Explanation of Equations

1. Equation for the Second Derivative with respect to x:

This equation represents the calculation of the second derivative of the function ψ with respect to the x direction at a grid point (i, j). It utilizes a finite difference approximation, where the second derivative is approximated by the difference between $\psi(i+1, j)$, $\psi(i, j)$, and $\psi(i-1, j)$ divided by Δx^2, where Δx is the grid spacing in the x direction.

$$\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{(i,j)} = \frac{\psi(i+1, j) - 2\psi(i, j) + \psi(i-1, j)}{\Delta x^2}$$

2. Backward Advection in the Semi-Lagrangian Scheme:

In the semi-Lagrangian scheme, backward advection is used to trace grid points backward in time. Given the current grid point (i, j), we calculate the position of the corresponding grid point at the previous time step by moving backward with the wave speed c multiplied by the time step Δt divided by the grid spacing Δx or Δy. After obtaining the old coordinates x_{old} and y_{old}, we calculate the indices i_{old} and j_{old} of the previous grid point using modulo operation and integer casting. Finally, the value of ψ at the current grid point (i, j) is updated using the value at the old grid point $(i_{\text{old}}, j_{\text{old}})$.

$$\begin{aligned}
&x_{\text{old}} = i - c\frac{\Delta t}{\Delta x}, \quad y_{\text{old}} = j - c\frac{\Delta t}{\Delta y} \\
&i_{\text{old}} = \text{int}(x_{\text{old}}) \mod N_x, \quad j_{\text{old}} = \text{int}(y_{\text{old}}) \mod N_y \\
&\psi_{\text{new}}(i, j) = \psi(i_{\text{old}}, j_{\text{old}})
\end{aligned}$$

3. Implicit Matrix for the Semi-Implicit Scheme:

In the semi-implicit scheme, an implicit method is used to solve the wave equation. The implicit matrix A is constructed based on the discretized equation for the wave equation. It is a tridiagonal matrix with the main diagonal elements equal to $1 + 2\alpha$ and the off-diagonal elements equal to $-\alpha$, where $\alpha = c\Delta t/\Delta x$ is a constant. Additionally, the periodic boundary conditions are incorporated into the matrix by setting the elements A_{1,N_x} and $A_{N_x,1}$ to $-\alpha$ to account for the periodic wrapping of the wave in the x direction.

$$\begin{aligned}
&A = \text{diag}(1 + 2\alpha) - \text{diag}(\alpha, k = 1) - \text{diag}(\alpha, k = -1) \\
&A_{1,N_x} = -\alpha, \quad A_{N_x,1} = -\alpha
\end{aligned}$$