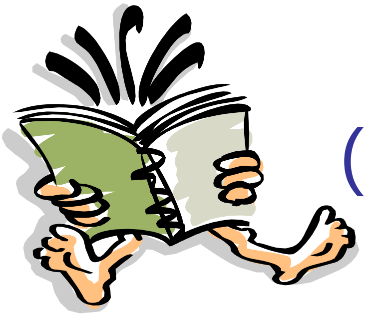
Analysis of Algorithms

Dynamic Programming



(Chapter 15)

Dynamic Programming

- An algorithm design technique (like divide and conquer)
- Divide and conquer
  - Partition the problem into independent subproblems
  - Solve the subproblems recursively
  - Combine the solutions to solve the original problem

Dynamic Programming

- Applicable when subproblems are **not** independent
  - Subproblems share subsubproblems

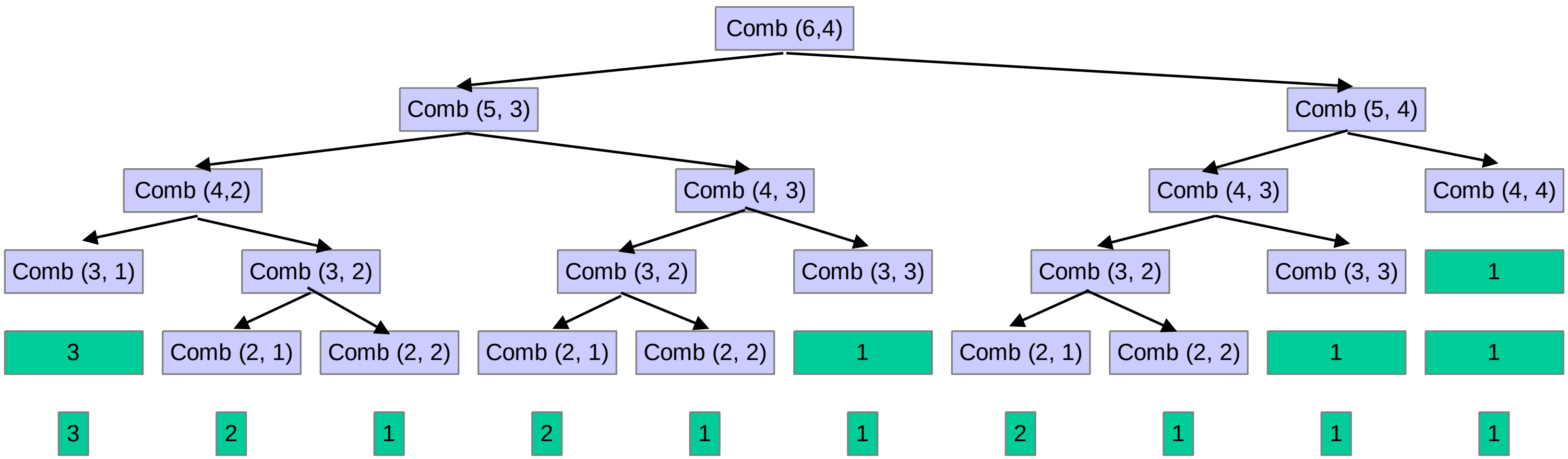
E.g.: Combinations:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{1} = 1 \qquad \binom{n}{n} = 1$$

  - A divide and conquer approach would repeatedly solve the common subproblems
  - Dynamic programming solves every subproblem just once and stores the answer in a table

Example: Combinations



$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Dynamic Programming

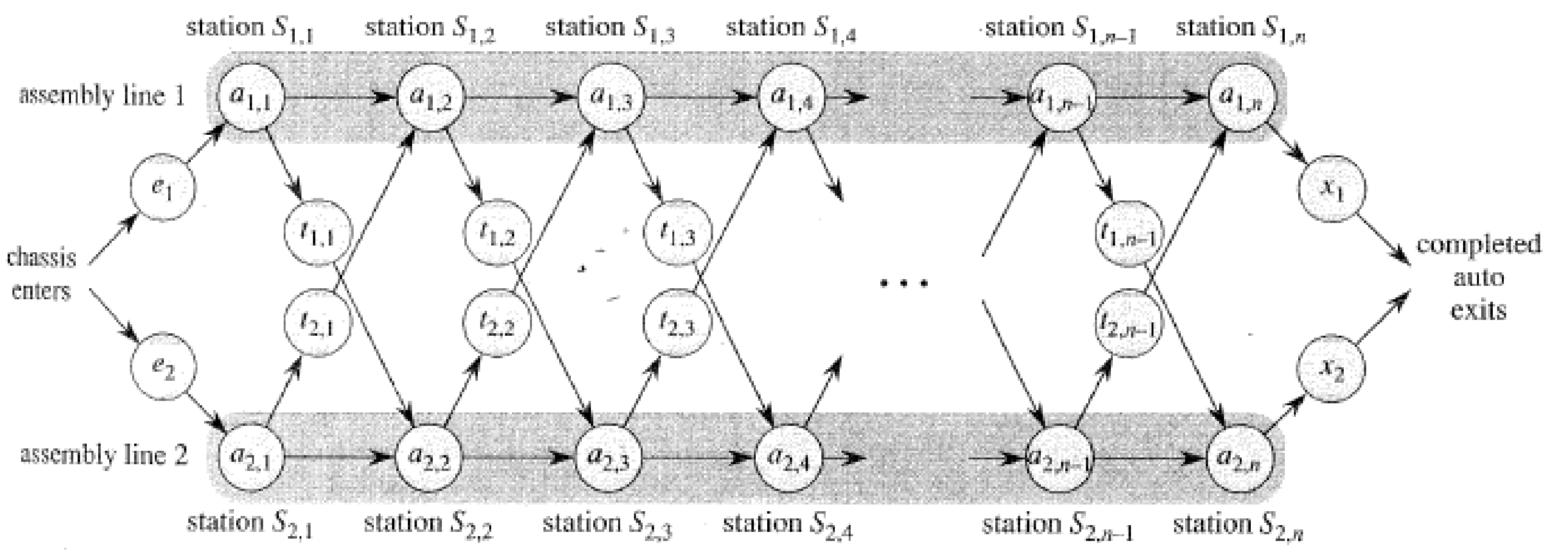
- Used for **optimization problems**
- **1- Breaks down the complex problem into simple subproblems.**
- **2- find the optimal solution of these subproblems**
- **3- store the result of these subproblems**
- **4- reuse them so that same subproblem is not calculated more than once.**
- **5- finally calculate the result of complex problem**
  – Our goal: **find an optimal solution**

Dynamic Programming Algorithm

1. **Characterize** the structure of an optimal solution
2. **Recursively** define the value of an optimal solution
3. **Compute** the value of an optimal solution in a bottom-up fashion
4. **Construct** an optimal solution from computed information (not always necessary)

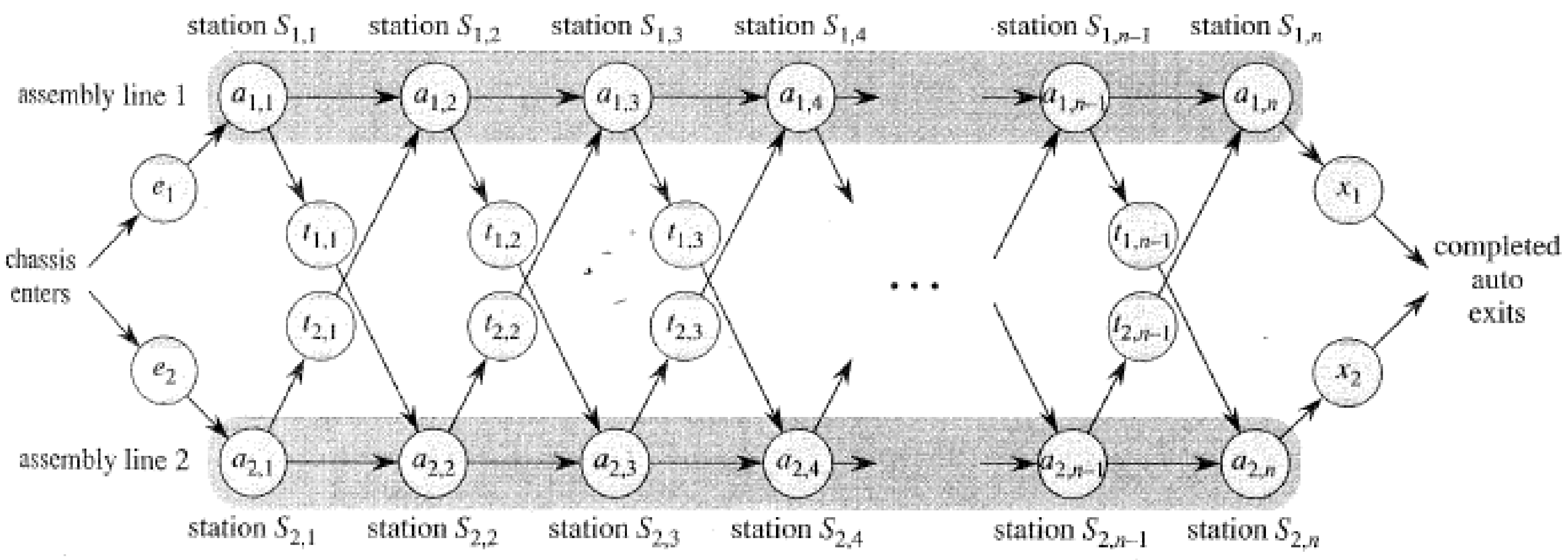
Assembly Line Scheduling

- Automobile factory with two assembly lines
  - Each line has **n** stations: $S_{1,1}, \ldots, S_{1,n}$ and $S_{2,1}, \ldots, S_{2,n}$
  - Corresponding stations $S_{1,j}$ and $S_{2,j}$ perform the same function but can take different amounts of time $a_{1,j}$ and $a_{2,j}$
  - Entry times are: e_1 and e_2; exit times are: x_1 and x_2



