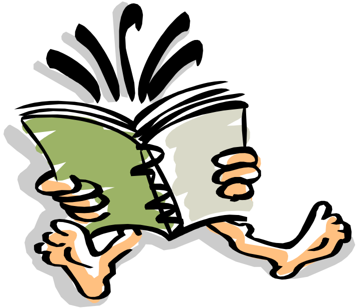
CS1101
Discrete Structures 1

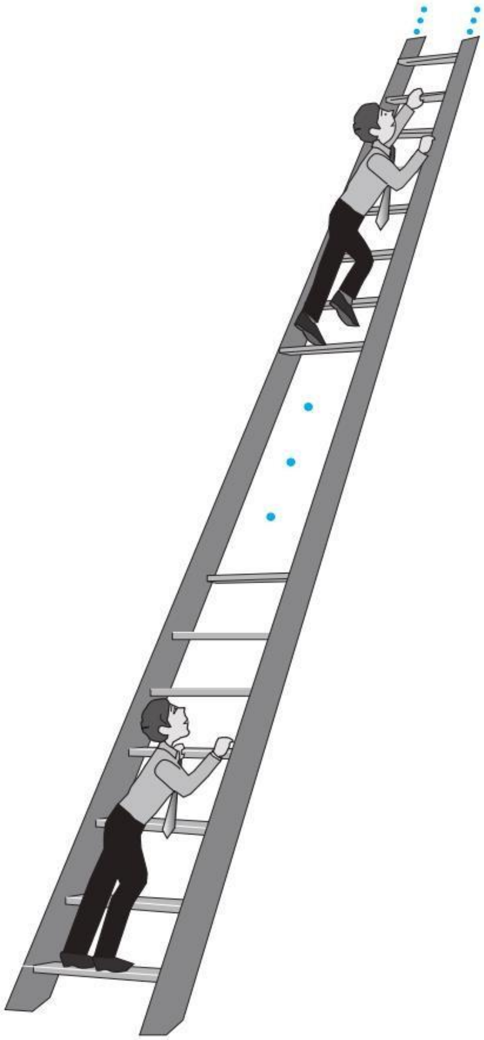
Chapter 05
Induction and Recursion



Chapter 5: Induction and Recursion

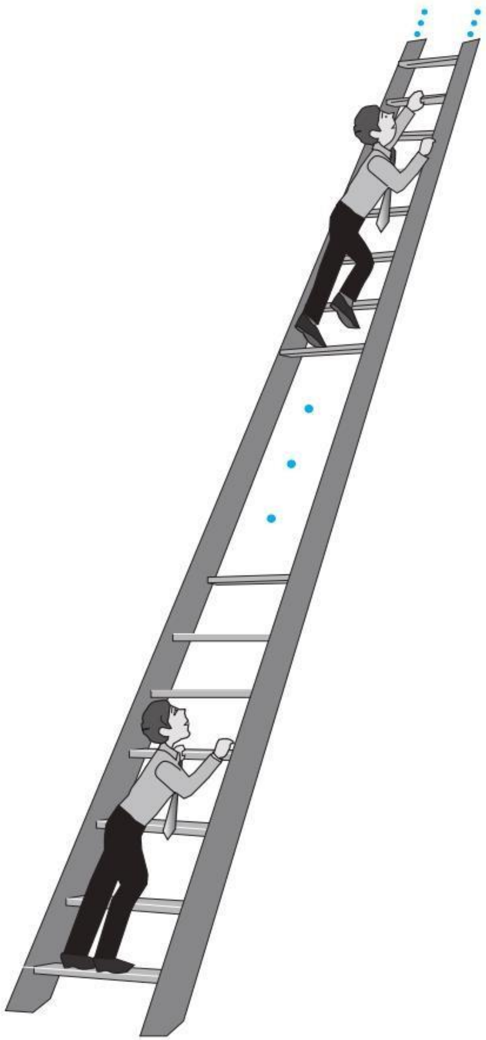
- Mathematical Induction.
- Strong Induction.
- Recursive Definitions.
- Recursive Algorithms.

