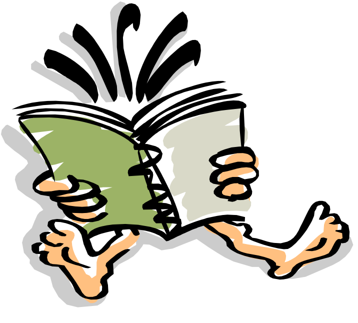
Discrete Structures 2

Chapter 6: Counting



Chapter 6: Counting

- 6.1 The Basics of Counting.
- 6.2 The Pigeonhole Principle.
- 6.3 Permutations and Combinations.
- 6.4 Binomial Coefficients and Identities.
- 6.5 Generalized Permutations and Combinations.
- 6.6 Generating Permutations and Combinations.

Chapter 6: Counting

- 6.1 The Basics of Counting.
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The Basic of Counting (1/24)

Introduction:

Suppose that a **password** on a computer system consists of **eight** characters. Each of these characters must be a **digit** or a **letter** of the alphabet. Each password must contain **at least one digit**.

How many such passwords are there?!

The Basic of Counting (2/24)

Multiplication (Product) Rule:

Suppose that a procedure can be **broken down** into a **sequence** of **two tasks**. If there are $\boldsymbol{n_1}$ ways to do the first task and for each of these ways of doing the first task, there are $\boldsymbol{n_2}$ ways to do the second task, then there are $\boldsymbol{n_1 n_2}$ ways to do the procedure.

The total number of ways to complete the operation is

$$n_1 \times n_2 \times \cdots \times n_k$$

The Basic of Counting (3/24)

Product Rule – Example 1:

A new company with just two employees, Sanchez and John, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

The Basic of Counting (3/24)

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Solution:

Sanchez	**John**
First, we have 12 offices Then, we select 1 from 12 offices	

The Basic of Counting (3/24)

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$n_1 = 12$ ways	

The Basic of Counting (3/24)

Product Rule – Example 1:

A new company with just two employees, Sanchez and John, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

<u>Solution:</u>

Sanchez	**John**
First, we have 12 offices Then, we select 1 from 12 offices	Second, we have 11 offices Then, we select 1 from 11 offices
$n_1 = 12$ ways	

Product Rule – Example 1:

A new company with just two employees, Sanchez and John, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

Sanchez	**John**
First, we have 12 offices Then, we select 1 from 12 offices	Second, we have 11 offices Then, we select 1 from 11 offices
$n_1 = 12$ ways	$n_2 = 11$ ways

The Basic of Counting (3/24)

Product Rule – Example 1:

A new company with just two employees, Sanchez and John, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

Sanchez	**John**
First, we have 12 offices Then, we select 1 from 12 offices	Second, we have 11 offices Then, we select 1 from 11 offices
$n_1 = 12$ ways	$n_2 = 11$ ways
Total = 12 × 11 = 132 ways to assign offices to these two employees.	

The Basic of Counting (4/24)

Product Rule – Example 2:

How many different bit strings of length seven are there?

The Basic of Counting (4/24)

Product Rule – Example 2:

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Bits #	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Value	either 0 or 1						
Ways	$n_1 = 2$						

The Basic of Counting (4/24)

Product Rule – Example 2:

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Bits #	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Value	either 0 or 1	either 0 or 1					
Ways	$n_1 = 2$	$n_2 = 2$					

Product Rule – Example 2:

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Bits #	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Value	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1
Ways	$n_1 = 2$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 2$	$n_6 = 2$	$n_7 = 2$

Product Rule – Example 2:

How many different bit strings of length seven are there?

<u>Solution:</u>

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Bits #	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Value	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1
Ways	$n_1 = 2$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 2$	$n_6 = 2$	$n_7 = 2$
Total $= 2^7 = 128$ different bit strings of length seven.							

Product Rule – Example 3:

In how many different ways can a true-false test consisting of 10 questions be answered?

The Basic of Counting (5/24)

Product Rule – Example 3:

In how many different ways can a true-false test consisting of 10 questions be answered?

