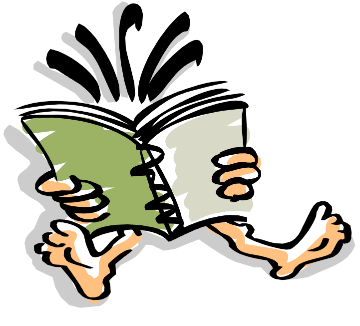
CS1101
Discrete Structures 1

Chapter 02
Basic Structures: Functions



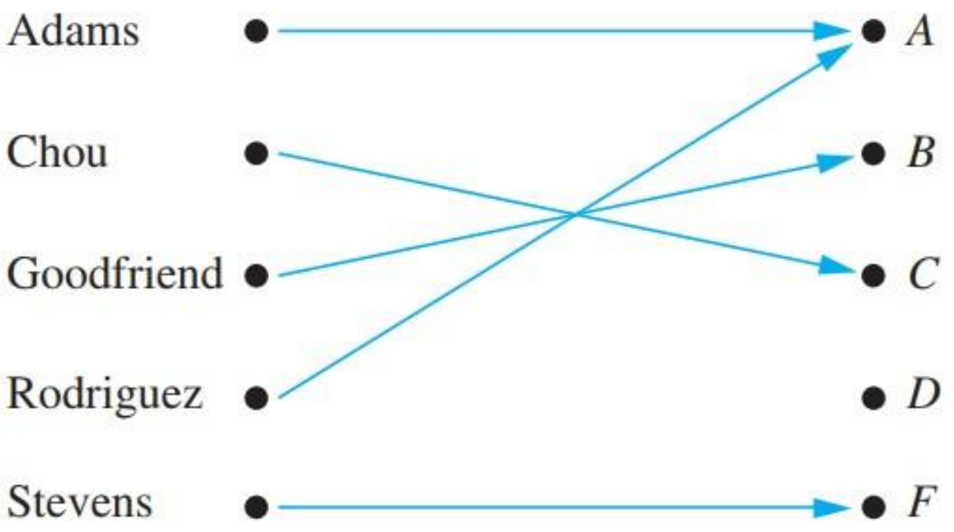
Function

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write $f: A \rightarrow B$.

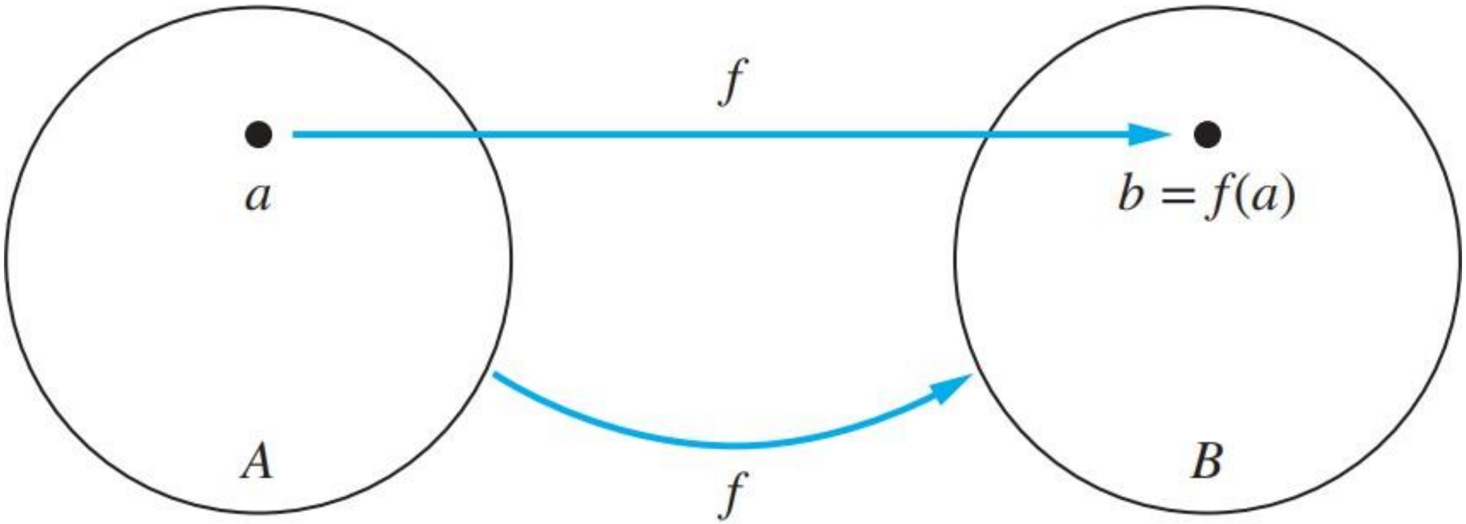
Function



Assignment of grades in a discrete mathematics class.

Functions (3/21)

The Function $f: A \to B$



The function f maps A to B.

Functions (3/21)