Mathematical Induction (1/10)



Infinite ladder

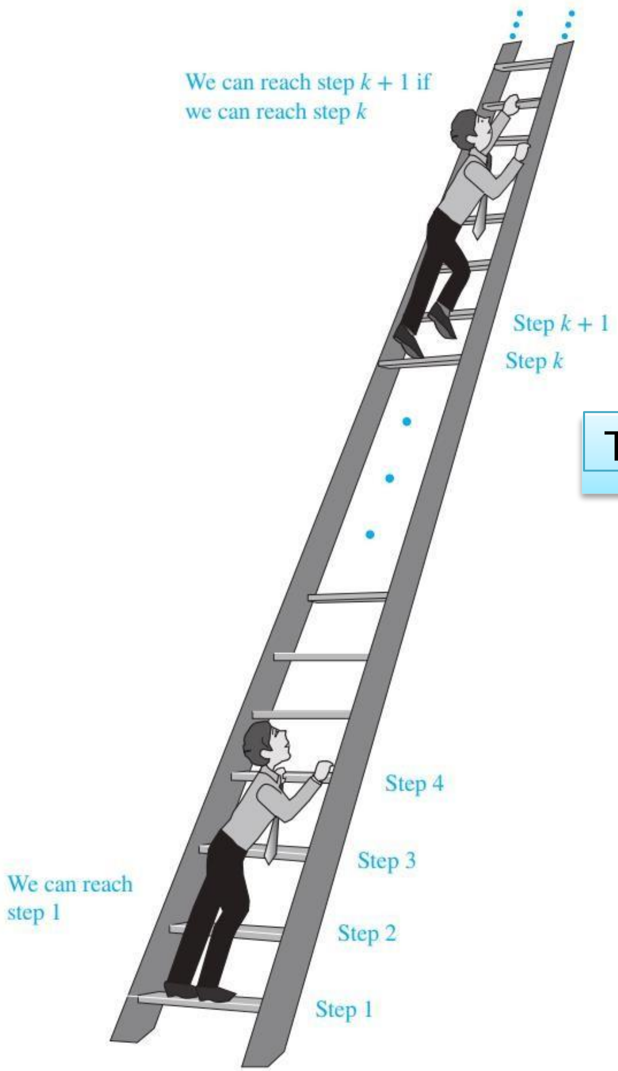
Mathematical Induction (1/10)



Infinite ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Mathematical Induction (1/10)

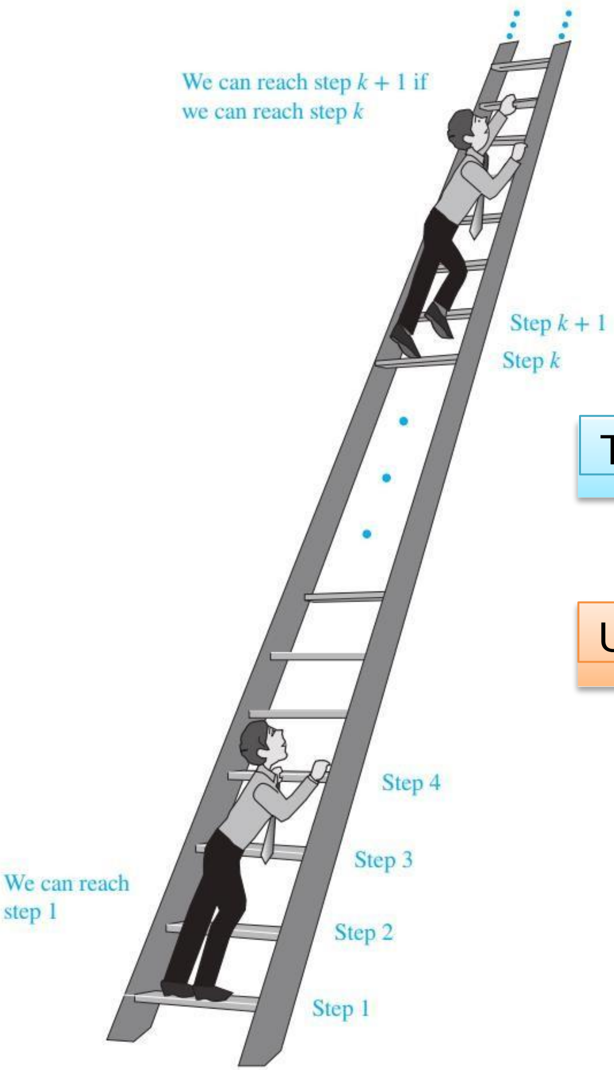


Infinite ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Therefore, we are able to reach every rung of this infinite ladder