Solution: Each of the 10 questions can be chosen in two ways, because each question is either true or false. Therefore, the product rule shows there are:

$2 \times 2 \times \cdots \times 2 = 2^{10} = 1024$ ways to answer the test.

The Basic of Counting (6/24)

Product Rule – Example 4:

The design for a Website is to consist of *four colors*, *three fonts*, and *three positions for an image*.

How many different designs are possible?

The Basic of Counting (6/24)

Product Rule – Example 4:

The design for a Website is to consist of *four colors*, *three fonts*, and *three positions for an image*.

How many different designs are possible?

<u>**Solution:**</u> From the product rule, $4 \times 3 \times 3 = 36$ different designs are possible.

The Basic of Counting (7/24)

Product Rule – Example 5:

How many bit strings of length 5, start and end with 1's?

Product Rule – Example 5:

How many bit strings of length 5, start and end with 1's?

Solution:

Bits #	1	2	3	4	5
Value	1	either 0 or 1	either 0 or 1	either 0 or 1	1
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 1$

The Basic of Counting (7/24)

Product Rule – Example 5:

How many bit strings of length 5, start and end with 1's?

Solution:

Bits #	1	2	3	4	5
Value	1	either 0 or 1	either 0 or 1	either 0 or 1	1
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 1$
Total $= 1 \times 2 \times 2 \times 2 \times 1 = 8$ different bit strings of length 5, start and end with 1's.					

The Basic of Counting (8/24)

Product Rule – Example 6:

How many different license plates are available if each plate contains a sequence of *three letters* followed by *three digits*.

The Basic of Counting (8/24)

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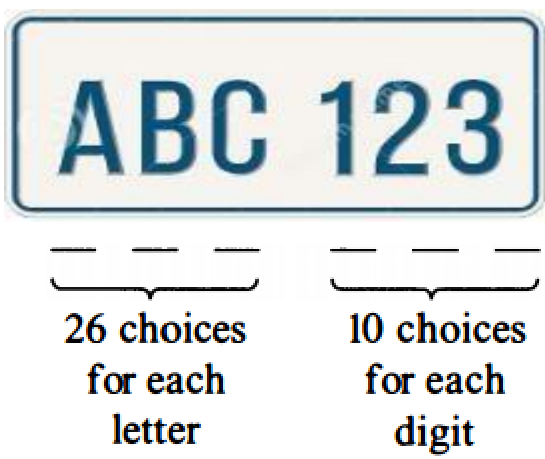
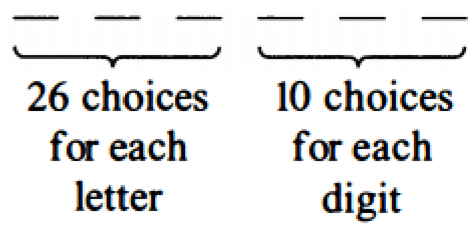
How many different license plates are available if each plate contains a sequence of *three letters* followed by *three digits*.



The Basic of Counting (8/24)

Product Rule – Example 6:

How many different license plates are available if each plate contains a sequence of *three letters* followed by *three digits*.



Solution:

There are 26 choices for each of the three letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

Solution:

Domain
1
2
3
.
m

$\longrightarrow$

Co-Domain
1
2
3
.
n

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

Solution:

ways	Domain
n	1
	2
	3
	.
	m

$\longrightarrow$

Co-Domain
1
2
3
.
n

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with *m* elements to a set with *n* elements?

Solution:

ways	Domain
n	1
n	2
	3
	.
	m

Co-Domain
1
2
3
.
n

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

Solution:

ways	Domain
n	1
n	2
n	3
.	.
n	m

Co-Domain
1
2
3
.
n

The Basic of Counting (9/24)

Product Rule – Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

Solution:

ways	Domain
n	1
n	2
n	3
.	.
n	m

Co-Domain
1
2
3
.
n

Hence, by the product rule there are $n \cdot n \cdot \ \ldots \cdot n = n^m$ functions from a set with m elements to one with n elements.

The Basic of Counting (10/24)

Product Rule – Counting Functions:

How many functions are there from a set with **3** elements to a set with **4** elements?

