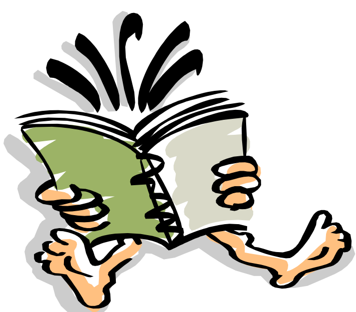
Discrete Structures 2

Chapter 9: Relations



Chapter 9: Relations

- Relations and Their Properties.
- Representing Relations.
- Closures of Relations.
- Equivalence Relations.
- Partial Orderings.

Introduction (1/2)

Relationships between elements of sets are represented using the structure called a **relation**, which is just a subset of the Cartesian product of the sets.

In mathematics, we study relationships such as those between a *positive integer and one that it divides*, *an integer and one that it is congruent to modulo 5*, *a real number and one that is larger than it*, *a real number* x *and the value* $f(x)$ *where* f *is a function*, and so on.

Introduction (2/2)

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

Definition 1:

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

A *binary relation* from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use the notation $a\,R\,b$ to denote that $(a,b) \in R$ and $a\,\not R\,b$ to denote that $(a,b) \notin R$. Moreover, when (a,b) belongs to R, a is said to be related to b by R.

Example 1:

Let $A = \{0, 1 , 2\}$ and $B = \{a , b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.

Roster notation (Roster form of set):

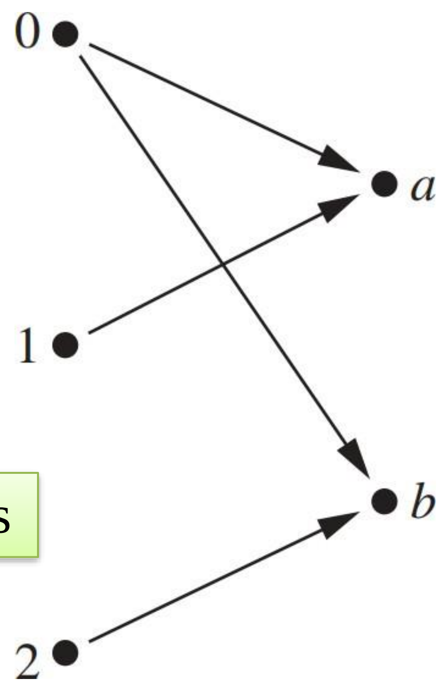
$R = \{(0, a), (0, b), (1, a), (2, b)\}$

Relations and Their Properties (3/30)

Example 1:

Let $A = \{0, 1\ , 2\}$ and $B = \{a\ , b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.



Using arrows

R	a	b
0	×	×
1	×	
2		×

Using table

Functions as Relations

Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B.

Relations on a Set

Definitions:

- A relation on the set A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.
- The identity relation I_A on a set A is the set $\{(a, a) | a \in A\}$
  - Ex. If $A = \{1, 2, 3\}$, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

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Set builder notation:

$$R = \{(a, b) | a \text{ divides } b\}$$

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

May change to be:

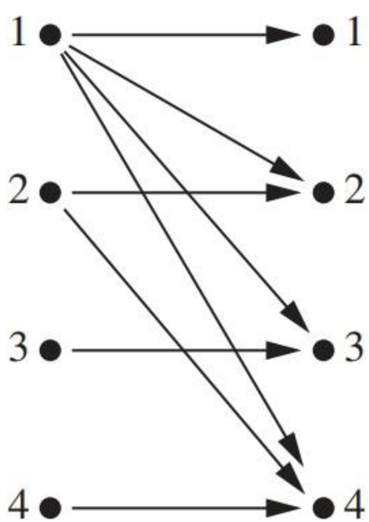
$a = b$
$a > b$
$a < b$
…

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

Solution:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Example 3:

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$R_1 = \{(a, b) | a < b\}$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = -b\}$

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | 0 \leq a + b \leq 1\}$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_1 = \{(a\ , b) | a < b\}$$

$$= \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_2 = \{(a, b) | a > b\}$$

$$= \{(0, -1), (1, 0), (1, -1), (2, 1), (2, 0), (2, -1)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_3 = \{(a\ , b) | a = b\}$$
$$= \{(-1, -1), (0, 0), (1, 1), (2, 2)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_4 = \{(a\ , b) | a = -b\}$$

$$= \{(-1, 1), (0, 0), (1, -1)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_3 = \{(a\,, b)|a = b\} = \{(-1, -1), (0, 0), (1, 1), (2, 2)\}$$

$$R_4 = \{(a\,, b)|a = -b\} = \{(-1, 1), (0, 0), (1, -1)\}$$

$$R_5 = \{(a\,, b)|a = b \text{ or } a = -b\}$$

$$= \{(-1, -1), (0, 0), (1, 1), (2, 2), (-1, 1), (1, -1)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_6 = \{(a\ , b) | 0 \leq a + b \leq 1\}$$

$$= \{(-1, 1), (-1, 2), (0, 0), (0, 1), (1, -1), (1, 0), (2, -1)\}$$

Example 4:

How many relations are there on a set with n elements?