Assembly Line Scheduling

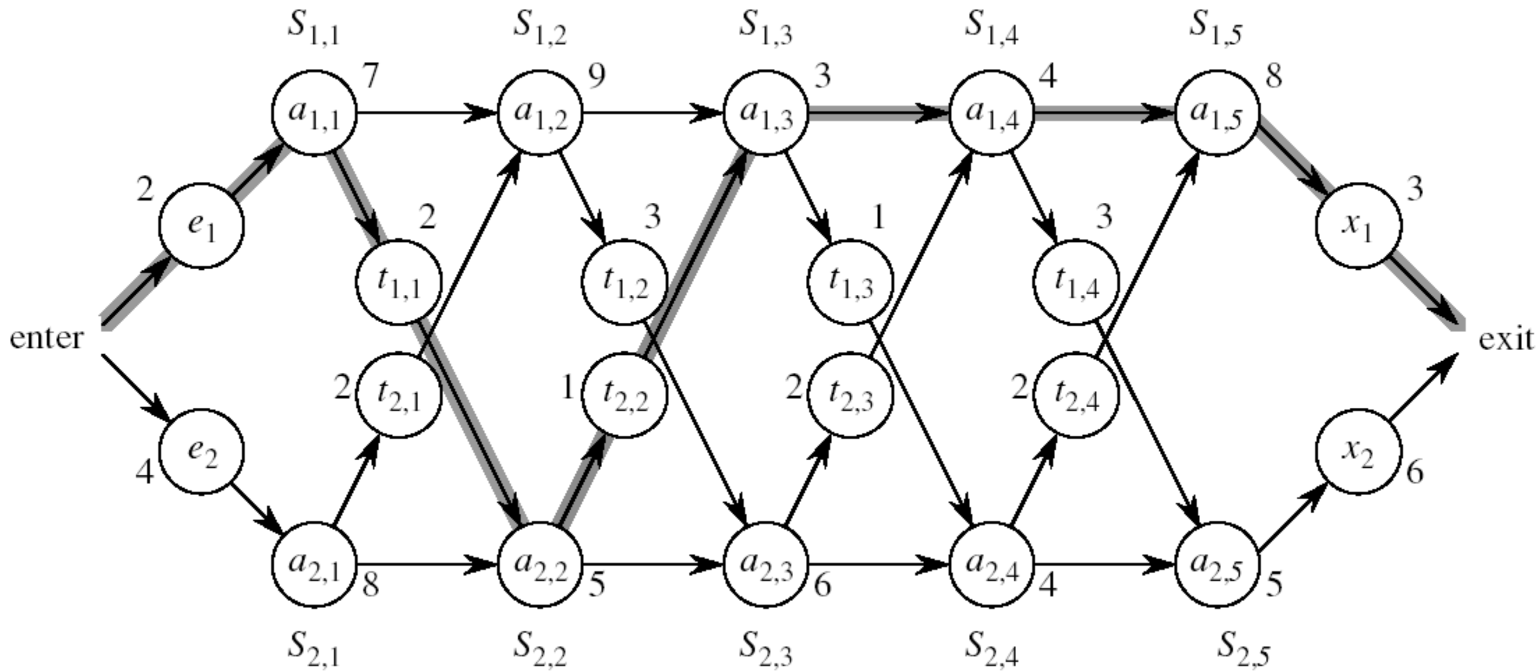
- After going through a station, can either:
  - stay on same line at no cost, or
  - transfer to other line: cost after $S_{i,j}$ is $t_{i,j}$, $j = 1, \ldots, n - 1$



Assembly Line Scheduling

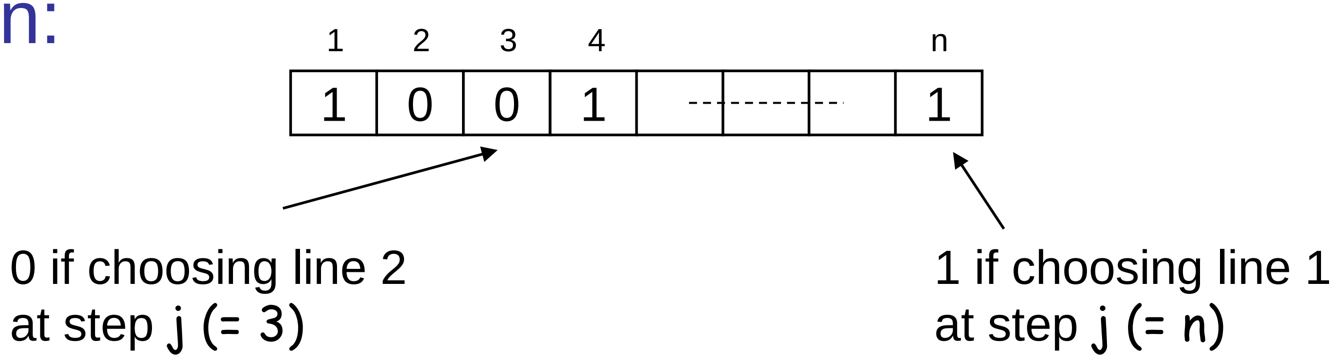
- Problem:

  what stations should be chosen from line 1 and which from line 2 in order to minimize the total time through the factory for one car?



One Solution

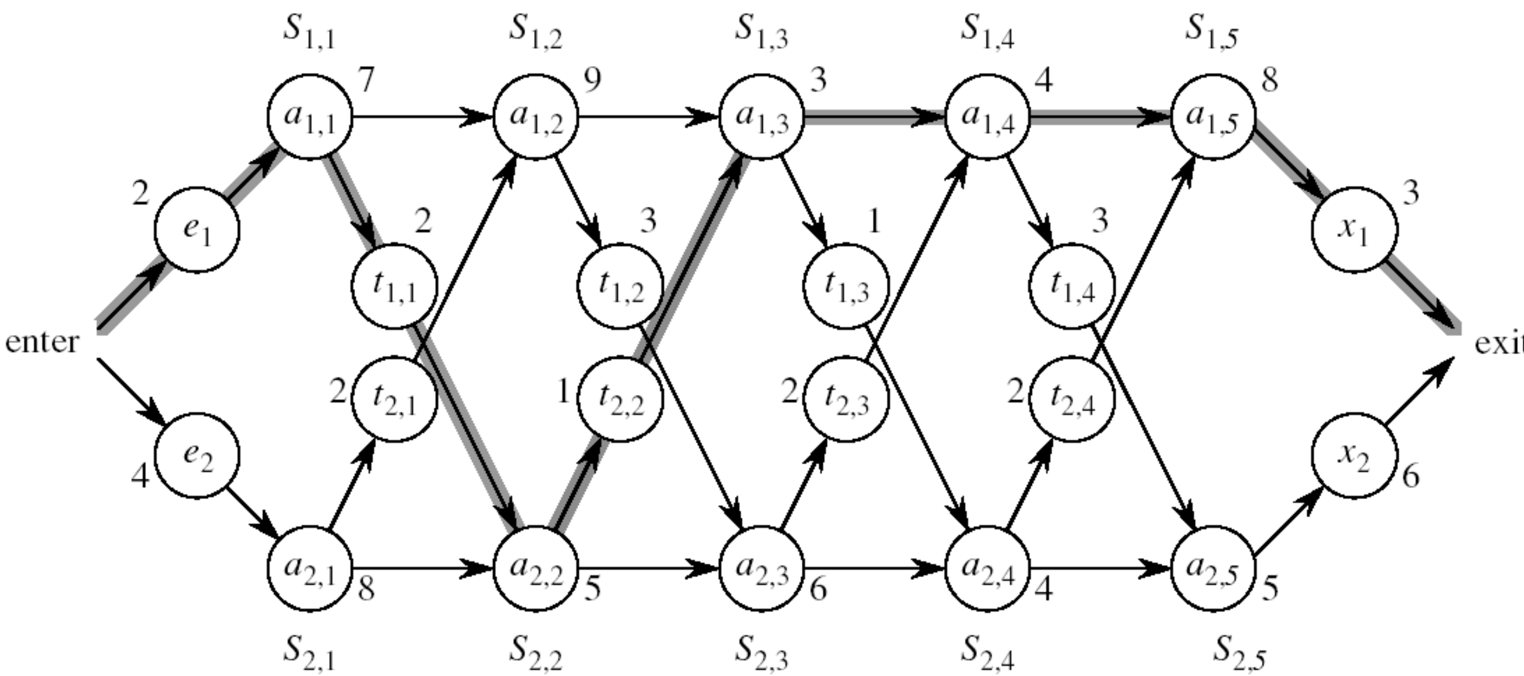
- Brute force
  - Enumerate all possibilities of selecting stations
  - Compute how long it takes in each case and choose the best one
- Solution:



  - There are 2^n possible ways to choose stations
  - Infeasible when n is large!!

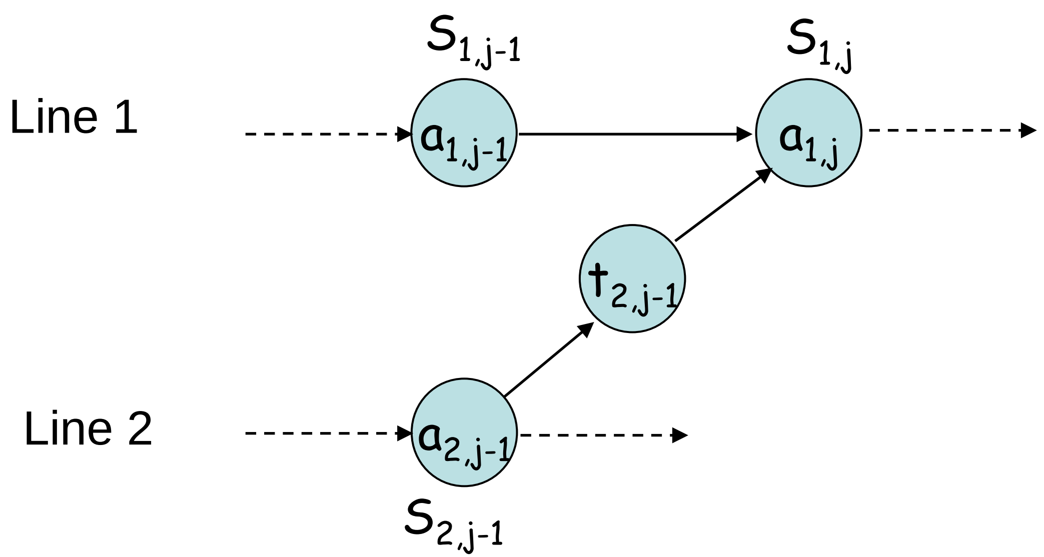
1. Structure of the Optimal Solution

- How do we compute the minimum time of going through a station?