The Basic of Counting (10/24)

Product Rule – Counting Functions:

How many functions are there from a set with **3** elements to a set with **4** elements?

Solution:

ways	Domain
	1
	2
	3

→

Co-Domain
1
2
3
4

The Basic of Counting (10/24)

Product Rule – Counting Functions:

How many functions are there from a set with **3** elements to a set with **4** elements?

Solution:

ways	Domain
4	1
4	2
4	3

$\longrightarrow$

Co-Domain
1
2
3
4

The Basic of Counting (10/24)

Product Rule – Counting Functions:

How many functions are there from a set with **3** elements to a set with **4** elements?

Solution:

ways	Domain
4	1
4	2
4	3

Co-Domain
1
2
3
4

Hence, by the product rule there are $4 \cdot 4 \cdot 4 = 4^3$ functions from a set with 3 elements to one with 4 elements.

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
	1
	2
	3
	.
	m

Co-Domain
1
2
.
.
n

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
	2
	3
	.
	m

$\longrightarrow$

Co-Domain
1
2
.
.
n

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
$(n-1)$	2
	3
	.
	m

Co-Domain
1
2
.
.
n

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
$(n-1)$	2
$(n-2)$	3
	.
	m

Co-Domain
1
2
.
.
n

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
$(n-1)$	2
$(n-2)$	3
.	.
$n - \quad m - 1$	m

Co-Domain
1
2
.
.
n

The Basic of Counting (11/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
$(n-1)$	2
$(n-2)$	3
.	.
$(n-(m-1))$	m

Co-Domain
1
2
.
.
n

By the product rule, there are $n(n-1)(n-2)\ldots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

The Basic of Counting (12/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with **4** elements to a set with **6** elements?

The Basic of Counting (12/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with **4** elements to a set with **6** elements?

Solution:

ways	Domain
	1
	2
	3
	4

→

Co-Domain
1
2
3
4
5
6

The Basic of Counting (12/24)

Product Rule – Counting One-to-One Functions:

How many one-to-one functions are there from a set with **4** elements to a set with **6** elements?

Solution:

$(n-(m-1))$

$(6-(4-1))$

ways	Domain
6	1
5	2
4	3
3	4

→

Co-Domain
1
2
3
4
5
6

By the product rule, there are $6 \times 5 \times 4 \times 3 = 360$ one-to-one functions from a set with 4 elements to one with 6 elements.

The Basic of Counting (13/24)

Product Rule – Counting One-to-One Functions:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad \textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad\quad \cdot \\
\qquad\quad \cdot \\
\qquad\quad \cdot \\
\qquad \textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad\quad k := k + 1
\end{array}
$$

The Basic of Counting (13/24)

Product Rule – Counting One-to-One Functions:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

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\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad \textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad \cdot \\
\qquad \cdot \\
\qquad \cdot \\
\qquad\quad \textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad\qquad k := k + 1
\end{array}
$$

The loop is traversed
n_1 times
$n_1 \times n_2$ times
$n_1 \times n_2 \times \cdots \times n_m$ times

The Basic of Counting (13/24)

Product Rule – Counting One-to-One Functions:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad \textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad \vdots \\
\qquad\quad \textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad\qquad k := k + 1
\end{array}
$$

The loop is traversed
n_1 times
$n_1 \times n_2$ times
$n_1 \times n_2 \times \cdots \times n_m$ times

The initial value of k is zero.

By the product rule, it follows that the nested loop is traversed $n_1 n_2 \cdots n_m$ times.

Hence, the final value of k is $n_1 n_2 \cdots n_m$

The Basic of Counting (14/24)

The product rule in terms of sets :

The product rule is often phrased in terms of sets in this way: If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , … , and an element in A_m. By the product rule it follows that

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

The Basic of Counting (15/24)

The Sum Rule:

If a task can be done either in n_1 ways or in n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The Basic of Counting (16/24)

The sum rule in terms of sets :

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

$$\text{if } A_1, A_2, \ldots, A_m \quad \text{disjoint}$$

$$\text{if } A_1, A_2 \ \ NOT \text{ disjoint}$$

$$\therefore\ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Basic of Counting (17/24)

Sum Rule – Example 1:

Suppose that either a member of the mathematics major or a student who is a physics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics majors and 83 physics majors and no one is both a mathematics and a physics major?