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a **subset** of $A \times A$.

Note: $|A \times A| = |A|^2 = n^2$

Example 4:

How many relations are there on a set with n elements?

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a **subset** of $A \times A$.

Note: $|A \times A| = |A|^2 = n^2$

Solution:

A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, there are 2^{n^2} subsets of $A \times A$.

Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these relations.

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Transitive

Reflexive and Irreflexive

A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is called ***irreflexive*** if $(a, a) \notin R$ for every element $a \in A$.

not reflexive* $\neq$ *irreflexive

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$\boxed{R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},}$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$\boxed{R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},}$$

$$R_6 = \{(3, 4)\}.$$

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive R_4 and R_6 are irreflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$\boxed{R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$\boxed{R_6 = \{(3, 4)\}}$.

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution:

R_3 and R_5 are reflexive

R_4 and R_6 are irreflexive

R_1 and R_2 are Not reflexive Not irreflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Example 2:

Is the "divides" relation on the set of positive integers reflexive?

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Is the "divides" relation on the set of positive integers reflexive?

Solution:

Because $a|a$ whenever a is a positive integer, the "divides" relation is **reflexive**.

Example 3:

Is the "divides" relation on the set of integers reflexive?

Example 3:

Is the "divides" relation on the set of integers reflexive?

Solution:

The relation is **not reflexive** because 0 does not divide 0.

Example 4:

Is the following relations on the integers are reflexive or not?

$R_1 = \{(a, b) | a \leq b\}$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = b + 1\}$

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | a + b \leq 3\}$

Example 4:

Is the following relations on the integers are reflexive or not?

Solution: R_1, R_3, and R_5 are reflexive

$$\boxed{R_1 = \{(a, b) | a \leq b\}}$$

$$R_2 = \{(a, b) | a > b\}$$

$$\boxed{R_3 = \{(a, b) | a = b\}}$$

$$R_4 = \{(a, b) | a = b + 1\}$$

$$\boxed{R_5 = \{(a, b) | a = b \text{ or } a = -b\}}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

Example 4:

Is the following relations on the integers are reflexive or not?

Solution: R_1, R_3, and R_5 are reflexive R_2, R_4, and R_6 are not reflexive

$R_1 = \{(a\,,b)|a \leq b\}$

$\boxed{R_2 = \{(a\,,b)|a > b\}}$ (Counter example, $2\ k\ \ 2$)

$R_3 = \{(a\,,b)|a = b\}$

$\boxed{R_4 = \{(a\,,b)|a = b + 1\}}$ (Counter example, $2 \neq 2 + 1$)

$R_5 = \{(a\,,b)|a = b \text{ or } a = -b\}$

$\boxed{R_6 = \{(a\,,b)|a + b \leq 3\}}$ (Counter example, $2 + 2 \nleq 3$)

Symmetric and Antisymmetric

A relation R on a set A is called ***symmetric***
if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a\ , b \in A$.

A relation R on a set A such that for all $a\ , b \in A$,
if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$
is called ***antisymmetric***.

Example 4:

Which of the following relations are symmetric and which are antisymmetric?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

$R_7 = \{(1,1), (2,2)\}$.

Example 4:

Which of the following relations are symmetric and which are antisymmetric?

Solution:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$, **symmetric**

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$, **symmetric**

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$, **antisymmetric**

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, **antisymmetric**

$R_6 = \{(3, 4)\}$. **antisymmetric**

$R_7 = \{(1,1), (2,2)\}$. **symmetric and antisymmetric**

Example 5:

Is the "divides" relation on the set of positive integers symmetric?

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Solution:

This relation is **not symmetric** because $1 \mid 2$, $2 \nmid 1$.

Example 6:

Is the "divides" relation on the set of positive integers antisymmetric?

Example 6:

Is the "divides" relation on the set of positive integers antisymmetric?

Solution:

This relation is **antisymmetric**.

To see this, note that if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

Transitive

A relation R on a set A is called ***transitive***
If whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$,
for all $a, b, c \in A$

Example 1:

Which of the following relations are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

$R_7 = \{(1,1), (2,2)\}$.