Mathematical Induction (1/10)



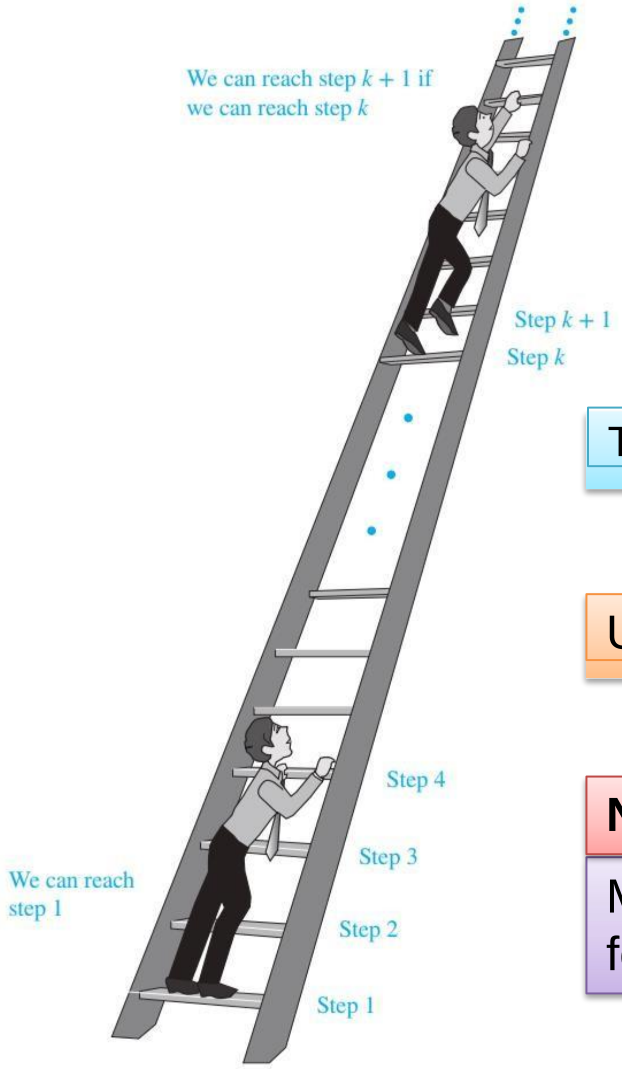
Infinite ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Therefore, we are able to reach every rung of this infinite ladder

Using proof technique called mathematical induction

Mathematical Induction (1/10)



Infinite ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Therefore, we are able to reach every rung of this infinite ladder

Using proof technique called mathematical induction

Note:

Mathematical induction is not a tool for discovering formulae or theorems.

Mathematical Induction (2/10)

Mathematical Induction definition:

Mathemaical induction can be used to prove statments that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

Principle of Mathematical Induction (1/4)

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function,

we complete **two** steps:

Basis Step
We verify that $P(1)$ is true.

Inductive Step
We show that the conditional statment
$P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Principle of Mathematical Induction (2/4)

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the ***inductive hypothesis*** (**IH**).

Principle of Mathematical Induction (2/4)

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k + 1)$ must also be true. The assumption that $P(k)$ is true is called the ***inductive hypothesis*** (**IH**).

$$\boxed{\forall k(P(k) \rightarrow P(k+1))}$$

Principle of Mathematical Induction (3/4)

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the ***inductive hypothesis*** (**IH**).

$$\boxed{\forall k(P(k) \rightarrow P(k+1))}$$

***Remark*:** In a proof by mathematical induction, it is **not** assumed that $P(k)$ is true for all positive integers! It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.