The Basic of Counting (17/24)

Sum Rule – Example 1:

Suppose that **either** a member of the mathematics major or a student who is a physics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are **37** members of the mathematics majors and **83** physics majors and **no one is both** a mathematics and a physics major?

Solution:

By the sum rule it follows that there are 37 + 83 = 120 possible ways.

The Basic of Counting (18/24)

Sum Rule – Example 2:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

The Basic of Counting (18/24)

Sum Rule – Example 2:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

By the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

The Basic of Counting (19/24)

Sum Rule – Example 3:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad\quad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\quad\quad k := k + 1 \\
\quad\quad\quad\quad . \\
\quad\quad\quad\quad . \\
\quad\quad\quad\quad . \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\quad\quad k := k + 1
\end{array}
$$

Sum Rule – Example 3:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k := 0
for i_1 := 1 to n_1
    k := k + 1
for i_2 := 1 to n_2
    k := k + 1
    .
    .
    .
for i_m := 1 to n_m
    k := k + 1
```

The loop is traversed
n_1 times
n_2 times
n_m times

The initial value of k is zero.

Because we only traverse one loop at a time, the sum rule shows that the final value of k, which is the number of ways to traverse one of the m loops is

$$n_1 + n_2 + \cdots + n_m$$

The Basic of Counting (20/24)

Counting Problems – Example 1:

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?

Counting Problems – Example 1:

Solution:

Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$.

Counting Problems – Example 1:

Solution:

Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits.

By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6$$

Counting Problems – Example 1:

Solution:

Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits.

By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6$$

# of uppercase letters = 26

+

# of digits = 10

The Basic of Counting (20/24)

Counting Problems – Example 1:

Solution:

Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$.

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776$
$= 1{,}867{,}866{,}560.$

Counting Problems – Example 1:

Solution:

Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$.

$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776$$
$$= 1{,}867{,}866{,}560.$$

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176$$
$$= 70{,}332{,}353{,}920.$$

$$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576$$
$$= 2{,}612{,}282{,}842{,}880.$$

The Basic of Counting (20/24)

Counting Problems – Example 1:

Solution:

By the sum rule, $P = P_6 + P_7 + P_8$.

$$P = P_6 + P_7 + P_8$$

$$= 1{,}867{,}866{,}560 + 70{,}332{,}353{,}920 + 2{,}612{,}282{,}842{,}880$$

$$= 2{,}684{,}483{,}063{,}360$$

The Basic of Counting (21/24)

Counting Problems – Example 2:

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
bride 	Select from 9 people	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
bride 	Select from 9 people	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	9	8	7	6	5

By the product rule, there are
$1 \times 9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$ ways that the bride must be in the picture.

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
bride 	Select from 9 people	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	9	8	7	6	5

By the product rule, there are
$1 \times 9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$ ways that the bride must be in the picture.

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
Select from 9 people	bride 	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
9	1	8	7	6	5

By the product rule, there are