Example 1:

Which of the following relations are transitive?

Solution:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$, **transitive**

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, **transitive**

$R_6 = \{(3, 4)\}$. **transitive**

$R_7 = \{(1,1), (2,2)\}$. **transitive**

Example 2:

Is the "divides" relation on the set of positive integers transitive?

Example 2:

Is the "divides" relation on the set of positive integers transitive?

Solution:

This relation is **transitive.**

Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$.

Hence, $c = (ak)l = a(kl)$, so a divides c.

It follows that this relation is **transitive**.

Relations and Their Properties (22/30)

Notes:

If $A = \emptyset$, then the empty relation R on the set A is ***reflexive, symmetric,*** and ***transitive*** **vacuously**.

For any set A, if the relation R on the set A is empty set,

i.e., $R = \emptyset$,

then it is ***irreflexive, transitive, symmetric,*** and ***antisymmetric.***

For any set A, if the relation R on the set A is universal set,

i.e., $R = U = A \times A$,

then it is ***Reflexive, transitive,*** and ***symmetric.***

Combining Relations

The relations

$R_1 = \{(1,1),(2,2),(3,3)\}$ and

$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

can be combined to obtain

$R_1 \cup R_2 =$

$R_1 \cap R_2 =$

$R_1 - R_2 =$

$R_2 - R_1 =$

$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 =$

Combining Relations

The relations
$R_1 = \{(1,1),(2,2),(3,3)\}$ and
$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$
can be combined to obtain

Solution:

$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$
$R_1 \cap R_2 = \{(1,1)\}$
$R_1 - R_2 = \{(2,2),(3,3)\}$
$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$
$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2$
$= \{(2,2),(3,3),(1,2),(1,3),(1,4)\}$

Representing Relations Using Matrices

A relation between finite sets can be represented using a **zero–one matrix**. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.

The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Example 1:

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R $(\mathbf{M}_R)$ if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Example 1:

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $\boxed{a > b.}$

What is the matrix representing R $(\mathbf{M}_R)$ if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution:

What is the matrix representing if

$R = \{(2, 1), (3, 1), (3, 2)\}$

Example 1:

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R $(\mathbf{M}_R)$ if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution:

What is the matrix representing if

$R = \{(2, 1), (3, 1), (3, 2)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Example 2:

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Example 2:

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution:

$$R = \{(1, b), (2, a), (2, c), (2, d), (3, a), (3, c), (3, e)\}$$

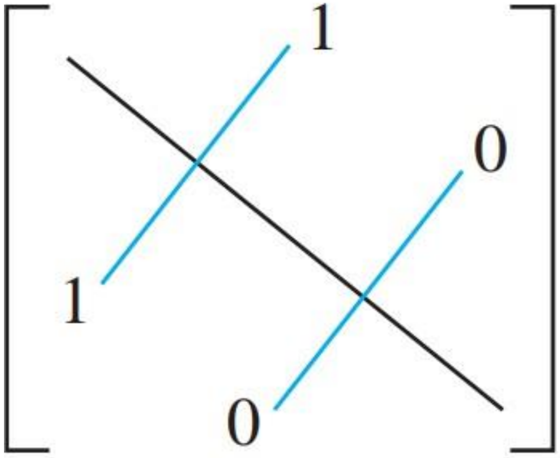
The zero–one Matrix for a Reflexive Relation.

(Off diagonal elements can be 0 or 1):

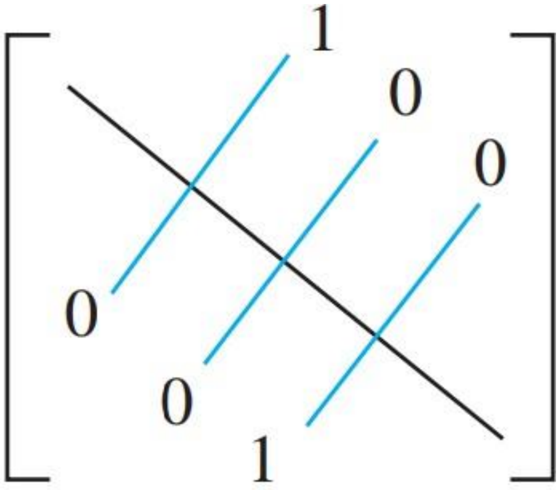
$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Matrices for Symmetric and Antisymmetric Relations

(Diagonal elements can be 0 or 1):



(a) Symmetric



(b) Antisymmetric

Example 3:

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Example 3:

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

Because all the diagonal elements of this matrix are equal to **1**, R is **reflexive**. Moreover, $\mathbf{M}_R$ is **symmetric**. It is also easy to see that R is **not antisymmetric**.

