COMP201

Computer Systems & Programming

Lecture #05 – Floating Point

KOÇ UNIVERSITY

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Recap: Bitwise Operators

- You're already familiar with many operators in C:
  - **Arithmetic operators:** +, -, *, /, %
  - **Comparison operators**: ==, !=, <, >, <=, >=
  - **Logical Operators**: &&, ||, !

- **Bitwise operators:**
  - **Logical operators:** &, |, ~, ^,
  - **Bit shift operators:** <<, >>

<u>COMP201 Topic 2</u>: How can a computer represent real numbers in addition to integer numbers?

Learning Goals

Understand the design and compromises of the floating point representation, including:

- Fixed point vs. floating point
- How a floating point number is represented in binary
- Issues with floating point imprecision
- Other potential pitfalls using floating point numbers in programs

Plan For Today

- Representing real numbers
- Fixed Point
- Floating Point

Disclaimer: Slides for this lecture were borrowed from

—Nick Troccoli's Stanford CS107 class

Lecture Plan

- Representing real numbers
- Fixed Point
- Floating Point

Real Numbers

- We previously discussed representing integer numbers using two's complement.
- However, this system does not represent real numbers such as 3/5 or 0.25.
- How can we design a representation for real numbers?

Real Numbers

Problem: unlike with the integer number line, where there are a finite number of values between two numbers, there are an *infinite* number of real number values between two numbers!

Integers between 0 and 2: 1

Real Numbers Between 0 and 2: 0.1, 0.01, 0.001, 0.0001, 0.00001,…

We need a fixed-width representation for real numbers. Therefore, by definition, *we will not be able to represent all numbers*.

Real Numbers

Problem: every number base has un-representable real numbers.

Base 10: $1/6_{10} = 0.16666666\ldots_{10}$

Base 2: $1/10_{10} = 0.000110011001100110011\ldots_2$

Therefore, by representing in base 2, *we will not be able to represent all numbers*, even those we can exactly represent in base 10.

Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

5	9	3	4	.	2	1	6
10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
1	0	1	1	.	0	1	1
2^3	2^2	2^1	2^0		2^{-1}	2^{-2}	2^{-3}

Lecture Plan

- Representing real numbers
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Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

1	0	1	1	.	0	1	1
8s	4s	2s	1s		1/2s	1/4s	1/8s

- **Pros:** arithmetic is easy! And we know exactly how much precision we have.

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

.	0	1	1	0	0	1	1
	1/2s	1/4s	1/8s	...			

1	0	1	1	0	.	1	1
16s	8s	4s	2s	1s		1/2s	1/4s

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

Base 10	Base 2
$5.07E30 =$	$10\underbrace{.....}_{\text{100 zeros}}0.1$
$9.86E\text{-}32 =$	$0.0\underbrace{.....}_{\text{100 zeros}}01$

To be able to store both these numbers using the same fixed point representation, the bitwidth of the type would need to be at least 207 bits wide!

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible "floating" decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Lecture Plan