$1 \times 9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$ ways that the bride must be in the picture.

The Basic of Counting (21/24)

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The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
Select from 9 people	Select from 8 people	bride 	Select from 7 people	Select from 6 people	Select from 5 people
9	8	1	7	6	5

By the product rule, there are
$6 \times (1 \times 9 \times 8 \times 7 \times 6 \times 5) = 6 \times 15{,}120$ ways that the bride must be in the picture.

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

b) both the bride and groom must be in the picture?

1	2	3	4	5	6
bride 	groom 	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

b) both the bride and groom must be in the picture?

1	2	3	4	5	6
bride 	groom 	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	1	8	7	6	5

The Basic of Counting (21/24)

Group of 10 people

We pick 6 people

Counting Problems – Example 2:

b) both the bride and groom must be in the picture?

1	2	3	4	5	6
bride 	groom 	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	1	8	7	6	5

By the product rule, there are $6 \times 5 \times (1 \times 1 \times 8 \times 7 \times 6 \times 5) = 30 \times 1{,}680$ ways that both the bride and groom must be in the picture.

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

c) exactly one of the bride and the groom is in the picture?

The bride must be in the picture	A	$\lvert A \rvert = 6 \times 15{,}120 = 90{,}720$
The groom must be in the picture	B	$\lvert B \rvert = 6 \times 15{,}120 = 90{,}720$

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

c) exactly one of the bride and the groom is in the picture?

The bride must be in the picture	A	$\|A\| = 6 \times 15{,}120 = 90{,}720$
The groom must be in the picture	B	$\|B\| = 6 \times 15{,}120 = 90{,}720$
Both the bride and groom must be in the picture	$A \cap B$	$\|A \cap B\| = 30 \times 1{,}680 = 50{,}400$

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

c) exactly one of the bride and the groom is in the picture?

The bride must be in the picture	A	$\lvert A\rvert = 6 \times 15{,}120 = 90{,}720$
The groom must be in the picture	B	$\lvert B\rvert = 6 \times 15{,}120 = 90{,}720$
Both the bride and groom must be in the picture	$A \cap B$	$\lvert A \cap B\rvert = 30 \times 1{,}680 = 50{,}400$
The bride in the picture and the groom is **<u>not</u>** in the picture	$A - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$
The groom in the picture and the bride is **<u>not</u>** in the picture	$B - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$

The Basic of Counting (21/24)

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

c) exactly one of the bride and the groom is in the picture?

The bride must be in the picture	A	$\|A\| = 6 \times 15{,}120 = 90{,}720$
The groom must be in the picture	B	$\|B\| = 6 \times 15{,}120 = 90{,}720$
Both the bride and groom must be in the picture	$A \cap B$	$\|A \cap B\| = 30 \times 1{,}680 = 50{,}400$
The bride in the picture and the groom is **<u>not</u>** in the picture	$A - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$
The groom in the picture and the bride is **<u>not</u>** in the picture	$B - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$
Exactly one of the bride and the groom is in the picture	Using the sum rule	$= 40{,}320 + 40{,}320 = 80{,}640$

The Basic of Counting (22/24)

The Subtraction Rule:

If a task can be done either in n_1 ways or in n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The Basic of Counting (22/24)

The principle of inclusion–exclusion:

The subtraction rule is also known as the ***principle of inclusion–exclusion***, especially when it is used to count the number of elements in the union of two sets.

$$\text{if } A_1, A_2 \ \ NOT \text{ disjoint}$$
$$\therefore |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *<u>either</u>* start with a 1 bit *<u>or</u>* end with the two bits 00 ?

