# 11th Material Subject: Special Distribution of Continuous Random Variable

## Undergraduate of Telecommunication Engineering

### MUH1F3 - PROBABILITY AND STATISTICS

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# WELCOME

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## LEARNING OBJECTIVES:

After careful study of this chapter, student should be able to do the following:

1. **Understand the assumptions for some common continuous probability distributions**
2. **Calculate continuous probability distribution to calculate probabilities in specific applications**
3. **Calculate probabilities and determine means and variances for some common continuous probability distributions**

# UNIFORM

**Uniform** random variables appear in situations where all values in a certain interval $(\mathbf{a}, \mathbf{b})$ have same probability.

- If **X** is a random variable that is uniformly distributed at interval $(\mathbf{a}, \mathbf{b})$ then:

$$\mathbf{X} \Rightarrow \mathbf{UNI}(\mathbf{a}, \mathbf{b}) \tag{1}$$

- The **Probability Density Function** of Uniform Distribution:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} \frac{1}{b-1} & , a \leq x \leq b \\ 0 & , \text{otherwise} \end{cases} \tag{2}$$

# UNIFORM

- The **Mean** of Uniform Distribution:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(X) = \frac{b+a}{2} \tag{3}$$

- The **Variance** of Uniform Distribution:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{2(b-a)}$$

$$E(X^2) = \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3}$$

# UNIFORM

$$\mathbf{Var}(\mathbf{X}) = \mathbf{E}(\mathbf{X}^2) - (\mathbf{E}(\mathbf{X}))^2 = \left(\frac{\mathbf{b}^2 + \mathbf{a}^2 + \mathbf{ab}}{\mathbf{3}}\right) - \left(\frac{\mathbf{b} + \mathbf{a}}{\mathbf{2}}\right)^2 = \frac{(\mathbf{b} - \mathbf{a})^2}{\mathbf{12}} \tag{4}$$

- The **Moment Generation Function** of Uniform Distribution:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} \frac{e^{bt} - e^{at}}{(b-a)t} & \text{, for } t \neq 0 \\ 1 & \text{, for } t = 0 \end{cases} \tag{5}$$

# UNIFORM

**Example:** Let **X** be a random variable that states the amount of fading which occurs due to the number of obstacles in the propagation environment. Fading that occurs uniformly distributed in the range of **80 dB** up to **95 dB**. Determine:

a. Probability Density Function (PDF) of **X**

b. Mean / Expected Value of **X**

c. Variance random variable of **X**

d. Moment Generation Function of **X**

e. $\mathbf{P(X \leq 85)}$, $\mathbf{P(X < 85)}$, $\mathbf{P(X > 85)}$ and $\mathbf{P(X \geq 85)}$

**Answer:**

a. The PDF can be written:

$$\mathbf{f_X(x)} = \begin{cases} \frac{1}{95-80} & , 80 \leq x \leq 95 \\ 0 & , \text{otherwise} \end{cases}$$

b. Mean / Expected Value of random variable **X**

$$E(X) = \frac{b+a}{2} = \frac{175}{2}$$

c. Variance of random variable **X**

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(95-80)^2}{12} = \frac{225}{12}$$

d. Moment Generation Function of random variable **X**

$$f_X(x) = \begin{cases} \frac{e^{95t} - e^{80t}}{15t} & \text{, for } t \neq 0 \\ 1 & \text{, for } t = 0 \end{cases}$$

## UNIFORM

d. $\mathbf{P}(\mathbf{X} \leq \mathbf{85})$, $\mathbf{P}(\mathbf{X} < \mathbf{85})$, $\mathbf{P}(\mathbf{X} > \mathbf{85})$ and $\mathbf{P}(\mathbf{X} \geq \mathbf{85})$

$$\mathbf{P}(\mathbf{X} \leq \mathbf{85}) = \mathbf{P}(\mathbf{X} < \mathbf{85}) = \int_{\mathbf{80}}^{\mathbf{85}} \frac{\mathbf{1}}{\mathbf{15}}\,\mathbf{dx} = \frac{\mathbf{5}}{\mathbf{15}}$$

$$\mathbf{P}(\mathbf{X} \geq \mathbf{85}) = \mathbf{P}(\mathbf{X} > \mathbf{85}) = \int_{\mathbf{85}}^{\mathbf{95}} \frac{\mathbf{10}}{\mathbf{15}}\,\mathbf{dx} = \frac{\mathbf{5}}{\mathbf{15}}$$