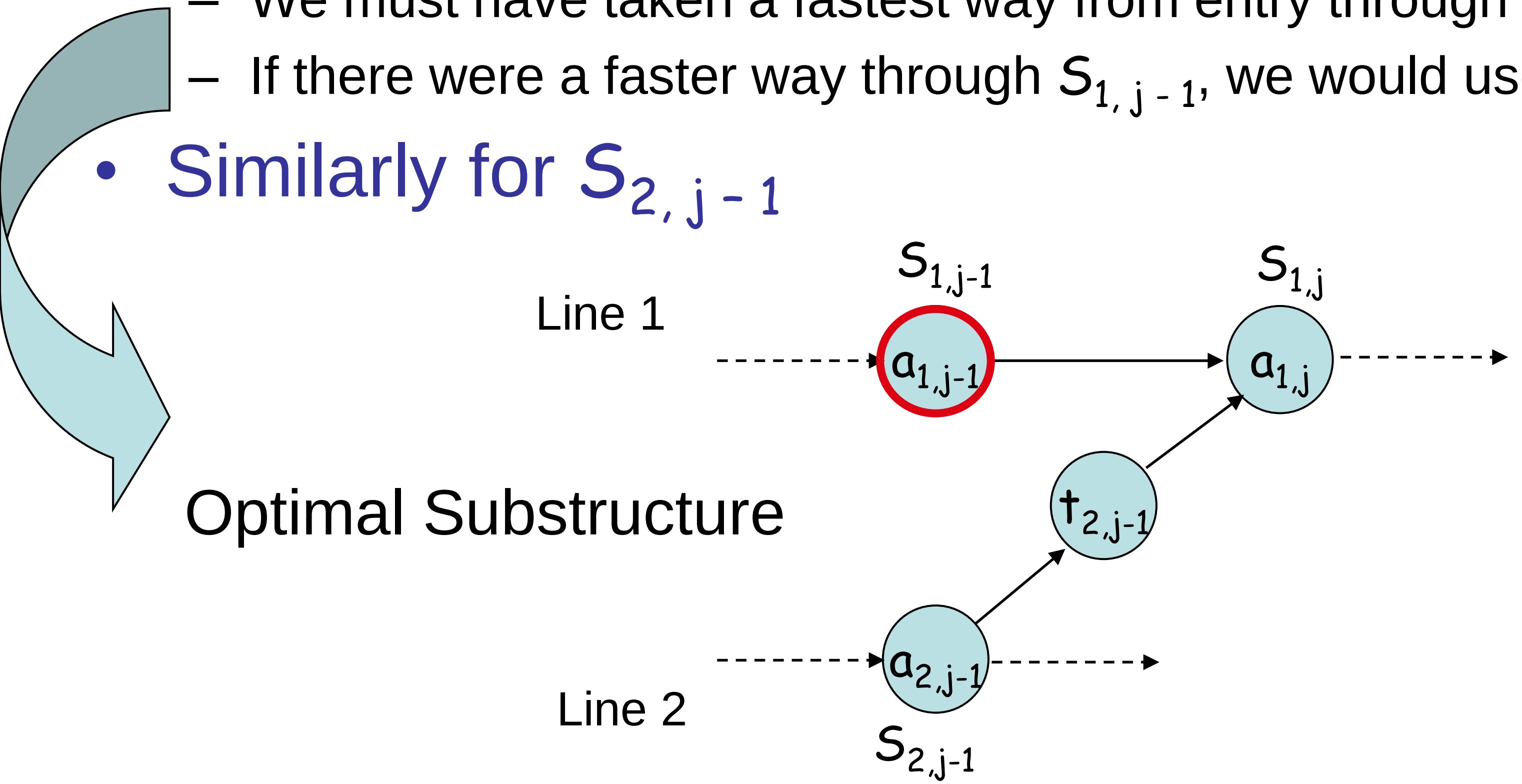
1. Structure of the Optimal Solution

- Let's consider all possible ways to get from the starting point through station $S_{1,j}$
  - We have two choices of how to get to $S_{1,j}$:
    - Through $S_{1,j-1}$, then directly to $S_{1,j}$
    - Through $S_{2,j-1}$, then transfer over to $S_{1,j}$



1. Structure of the Optimal Solution

- Suppose that the fastest way through $S_{1,j}$ is through $S_{1,j-1}$
  - We must have taken a fastest way from entry through $S_{1,j-1}$
  - If there were a faster way through $S_{1,j-1}$, we would use it instead
- Similarly for $S_{2,j-1}$



Optimal Substructure

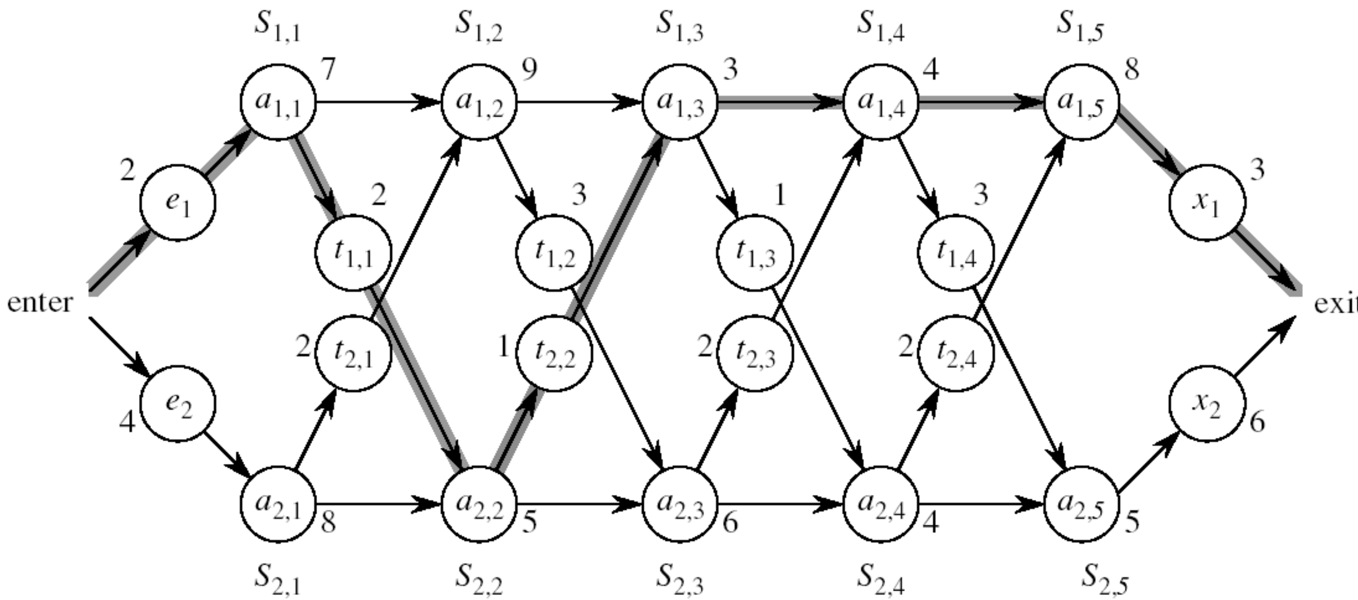
- **Generalization**: an optimal solution to the problem "*find the fastest way through* $S_{1,j}$" contains within it an optimal solution to subproblems: "*find the fastest way through* $S_{1,j-1}$ *or* $S_{2,j-1}$".

- This is referred to as the **optimal substructure** property

- We use this property to construct an optimal solution to a problem from optimal solutions to subproblems

2. A Recursive Solution

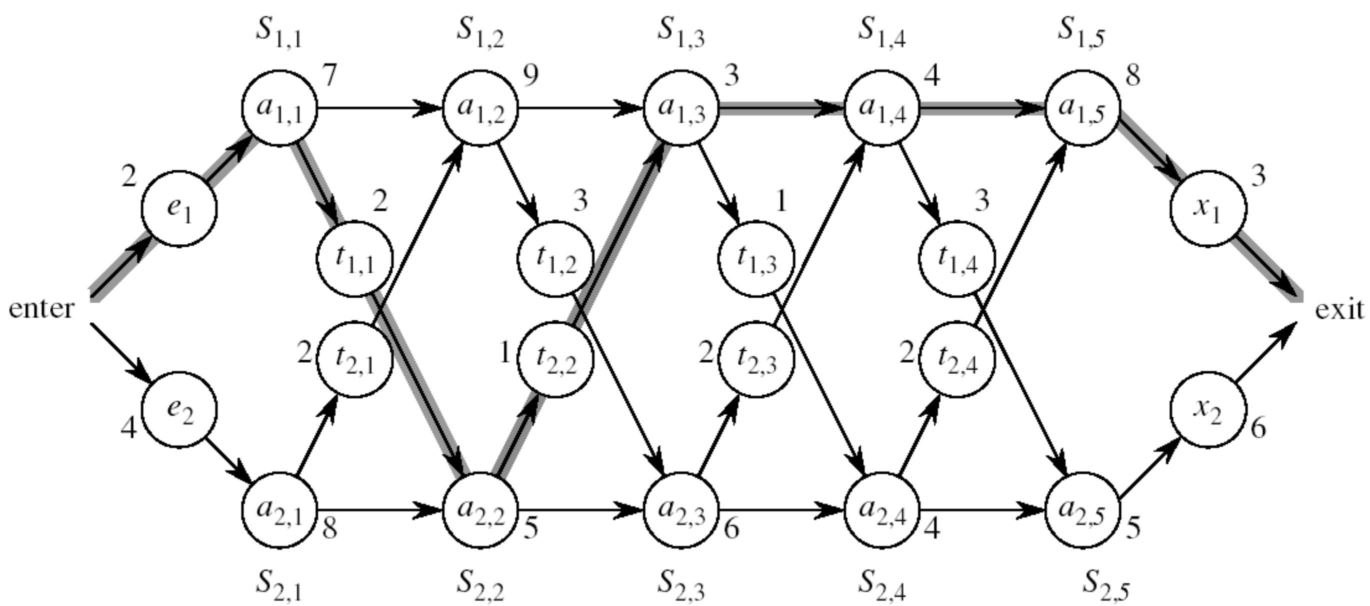
- Define the value of an optimal solution in terms of the optimal solution to subproblems