Principle of Mathematical Induction (4/4)

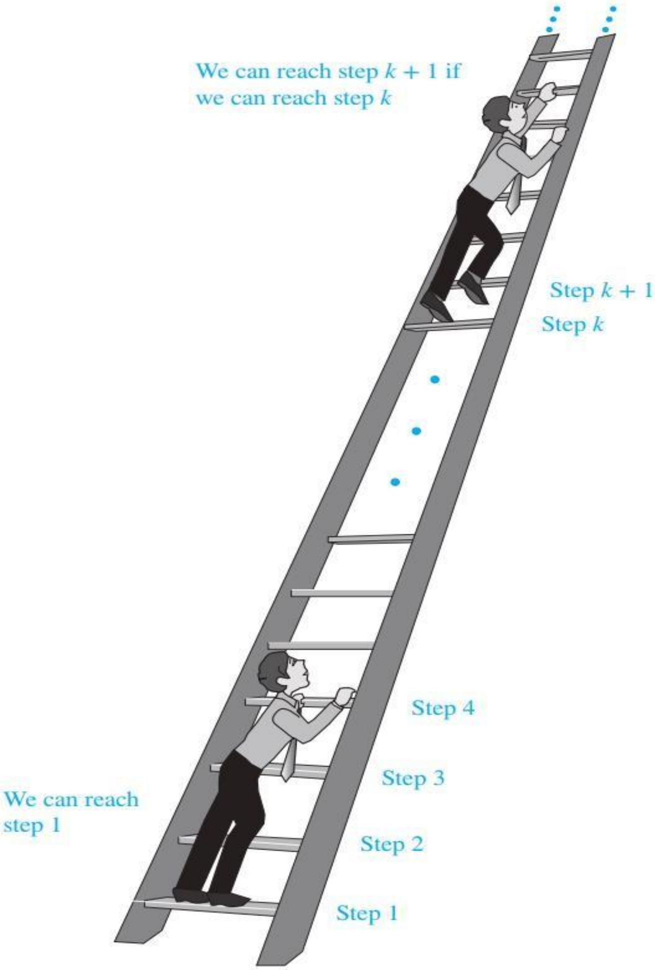
Expressed as a rule of inference,
this proof technique can be stated as:

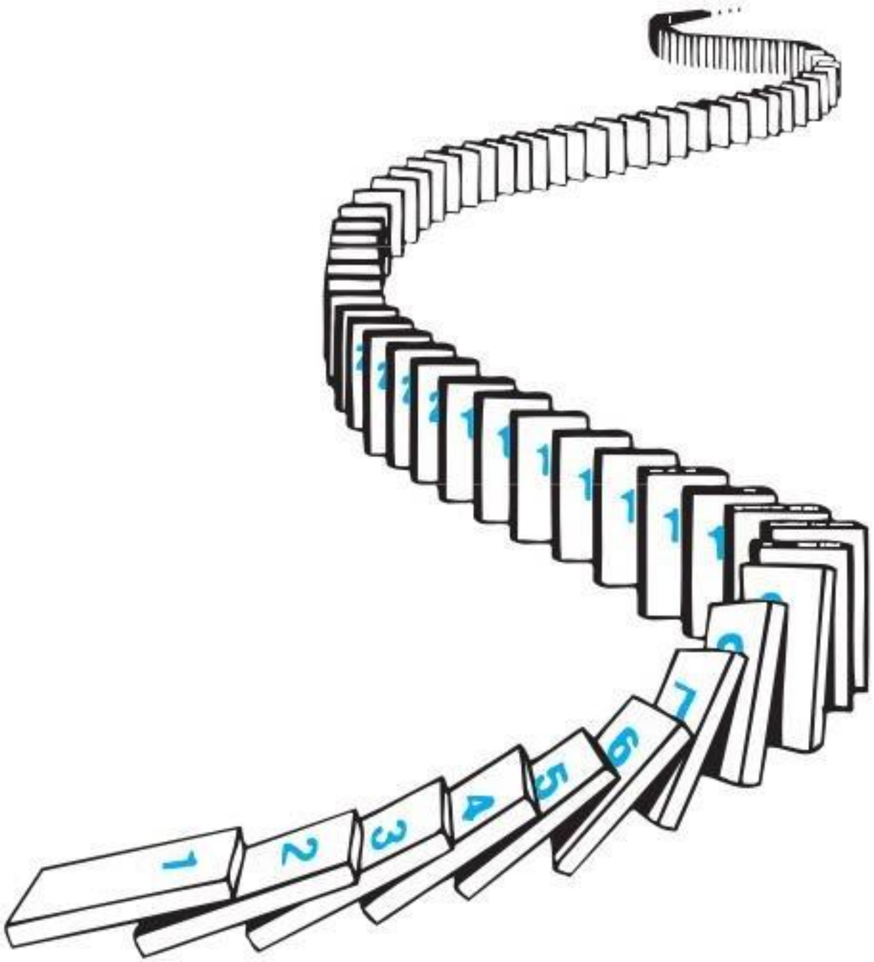
$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

