cos\frac{\pi n t}{2}$$

$$= \frac{1}{4}\left(2t.\ \frac{-2}{\pi n}\cos\frac{\pi n t}{2} - \int -\frac{2}{\pi n}\cos\frac{\pi n t}{2}.\ 2\,dt\right)\Bigg|_{-2}^{2}$$

$$u = 2t$$

$$du = 2\,dt$$

$$= \frac{1}{4}\left(-\frac{4t}{\pi n}\cos\frac{\pi n t}{2} + \frac{4}{\pi n}\int\cos\frac{\pi n t}{2}\, dt\right)\Bigg|_{-2}^{2}$$

$$= \frac{1}{4}\left(-\frac{4t}{\pi n}\cos\frac{\pi n t}{2} + \frac{4}{\pi n}.\ \frac{2}{\pi n}.\ \sin\frac{\pi n t}{2}\right)\Bigg|_{-2}^{2}$$

$$\boxed{\begin{aligned}\cos -x &= \cos x\\ \sin -x &= -\sin x\end{aligned}}$$

$$b_n = \frac{1}{4}\left(-\frac{4t}{\pi n}\cos\frac{\pi n t}{2} + \frac{8}{\pi^2 n^2}.\ \sin\frac{\pi n t}{2}\right)$$

$$= \frac{1}{4}\left[\left(-\frac{8}{\pi n}\cos\pi n + \frac{8}{\pi^2 n^2}\sin\pi n\right) - \left(\frac{8}{\pi n}\cos -\pi n + \frac{8}{\pi^2 n^2}\sin -\pi n\right)\right]$$

$$= \frac{1}{4}\left(-\frac{8}{\pi n}\cos\pi n + \frac{8}{\pi^2 n^2}\sin\pi n - \frac{8}{\pi n}\cos\pi n + \frac{8}{\pi^2 n^2}\sin\pi n\right)$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{16}^{4}}{\pi^2 n^2}\sin\pi n - \frac{\cancel{16}^{4}}{\pi n}\cos\pi n\right)$$

$$f(t) = a_0 + a_1\cos\frac{\pi t}{P} + \dots + a_n\cos\frac{n\pi t}{P}$$
$$+ b_1\sin\frac{\pi t}{P} + \dots + b_n\sin\frac{n\pi t}{P}$$

$$b_n = \frac{4}{\pi n}\left(\frac{1}{\pi n}\sin\pi n - \cos\pi n\right)$$

d. $b_1 = \frac{4}{\pi}$ $\quad b_2 = -\frac{2}{\pi}$ $\quad b_3 = \frac{4}{3\pi}$

$$f(t) = a_0 + a_1\cos\frac{\pi t}{P} + \dots + a_n\cos\frac{n\pi t}{P} + b_1\sin\frac{\pi t}{P} + \dots + b_n\sin\frac{n\pi t}{P}$$

$$f(t) = \frac{4}{\pi}\sin\frac{\pi t}{4} - \frac{2}{\pi}\sin\frac{\pi t}{2} + \frac{4}{3\pi}\sin\frac{3\pi t}{4} + \dots b_n\sin\frac{n\pi t}{P}$$

4. Transformasi Fourier

a. (10 poin) Hitung Transformasi Fourier dari

$$f(t) = \begin{cases} 2, & 0 < t < 3 \\ 0, & untuk\ t\ lainnya \end{cases}$$

$f(t)$	$F(iw)$	$f(t)$	$F(iw)$
$ag(t) + bh(t)$	$aG(iw) + bH(iw)$	$f(t - t_0)$	$F(w - t_0)$
$\sin(at)\,u(t)$	$\frac{a}{(iw)^2 + a^2}$	$e^{at}u(t)$	$\frac{1}{iw - a}$
$\cos(at)\,u(t)$	$\frac{iw}{(iw)^2 + a^2}$	$u(t)$	$\frac{1}{iw} + \pi\delta(iw)$

Berdasarkan Tabel di atas tentukan invers Transformasi Fourier dari fungsi berikut:

b. (5 poin)

$$F(iw) = \frac{3}{(iw)^2 + 16}$$

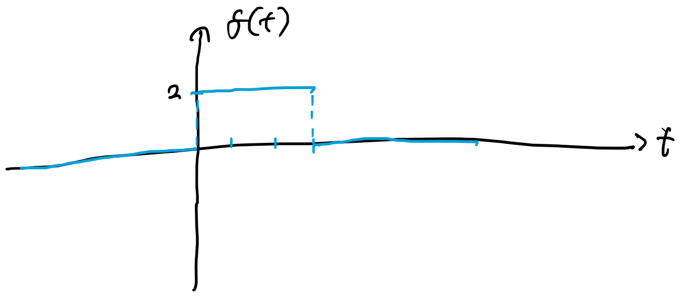
c. (5 poin)

$$F(iw) = \frac{1}{i}\frac{iw}{(iw)^2 + (-1)}$$

d. (5 poin)

$$F(iw) = \frac{i^2 w + 2}{(iw)^2 + 9}$$

a. $f(t) \begin{cases} 2, & 0 < t < 3 \\ 0, & \text{lainnya} \end{cases}$



$f(t) = 2u(t-0) - 2u(t-3)$

$f(t) = 2u(t) - 2u(t-3)$

b. $F(iw) = \frac{3}{(iw)^2 + 16} = \frac{3}{4}\ \frac{4}{(iw)^2 + 4^2}$

$f(t) = \frac{3}{4} \sin 4t \,.\, u(t)$

c. $F(iw) = \frac{1}{i} \cdot \frac{iw}{(iw)^2 + (-1)} = \frac{1}{i} \cdot \frac{iw}{(iw)^2 + i^2}$

$= \frac{1}{i} \cdot \cos it \,.\, u(t)$

d. $F(iw) = \dfrac{i^2 w + 2}{(iw)^2 + 9} = \dfrac{i^2 w}{(iw)^2 + 3^2} + \dfrac{2}{(iw)^2 + 3^2}$

$$= i \cdot \frac{iw}{(iw)^2 + 3^2} + \frac{2}{3} \cdot \frac{3}{(iw)^2 + 3^2}$$

$$f(t) = i\cos 3t \cdot u(t) + \frac{2}{3} \sin 3t \cdot u(t)$$