# 1 Finite-Difference Methods for the One Dimensional Wave Equation

A one-dimensional for of the wave equation can be found as follows,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} + f(x,t), & (x,t) \in (a,b)\times(T_1,T_2], \\ u(x,T_1) = h(x), & x \in [a,b], \\ \frac{\partial}{\partial t}u(x,T_1) = s(x), & x \in [a,b], \\ u(a,t) = f_1(t), & t \in [T_1,T_2], \\ u(b,t) = f_2(t), & t \in [T_1,T_2]. \end{cases} \tag{1}$$

Moreover, we derive the general scheme in the main article a follows,

$$\begin{aligned} -\alpha c_i^2\lambda^2\left(u_{i-1}^{n+1}+u_{i+1}^{n+1}\right) + \left(1+2\alpha c_i^2\lambda^2\right)u_i^{n+1} &= \beta c_i^2\lambda^2\left(u_{i-1}^{n}+u_{i+1}^{n}\right) + 2\left(1-\beta c_i^2\lambda^2\right)u_i^n \\ &+ \gamma c_i^2\lambda^2\left(u_{i-1}^{n-1}+u_{i+1}^{n-1}\right) - \left(1+2\gamma c_i^2\lambda^2\right)u_i^{n-1} + \tau^2 f_i^n. \end{aligned} \tag{2}$$

**Theorem 1** (Stability)**.** *Assume that the solution of the one dimensional acoustic wave equation* (1) *is sufficiently smooth. The stability criteria of the numerical scheme introduced in* (2) *can be derived as follows,*

$$\tau < \frac{1}{\sqrt{(1-4\alpha)}}\frac{h}{\max\limits_{0<m\le N_x} c_m} \tag{3}$$

*where* $\alpha = \gamma$*.*

*Proof.* The Fourier transform of a function $u(x)$, which is defined on $\mathbb{R}$, is uniquely defined as follows,

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}u(x)dx. \tag{4}$$

In addition, the Fourier inversion formula is given by the following equation,

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\hat{u}(x)d\omega. \tag{5}$$

Furthermore, the Fourier transform of a gird function such as $u_m$ for all integers $m$ is available as follows [**?**],

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-imh\xi}u_m h, \quad \xi \in \left[-\frac{\pi}{h},\frac{\pi}{h}\right], \tag{6}$$

where $h = \Delta x$. In addition, the Fourier inversion formula of $v$ for all integers $m$,

$$u_m = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\hat{u}(\xi)d\xi. \tag{7}$$

As a result, the Fourier inversion formula for $u_m^n$ for all $m$ and $n$ can be found as follows,

$$u_m^n = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\hat{u}^n(\xi)d\xi, \tag{8}$$

where,

$$\hat{u}^n(\xi) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-imh\xi}u_m^n, \quad \xi \in \left[-\frac{\pi}{h},\frac{\pi}{h}\right], \tag{9}$$

First, let's perform the stability analyze of the explicit scheme using *Von Neumann stability analysis*. Consider the following explicit scheme,

$$u_m^{n+1} = c_m^2\lambda^2\left(u_{m-1}^n+u_{m+1}^n\right) + 2\left(1-c_m^2\lambda^2\right)u_m^n - u_m^{n-1} + \tau^2 f_{Nm}^n. \tag{10}$$

Without loss of generality, let $f_m^n = 0$ for all values of $m$ and $n$. It follows from using the Fourier inversion formula (8) and (10),

$$\begin{aligned} u_m^{n+1} &= c_m^2\lambda^2 \left( \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(m-1)h\xi} \hat{u}^n(\xi) d\xi + \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(m+1)h\xi} \hat{u}^n(\xi) d\xi \right) \\ &+ 2\left(1 - c_m^2\lambda^2\right) \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{u}^n(\xi) d\xi - \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{u}^{n-1}(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \left[ c_m^2\lambda^2 (e^{-ih\xi} + e^{ih\xi}) \hat{u}^n(\xi) + 2\left(1 - c_m^2\lambda^2\right) \hat{u}^n(\xi) - \hat{u}^{n-1}(\xi) \right] d\xi, \end{aligned} \tag{11}$$

for all $1 \le m \le N_x$ and $1 \le n \le N_t$.