when the domain is the set of positive integers.

***Remark*:** In a proof by mathematical induction, for basis step, we **not always** start at the integer 1. In such a case, the basis step begins at a starting point *b* where *b* is an integer.

Mathematical Induction (4/10)





Mathematical Induction (5/10)

Notes for Proofs by Mathematical Induction (1/3)

- Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
  - ✓ for all positive integers n, let $b = 1$, and
  - ✓ for all nonnegative integers n, let $b = 0$, and so on …

- Write out the words "Basis Step." Then show that $P(b)$ is true.

- Write out the words "Inductive Step" and state, and clearly identify, the inductive hypothesis, in the form "Assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."

Notes for Proofs by Mathematical Induction (2/3)

- State what needs to be proved under the assumption that the inductive hypothesis (IH) is true.
  - ✓ That is, write out what $P(k+1)$ says.

- Show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
  - ✓ The most difficult part of a mathematical induction proof.
  - ✓ This completes the inductive step.

Mathematical Induction (5/10)

Notes for Proofs by Mathematical Induction (3/3)

- After completing the basis step and the inductive step, state the conclusion, namely, “By mathematical induction, $P(n)$ is true for all integers n with $n \geq b$”.

Example 1:

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i = 1 + 2 + 3 \cdots + n = \frac{n(n+1)}{2}$$

For all positive integers n . (i.e., $n \geq 1$)

Example 1 – Answer (1/4):

Let $P(n)$ be the proposition that

$$1 + 2 \ + 3 \ \cdots \ + n \ = \frac{n(n+1)}{2}$$

1) Basis Step:

If $\boldsymbol{n \ = \ 1}$. $P(1)$ is **true**, because $1 = \frac{(1)(2)}{2}$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer $k,$ i.e.: $P(k)$

$$"1 + 2 \ + 3 \ \cdots \ + k = \frac{k(k+1)}{2}".$$

Example 1 – Answer (2/4):

$P(k)$ $\boxed{"1 + 2 \ + 3 \ \cdots \ + k = \frac{k(k+1)}{2} ".}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i.e., we need to show that $P(k+1)$ is also true.

$$1 + 2 \ + 3 \ \cdots \ + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

Example 1 – Answer (3/4):

$P(k)$

$$\text{"}1 + 2 \;+ 3 \;\cdots\; + k = \frac{k(k+1)}{2}\text{"}.$$

We **add** $(\boldsymbol{k} + \mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$1 + 2 \;+ 3 \;\cdots\; + k + \boxed{(k+1)} \overset{\text{IH}}{=} \frac{k(k+1)}{2} + \boxed{(k+1)}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Example 1 – Answer (3/4):

$P(k)$

$$\text{"}1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}\text{"}.$$

We **add** $(\boldsymbol{k} + \mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$1 + 2 + 3 \cdots + k + \boxed{(k+1)} \overset{\text{IH}}{=} \frac{k(k+1)}{2} + \boxed{(k+1)}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

- **This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.**
- **This completes the inductive step.**

Example 1 – Answer (4/4):

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$1 + 2 \ + 3 \ \cdots \ + n \ = \frac{n(n+1)}{2}$$

for all positive integers n.