1	**--**	**--**	**--**	**--**

or

--	**--**	**--**	**0**	**0**

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

1	--	--	--	--

or

--	--	--	**0**	**0**

1	0	1	0	1
1	0	0	0	0
1	1	1	0	0

- -

1	1	1	**0**	**0**
1	0	0	**0**	**0**
0	1	1	**0**	**0**

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *<u>either</u>* start with a 1 bit *<u>or</u>* end with the two bits 00 ?

1	--	--	--	--

or

--	--	--	**0**	**0**

1	0	1	0	1
1	0	0	0	0
1	1	1	0	0

1	1	1	**0**	**0**
1	0	0	**0**	**0**
0	1	1	**0**	**0**

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

1	--	--	--	--	A
--	--	--	0	0	B
1	--	--	0	0	$A \cap B$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

1	--	--	--	--

A

Solution:

Bits #	**1**	**2**	**3**	**4**	**5**
Value	1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 2$

Total $= 1 \times 2 \times 2 \times 2 \times 2 = 2^4$ different bit strings of length 5, start with 1.

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

--	--	--	0	0	B

Solution:

Bits #	1	2	3	4	5
Value	either 0 or 1	either 0 or 1	either 0 or 1	0	0
Ways	$n_1 = 2$	$n_2 = 2$	$n_3 = 2$	$n_4 = 1$	$n_5 = 1$

Total $= 2 \times 2 \times 2 \times 1 \times 1 = 2^3$ different bit strings of length 5, end with the two bits 00.

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

1	--	--	0	0

$A \cap B$

Solution:

Bits #	**1**	**2**	**3**	**4**	**5**
Value	1	either 0 or 1	either 0 or 1	0	0
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 1$	$n_5 = 1$

Total $= 1 \times 2 \times 2 \times 1 \times 1 = 2^2$ different bit strings of length 5, start with a 1 bit *AND* end with the two bits 00.

The Basic of Counting (23/24)

Subtraction Rule – Example 1:

How many bit strings of length **five** *either* start with a 1 bit *or* end with the two bits 00 ?

Solution:

The total number of bit strings of length five either start with a 1 bit or end with the two bits 00 is:

$$= 2^4 + 2^3 - 2^2 = 16 + 8 - 4 = 20$$

The Basic of Counting (24/24)

Subtraction Rule – Example 2:

A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Subtraction Rule – Example 2:

A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution:

220 computer science	$\lvert A \rvert$
147 business	$\lvert B \rvert$
51 both	$\lvert A \cap B \rvert$
Total 350 applications	$\lvert U \rvert$

The Basic of Counting (24/24)

Subtraction Rule – Example 2:

Solution:

220 computer science	$\lvert A\rvert$
147 business	$\lvert B\rvert$
51 both	$\lvert A\cap B\rvert$
Total 350 applications	$\lvert U\rvert$

The applicants majored neither in computer science nor in business

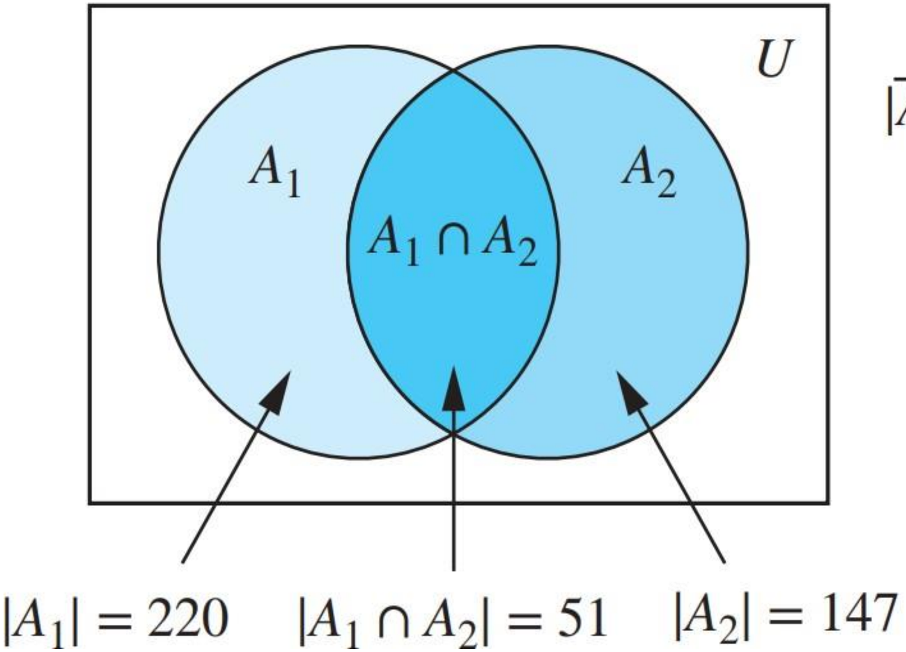
$$= |U| - |A \cup B|$$

$$= |U| - (|A| + |B| - |A \cap B|) = 350 - (220 + 147 - 51) = 34$$

Subtraction Rule – Example 2:

Solution:

Venn diagram



$$
\begin{aligned}
|\overline{A_1 \cup A_2}| &= |U| - |A_1 \cup A_2| \\
&= |U| - (|A_1| + |A_2| - |A_1 \cap A_2|) \\
&= 350 - (220 + 147 - 51) \\
&= 350 - 316 \\
&= 34
\end{aligned}
$$

Permutations and Combinations (1/16)

Introduction

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements' matters.

Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Permutations and Combinations (2/16)

Permutation (1/2)

A **permutation** of a set of distinct objects is an *ordered* arrangement of these objects. For instant, find the number of ordered sequences of the elements of a set. Consider a set of elements, such as $S = \{a, b, c\}$.

A permutation of the elements is an ordered sequence of the elements. For example, abc, acb, bac, bca, cab, and cba are all of the permutations of the elements of S.

$$3 \times 2 \times 1 = 6$$

Permutations and Combinations (2/16)

Permutation (2/2)

The number of permutations of n different elements is $n!$ where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

Permutations and Combinations (2/16)

Permutation (2/2)