# EXPONENTIAL

- The random variable **X** that equals the distance between successive events from a Poisson process with mean number of events $\lambda > \mathbf{0}$ per unit interval is an **Exponential** random variable with parameter $\lambda$.

$$\mathbf{X} \Rightarrow \mathbf{EXP}(\lambda) \tag{6}$$

- The probability density function of **X** is:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} \lambda e^{-\lambda} & , 0 \leq x < \lambda \\ 0 & , \textbf{otherwise} \end{cases} \tag{7}$$

- The **Mean** of Exponential Distribution:

$$\mathbf{E}(\mathbf{X}) = \frac{\mathbf{1}}{\lambda} \tag{8}$$

# EXPONENTIAL

- The **Variance** of Exponential Distribution:

$$\mathbf{Var}(\mathbf{X}) = \frac{\mathbf{1}}{\lambda^{\mathbf{2}}} \tag{9}$$

- The **Moment Generation Function** of Binomial Distribution:

$$\mathbf{E}(\mathbf{e}^{\mathbf{tx}}) = \frac{\lambda}{\lambda - \mathbf{t}} \quad \text{, for } \mathbf{t} < \lambda \tag{10}$$

# EXPONENTIAL

**Example:** The average of telephone calls coming to the information center has a waiting time of 5 minutes / call. If **X** is random variable which states the average length of waiting time between call, specify:

a. Probability Density Function (PDF) of random variable **X**
b. Mean / Expected Value random variable **X**
c. Variance random variable **X**
d. Moment Generation Function random variable **X**
e. The probability of waiting no more than 1 minute
f. The probability of waiting between 1 minute - 3 minutes
g. The probability of waiting more than 2 minute

# EXPONENTIAL

**Answer:**

a. The PDF can be written:

$$\mathbf{f_X(x)} = \begin{cases} 5e^{-5} & , 0 \leq x < \infty \\ 0 & , \textbf{otherwise} \end{cases} \tag{11}$$

b. Mean / Expected Value of random variable **X**

$$\mathbf{E(X)} = \frac{\mathbf{1}}{\lambda} = \frac{\mathbf{1}}{\mathbf{5}}$$

c. Variance of random variable **X**

$$\mathbf{Var(X)} = \frac{\mathbf{1}}{\lambda^{\mathbf{2}}} = \frac{\mathbf{1}}{\mathbf{25}}$$

d. Moment Generation Function of random variable **X**

$$\mathbf{E(e^{tx})} = \frac{\lambda}{\lambda \mathbf{t}} = \frac{\mathbf{5}}{\mathbf{5-t}}, \quad \text{, for } \ \mathbf{t} < \lambda$$

# EXPONENTIAL

e. The probability of waiting no more than 1 minute

$$P(t \leq 1) = \int_0^1 5 \cdot e^{-5x}\, dx = -e^{-5x}\Big|_0^1 = 0.9933$$

f. The probability of waiting between 1 minute - 3 minutes

$$P(1 \leq t \leq 3) = \int_1^2 5 \cdot e^{-5x}\, dx = -e^{-5x}\Big|_1^3 = 0.0067$$

g. The probability of waiting more than 2 minute

$$P(t > 2) = \int_2^\infty 5 \cdot e^{-5x}\, dx = -e^{-5x}\Big|_2^\infty = 4.54 \times 10^{-5}$$

# NORMAL

Undoubtedly, the most widely used model for a continuous measurement is a **Normal** random variable. The **Normal** distribution often referred to as the Gauss distribution, taken from the German mathematician Carl Friedrich Gauss.