Example 2:

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For all positive integers n . (i.e., $n \geq 1$)

Example 2 – Answer (1/4):

Let $P(n)$ be the proposition that

$$1^2 + 2^2 \ + 3^2 \ \cdots \ + n^2 \ = \frac{n(n+1)(2n+1)}{6}$$

1) <u>Basis Step:</u>

If $\boldsymbol{n} \ = \ \mathbf{1}$. $P(1)$ is **true**, because $1^2 = 1 = \frac{(1)(2)(3)}{6}$

This completes the basis step.

2) <u>Inductive Step:</u>

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$$"1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 = \frac{k(k+1)(2k+1)}{6} \ ".$$

Mathematical Induction (7/10)

Example 2 – Answer (2/4):

$P(k)$
$$"1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 = \frac{k(k+1)(2k+1)}{6}".$$

We **need to show** that if $P(k)$ is true, then $P(k + 1)$ is true.

i. e. : we need to show that $P(k + 1)$ is also true.

$$1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 + (k+1)^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

$$1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Example 2 – Answer (3/4):

$P(k)$
"$1^2 + 2^2 + 3^2 \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$".

We **add** $(\boldsymbol{k}+\mathbf{1})^{\mathbf{2}}$ to both sides of the equation in $P(k)$ we obtain

$$1^2 + 2^2 + 3^2 \cdots + k^2 + (k+1)^2 \overset{\text{IH}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

Example 2 – Answer (3/4):

$P(k)$
$$\text{"}1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 = \frac{k(k+1)(2k+1)}{6}\text{"}.$$

We **add** $(\boldsymbol{k}+\mathbf{1})^{\mathbf{2}}$ to both sides of the equation in $P(k)$ we obtain

$$1^2 + 2^2 \ + 3^2 \ \cdots \ + k^2 + (k+1)^2 \overset{\text{IH}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

- **This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.**
- **This completes the inductive step.**

Example 2 – Answer (4/4):

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$1^2 + 2^2 \ + 3^2 \ \cdots \ + n^2 \ = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Example 3:

Use mathematical induction to prove that

$$n < 2^n$$

For all positive integers n . (i.e., $n \geq 1$)

Example 3 – Answer (1/4):

Let $P(n)$ be the proposition that

$$n < 2^n$$

1) Basis Step:

If $\boldsymbol{n = 1}$. $P(1)$ is **true**, because $1 < 2^1$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$$k < 2^k$$

Example 3 – Answer (2/4):

$P(k)$ $\boxed{k < 2^k}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i.e., we need to show that $P(k+1)$ is also true.

$$(k+1) < 2^{k+1}$$

Example 3 – Answer (3/4):

$P(k)$ $\boxed{k < 2^k}$

We **add** $(\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$(k + \boxed{1}) \overset{\text{IH}}{<} 2^k + \boxed{1}$$

Example 3 – Answer (3/4):

$P(k)$ $\boxed{k < 2^k}$

We **add** $(\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$(k+1) \overset{\text{IH}}{<} 2^k + \boxed{1}$$

Because the integer $k \geq 1$. Therefore, $2^k > 1$

$$(k+1) < 2^k + \boxed{2^k}$$

Example 3 – Answer (3/4):

$P(k)$ $\boxed{k < 2^k}$

We **add** $(\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$(k+1) \overset{\text{IH}}{<} 2^k + 1$$

$$(k+1) < 2^k + 2^k$$

$$(k+1) < 2 \cdot 2^k$$

$$(k+1) < 2^{k+1}$$

Mathematical Induction (8/10)

Example 3 – Answer (3/4):

$P(k)$ $\boxed{k < 2^k}$

We **add** $(\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$(k+1) \overset{\text{IH}}{<} 2^k + 1$$

$$(k+1) < 2^k + 2^k$$

$$(k+1) < 2 \cdot 2^k$$

$$(k+1) < 2^{k+1}$$

- **This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.**
- **This completes the inductive step.**

Example 3 – Answer (4/4):

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$n < 2^n$$

for all positive integers n.