The Function $f: A \to B$

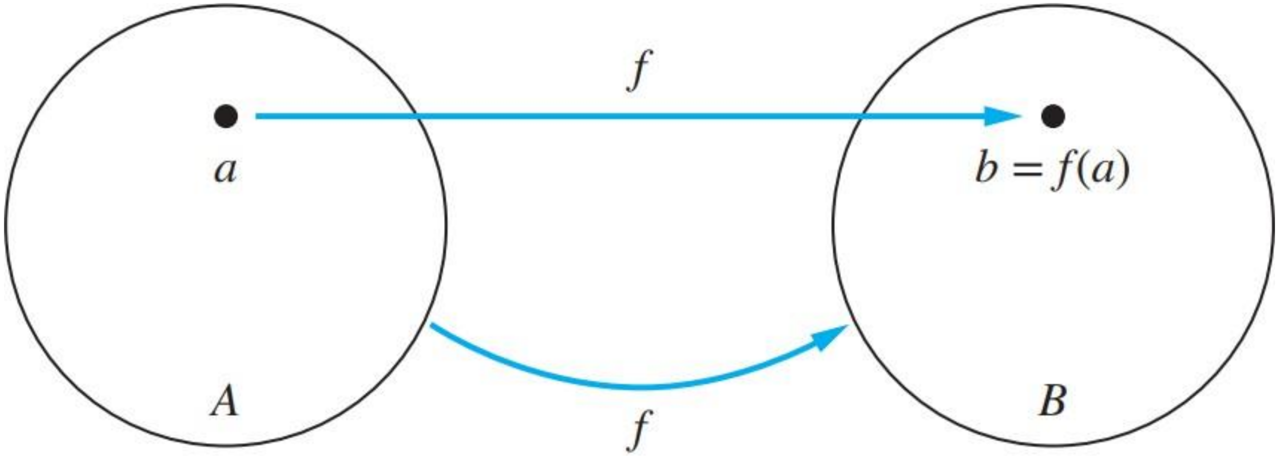
Domain: A

Co-Domain: B

$f(a) = b$

b is the ***image*** of a
a is a ***preimage*** of b

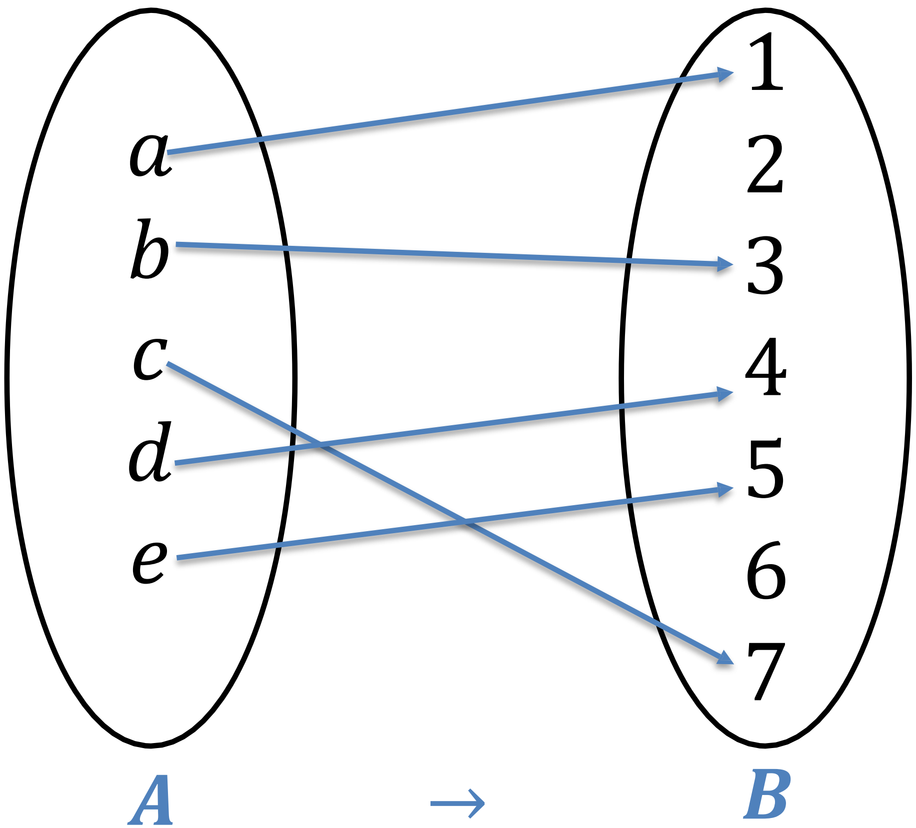
The **range**, or image, of f is the ***set of all images*** of elements of A.



The function f maps A to B.

Functions (4/21)

The Function $f: A \to B$



$\text{Domain} = \{a, b, c, d, e\}$

$\text{Co-Domain} = \{1,2,3,4,5,6,7\}$

$\text{Range} = \{1,3,4,5,7\}$

Functions (5/21)

Definition

Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined for all $x \in A$ by

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$,

$(f_1 f_2)(x) = f_1(x) f_2(x)$.

Functions (6/21)

Example

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x,$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2 (x - x^2) = x^3 - x^4.$$

Functions (7/21)

Definition

Let f be a function from A to B and let S be a subset of A.

The image of S under the function f is the subset of B that consists of the images of the elements of S.

We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\,(t = f(s))\}.$$

or shortly $\{f(s) \mid s \in S\}$.

Functions (8/21)

Example

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.

$S = \{b, c, d\} \subseteq A$

The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$

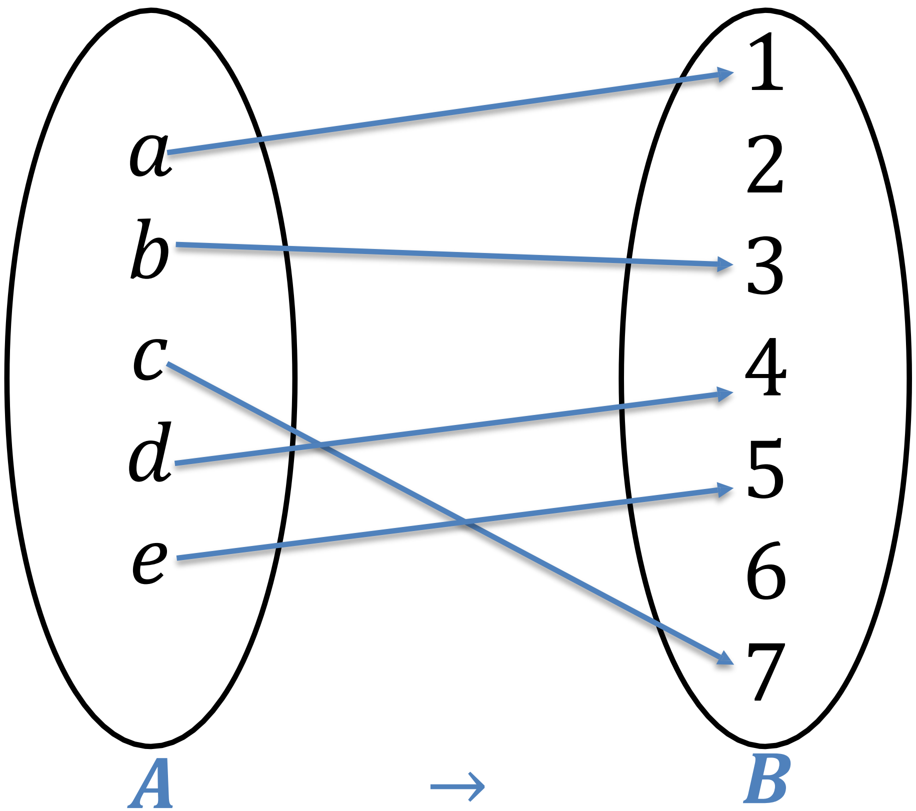
Functions (9/21)

One-to-One **function (injective)**

A function f is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

Functions (9/21)

One-to-One function (injective)



$f(a) = 1$

$f(b) = 3$

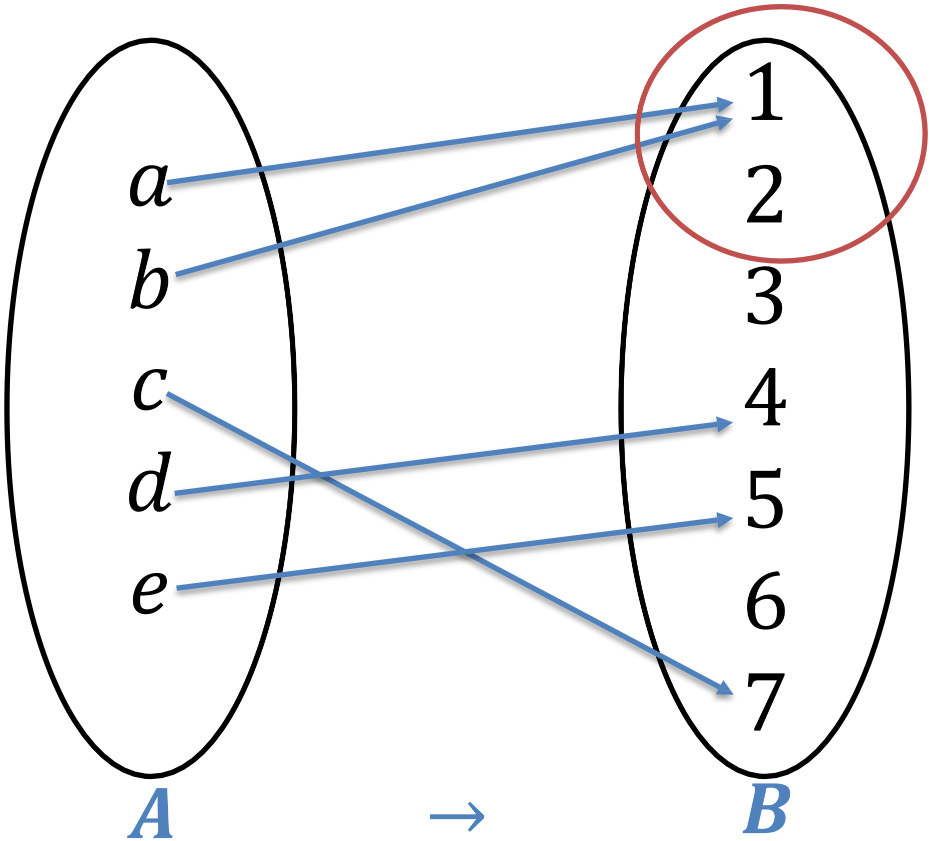
$f(c) = 7$

$f(d) = 4$

$f(e) = 5$

Functions (9/21)