On the other hand, the Fourier inversion formula for $u_m^{n+1}$ can be found as follows,

$$u_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{u}^{n+1}(\xi) d\xi. \tag{12}$$

It follows from comparison of (11) and (12) and also the uniqueness of the Fourier transform,

$$\hat{u}^{n+1}(\xi) = \left[ c_m^2\lambda^2 (e^{-ih\xi} + e^{ih\xi}) + 2\left(1 - c_m^2\lambda^2\right) \right] \hat{u}^n(\xi) - \hat{u}^{n-1}(\xi). \tag{13}$$

Define *the amplification factor* as follows,

$$g = \frac{\hat{u}^{n+1}(\xi)}{\hat{u}^n(\xi)}. \tag{14}$$

As a result of (13) and (14),

$$g + g^{-1} = 2 - 4c_m^2\lambda^2 \sin^2\frac{\theta}{2}, \tag{15}$$

where $\theta = h\xi$. It can be concluded that,

$$g^2 - 2\varphi g + 1 = 0. \tag{16}$$

where $\varphi = 1 - 2c_m^2\lambda^2 \sin^2\frac{\theta}{2}$. The roots of the last quadratic equation are

$$g_{\pm} = \varphi \pm \sqrt{\varphi^2 - 1} \tag{17}$$

In case that $\varphi^2 - 1 \ge 0$, the quadratic equation will have two real roots and it follows form the definition of $\varphi$ that $\varphi < -1$. This means that one of the roots is $g_-$,

$$g_- = \varphi - \sqrt{\varphi^2 - 1} < \varphi < -1, \tag{18}$$

which would lead to instability.

However, in case that $\varphi^2 - 1 < 0$ or $|\varphi| < 1$ and the roots of the quadratic equation are complex,

$$|g_{\pm}| = \left| \varphi \pm i\sqrt{1 - \varphi^2} \right| = \sqrt{\varphi^2 + 1 - \varphi^2} = 1. \tag{19}$$

It can be concluded that the stability is equivalent to $|\varphi| < 1$. Therefore,

$$c_m^2\lambda^2 \sin^2\frac{\theta}{2} < 1, \tag{20}$$

for any $\theta$. This means that $c_m\lambda < 1$ for $0 \le m \le N_x$.

Following a stability analysis, the stability requirement here can be derived as follows,

$$\tau \max_{0 \le m \le N_x} c_m < h. \tag{21}$$

Von Neumann stability analysis also can be applied to the general implicit scheme,

$$
\begin{aligned}
-\alpha c_m^2\lambda^2\left(u_{N_x-1}^{n+1}+u_{m+1}^{n+1}\right)+\left(1+2\alpha c_m^2\lambda^2\right)u_{N_x}^{n+1} &= \beta c_m^2\lambda^2\left(u_{N_x-1}^{n}+u_{m+1}^{n}\right)+2\left(1-\beta c_m^2\lambda^2\right)u_{N_x}^{n}\\
&+\gamma c_m^2\lambda^2\left(u_{N_x-1}^{n-1}+u_{m+1}^{n-1}\right)-\left(1+2\gamma c_m^2\lambda^2\right)u_{N_x}^{n-1}+f_{N_x}^{n}\tau^2. \qquad (22)
\end{aligned}
$$

The analyze of stability is not sensitive to $f(x,y)$ due to the fact that $f(x,t)$ is known for $(x,t)\in[a,b]\times[T_1,T_2]$ and $f$ is assumed bounded. Therefore, without loss of generality, $f$ can be regarded zero for convenience. Similar to the previous case, it can be concluded that,

$$
\begin{aligned}
\left[-\alpha c_m^2\lambda^2\left(e^{-ih\xi}+e^{ih\xi}\right)+\left(1+2\alpha c_m^2\lambda^2\right)\right]\hat{u}^{n+1}(\xi) &= \left[\beta c_m^2\lambda^2\left(e^{-ih\xi}+e^{ih\xi}\right)+\left(2-2\beta c_m^2\lambda^2\right)\right]+\hat{u}^{n}(\xi)\\
&+\left[\gamma c_m^2\lambda^2\left(e^{-ih\xi}+e^{ih\xi}\right)-\left(1+2\gamma c_m^2\lambda^2\right)\right]\hat{u}^{n-1}(\xi). \qquad (23)
\end{aligned}
$$