Example 4:

Use mathematical induction to prove that

$$2^n < n!$$

For every integer integers n with $n \geq 4$.

Example 4 – Answer (1/5):

Let $P(n)$ be the proposition that

$$2^n < n! \qquad \boxed{n \geq 4}$$

1) Basis Step:

If $\boldsymbol{n} = \mathbf{4}$. $P(4)$ is **true**, because $(2^4 = 16) < (4! = 24)$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer $k \geq 4$, i.e.: $P(k)$

$$2^k < k!$$

Example 4 – Answer (2/5):

$P(k)$ $\boxed{2^k < k!}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i.e., we need to show that $P(k+1)$ is also true.

$\boxed{k \geq 4}$

$\boxed{2^{k+1} < (k+1)!}$

$$2^{k+1} < (k+1)!$$

Mathematical Induction (9/10)

Example 4 – Answer (3/5):

$P(k)$ $\boxed{2^k < k!}$

$\boxed{k \geq 4}$

We are **multiple** both sides of the equation in $P(k)$ by $(\mathbf{2})$, we obtain

$$2^k \overset{\text{IH}}{<} k!$$

$$2 \cdot 2^k < 2 \cdot k!$$

By definition of exponent

$2^{k+1} = 2 \cdot 2^k$

Mathematical Induction (9/10)

Example 4 – Answer (3/5):

$P(k)$ $\boxed{2^k < k!}$

$\boxed{k \geq 4}$

We are **multiple** both sides of the equation in $P(k)$ by $(\mathbf{2})$, we obtain

$$2^k \overset{\text{IH}}{<} k!$$

$$2 \cdot 2^k < 2 \cdot k!$$

$$2^{k+1} < 2 \cdot k!$$

By definition of exponent

$2^{k+1} = 2 \cdot 2^k$

Example 4 – Answer (4/5):

$P(k)$ $\boxed{2^k < k!}$

$\boxed{k \geq 4}$

We are **multiple** both sides of the equation in $P(k)$ by $(\mathbf{2})$, we obtain

$$2^{k+1} < \boxed{2} \cdot k!$$

Because the integer $k \geq 4$. Therefore, $2 < k + 1$

$$2^{k+1} < \boxed{(k+1)} \cdot k!$$

Example 4 – Answer (4/5):

$P(k)$ $\boxed{2^k < k!}$

$\boxed{k \geq 4}$

We are **multiple** both sides of the equation in $P(k)$ by $(\mathbf{2})$, we obtain

$$2^{k+1} < 2 \cdot k!$$

$$2^{k+1} < \boxed{(k+1) \cdot k!}$$

By definition of factorial function.

$$2^{k+1} < \boxed{(k+1)!}$$

Example 4 – Answer (4/5):

$P(k)$ $\boxed{2^k < k!}$

$\boxed{k \geq 4}$

We are **multiple** both sides of the equation in $P(k)$ by $(\mathbf{2})$, we obtain

$$2^{k+1} < 2 \cdot k!$$

$$2^{k+1} < (k+1) \cdot k!$$

$$2^{k+1} < (k+1)!$$

- **This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.**
- **This completes the inductive step.**

Example 4 – Answer (5/5):

So, by mathematical induction we know that $P(n)$ is true for all positive integers $n \geq 4$.

That is, we proven that

$$2^n < n!$$

for all positive integers $n \geq 4$.

Example 5:

Use mathematical induction to prove that

$$n^3 - n \text{ is divisible by } 3$$

For every positive integer integers n. (i.e., $n \geq 1$)

Example 5 – Answer (1/4):

Let $P(n)$ be the proposition that

$$\text{" } n^3 - n \text{ is divisible by 3 "} \qquad \boxed{n \geq 1}$$

1) Basis Step:

If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is **true**, because $(1^3 - 1 = 0)$ is divisible by 3.
This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer $k \geq 1$, i.e.: $P(k)$

$$k^3 - k \text{ is divisible by 3}$$

Example 5 – Answer (2/4):

$P(k)$

$\boxed{k^3 - k \text{ is divisible by } 3}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i.e., we need to show that $P(k+1)$ is also true.