NOT ***One-to-One*** **function (Not injective)**



$$\boxed{\begin{array}{c} f(a) = 1 \\ f(b) = 1 \end{array}}$$

$$f(c) = 7$$

$$f(d) = 4$$

$$f(e) = 5$$

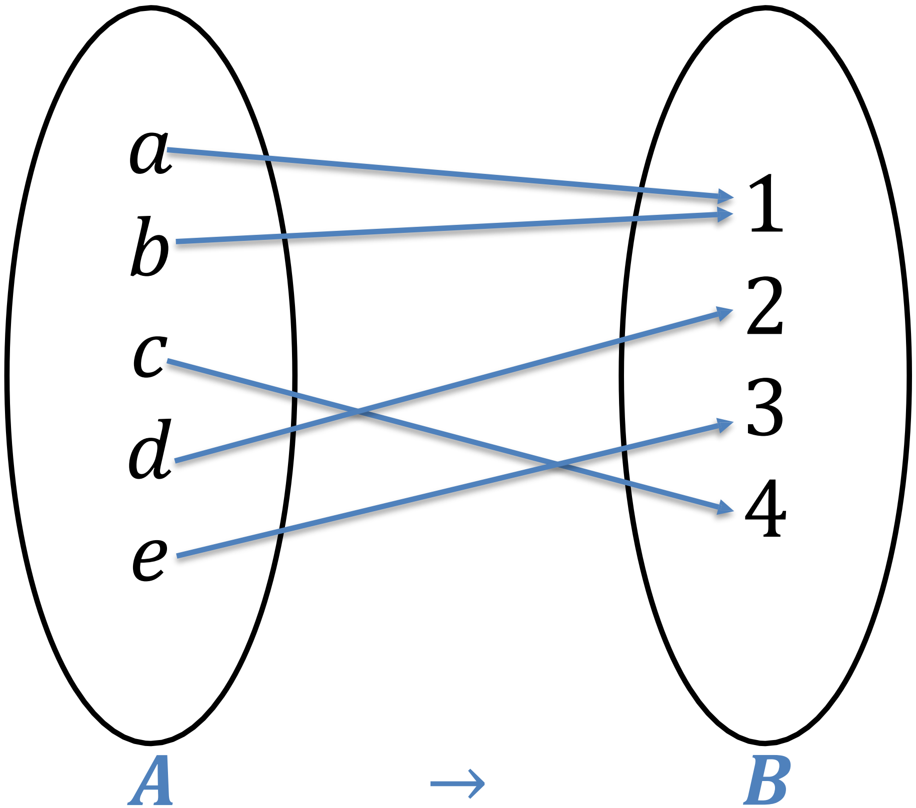
Functions (10/21)

onto function (surjective)

A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

Functions (10/21)

onto function (surjective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

$f(d) = 2$

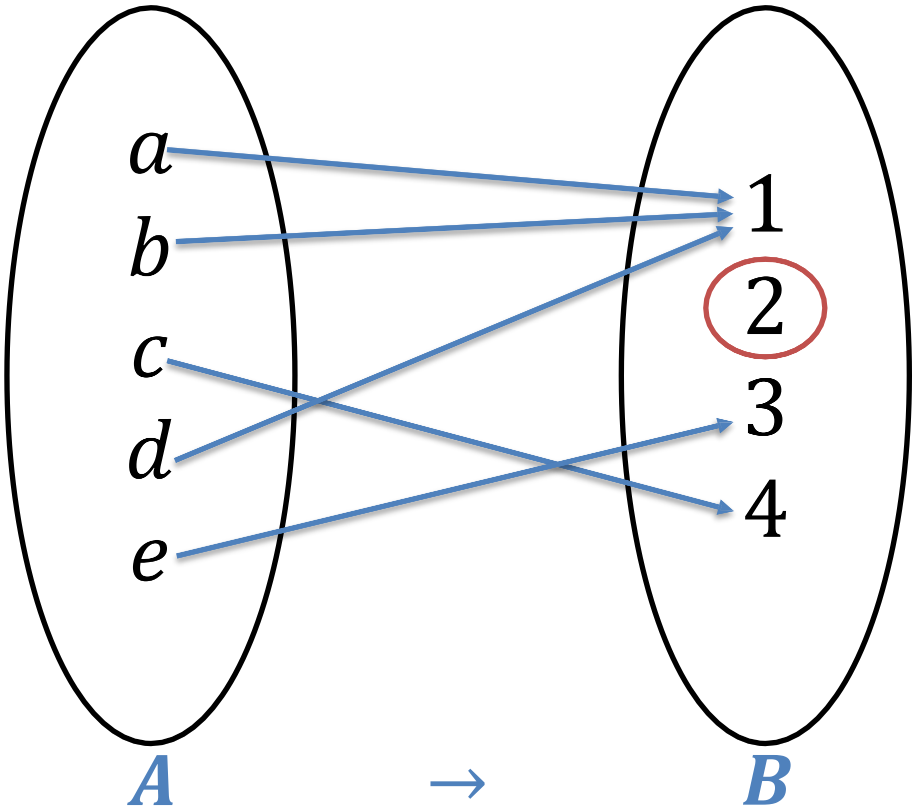
$f(e) = 3$

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

Functions (10/21)

NOT *onto* **function (Not surjective)**



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

$f(d) = 1$

$f(e) = 3$

Co-Domain = {1,2,3,4}

Range = {1,3,4}

Functions (11/21)

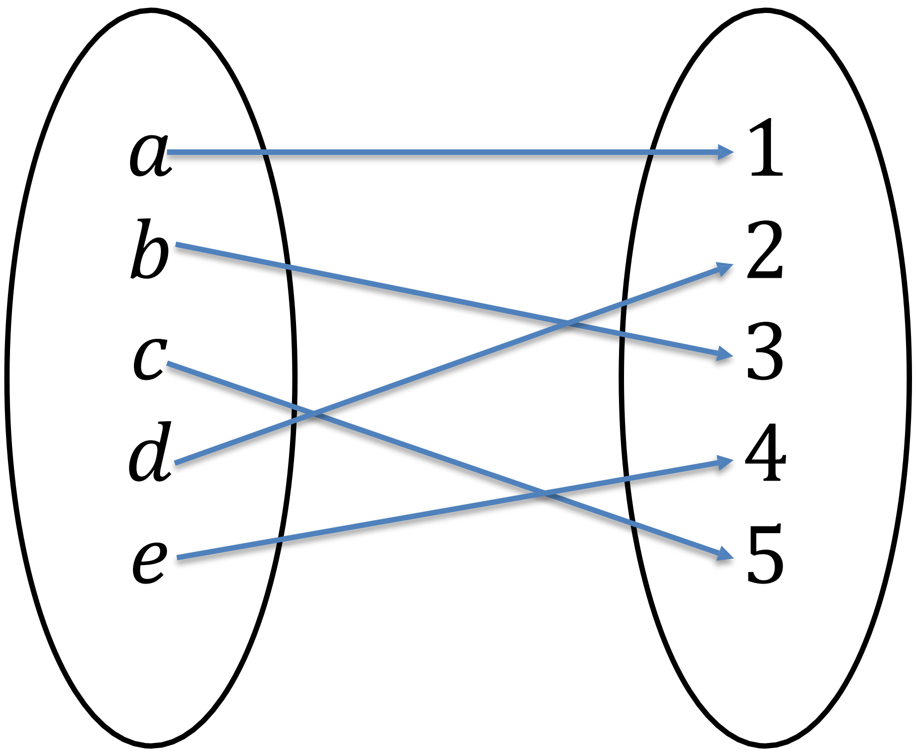
One-to-one correspondence (bijection)

The function *f* is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.