The Boolean Operations

The Boolean operations *join* and *meet* can be used to find the matrices representing the union and the intersection of two relations.

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

join *meet*

Example 4:

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Example 4:

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution:

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Boolean Product

In particular, suppose that R is a relation from A to B and S is a relation from B to C.

Let the zero–one matrices as follows:

For $S \circ R$, $M_{S \circ R} = [t_{ij}]$,

For R, $M_R = [r_{ij}]$, and

For S, $M_S = [s_{ij}]$.

$t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k.

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$$\boxed{\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S}$$

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Example 5:

Find the matrix representing the relations $S \circ R$, where the matrices representing S and R are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

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Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$$

Example 5:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{1} & & \\ & & \\ & & \end{pmatrix}$$

$$(1 \text{ A } 0) \text{ V } (0 \text{ A } 0) \text{ V } (1 \text{ A } 1) = 0 \text{ V } 0 \text{ V } 1 = 1$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 5:

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$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 6:

Find the matrix representing the relations $\boldsymbol{R^2}$, where the matrix representing $\boldsymbol{R}$ is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example 6:

Find the matrix representing the relations $\boldsymbol{R^2}$, where the matrix representing $\boldsymbol{R}$ is

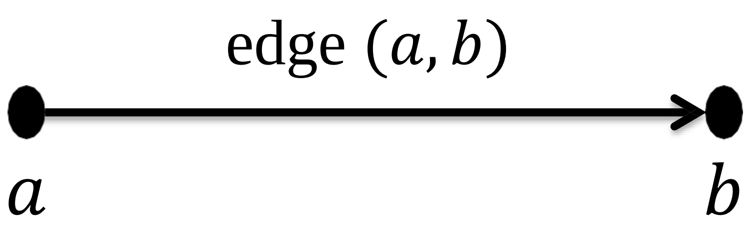
$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution:

$$R^2 = R \circ R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

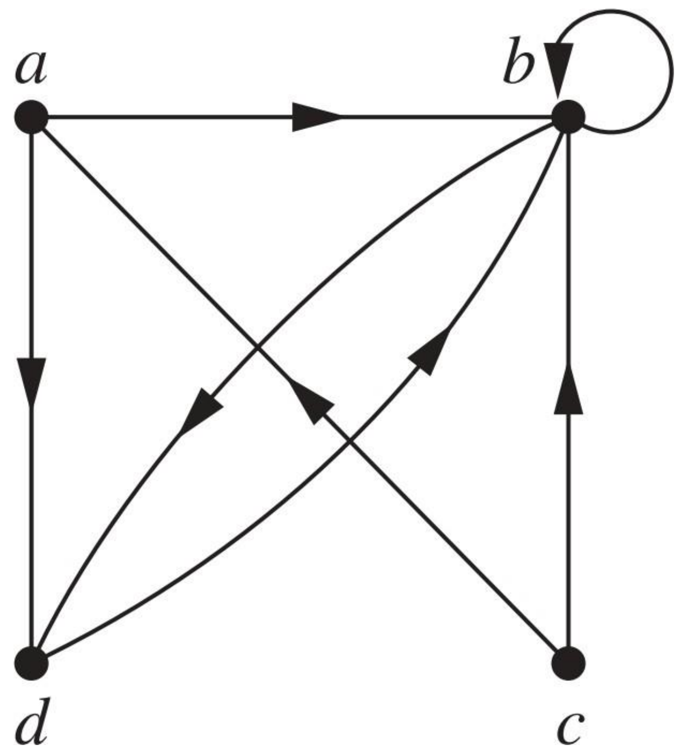
Representing Relations Using Digraphs

A **directed graph**, or **digraph**, consists of a set V of ***vertices*** (or ***nodes***) together with a set E of ordered pairs of elements of V called **edges**. The vertex a is called the ***initial vertex*** of the edge (a, b), and the vertex b is called the ***terminal vertex*** of this edge.