It follows from dividing the equation (23) by $\hat{u}^n(\xi)$ and the theorem assumption that $\alpha=\gamma$,

$$
\begin{aligned}
\left[-\alpha c_m^2\lambda^2\left(e^{-i\theta}+e^{i\theta}\right)+\left(1+2\alpha c_m^2\lambda^2\right)\right]g &= \left[\beta c_m^2\lambda^2\left(e^{-i\theta}+e^{i\theta}\right)+\left(2-2\beta c_m^2\lambda^2\right)\right]\\
&+\left[\alpha c_m^2\lambda^2\left(e^{-i\theta}+e^{i\theta}\right)-\left(1+2\alpha c_m^2\lambda^2\right)\right]g^{-1}, \qquad (24)
\end{aligned}
$$

where $g$ is the amplification factor and $\theta=h\xi$. The equation (24) can be simplified as follows,

$$g^2-2\left[\frac{1-2\beta\varphi}{1+4\alpha\varphi}\right]g+1=0, \tag{25}$$

where $\varphi=c_m^2\lambda^2\sin^2\frac{\theta}{2}$. It follows that,

$$g^2+2\left(\frac{2(1-2\alpha)\varphi-1}{1+4\alpha\varphi}\right)g+1=0, \tag{26}$$

In case that

$$\left(\frac{2(1-2\alpha)\varphi-1}{1+4\alpha\varphi}\right)^2-1\geq 0,$$

the quadratic equation (26) will have two distinct real roots and one of them is greater than 1. This would lead to instability.

Therefore, the following case is considered in which the roots of quadratic equation (26) are complex.

$$\left(\frac{2(1-2\alpha)\varphi-1}{1+4\alpha\varphi}\right)^2-1<0. \tag{27}$$

As a result,

$$-16\alpha\varphi^2+4\varphi^2-4\varphi<0. \tag{28}$$

Since $\varphi$ is non-zero,

$$(1-4\alpha)\varphi<1. \tag{29}$$

This means,

$$(1-4\alpha)c_m^2\lambda^2\sin^2\frac{\theta}{2}<1, \tag{30}$$

for any $\theta$. Therefore,

$$\max_{0\leq m\leq N_x} c_m\lambda<\frac{1}{\sqrt{(1-4\alpha)}} \text{ or } \tau<\frac{1}{\sqrt{(1-4\alpha)}}\frac{h}{\max\limits_{0<m\leq N_x} c_m}. \tag{31}$$

□

**Corollary 1.1.** *The stability of the implicit method introduced in theorem* (1) *can be derived by setting* $\alpha = 1/4$ *in* (3). *It follows that,*

$$c_m \lambda < \infty, \tag{32}$$

*which means that the implicit scheme is unconditionally stable.*

**Theorem 2** (Convergence)**.** *The numerical scheme* (2) *is the second order in both time and space considering* $h = h_x = h_y$.

*Proof.* Without loss of generality, assume $f_i^n = 0$ for all values of $i$ and $n$ in the equation (2). It follows that,

$$\frac{1}{\tau^2}\delta_t^2 u_i^n = \frac{1}{h^2} c_i^2 \left(\alpha \delta_x^2 u_i^{n+1} + \beta \delta_x^2 u_i^n + \gamma \delta_x^2 u_i^{n-1}\right). \tag{33}$$

We know that

$$\begin{cases} \dfrac{1}{\tau^2}\delta_t^2 u_i^n = \dfrac{d^2}{dt^2}u_i^n + \dfrac{1}{12}\dfrac{d^4}{dt^4}u_i^n \tau^2 + \dfrac{1}{360}\dfrac{d^6}{dt^6}u_i^n \tau^4 + \mathscr{O}(\tau^6), \\ \dfrac{1}{h^2}\delta_x^2 u_i^n = \dfrac{d^2}{dx^2}u_i^n + \dfrac{1}{12}\dfrac{d^4}{dx^4}u_i^n h^2 + \dfrac{1}{360}\dfrac{d^6}{dx^6}u_i^n h^4 + \mathscr{O}(h^6), \end{cases} \tag{34}$$

It follows from (34),

$$\begin{aligned} &\frac{d^2}{dt^2}u_i^n + \frac{1}{12}\frac{d^4}{dt^4}u_i^n\tau^2 + \frac{1}{360}\frac{d^6}{dt^6}u_i^n\tau^4 + \mathscr{O}(\tau^6) = c_i^2\left[\alpha\frac{d^2}{dx^2}u_i^{n+1} + \beta\frac{d^2}{dx^2}u_i^n + \gamma\frac{d^2}{dx^2}u_i^{n-1}\right] \\ &+ \frac{c_i^2}{12}\left[\alpha\frac{d^4}{dx^4}u_i^{n+1} + \beta\frac{d^4}{dx^4}u_i^n + \frac{\gamma}{\Gamma}\frac{d^4}{dx^4}u_i^{n-1}\right]h^2 + \frac{c_i^2}{360}\left[\alpha\frac{d^6}{dx^6}u_i^{n+1} + \beta\frac{d^6}{dx^6}u_i^n + \frac{\gamma}{\Gamma}\frac{d^6}{dx^6}u_i^{n-1}\right]h^4 + \mathscr{O}(h^6). \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} &\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2\left(\alpha u_i^{n+1} + \beta u_i^n + \gamma\, u_i^{n-1}\right) = \frac{d^2}{dt^2}u_i^n - c_i^2\frac{d^2}{dx^2}\left[\alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1}\right] \\ &+ \frac{1}{12}\left[\frac{d^4}{dt^4}u_i^n\tau^2 - c_i^2\frac{d^4}{dx^4}\left(\alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1}\right)h^2\right] \\ &+ \frac{1}{360}\left[\frac{d^6}{dt^6}u_i^n\tau^4 - c_i^2\frac{d^6}{dx^6}\left(\alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1}\right)h^4\right] + \mathscr{O}(h^6 + \tau^6). \end{aligned} \tag{36}$$