Functions (11/21)

One-to-one correspondence (bijection)

$|A| = |B|$



$A \rightarrow B$

$f(a) = 1$

$f(b) = 3$

$f(c) = 5$

$f(d) = 2$

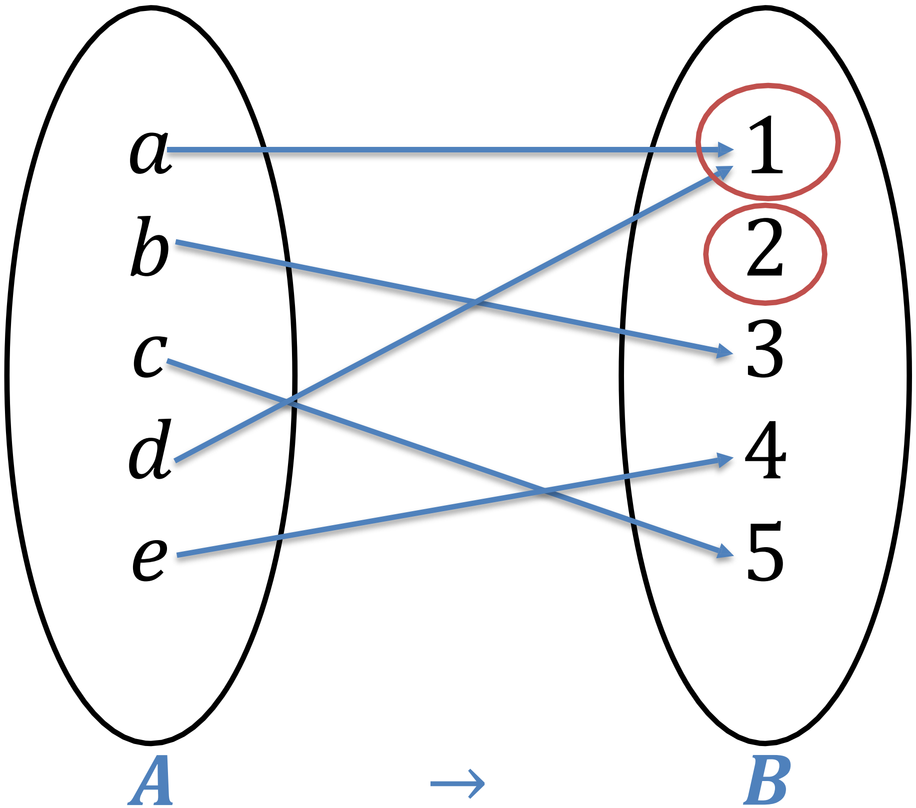
$f(e) = 4$

Co-Domain = {1,2,3,4,5}

Range = {1,2,3,4,5}

Functions (11/21)

NOT ***One-to-one correspondence*** **(Not bijection)**



$f(a) = 1$

$f(b) = 3$ **NOT one-to-one**

$f(c) = 5$ **NOT onto**

$f(d) = 1$

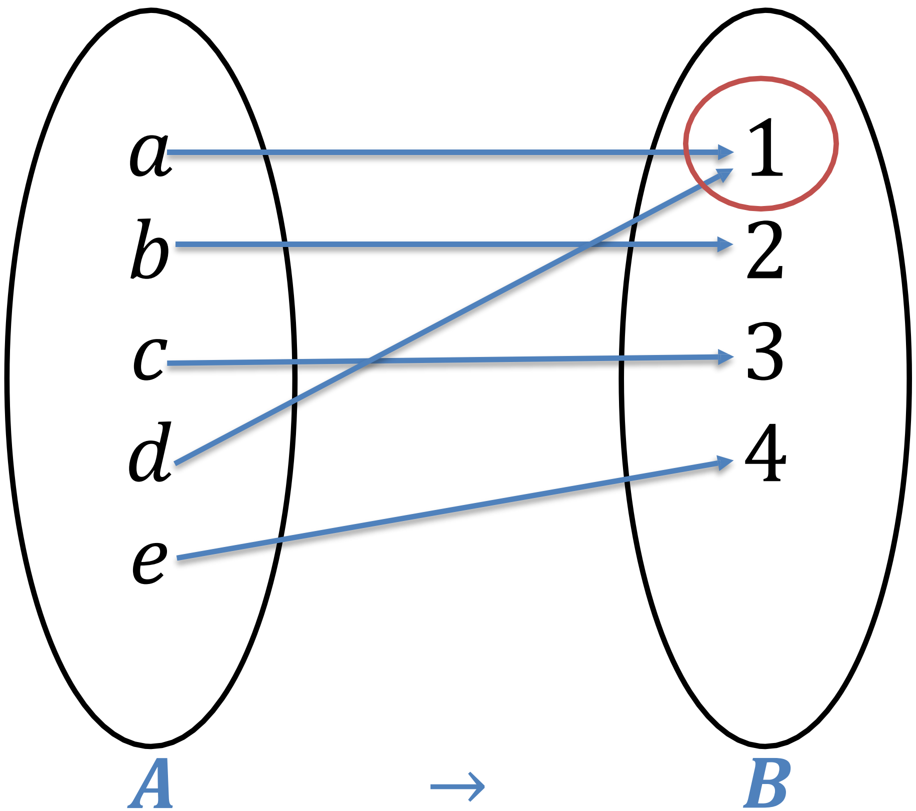
$f(e) = 4$

Co-Domain = {1,2,3,4,5}

Range = {1,3,4,5}

Functions (11/21)

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 2$

$f(c) = 3$

$f(d) = 1$

$f(e) = 4$

Onto

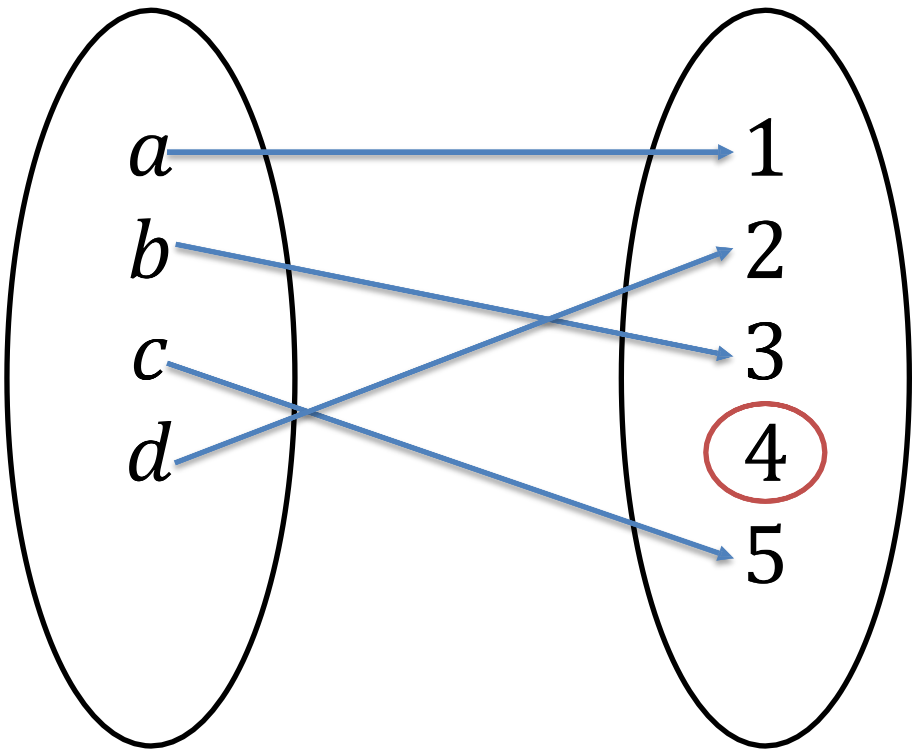
NOT one-to-one

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

Functions (11/21)

NOT ***One-to-one correspondence*** **(Not bijection)**



$A \rightarrow B$

$f(a) = 1$

$f(b) = 3$

$f(c) = 5$

$f(d) = 2$

One-to-one

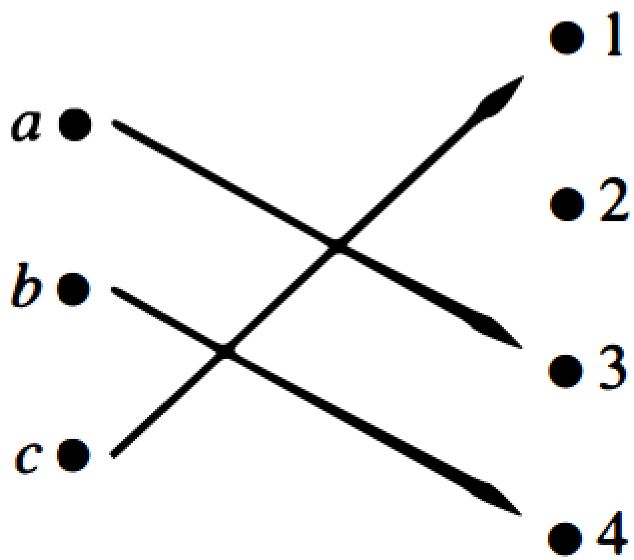
NOT onto

Co-Domain = $\{1,2,3,4,5\}$

Range = $\{1,2,3,5\}$

Functions (12/21)