Example 1:

$$R = \{(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(d,b)\}$$



Example 2:

The directed graph of the relation

$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$

on the set $\{1,2,3,4\}$ is

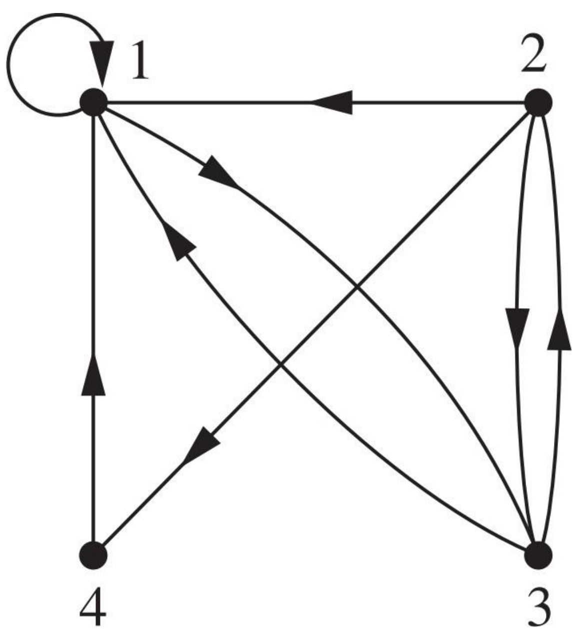
Example 2:

The directed graph of the relation

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$

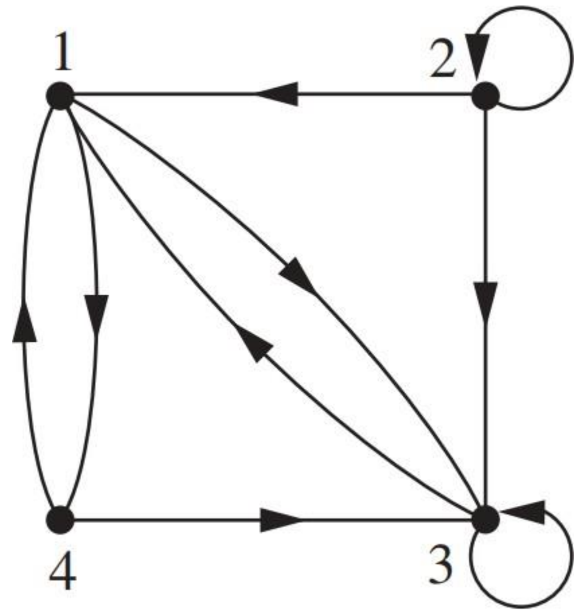
on the set $\{1, 2, 3, 4\}$ is

Solution:



Example 3:

What are the ordered pairs in the relation R represented by the directed graph shown in



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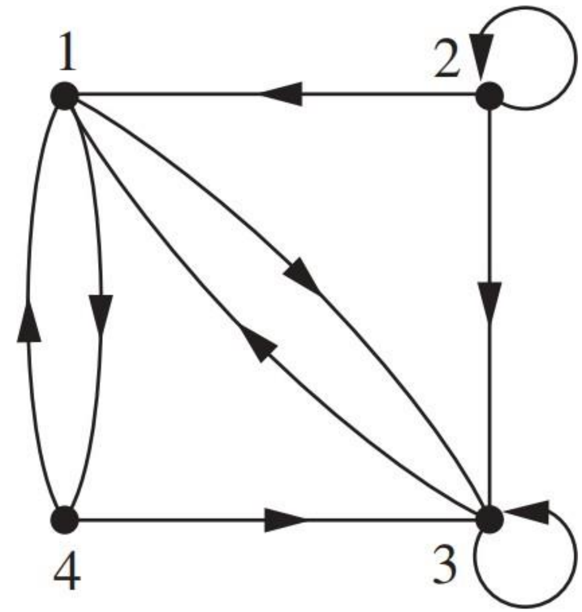

Solution:

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$$

Equivalence Relations (1/3)

Definition

A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Example 1:

Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Example 1:

Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\boxed{\{(0,0),(1,1),(2,2),(3,3)\}}$ **Equivalence**

b) $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)\}$

c) $\boxed{\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}}$ **Equivalence**

d) $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

e) $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,2),(3,3)\}$

Example 2:

Show that a relation on the set of real numbers

$R = \{(a, b) \mid (a - b) \text{ is an integer}\}$ is an equivalence relation.

Equivalence Relations (3/3)

Example 2:

Show that a relation on the set of real numbers

$R = \{(a, b) \mid (a - b) \text{ is an integer}\}$ is an equivalence relation.

Solution:

$a - a = 0$ is an integer, then $(a, a) \in R$ for all a.
So, R is **reflexive**.

If $(a, b) \in R$, then $a - b$ is an integer, therefore, $b - a$ is also an integer, i.e., $(b, a) \in R$. So, R is **symmetric**.

If (a, b) and $(b, c) \in R$, then $a - b$ and $b - c$ are integers, therefore, $a - b + b - c = a - c$ is also an integer, i.e., $(a, c) \in R$. So, R is **transitive**.