2. A Recursive Solution (cont.)

- Definitions:
  - f^* : the fastest time to get through the entire factory
  - $f_i[j]$: the fastest time to get from the starting point through station $S_{i,j}$

$$f^* = \min(f_1[n] + x_1,\ f_2[n] + x_2)$$

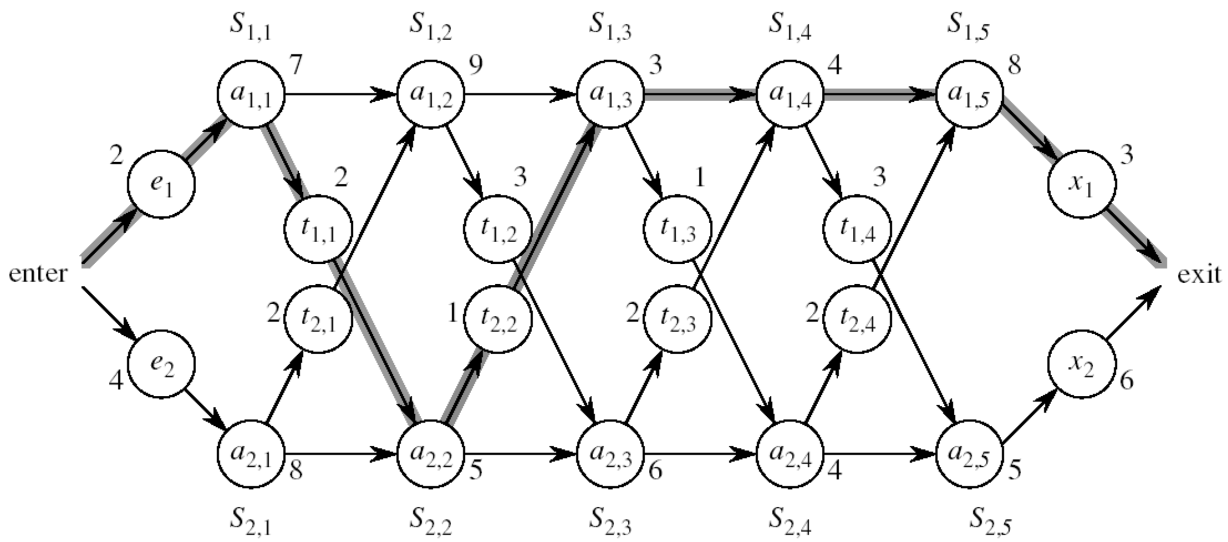


2. A Recursive Solution (cont.)

- Base case: $j = 1$, $i=1,2$ (getting through station 1)

  $f_1[1] = e_1 + a_{1,1}$

  $f_2[1] = e_2 + a_{2,1}$



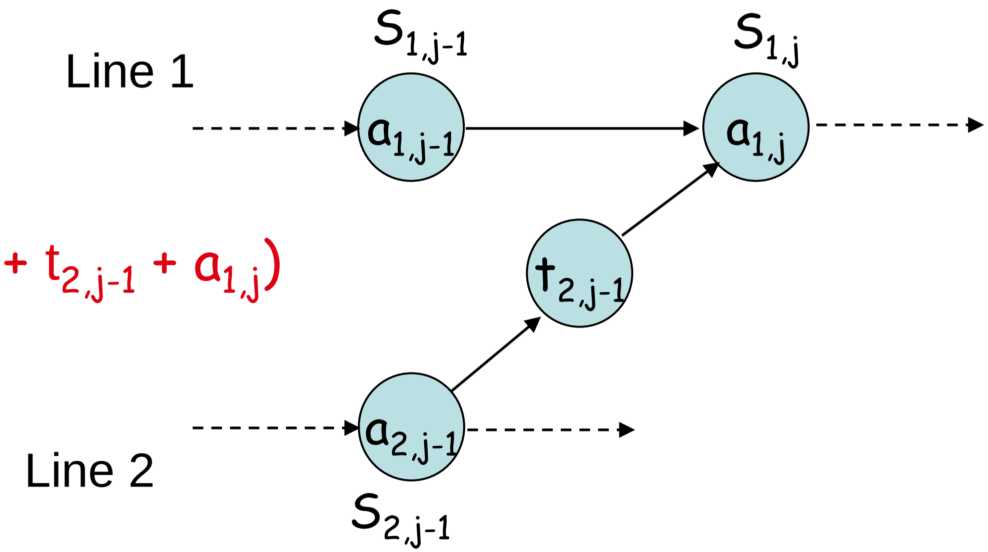
2. A Recursive Solution (cont.)

- General Case: $j = 2, 3, \ldots, n$, and $i = 1, 2$
- Fastest way through $S_{1,j}$ is either:
  - the way through $S_{1,j-1}$ then directly through $S_{1,j}$, or

    $f_1[j - 1] + a_{1,j}$
  - the way through $S_{2,j-1}$, transfer from line 2 to line 1, then through $S_{1,j}$

    $f_2[j - 1] + t_{2,j-1} + a_{1,j}$

$$f_1[j] = \min(f_1[j - 1] + a_{1,j}, f_2[j - 1] + t_{2,j-1} + a_{1,j})$$



2. A Recursive Solution (cont.)

$$f_1[j] = \begin{cases} e_1 + a_{1,1} & \text{if } j = 1 \\ \min(f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}) & \text{if } j \geq 2 \end{cases}$$

$$f_2[j] = \begin{cases} e_2 + a_{2,1} & \text{if } j = 1 \\ \min(f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}) & \text{if } j \geq 2 \end{cases}$$

3. Computing the Optimal Solution

$$f^* = \min(f_1[n] + x_1,\ f_2[n] + x_2)$$

$$f_1[j] = \min(f_1[j-1] + a_{1,j},\ f_2[j-1] + t_{2,j-1} + a_{1,j})$$

$$f_2[j] = \min(f_2[j-1] + a_{2,j},\ f_1[j-1] + t_{1,j-1} + a_{2,j})$$

	1	2	3	4	5
$f_1[j]$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$
$f_2[j]$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$

4 times 2 times

- Solving top-down would result in exponential running time

3. Computing the Optimal Solution

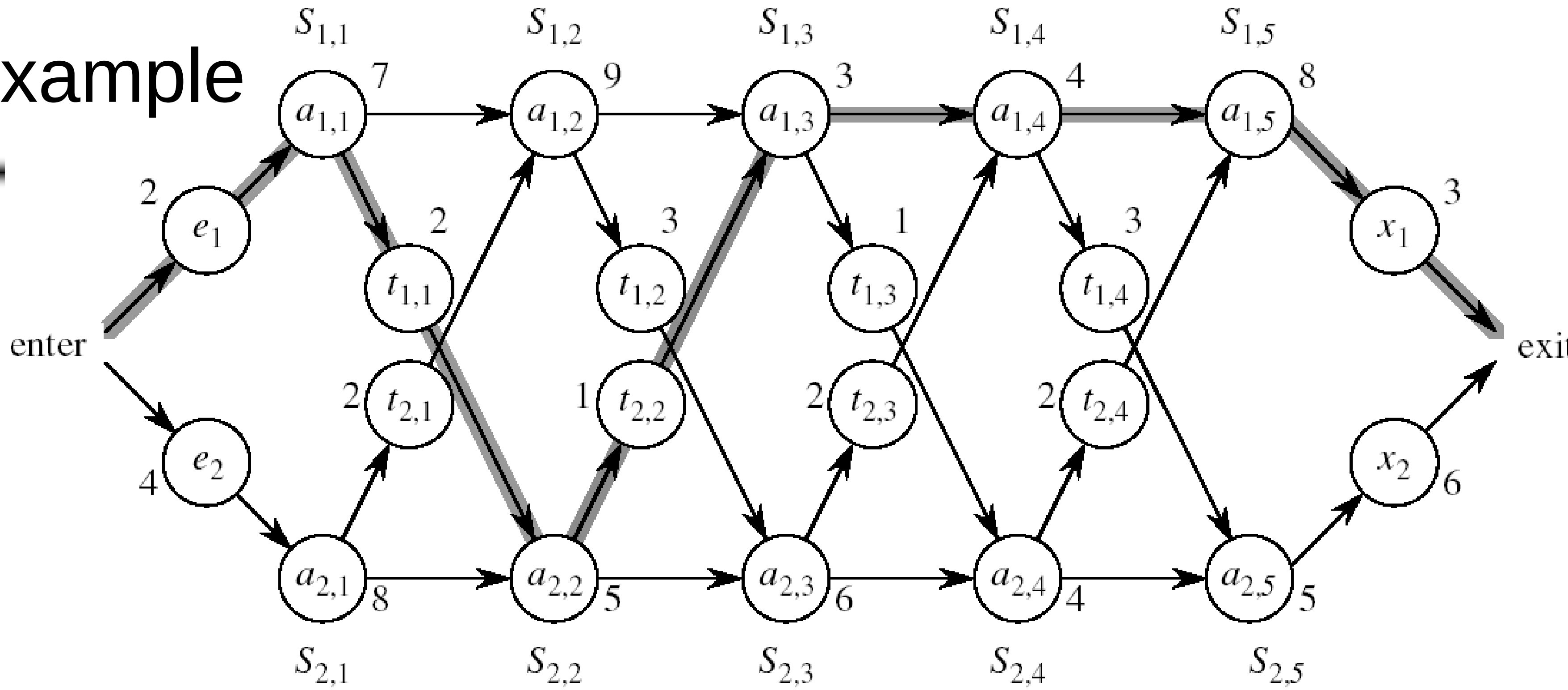
- For $j \geq 2$, each value $f_i[j]$ depends only on the values of $f_1[j - 1]$ and $f_2[j - 1]$
- Idea: compute the values of $f_i[j]$ as follows:

in increasing order of j

	1	2	3	4	5
$f_1[j]$					
$f_2[j]$					

- Bottom-up approach
  - First find optimal solutions to subproblems
  - Find an optimal solution to the problem from the subproblems