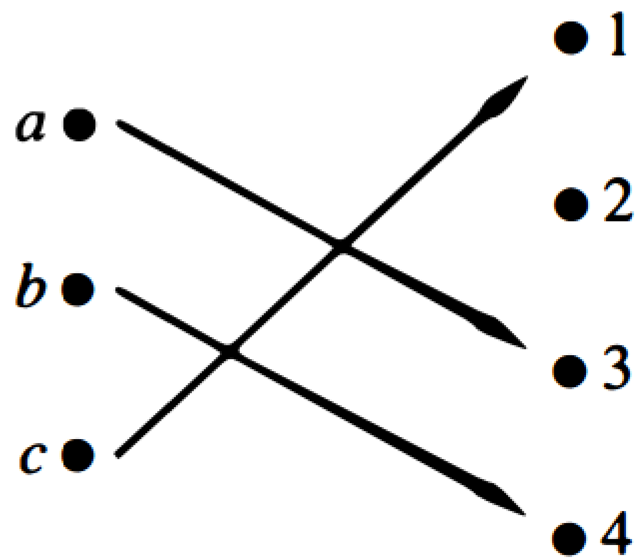
Examples



$A \quad \rightarrow \quad B$

Functions (12/21)

Examples



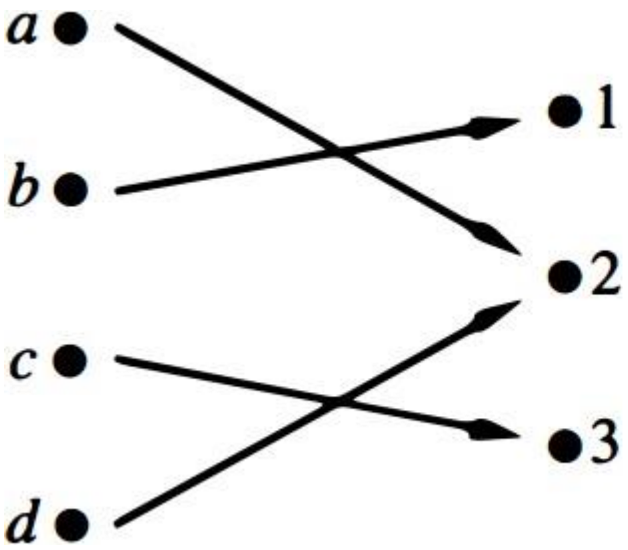
$A \rightarrow B$

One-to-one

NOT onto

Functions (12/21)

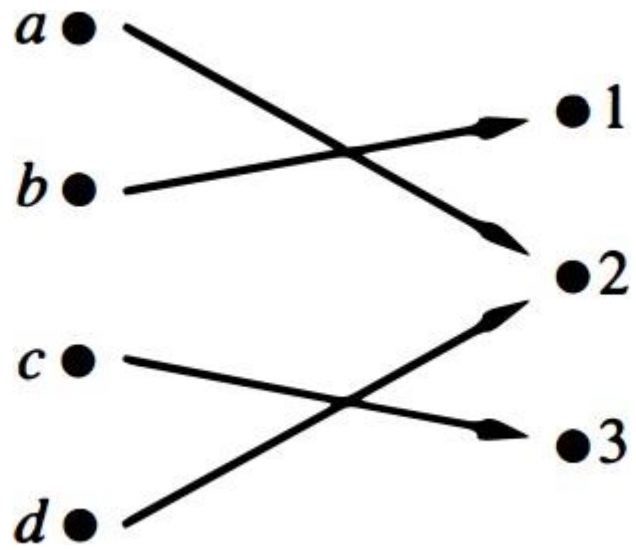
Examples



$A \quad \to \quad B$

Functions (12/21)

Examples



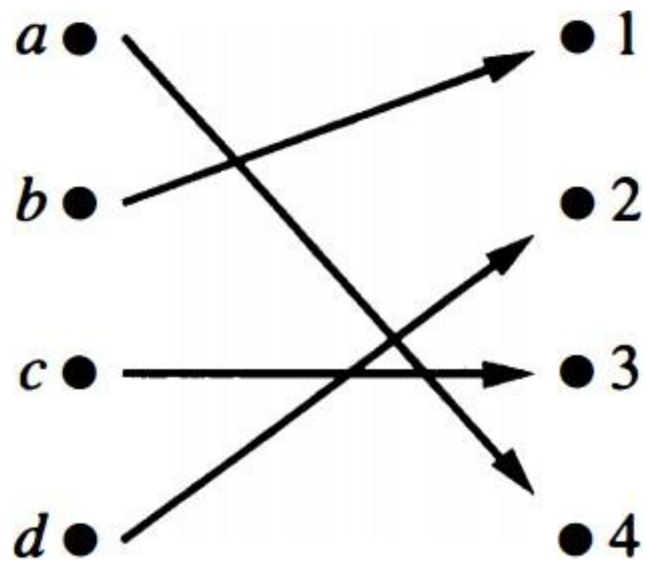
NOT One-to-one

Onto

A → ***B***

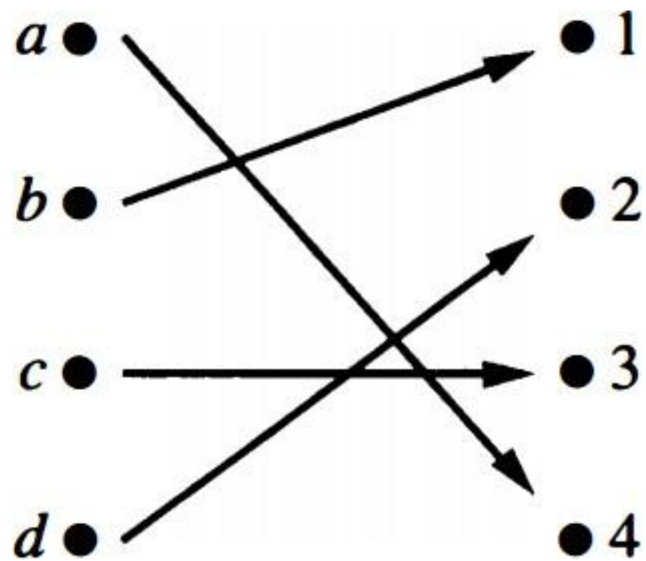
Functions (12/21)

Examples



$A \rightarrow B$

Examples



One-to-one

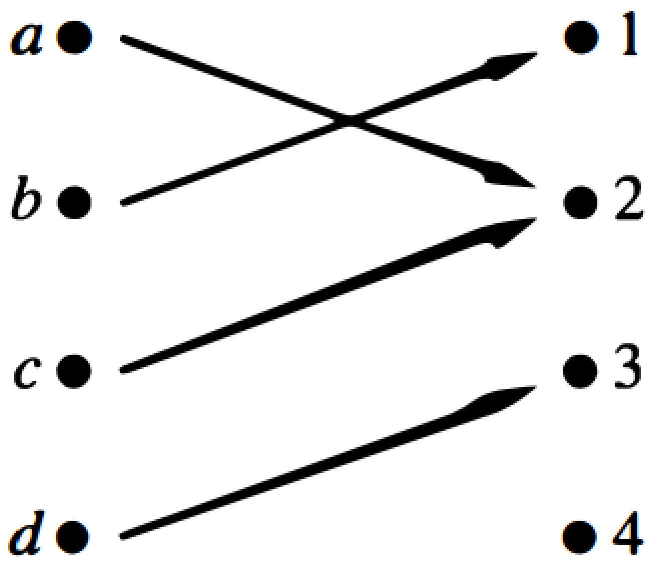
Onto

∴ bijection

$A \rightarrow B$

Functions (12/21)