- A Normal random variable **X** is with an average value $\mathbf{E}(\mathbf{x}) = \mu$ and variance $\mathbf{Var}(\mathbf{x}) = \sigma^{\mathbf{2}}$:

$$\mathbf{X} \Rightarrow \mathbf{NOR}(\mu, \sigma^{\mathbf{2}}) \tag{12}$$

- The **Probability Density Function** of Normal Distribution:

$$\mathbf{f_X}(\mathbf{x}) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & , -\infty < x < \infty \\ 0 & , \textbf{otherwise} \end{cases} \tag{13}$$

(14)

# NORMAL

- The **Mean** of Normal Distribution:

$$\mathbf{E}(\mathbf{X}) = \mu \tag{15}$$

- The **Variance** of Normal Distribution:

$$\mathbf{Var}(\mathbf{X}) = \sigma^2 \tag{16}$$

- The **Moment Generation Function** of Normal Distribution:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = e^{\mu t + \frac{\sigma^2 t^2}{2}} \tag{17}$$

# NORMAL

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **Standard Normal** random variable and is denoted as **Z**. The cumulative distribution function of a standard normal random variable is denoted as:

$$\Phi(z) = P(Z \leq z) \tag{18}$$

Probabilities that are not of the form $\Phi(z) = P(Z \leq z)$ are found by using the basic rules of probability and the symmetry of the normal distribution along with Appendix Table. If $\mathbf{X}$ is a normal random variable with $\mathbf{E}(\mathbf{X}) = \mu$ and $\mathbf{Var}(\mathbf{X}) = \sigma^2$, the random variable:

$$\mathbf{Z} = \frac{\mathbf{X} - \mu}{\sigma} \tag{19}$$

is a normal random variable with $\mathbf{E}(\mathbf{Z}) = 0$ and $\mathbf{Var}(\mathbf{Z}) = 1$. That is, $\mathbf{Z}$ is a standard normal random variable.

# NORMAL

**Example:** lamp life produced by PT. PIJAR JAYA normally distributed with an average of 1000 hours and standard deviations 100 hours. If X is a random variable state the lamp's lifespan as in the conditions above, determine:

a. The Probability Density Function of **X**

b. Mean, Standard Deviation and Variance of **X**

c. $\mathbf{f_X(900)}$, $\mathbf{P(X \leq 850)}$, $\mathbf{P(X \geq 1254)}$, and $\mathbf{P(975 < x < 1378)}$

d. If the company advertises that 95% of the lights can ignite for at least 900 hours. Are these advertisements honest?

# NORMAL

**Answer:**

a. The Probability Density Function of **X**

$$\mathbf{f_X(x)} = \begin{cases} \frac{1}{100\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1000}{100}\right)^2} & , -\infty < x < \infty \\ 0 & , \textbf{otherwise} \end{cases}$$

b. Mean, Standard Deviation and Variance of **X**

$$\mu = \mathbf{1000} \quad , \ \sigma = \mathbf{100} \quad \textbf{and} \ \ \sigma^2 = \mathbf{100^2} = \mathbf{10000}$$

# NORMAL

**Answer:**

c. $f_X(900)$, $P(X \leq 850)$, $P(X \geq 1254)$, and $P(975 < x < 1378)$

$$f_X(900) = \frac{1}{100\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{900-1000}{100}\right)^2} = 0.0024$$

$$P(X \leq 850) = P\left(Z \leq \frac{850-1000}{100}\right) = \Phi(-1.5) = 0.0681$$

$$P(X > 1254) = 1 - P(X < 1254) = 1 - P\left(Z \leq \frac{1254-1000}{100}\right) = 1 - \Phi(2.54) = 0.00554$$

$$P(975 < x < 1254) = P(X < 1254) - P(X < 975)$$

$$= P\left(Z \leq \frac{1254-1000}{100}\right) - P\left(Z \leq \frac{975-1000}{100}\right) = \Phi(2.54) - \Phi(-0.25) = 0.59863$$

d. If the company advertises that 95% of the lights can ignite for at least 900 hours. Are these advertisements honest?

Advertising is said to be honest if:

$$P(X \geq 900) = 0.95$$

Lets count:

$$P(X \geq 900) = 1 - P(X < 900) = 1 - P\left(Z < \frac{900 - 1000}{100}\right) = 1 - \Phi(1) = 1 - 0.15866 = 0.84134$$

It turns out that there are only about 84.134% of light that can light up for at least 900 hours, so the advertisement is dishonest.

*Thank You*