Example



$$f_1[j] = \begin{cases} e_1 + a_{1,1}, & \text{if } j = 1 \\ \min(f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}) & \text{if } j \geq 2 \end{cases}$$

	1	2	3	4	5
$f_1[j]$	9	18[1]	20[2]	24[1]	32[1]
$f_2[j]$	12	16[1]	22[2]	25[1]	30[2]

$f^* = 35$[1]

FASTEST-WAY(a, t, e, x, n)

$f_1[1] \leftarrow e_1 + a_{1,1}$
$f_2[1] \leftarrow e_2 + a_{2,1}$

Compute initial values of f_1 and f_2

for $j \leftarrow 2$ **to** n

O(N)

    do if $f_1[j - 1] + a_{1,j} \le f_2[j - 1] + t_{2, j-1} + a_{1, j}$
        then $f_1[j] \leftarrow f_1[j - 1] + a_{1, j}$
            $l_1[j] \leftarrow 1$
        else $f_1[j] \leftarrow f_2[j - 1] + t_{2, j-1} + a_{1, j}$
            $l_1[j] \leftarrow 2$

Compute the values of $f_1[j]$ and $l_1[j]$

    if $f_2[j - 1] + a_{2, j} \le f_1[j - 1] + t_{1, j-1} + a_{2, j}$
        then $f_2[j] \leftarrow f_2[j - 1] + a_{2, j}$
            $l_2[j] \leftarrow 2$
        else $f_2[j] \leftarrow f_1[j - 1] + t_{1, j-1} + a_{2, j}$
            $l_2[j] \leftarrow 1$

Compute the values of $f_2[j]$ and $l_2[j]$

FASTEST-WAY(a, t, e, x, n) (cont.)

```
if f1[n] + x1 ≤ f2[n] + x2
    then f* = f1[n] + x1
         l* = 1
    else f* = f2[n] + x2
         l* = 2
```

Compute the values of the fastest time through the entire factory

4. Construct an Optimal Solution

Alg.: PRINT-STATIONS(l, n)

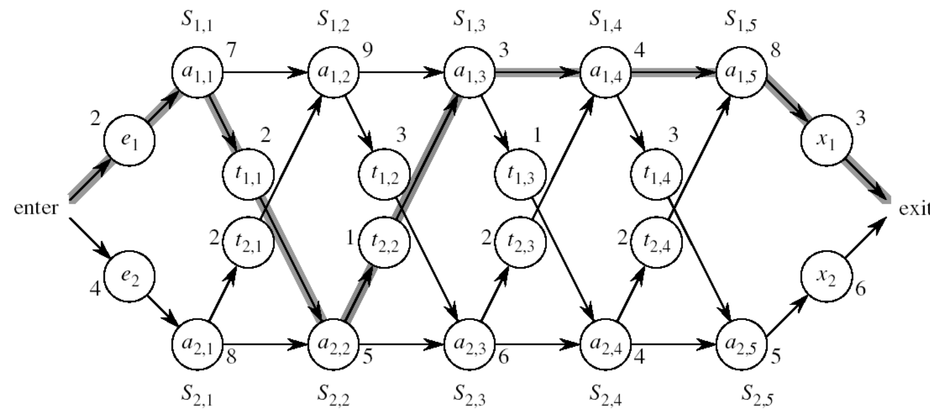
i ← l*

print "line " i ", station " n

for j ← n **downto** 2

do i ← $l_i[j]$

print "line " i ", station " j - 1



	1	2	3	4	5
$f_1[j]/l_1[j]$	9	18[1]	20[2]	24[1]	32[1]
$f_2[j]/l_2[j]$	12	16[1]	22[2]	25[1]	30[2]

l* = 1

Matrix-Chain Multiplication

Problem: given a sequence $\langle A_1, A_2, \ldots, A_n \rangle$, compute the product:

$$A_1 \cdot A_2 \cdots A_n$$

- Matrix compatibility:

$$C = A \cdot B$$

$col_A = row_B$

$row_C = row_A$

$col_C = col_B$

$$C = A_1 \cdot A_2 \cdots A_i \cdot A_{i+1} \cdots A_n$$

$col_i = row_{i+1}$

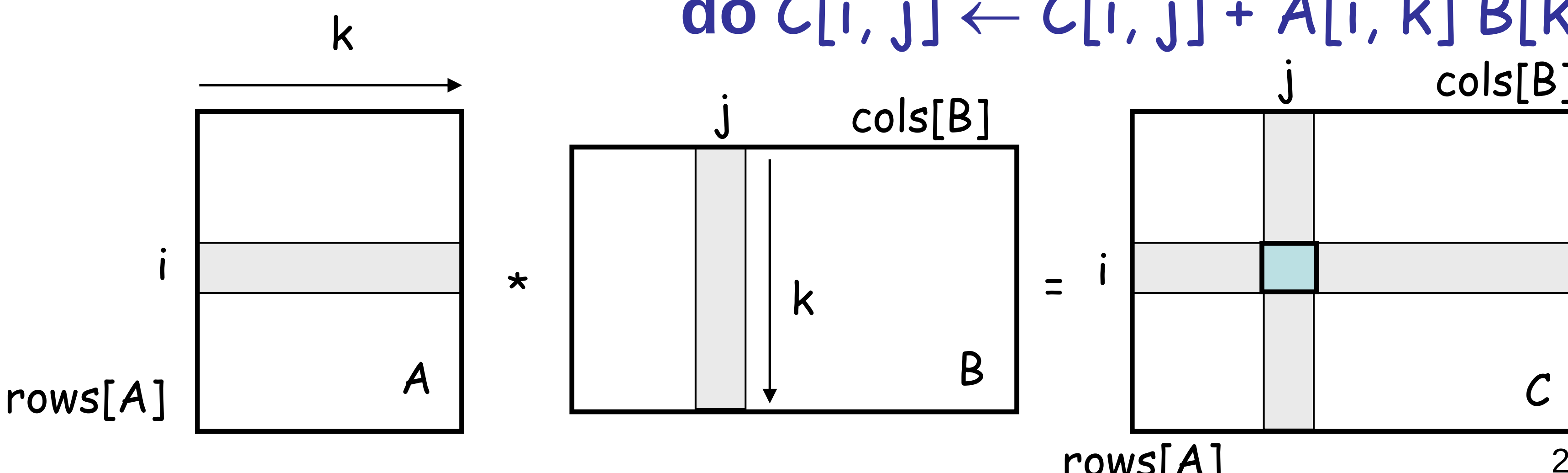
$row_C = row_{A1}$

$col_C = col_{An}$

MATRIX-MULTIPLY(A, B)

```
if columns[A] ≠ rows[B]
   then error "incompatible dimensions"
   else for i ← 1 to rows[A]
            do for j ← 1 to columns[B]
                  do C[i, j] = 0
                     for k ← 1 to columns[A]
                        do C[i, j] ← C[i, j] + A[i, k] B[k, j]
```

$$\overbrace{A[i,k]\,B[k,j]}^{\text{rows}[A]\cdot\text{cols}[A]\cdot\text{cols}[B]\ \text{multiplications}}$$



Matrix-Chain Multiplication

- In what order should we multiply the matrices?

$$A_1 \cdot A_2 \cdots A_n$$

- Parenthesize the product to get the order in which matrices are multiplied
- *E.g.:*

$$A_1 \cdot A_2 \cdot A_3 = ((A_1 \cdot A_2) \cdot A_3)$$
$$= (A_1 \cdot (A_2 \cdot A_3))$$

- Which one of these orderings should we choose?
  - The order in which we multiply the matrices has a significant impact on the cost of evaluating the product

Example

$$A_1 \cdot A_2 \cdot A_3$$

- A_1: 10 x 100
- A_2: 100 x 5
- A_3: 5 x 50

1. $((A_1 \cdot A_2) \cdot A_3)$: $A_1 \cdot A_2 = 10 \times 100 \times 5 = 5{,}000$ (10 x 5)

   $((A_1 \cdot A_2) \cdot A_3) = 10 \times 5 \times 50 = 2{,}500$

   Total: 7,500 scalar multiplications

2. $(A_1 \cdot (A_2 \cdot A_3))$: $A_2 \cdot A_3 = 100 \times 5 \times 50 = 25{,}000$ (100 x 50)

   $(A_1 \cdot (A_2 \cdot A_3)) = 10 \times 100 \times 50 = 50{,}000$

   Total: 75,000 scalar multiplications

one order of magnitude difference!!

Matrix-Chain Multiplication: Problem Statement

- Given a chain of matrices $\langle A_1, A_2, \ldots, A_n \rangle$, where A_i has dimensions $p_{i-1} \times p_i$, fully parenthesize the product $A_1 \cdot A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications.

$$\begin{array}{ccccccccc} A_1 & \cdot & A_2 & \cdots & A_i & \cdot & A_{i+1} & \cdots & A_n \\ p_0 \times p_1 & & p_1 \times p_2 & & p_{i-1} \times p_i & & p_i \times p_{i+1} & & p_{n-1} \times p_n \end{array}$$

What is the number of possible parenthesizations?