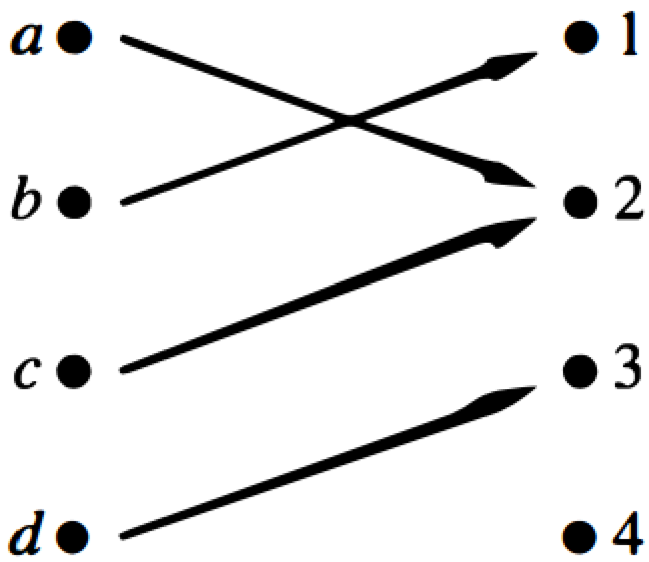
Examples



$A \rightarrow B$

Functions (12/21)

Examples

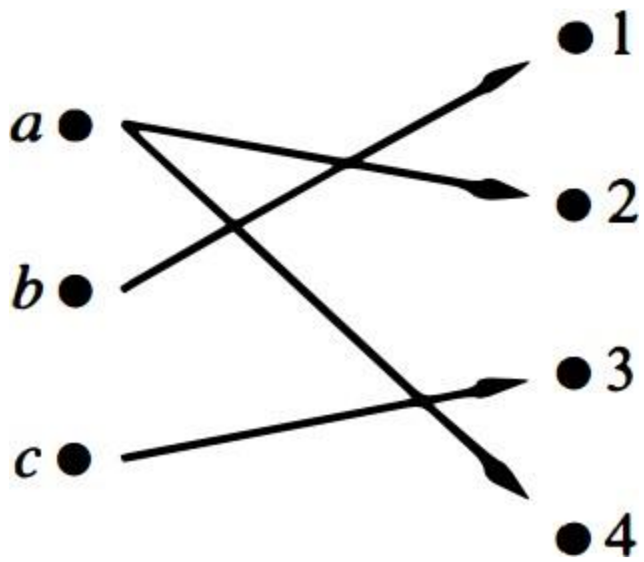


NOT One-to-one

NOT Onto

A → ***B***

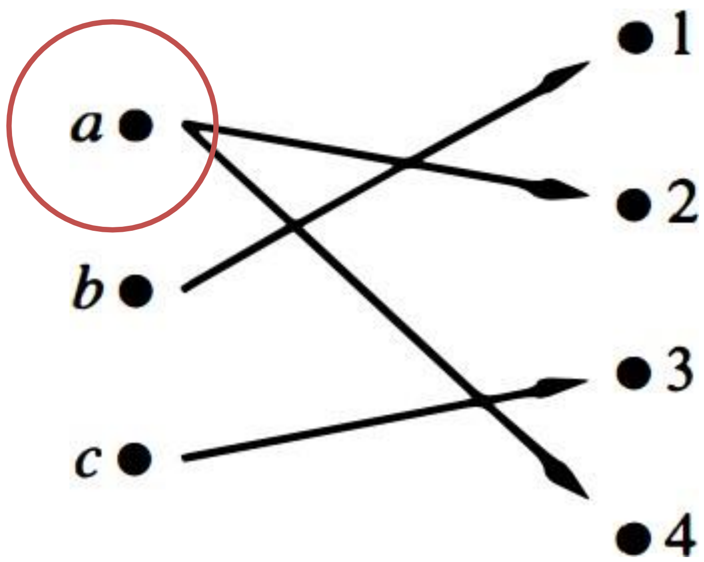
Examples



$A \quad \rightarrow \quad B$

Functions (12/21)

Examples



NOT a function from *A* to *B*

A* → *B

Functions (13/21)

Examples

Determine whether the function $f(x) = x + 1$ from the set of integers to the set of integers is one-to-one.

Functions (13/21)

Examples (Answer)

Determine whether the function $f(x) = x + 1$ from the set of integers to the set of integers is one-to-one.

$f(a) = a + 1$ and $f(b) = b + 1$

$f(x)$ is one–to–one (if $f(a) = f(b)$ and a equal b then).

$$a + 1 = b + 1$$

$$a = b$$

$$\therefore f(x) \text{ is one–to–one}$$

Functions (14/21)

Examples

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Functions (14/21)

Examples (Answer)

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

$f(a) = a^2$ and $f(b) = b^2$

$f(x)$is one–to–one (if $f(a) = f(b)$ and a equal b then).

$$a^2 = b^2$$

$$\pm a = \pm b$$

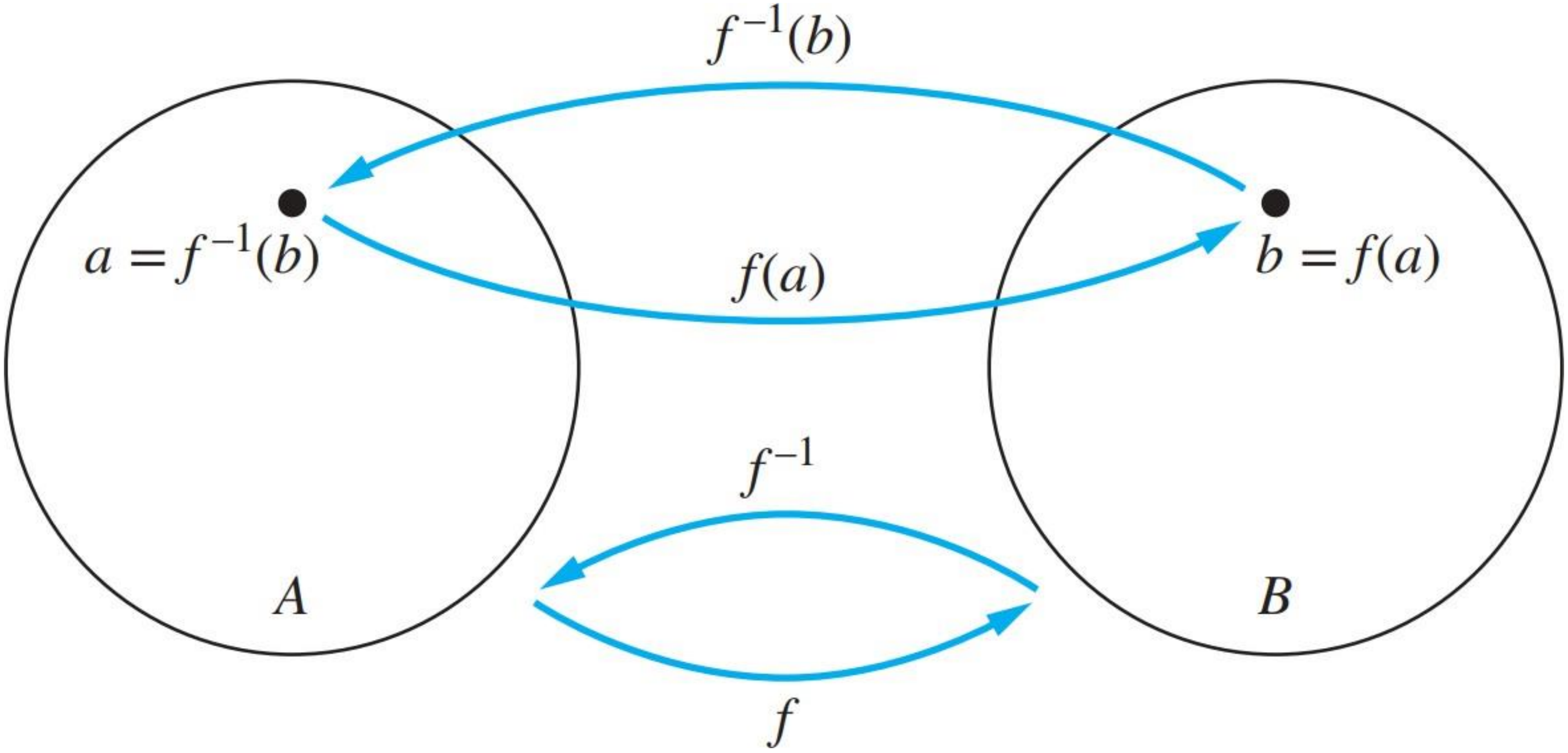
$$a \text{ may be not equal } b$$

$$\therefore f(x) \text{ is NOT one–to–one}$$

Inverse Functions

Let f be a *one-to-one correspondence* from the set A to the set B. The **inverse** function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by $\boldsymbol{f^{-1}}$. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Functions