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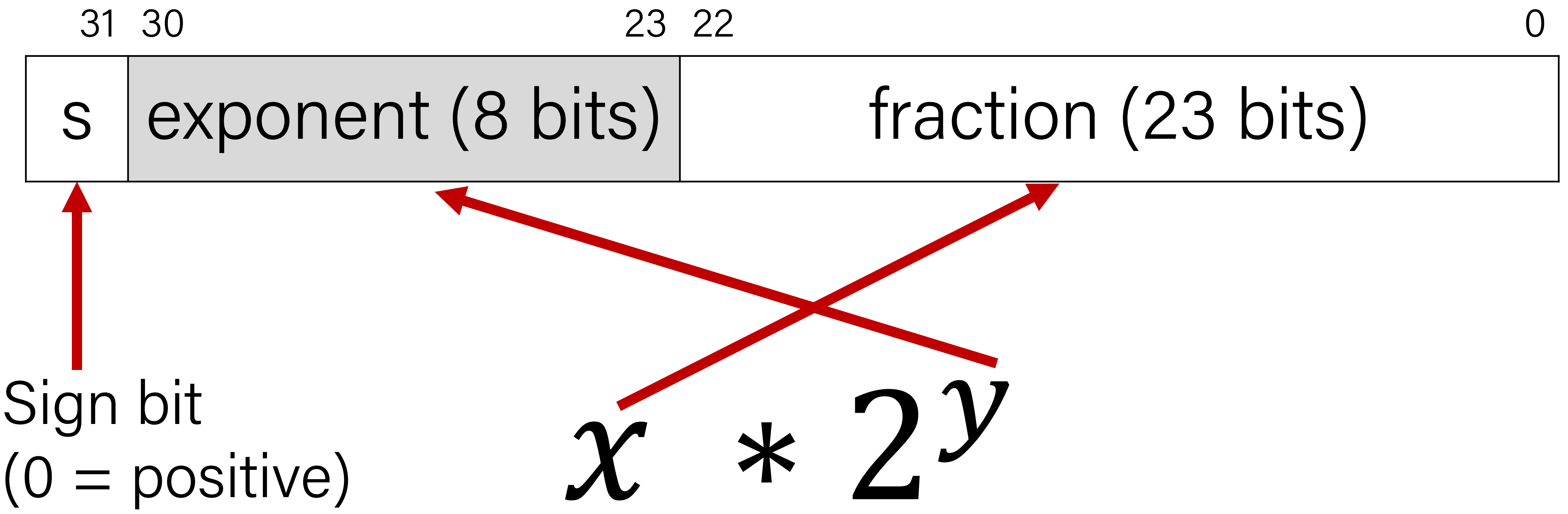
IEEE Floating Point

Let’s aim to represent numbers of the following scientific-notation-like format:

$$x * 2^y$$

With this format, 32-bit floats represent numbers in the range ~1.2 x10^{-38} to ~3.4 x10^{38}! Is every number between those representable? **No**.

IEEE Single Precision Floating Point



Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	?
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	?

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	127
11111101	126
11111100	125
…	…
00000011	-124
00000010	-125
00000001	-126
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

- The exponent is **not** represented in two's complement.
- Instead, exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive). This makes bit-level comparison fast.
- **Actual value = binary value – 127 ("bias")**

11111110	254 – 127 = 127
11111101	253 – 127 = 126
…	…
00000010	2 – 127 = -125
00000001	1 – 127 = -126

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x * 2^{y}$$

- We could just encode whatever x is in the fraction field. But there's a trick we can use to make the most out of the bits we have.

An Interesting Observation

In Base 10:

$42.4 \times 10^5 = 4.24 \times 10^6$

$324.5 \times 10^5 = 3.245 \times 10^7$

$0.624 \times 10^5 = 6.24 \times 10^4$

We tend to adjust the exponent until we get down to one place to the left of the decimal point.

In Base 2:

$10.1 \times 2^5 = 1.01 \times 2^6$

$1011.1 \times 2^5 = 1.0111 \times 2^8$

$0.110 \times 2^5 = 1.10 \times 2^4$

Observation: in base 2, this means there is always a 1 to the left of the decimal point!

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x * 2^y$$

- We can adjust this value to fit the format described previously. Then, x will always be in the format 1.XXXXXXXXX...
- Therefore, in the fraction portion, we can encode just what is *to the right* of the decimal point! This means we get one more digit for precision.

Value encoded = 1._[FRACTION BINARY DIGITS]_

Practice

Sign	Exponent						Fraction			
0	0	…	0	0	0	1	0	1	0	…

Is this number:

A) **Greater than 0?**

B) **Less than 0?**

Is this number:

A) **Less than -1?**

B) **Between -1 and 1?**

C) **Greater than 1?**

Skipping Numbers

- We said that it's not possible to represent *all* real numbers using a fixed-width representation. What does this look like?