$$(k+1)^3 - (k+1) \text{ is divisible by } 3$$

Example 5 – Answer (3/4):

$P(k)$

$\boxed{k^3 - k \text{ is divisible by } 3}$

Note that

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

$$= k^3 + 3k^2 + 3k - k$$

$$= k^3 - k + 3k^2 + 3k$$

$$= (k^3 - k) + 3(k^2 + k)$$

Example 5 – Answer (3/4):

$P(k)$

$k^3 - k$ is divisible by 3

Note that

$$(k+1)^3-(k+1) = (k^3+3k^2+3k+1)-(k+1)$$

$$= k^3+3k^2+3k-k$$

$$= k^3-k+3k^2+3k$$

$$= \boxed{(k^3-k)} + 3(k^2+k)$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3

Mathematical Induction (10/10)

Example 5 – Answer (3/4):

$P(k)$

$\boxed{k^3 - k \text{ is divisible by } 3}$

Note that

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

$$= k^3 + 3k^2 + 3k - k$$

$$= k^3 - k + 3k^2 + 3k$$

$$= (k^3 - k) + \boxed{3(k^2 + k)}$$

The second term is divisible by 3 because it is 3 times an integer.

Mathematical Induction (10/10)

Example 5 – Answer (3/4):

$P(k)$

$k^3 - k$ is divisible by 3

Note that

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

$$= k^3 + 3k^2 + 3k - k$$

$$= k^3 - k + 3k^2 + 3k$$

$$= (k^3 - k) + 3(k^2 + k)$$

- **So, $(k+1)^3 - (k+1)$ is divisible by 3**
- **This completes the inductive step.**

Example 5 – Answer (4/4):

So, by mathematical induction we know that $P(n)$ is true for all positive integers $n \geq 1$.

That is, we proven that

$$\text{” } n^3 - n \text{ is divisible by 3 ”}$$

for all positive integers $n \geq 1$.

Strong Induction (1/4)

Introduction (1/2)

Strong induction is another form of mathematical induction, which can often be used when we cannot easily prove a result using mathematical induction.

The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. However, the inductive steps in these two proof methods are different.

Introduction (2/2)

In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers j not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

Strong Induction (2/4)

Second Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function,

Complete Induction

Strong Induction

we complete **two** steps:

Basis Step
We verify that $P(1)$ is true.

Inductive Step
We show that the conditional statment
$[P(1)\ \mathrm{A}\ P(2)\ \mathrm{A} \cdots \mathrm{A}\ P(k)] \rightarrow P(k+1)$
is true for all positive integers k.

Strong Induction (3/4)

Example 1:

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Example 1 – Answer (1/5):

Let $P(n)$ be the proposition that be the proposition that n can be written as the product of primes.

$\boxed{n > 1}$

1) <u>Basis Step:</u>

If $\boldsymbol{n} = \mathbf{2}$. $P(2)$ is **true**, because 2 can be written as the product of one prime, itself. This completes the basis step.

2) <u>Inductive Step:</u>

The inductive hypothesis (IH) is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k.

Strong Induction (4/4)

Example 1 – Answer (2/5):

To complete the inductive step, it must be shown that $P(k + 1)$ is true under this assumption, that is, that $k + 1$ is the product of primes.

Strong Induction (4/4)

Example 1 – Answer (3/5):

There are two cases to consider, namely,

1) when $k + 1$ is prime and
2) when $k + 1$ is composite.

Strong Induction (4/4)

Example 1 – Answer (4/5):

1) If $k + 1$ is prime

We immediately see that $P(k + 1)$ is true. (as we shown $P(2)$)

Example 1 – Answer (5/5):

2) If $k+1$ is composite

It can be written as the product of two positive integers a and b with $2 \leq a < k+1$ and $2 \leq b < k+1$

Strong Induction (4/4)

Example 1 – Answer (5/5):

2) Inductive Step:

The inductive hypothesis (IH) is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k.

2) If $k + 1$ is composite

It can be written as the product of two positive integers a and b with $2 \leq a < k + 1$ and $2 \leq b < k + 1$

Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k + 1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.