Invertible

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Invertible – Example

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Invertible – Example

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

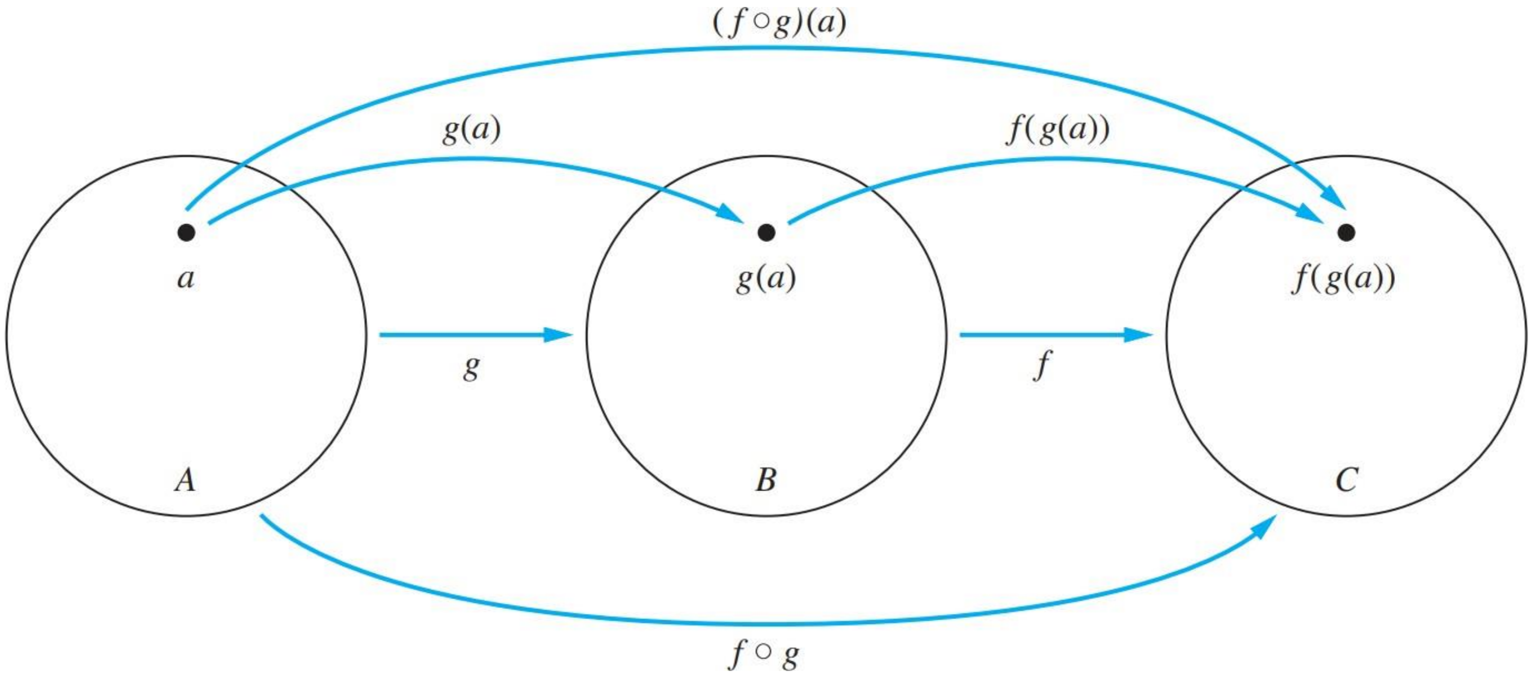
Answer:

The function f is invertible because it is a one-to-one correspondence.

The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Composition of the Functions *f* and *g*

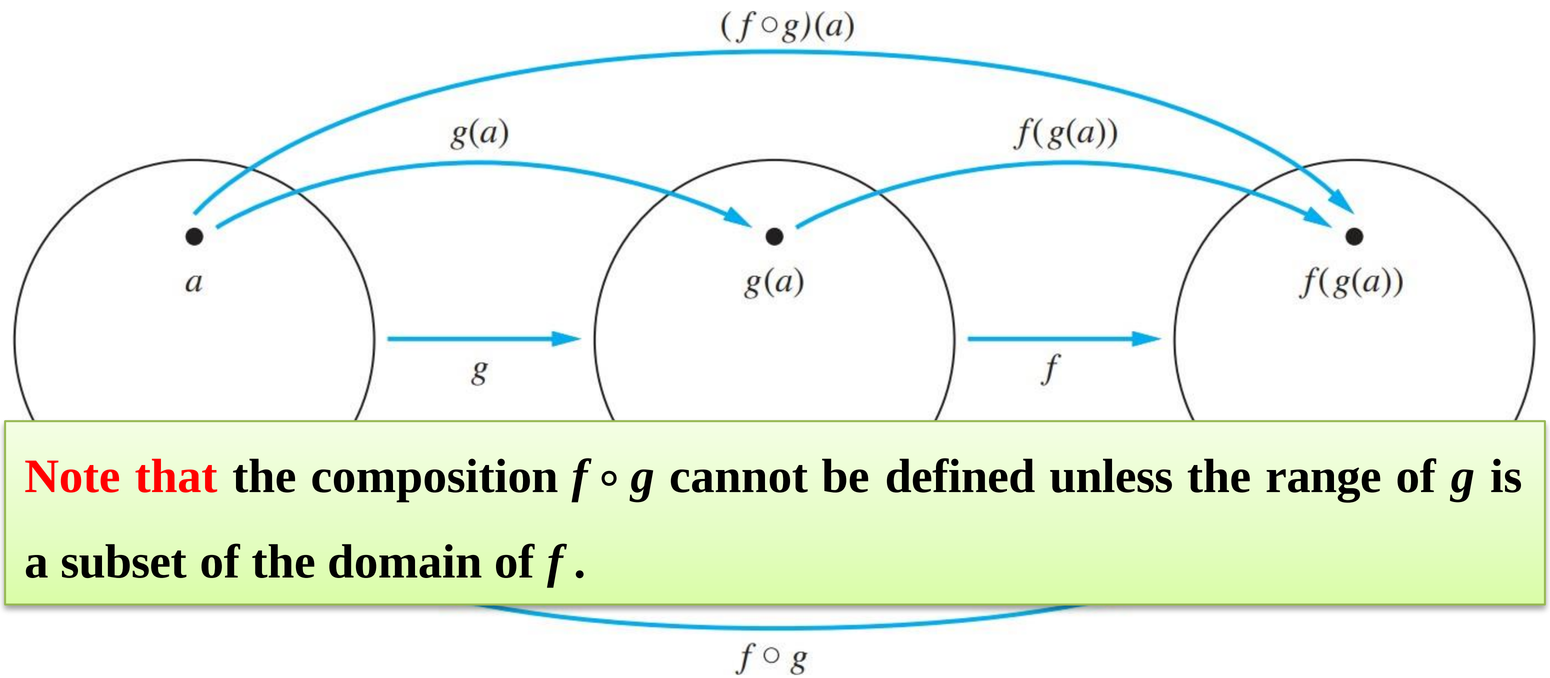
Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Functions (18/21)

Composition of the Functions *f* and *g*

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Composition Example 1

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Composition Example 1

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

Answer:

1)The composition of f and g (i.e., $(f \circ g)$)

$(f \circ g)(a) = 2$, $(f \circ g)(b) = 1$, $(f \circ g)(c) = 3$

2) The composition of g and f (i.e., $(g \circ f)$) **cannot be defined** because the range of f is NOT a subset of the domain of g .