- Exhaustively checking all possible parenthesizations is not efficient!
- It can be shown that the number of parenthesizations grows as $\Omega(4^n/n^{3/2})$

  (see page 333 in your textbook)

1. The Structure of an Optimal Parenthesization

- Notation:

$$A_{i \dots j} = A_i \, A_{i+1} \cdots A_j, \; i \le j$$

- Suppose that an optimal parenthesization of $A_{i \dots j}$ splits the product between A_k and A_{k+1}, where $i \le k < j$

$$\begin{aligned} A_{i \dots j} &= A_i \, A_{i+1} \cdots A_j \\ &= A_i \, A_{i+1} \cdots A_k \, A_{k+1} \cdots A_j \\ &= A_{i \dots k} \, A_{k+1 \dots j} \end{aligned}$$

Optimal Substructure

$$A_{i\ldots j} = A_{i\ldots k}\, A_{k+1\ldots j}$$

- The parenthesization of the “prefix” $A_{i\ldots k}$ must be an optimal parentesization
- If there were a less costly way to parenthesize $A_{i\ldots k}$, we could substitute that one in the parenthesization of $A_{i\ldots j}$ and produce a parenthesization with a lower cost than the optimum $\Rightarrow$ contradiction!
- *An optimal solution to an instance of the matrix-chain multiplication contains within it optimal solutions to subproblems*

2. A Recursive Solution

- Subproblem:

  determine the minimum cost of parenthesizing

  $A_{i...j} = A_i A_{i+1} \cdots A_j$ for $1 \le i \le j \le n$

- Let $m[i, j]$ = the minimum number of multiplications needed to compute $A_{i...j}$
  - full problem ($A_{1..n}$): $m[1, n]$
  - $i = j$: $A_{i...i} = A_i \Rightarrow m[i, i] =$ 0, for i = 1, 2, …, n

2. A Recursive Solution

- Consider the subproblem of parenthesizing

$$A_{i\ldots j} = A_i\, A_{i+1} \cdots A_j \qquad \text{for } 1 \le i \le j \le n$$

$$= \underbrace{A_{i\ldots k}}_{m[i,\,k]}\ \underbrace{A_{k+1\ldots j}}_{m[k+1,j]} \quad p_{i-1}p_kp_j \qquad \text{for } i \le k < j$$

- Assume that the optimal parenthesization splits the product $A_i\, A_{i+1} \cdots A_j$ at k ($i \le k < j$)

$$m[i, j] = \underbrace{m[i, k]}_{\text{min \# of multiplications to compute } A_{i\ldots k}} + \underbrace{m[k+1, j]}_{\text{min \# of multiplications to compute } A_{k+1\ldots j}} + \underbrace{p_{i-1}p_kp_j}_{\text{\# of multiplications to compute } A_{i\ldots k}A_{k\ldots j}}$$

2. A Recursive Solution (cont.)

$$m[i, j] = m[i, k] \quad + \quad m[k+1, j] \quad + \quad p_{i-1}p_kp_j$$

- We do not know the value of k
  - There are j – i possible values for k: k = i, i+1, …, j-1
- Minimizing the cost of parenthesizing the product $A_i\ A_{i+1} \cdots A_j$ becomes:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

3. Computing the Optimal Costs

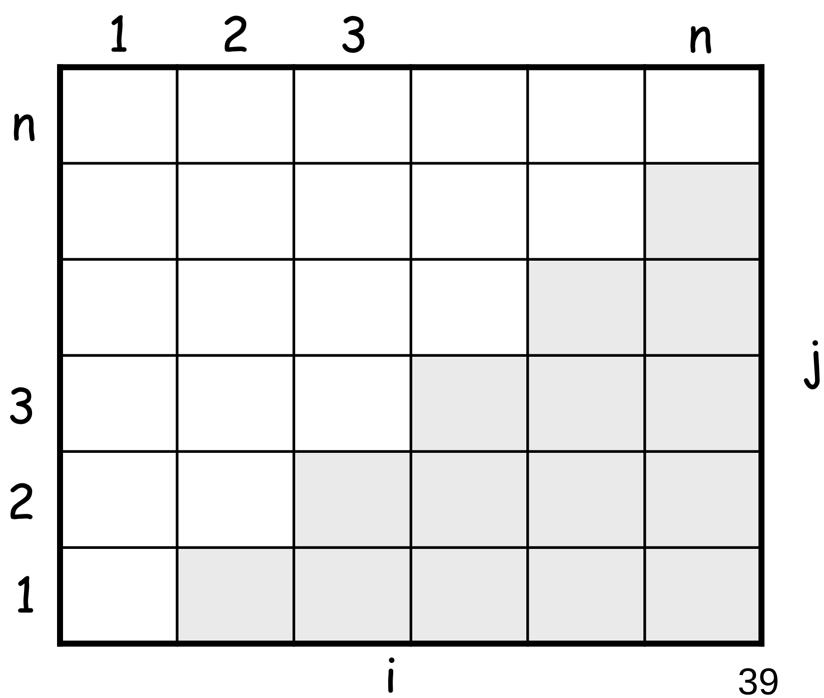
$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_k p_j\} & \text{if } i < j \end{cases}$$

- Computing the optimal solution recursively takes exponential time!
- How many subproblems?

  $\Rightarrow \Theta(n^2)$

  – Parenthesize $A_{i\ldots j}$ for $1 \le i \le j \le n$

  – One problem for each choice of i and j



3. Computing the Optimal Costs (cont.)

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- How do we fill in the tables m[1..n, 1..n]?
  - Determine which entries of the table are used in computing $m[i, j]$

$$A_{i\ldots j} = A_{i\ldots k} \, A_{k+1\ldots j}$$

  - Subproblems' size is one less than the original size
  - **Idea:** fill in m such that it corresponds to solving problems of increasing length

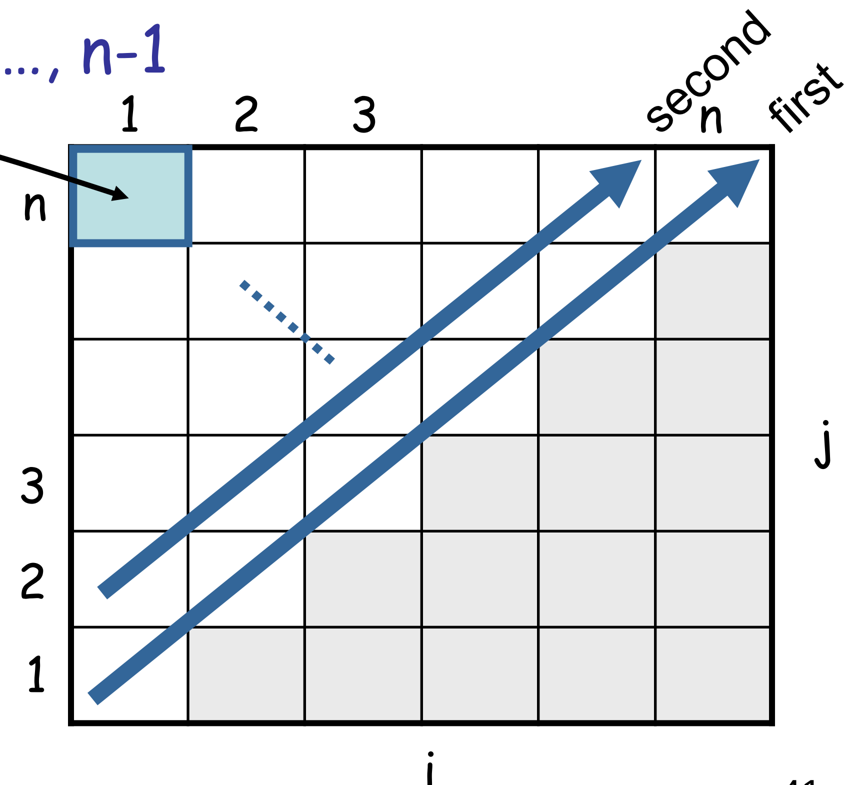
3. Computing the Optimal Costs (cont.)

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_k p_j\} & \text{if } i < j \end{cases}$$

- Length = 1: $i = j$, $i = 1, 2, \ldots, n$
- Length = 2: $j = i + 1$, $i = 1, 2, \ldots, n-1$

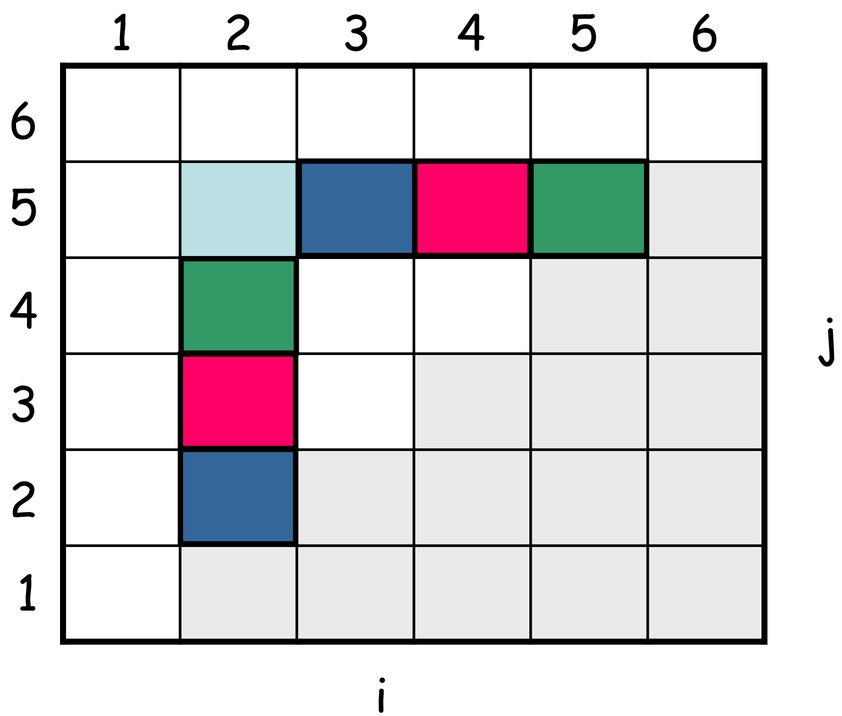
m[1, n] gives the optimal solution to the problem

Compute rows from bottom to top and from left to right



Example: $\min\{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\}$

$$m[2, 5] = \min \begin{cases} m[2, 2] + m[3, 5] + p_1p_2p_5 & k = 2 \\ m[2, 3] + m[4, 5] + p_1p_3p_5 & k = 3 \\ m[2, 4] + m[5, 5] + p_1p_4p_5 & k = 4 \end{cases}$$



- Values $m[i, j]$ depend only on values that have been previously computed

Example $\min\{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\}$

	1	2	3
3	2 7500	2 25000	0
2	1 5000	0	
1	0		

Compute $A_1 \cdot A_2 \cdot A_3$

- A_1: 10 x 100 (p_0 x p_1)
- A_2: 100 x 5 (p_1 x p_2)
- A_3: 5 x 50 (p_2 x p_3)