Float Converter

- https://www.h-schmidt.net/FloatConverter/IEEE754.html

Floats and Graphics

- https://www.shadertoy.com/view/4tVyDK

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible “floating” decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Representing Zero

The float representation of zero is all zeros (with any value for the sign bit)

Sign	Exponent	Fraction
`any`	`All zeros`	`All zeros`

- This means there are two representations for zero! ☹

Representing Small Numbers

If the exponent is all zeros, we switch into “denormalized” mode.

Sign	Exponent	Fraction
`any`	`All zeros`	`Any`

- We now treat the exponent as -126, and the fraction as *without* the leading 1.
- This allows us to represent the smallest numbers as precisely as possible.

Representing Exceptional Values

If the exponent is all ones, and the fraction is all zeros, we have +- infinity.

Sign	Exponent	Fraction
`any`	`All ones`	`All zeros`

- The sign bit indicates whether it is positive or negative infinity.
- Floats have built-in handling of over/underflow!
  - Infinity + anything = infinity
  - Negative infinity + negative anything = negative infinity
  - Etc.

Representing Exceptional Values

If the exponent is all ones, and the fraction is nonzero, we have
Not a Number (NaN)

Sign	Exponent						Fraction
any	1	…	…	…	…	1	Any nonzero

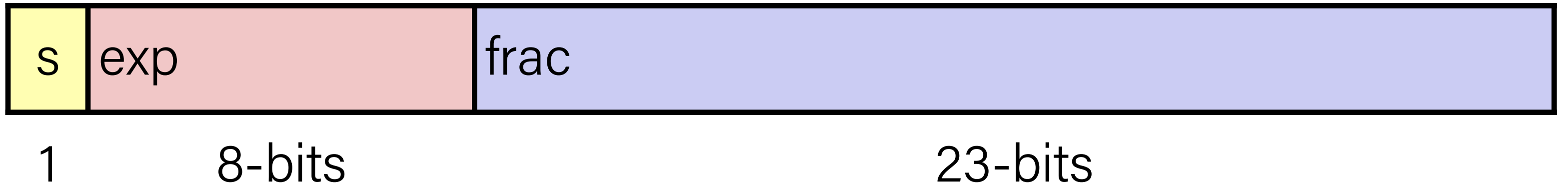
- NaN results from computations that produce an invalid mathematical result.
  - Sqrt(negative)
  - Infinity / infinity
  - Infinity + -infinity
  - Etc.

Number Ranges

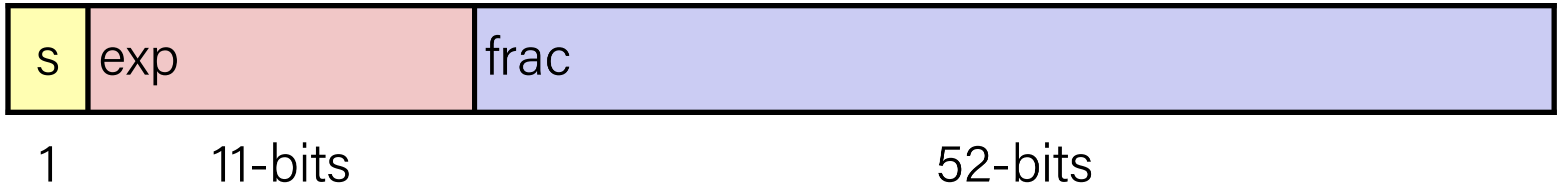
- 32-bit integer (type **int**):
  - › -2,147,483,648 to 2147483647
  - › Every integer in that range can be represented
- 64-bit integer (type **long**):
  - › −9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
- 32-bit floating point (type **float**):
  - – ~1.2 $x10^{-38}$ to ~3.4 $x10^{38}$
  - – Not all numbers in the range can be represented (not even all integers in the range can be represented!)
  - – Gaps can get quite large! (larger the exponent, larger the gap between successive fraction values)
- 64-bit floating point (type **double**):
  - – ~2.2 $x10^{-308}$ to ~1.8 $x10^{308}$