Composition Example 2

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Composition Example 2

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

Answer:

1) The composition of f and g (i.e., $(f \circ g)$)

$(f \circ g)(x) = f\big(g(x)\big) = 2(3x + 2) + 3 = 6x + 7$

2) The composition of g and f (i.e., $(g \circ f)$)

$(g \circ f)(x) = g\big(f(x)\big) = 3(2x + 3) + 2 = 6x + 11$

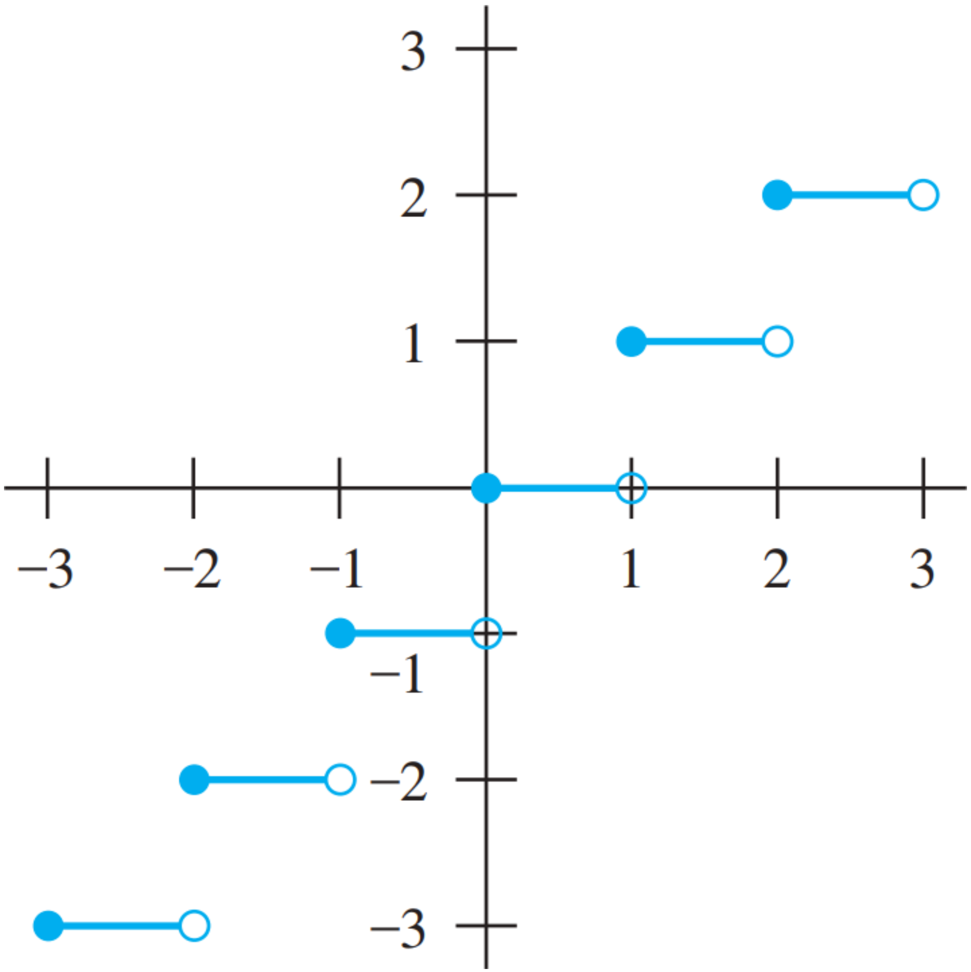
The Graphs of Functions

Let f be a function from A to B. The graph of the function f is the set of ordered pairs $\{(a, b) | \ a \in A \text{ and } b \in B\}$.

The graph of $f(x) = x^2$ from Z to Z.

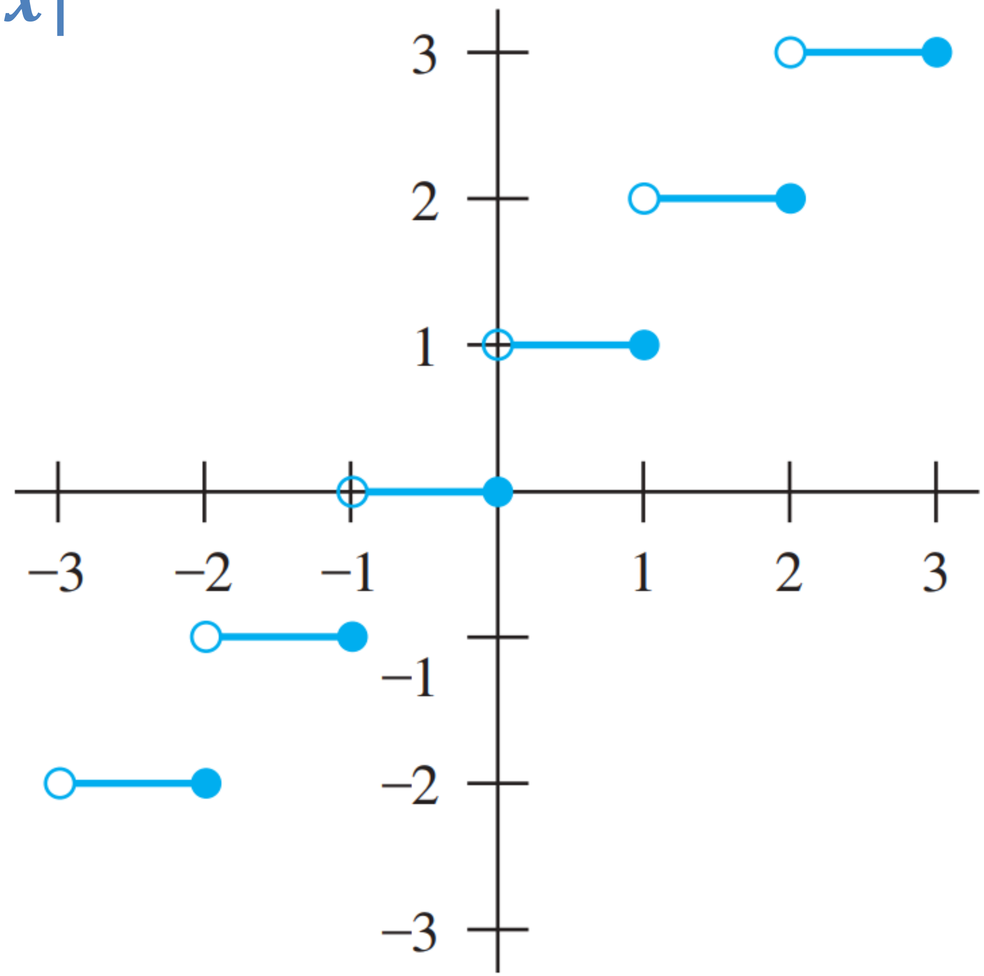
Some Important Functions (1/4)

Floor function $y = \lfloor x \rfloor$



Some Important Functions (2/4)

Ceiling function $y = \lceil x \rceil$



Some Important Functions (3/4)

Useful Properties

$$\lfloor -x \rfloor = -\lceil x \rceil$$
$$\lceil -x \rceil = -\lfloor x \rfloor$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$
$$\lceil x + n \rceil = \lceil x \rceil + n$$

Some Important Functions (4/4)

Examples

$\lfloor 0.5 \rfloor =$

$\lceil 0.5 \rceil =$

$\lceil 3 \rceil =$

$\lfloor -0.5 \rfloor =$

$\lceil -1.2 \rceil =$

$\lfloor 1.1 \rfloor =$

$\lfloor 0.3 + 2 \rfloor =$

$\left\lceil 1.1 + \lceil 0.5 \rceil \right\rceil =$

Some Important Functions (4/4)

Examples-Answer

$\lfloor 0.5 \rfloor = 0$

$\lceil 0.5 \rceil = 1$

$\lceil 3 \rceil = 3$

$\lfloor -0.5 \rfloor = -\lceil 0.5 \rceil = -1$

$\lceil -1.2 \rceil = -1$

$\lfloor 1.1 \rfloor = 1$

$\lfloor 0.3 + 2 \rfloor = 2$

$\left\lceil 1.1 + \lceil 0.5 \rceil \right\rceil = 3$