$m[i, i] = 0$ for $i = 1, 2, 3$

$m[1, 2] = m[1, 1] + m[2, 2] + p_0p_1p_2$ (A_1A_2)

$= 0 + 0 + 10 * 100 * 5 = 5{,}000$

$m[2, 3] = m[2, 2] + m[3, 3] + p_1p_2p_3$ (A_2A_3)

$= 0 + 0 + 100 * 5 * 50 = 25{,}000$

$$m[1, 3] = \min \begin{cases} m[1, 1] + m[2, 3] + p_0p_1p_3 = 75{,}000 & (A_1(A_2A_3)) \\ m[1, 2] + m[3, 3] + p_0p_2p_3 = 7{,}500 & ((A_1A_2)A_3) \end{cases}$$

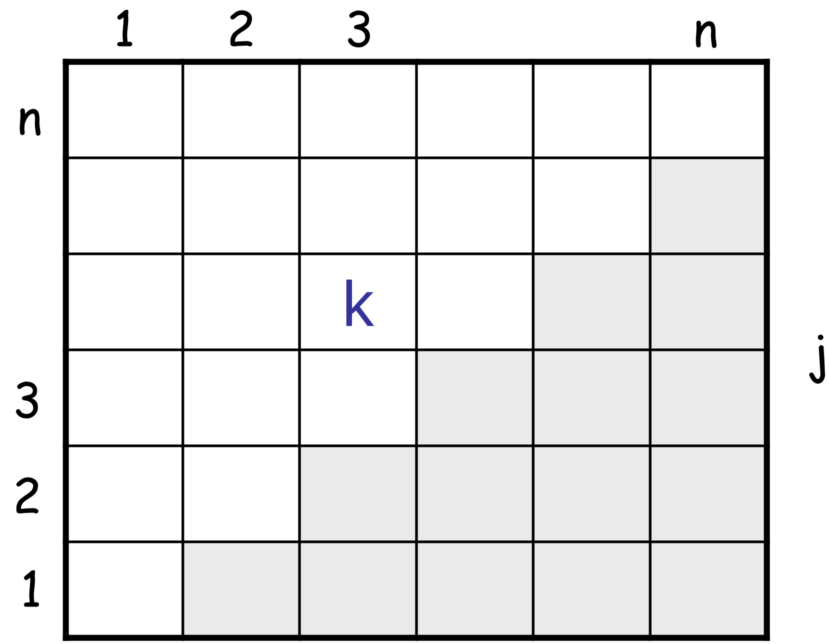
Matrix-Chain-Order(p)

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] − 1
for i ← 1 to n
     do m[i, i] ← 0
for l ← 2 to n                          ▷ l is the chain length.
     do for i ← 1 to n − l + 1
            do j ← i + l − 1
               m[i, j] ← ∞
               for k ← i to j − 1
                   do q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
                      if q < m[i, j]
                         then m[i, j] ← q
                              s[i, j] ← k
return m and s
```

$O(N^3)$

4. Construct the Optimal Solution

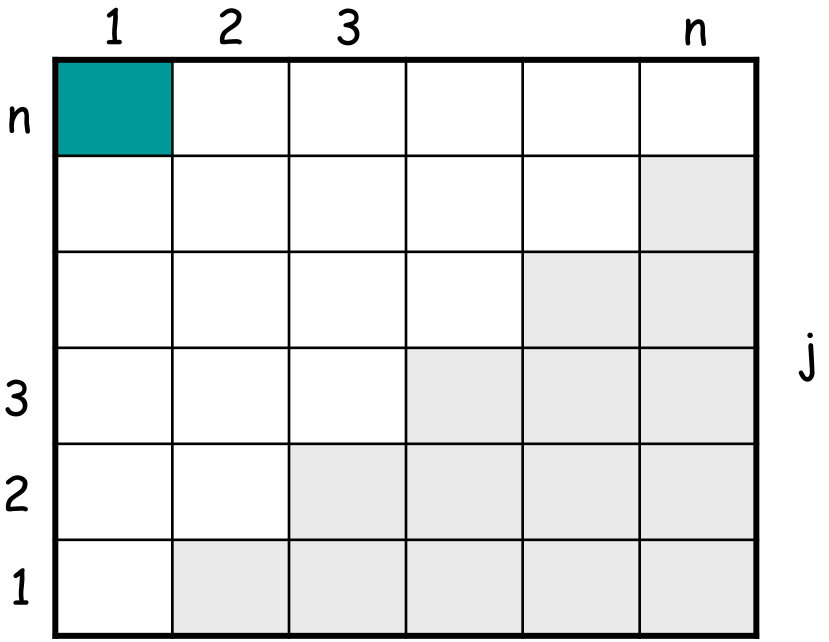
- In a similar matrix s we keep the optimal values of k
- $s[i, j]$ = a value of k such that an optimal parenthesization of $A_{i..j}$ splits the product between A_k and A_{k+1}



4. Construct the Optimal Solution

- $s[1, n]$ is associated with the entire product $A_{1..n}$
  - The final matrix multiplication will be split at k = $s[1, n]$

    $A_{1..n} = A_{1..s[1, n]} \cdot A_{s[1, n]+1..n}$
  - For each subproduct recursively find the corresponding value of k that results in an optimal parenthesization



4. Construct the Optimal Solution

- $s[i, j]$ = value of k such that the optimal parenthesization of $A_i\, A_{i+1} \cdots A_j$ splits the product between A_k and A_{k+1}

j \ i	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

- $s[1, n] = 3 \Rightarrow A_{1..6} = A_{1..3}\, A_{4..6}$
- $s[1, 3] = 1 \Rightarrow A_{1..3} = A_{1..1}\, A_{2..3}$
- $s[4, 6] = 5 \Rightarrow A_{4..6} = A_{4..5}\, A_{6..6}$

4. Construct the Optimal Solution (cont.)

PRINT-OPT-PARENS(s, i, j)

if i = j

then print "A"$_i$

else print "("

PRINT-OPT-PARENS(s, i, $s[i, j]$)

PRINT-OPT-PARENS(s, $s[i, j] + 1$, j)

print ")"

j \ i	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

Example: $A_1 \cdot \cdot \cdot A_6$ $((A_1(A_2A_3))((A_4A_5)A_6))$

PRINT-OPT-PARENS(s, i, j)

if $i = j$

  then print "A"$_i$

  else print "("

    PRINT-OPT-PARENS(s, i, $s[i, j]$)

    PRINT-OPT-PARENS(s, $s[i, j] + 1$, j)

    print ")"

s[1..6, 1..6]

j \ i	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

P-O-P(s, 1, 6) $s[1, 6] = 3$

$i = 1, j = 6$ "("   P-O-P (s, 1, 3) $s[1, 3] = 1$

$i = 1, j = 3$ "("   P-O-P(s, 1, 1) $\Rightarrow$ "A_1"

P-O-P(s, 2, 3) $s[2, 3] = 2$

$i = 2, j = 3$ "("   P-O-P (s, 2, 2) $\Rightarrow$ "A_2"

P-O-P (s, 3, 3) $\Rightarrow$ "A_3"

")"

")"   …

Memoization

- Top-down approach with the efficiency of typical dynamic programming approach
- Maintaining an entry in a table for the solution to each subproblem
  - **memoize** the inefficient recursive algorithm
- When a subproblem is first encountered its solution is computed and stored in that table
- Subsequent “calls” to the subproblem simply look up that value

Memoized Matrix-Chain

Alg.: MEMOIZED-MATRIX-CHAIN(p)

1. $n \leftarrow length[p] - 1$
2. **for** $i \leftarrow 1$ **to** n
3. **do for** $j \leftarrow i$ **to** n
4. **do** $m[i, j] \leftarrow \infty$
5. **return** LOOKUP-CHAIN($p, 1, n$)

Initialize the m table with large values that indicate whether the values of $m[i, j]$ have been computed

Top-down approach

Memoized Matrix-Chain

Alg.: LOOKUP-CHAIN(p, i, j)

Running time is $O(n^3)$