The order of the explicit scheme follows from setting $\alpha = \gamma = 0$ and $\beta = 1$ in (36),

$$\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2 u_i^n = \frac{d^2}{dt^2}u_i^n - c_i^2\frac{d^2}{dx^2}u_i^n + \frac{1}{12}\left[\frac{d^4}{dt^4}u_i^n\tau^2 - c_i^2\frac{d^4}{dx^4}u_i^n h^2\right] + \mathscr{O}(h^4 + \tau^4). \tag{37}$$

It follows from (37),

$$\begin{aligned} \frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2 u_i^n &= \frac{d^2}{dt^2}u_i^n - c_i^2\frac{d^2}{dx^2}u_i^n + \lim_{\tau\to 0}\left[\frac{1}{12}\left[\frac{d^4}{dt^4}u_i^n\tau^2 - c_i^2\frac{d^4}{dx^4}u_i^n h^2\right] + \mathscr{O}(h^4 + \tau^4)\right], \\ &= -\frac{c_i^2}{12}\frac{d^4}{dx^4}u_i^n h^2 + \mathscr{O}(h^4), \end{aligned} \tag{38}$$

and setting $h = \mu\tau$ where $\mu$ is a known integer (this condition is considered due to stability criterion (21)).

$$\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2 u_i^n = \frac{d^2}{dt^2}u_i^n - c_i^2\frac{d^2}{dx^2}u_i^n + \frac{1}{12}\left[\frac{d^4}{dt^4}u_i^n - \mu^2 c_i^2\frac{d^4}{dx^4}u_i^n\right]\tau^2 + \mathscr{O}(\tau^4), \tag{39}$$

Therefore, the explicit scheme is second order respect to both $h$ and $\tau$ and also the total order of this scheme is 2.

The order of the implicit scheme follows from setting $\alpha = \gamma = \frac{1}{4}$ and $\beta = \frac{1}{2}$ in (36),

$$\begin{aligned} &\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\left(\frac{1}{4}\delta_x^2 u_i^{n+1} - \frac{1}{2}\delta_x^2 u_i^n - \frac{1}{4}\delta_x^2 u_i^{n-1}\right) = \\ &\frac{1}{12}\left[\frac{d^4}{dt^4}u_i^n\tau^2 - c_i^2\left(\frac{1}{4}\frac{d^4}{dx^4}u_i^{n+1} + \frac{1}{2}\frac{d^4}{dx^4}u_i^n + \frac{1}{4}\frac{d^4}{dx^4}u_i^{n-1}\right)h^2\right] + \mathscr{O}(h^4 + \tau^4). \end{aligned} \tag{40}$$

Similarly, after taking limits,

Similarly, it follows from applying limits on (40)

$$\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2\left(\frac{1}{4}u_i^{n+1} - \frac{1}{2}u_i^n - \frac{1}{4}u_i^{n-1}\right) = -\frac{c_i^2}{12}\frac{d^4}{dx^4}\left(\frac{1}{4}u_i^{n+1} + \frac{1}{2}u_i^n + \frac{1}{4}u_i^{n-1}\right)h^2 + \mathscr{O}(h^4) \tag{41}$$

$$\frac{1}{\tau^2}\delta_t^2 u_i^n - \frac{1}{h^2}c_i^2\delta_x^2\left(\frac{1}{4}u_i^{n+1} - \frac{1}{2}u_i^n - \frac{1}{4}u_i^{n-1}\right) = \frac{1}{12}\frac{d^4}{dt^4}u_i^n\tau^2 + \mathscr{O}(\tau^4). \tag{42}$$

Hence, the implicit scheme is second order respect to both $h$ and $\tau$ and the total order of this scheme is also 2. $\square$