Precision options

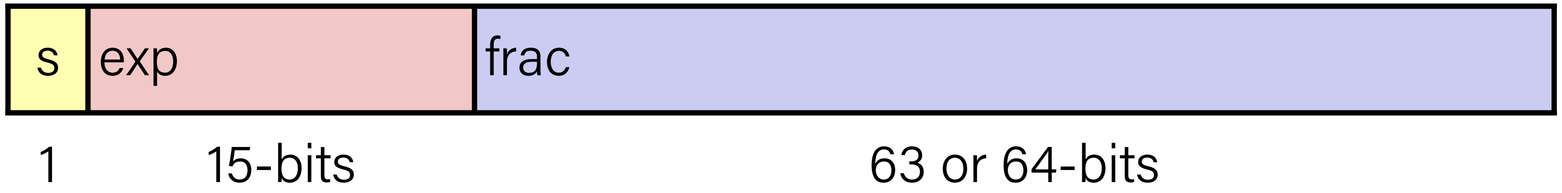
- Single precision: 32 bits



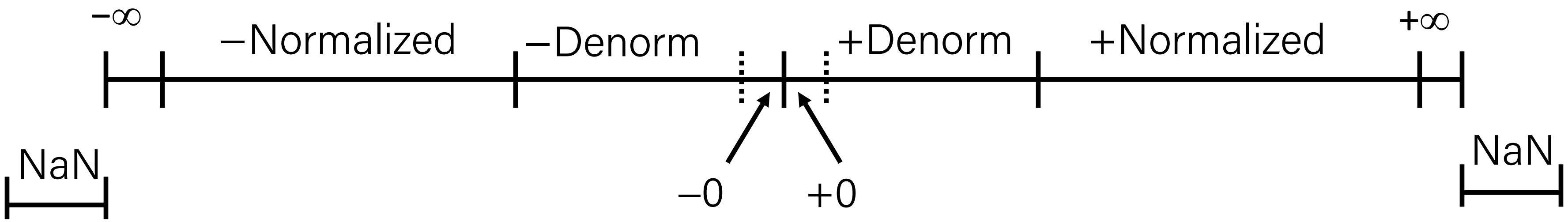
- Double precision: 64 bits



- Extended precision: 80 bits (Intel only)



Visualization: Floating Point Encodings



Additional Reading

What Every Computer Scientist Should Know About Floating-Point Arithmetic

DAVID GOLDBERG

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Floating-point arithmetic is considered an esotoric subject by many people. This is rather surprising, because floating-point is ubiquitous in computer systems: Almost every language has a floating-point datatype; computers from PCs to supercomputers have floating-point accelerators; most compilers will be called upon to compile floating-point algorithms from time to time; and virtually every operating system must respond to floating-point exceptions such as overflow This paper presents a tutorial on the aspects of floating-point that have a direct impact on designers of computer systems. It begins with background on floating-point representation and rounding error, continues with a discussion of the IEEE floating-point standard, and concludes with examples of how computer system builders can better support floating point.

Categories and Subject Descriptors: (Primary) C.0 [**Computer Systems Organization**]: General—*instruction set design*; D.3.4 [**Programming Languages**]: Processors—*compilers*, *optimization*; G.1.0 [**Numerical Analysis**]: General—*computer arithmetic*, *error analysis*, *numerical algorithms* (Secondary) D.2.1 [Software Engineering]: Requirements/Specifications—*languages*; D.3.1 [**Programming Languages**]: Formal Definitions and Theory—*semantics* D.4.1 [**Operating Systems**]: Process Management—*synchronization*

General Terms: Algorithms, Design, Languages

Additional Key Words and Phrases: denormalized number, exception, floating-point, floating-point standard, gradual underflow, guard digit, NaN, overflow, relative error, rounding error, rounding mode, ulp, underflow

What Every Computer Scientist Should Know About Floating-Point Arithmetic,
David Goldberg, ACM Computing Surveys, 23(1), 1991

Recap

- Representing real numbers
- Fixed Point
- Floating Point

Next time: *More on floating points. How can a computer perform arithmetic operating on floating points?*