```
if m[i, j] < ∞
    then return m[i, j]
if i = j
    then m[i, j] ← 0
    else for k ← i to j - 1
            do q ← LOOKUP-CHAIN(p, i, k) +
                   LOOKUP-CHAIN(p, k+1, j) + p_{i-1} p_k p_j
               if q < m[i, j]
                  then m[i, j] ← q
return m[i, j]
```

Dynamic Progamming vs. Memoization

- Advantages of dynamic programming vs. memoized algorithms
  - No overhead for recursion, less overhead for maintaining the table
  - The regular pattern of table accesses may be used to reduce time or space requirements
- Advantages of memoized algorithms vs. dynamic programming
  - Some subproblems do not need to be solved

Elements of Dynamic Programming

- Optimal Substructure
  - An optimal solution to a problem contains within it an optimal solution to subproblems
  - Optimal solution to the entire problem is build in a bottom-up manner from optimal solutions to subproblems
- Overlapping Subproblems
  - If a recursive algorithm revisits the same subproblems over and over $\Rightarrow$ the problem has overlapping subproblems

Parameters of Optimal Substructure

- How many subproblems are used in an optimal solution for the original problem
  - Assembly line: One subproblem (the line that gives best time)
  - Matrix multiplication: Two subproblems (subproducts $A_{i..k}$, $A_{k+1..j}$)
- How many choices we have in determining which subproblems to use in an optimal solution
  - Assembly line: Two choices (line 1 or line 2)
  - Matrix multiplication: $j - i$ choices for k (splitting the product)

Parameters of Optimal Substructure

- Intuitively, the running time of a dynamic programming algorithm depends on two factors:
  - Number of subproblems overall
  - How many choices we look at for each subproblem
- Assembly line
  - $\Theta(n)$ subproblems (n stations)
  - 2 choices for each subproblem

  $\Theta(n)$ overall
- Matrix multiplication:
  - $\Theta(n^2)$ subproblems ($1 \leq i \leq j \leq n$)
  - At most $n-1$ choices

  $\Theta(n^3)$ overall

Longest Common Subsequence

- Given two sequences

  $X = \langle x_1, x_2, \dots, x_m \rangle$

  $Y = \langle y_1, y_2, \dots, y_n \rangle$

  find a maximum length common subsequence (LCS) of X and Y

- *E.g.:*

  $X = \langle A, B, C, B, D, A, B \rangle$

- Subsequences of X:
  - A subset of elements in the sequence taken in order

  $\langle A, B, D \rangle$, $\langle B, C, D, B \rangle$, etc.

Example

$X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

$X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

- $\langle B, C, B, A \rangle$ and $\langle B, D, A, B \rangle$ are longest common subsequences of X and Y (length = 4)

- $\langle B, C, A \rangle$, however is not a LCS of X and Y

Brute-Force Solution

- For every subsequence of X, check whether it’s a subsequence of Y
- There are 2^m subsequences of X to check
- Each subsequence takes $\Theta(n)$ time to check
  - scan Y for first letter, from there scan for second, and so on
- Running time: $\Theta(n2^m)$

Making the choice

$X = \langle A, B, D, E \rangle$

$Y = \langle Z, B, E \rangle$

- Choice: include one element into the common sequence (E) and solve the resulting subproblem

$X = \langle A, B, D, G \rangle$

$Y = \langle Z, B, D \rangle$

- Choice: exclude an element from a string and solve the resulting subproblem

Notations

- Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$ we define the i-th prefix of X, for i = 0, 1, 2, …, m

  $X_i = \langle x_1, x_2, \ldots, x_i \rangle$

- $c[i, j]$ = the length of a LCS of the sequences $X_i = \langle x_1, x_2, \ldots, x_i \rangle$ and $Y_j = \langle y_1, y_2, \ldots, y_j \rangle$

A Recursive Solution

Case 1: $x_i = y_j$

e.g.: $X_i = \langle A, B, D, E \rangle$

$Y_j = \langle Z, B, E \rangle$

$$c[i, j] = c[i - 1, j - 1] + 1$$

– Append $x_i = y_j$ to the LCS of X_{i-1} and Y_{j-1}

– Must find a LCS of X_{i-1} and Y_{j-1} $\Rightarrow$ optimal solution to a problem includes optimal solutions to subproblems

A Recursive Solution

Case 2: $x_i \neq y_j$

e.g.: $X_i = \langle A, B, D, G \rangle$

$Y_j = \langle Z, B, D \rangle$

$$c[i, j] = \max \{ c[i - 1, j], c[i, j-1] \}$$

– Must solve two problems

- find a LCS of X_{i-1} and Y_j: $X_{i-1} = \langle A, B, D \rangle$ and $Y_j = \langle Z, B, D \rangle$
- find a LCS of X_i and Y_{j-1}: $X_i = \langle A, B, D, G \rangle$ and $Y_j = \langle Z, B \rangle$

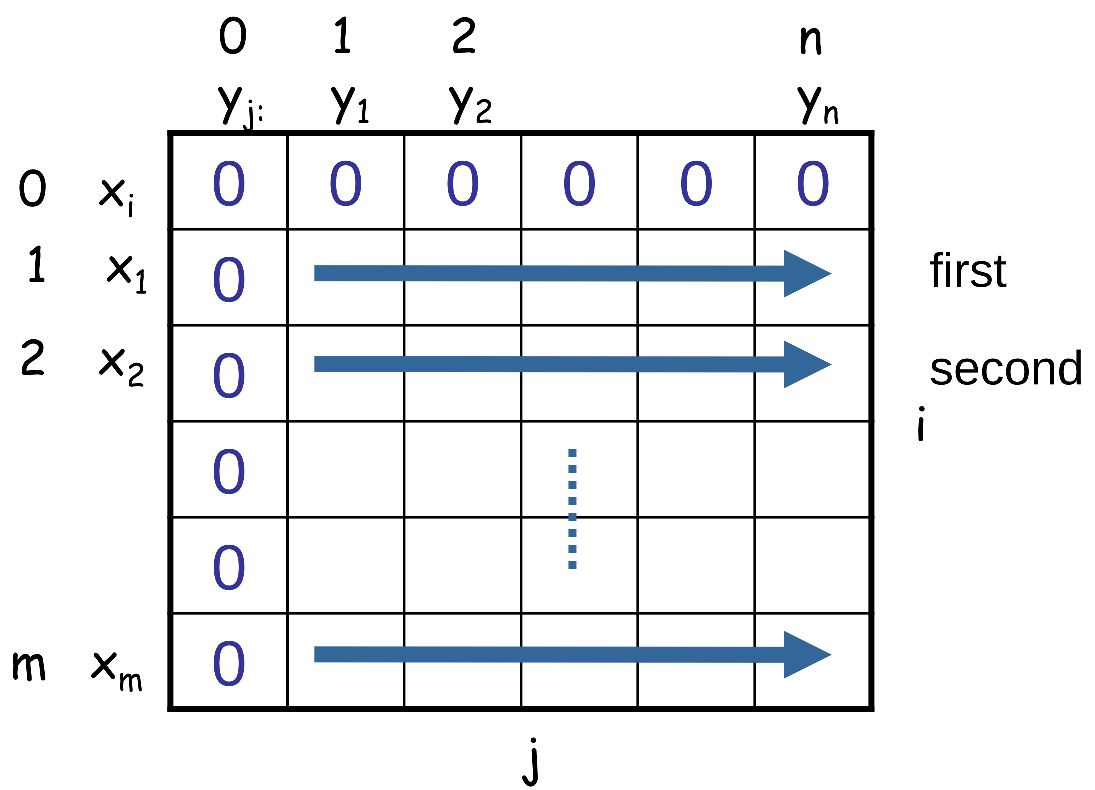
- Optimal solution to a problem includes optimal solutions to subproblems

Overlapping Subproblems

- To find a LCS of X and Y
  - we may need to find the LCS between X and Y_{n-1} and that of X_{m-1} and Y
  - Both the above subproblems has the subproblem of finding the LCS of X_{m-1} and Y_{n-1}
- Subproblems share subsubproblems

3. Computing the Length of the LCS

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$



Additional Information

$$c[i, j] = \begin{cases} 0 & \text{if } i,j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

b & c:

		0	1	2	3		n
		y_j:	A	C	D		F
0	x_i	0	0	0	0	0	0
1	A	0					
2	B	0			c[i-1,j]		
3	C	0		↖ c[i,j-1]	↑		
		0					
m	D	0					

i

j

A matrix b[i, j]:

- For a subproblem [i, j] it tells us what choice was made to obtain the optimal value
- If $x_i = y_j$
  b[i, j] = "↖"
- Else, if
  c[i - 1, j] ≥ c[i, j-1]
  b[i, j] = " ↑ "
  else
  b[i, j] = " ← "

LCS-LENGTH(X, Y, m, n)

```
for i ← 1 to m
    do c[i, 0] ← 0
for j ← 0 to n
    do c[0, j] ← 0
for i ← 1 to m
    do for j ← 1 to n
        do if xi = yj
            then c[i, j] ← c[i - 1, j - 1] + 1
                 b[i, j ] ← "↖"
            else if c[i - 1, j] ≥ c[i, j - 1]
                then c[i, j] ← c[i - 1, j]
                     b[i, j] ← "↑"
                else c[i, j] ← c[i, j - 1]
                     b[i, j] ← "←"
return c and b
```

The length of the LCS if one of the sequences is empty is zero

Case 1: $x_i = y_j$

Case 2: $x_i \neq y_j$

Running time: $\Theta(mn)$

Example

$X = \langle A, B, C, B, D, A \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

If $x_i = y_j$

$b[i, j] =$ " $\nwarrow$ "

Else if

$c[i - 1, j] \geq c[i, j-1]$

$b[i, j] =$ " $\uparrow$ "

else

$b[i, j] =$ " $\leftarrow$ "

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	$\uparrow$ 0	$\uparrow$ 0	$\uparrow$ 0	$\nwarrow$ 1	$\leftarrow$ 1	$\nwarrow$ 1
2	B	0	$\nwarrow$ 1	$\leftarrow$ 1	$\leftarrow$ 1	$\uparrow$ 1	$\nwarrow$ 2	$\leftarrow$ 2
3	C	0	$\uparrow$ 1	$\uparrow$ 1	$\nwarrow$ 2	$\leftarrow$ 2	$\uparrow$ 2	$\uparrow$ 2
4	B	0	$\nwarrow$ 1	$\uparrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\nwarrow$ 3	$\leftarrow$ 3
5	D	0	$\uparrow$ 1	$\nwarrow$ 2	$\uparrow$ 2	$\uparrow$ 2	$\uparrow$ 3	$\uparrow$ 3
6	A	0	$\uparrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\nwarrow$ 3	$\uparrow$ 3	$\nwarrow$ 4
7	B	0	$\nwarrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\uparrow$ 3	$\nwarrow$ 4	$\uparrow$ 4

4. Constructing a LCS

- Start at b[m, n] and follow the arrows
- When we encounter a "↖" in b[i, j] $\Rightarrow x_i = y_j$ is an element of the LCS

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2	B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3	C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

PRINT-LCS(b, X, i, j)

Running time: $\Theta(m + n)$

if $i = 0$ or $j = 0$
  then return
if $b[i, j] =$ "↖"
  then PRINT-LCS(b, X, $i - 1$, $j - 1$)
    print x_i
elseif $b[i, j] =$ "↑"
    then PRINT-LCS(b, X, $i - 1$, j)
    else PRINT-LCS(b, X, i, $j - 1$)

Initial call: PRINT-LCS(b, X, $length[X]$, $length[Y]$)

Improving the Code

- What can we say about how each entry $c[i, j]$ is computed?
  - It depends only on $c[i - 1, j - 1]$, $c[i - 1, j]$, and $c[i, j - 1]$
  - Eliminate table b and compute in $O(1)$ which of the three values was used to compute $c[i, j]$
  - We save $\Theta(mn)$ space from table b
  - However, we do not asymptotically decrease the auxiliary space requirements: still need table c

Improving the Code

- If we only need the length of the LCS
  - LCS-LENGTH works only on two rows of c at a time
    - The row being computed and the previous row
  - We can reduce the asymptotic space requirements by storing only these two rows