The number of permutations of n different elements is $n!$ where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

For instance, the number of permutations of the four letters a, b, c, and d will be $4! = 24$.

Permutations and Combinations (3/16)

Example 1:

How many permutations of the letters ABCDEFGH contain the string ABC ?

Permutations and Combinations (3/16)

Example 1:

How many permutations of the letters ABCDEFGH contain the string ABC ?

Solution:

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.

Permutations and Combinations (4/16)

r-Permutation

We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

The number of permutations of subsets of $\boldsymbol{r}$ elements selected from a set of $\boldsymbol{n}$ different elements is

$$P(n,r) = P_r^n = {}_nP_r$$

$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

where $1 \leq r \leq n$

Permutations and Combinations (5/16)

Example 1:

Consider a set of elements, such as $S = \{a, b, c, d, e\}$.

What is the number of permutation of subsets of **3** elements selected from S is?

Permutations and Combinations (5/16)

Example 1:

Consider a set of elements, such as $S = \{a, b, c, d, e\}$.

What is the number of permutation of subsets of **3** elements selected from S is?

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

Solution:

$\boldsymbol{r = 3}$, $\boldsymbol{n = 5}$

$$P(5,3) = \frac{5!}{(5-3)!}$$

$$= \frac{5!}{(2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

Permutations and Combinations (6/16)

Example 2:

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations and Combinations (6/16)

Example 2:

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solution:

$\boldsymbol{r = 3}$, $\boldsymbol{n = 100}$

$$P(100,3) = \frac{100!}{(100-3)!}$$

$$= \frac{100!}{(97)!} = 100 \times 99 \times 98 = 970{,}200$$

Permutations and Combinations (7/16)

Combinations:

We now turn our attention to counting **unordered** selections of objects. To find the number of subsets of a particular size of a set with n elements, where n is a positive integer. An ***r*-combination** of elements of a set is an unordered selection of r elements from the set.

$$C(n,r) = C^n = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$C(n,r) = C(n,n-r)$$

Permutations and Combinations (8/16)

Example 1:

How many possible selections of 3 balls from a box contains 10 colored balls ?

Permutations and Combinations (8/16)

Example 1:

How many possible selections of 3 balls from a box contains 10 colored balls ?

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Solution:

$\boldsymbol{r = 3}$, $\boldsymbol{n = 10}$

$$C(10,3) = \frac{10!}{3!\,(10-3)!}$$

$$= \frac{10!}{3!\,7!} = 120$$

Permutations and Combinations (9/16)

Example 2:

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school ?

Permutations and Combinations (9/16)

Example 2:

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school ?

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Solution:

$\boldsymbol{r = 5}$, $\quad\quad \boldsymbol{n = 10}$

$$C(10,5) = \frac{10!}{5!\,(10-5)!}$$

$$= \frac{10!}{5!\,5!} = 252$$

Permutations and Combinations (10/16)

Example 3:

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Permutations and Combinations (10/16)

Example 3:

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution:

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!\,6!} \cdot \frac{11!}{4!\,7!} = 84 \cdot 330 = 27{,}720$$

Permutations and Combinations (11/16)

Example 4:

How many bit strings of length n contain exactly r 1s?

Permutations and Combinations (11/16)

Example 4:

How many bit strings of length n contain exactly r 1s?

Location	1	2	3	4		…		$n-2$	$n-1$	n
Bit										

Permutations and Combinations (11/16)

Example 4:

How many bit strings of length n contain exactly r 1s?

Location	1	2	3	4		…		$n-2$	$n-1$	n
Bit	1	1	0	1		…	0	1	0	1

Permutations and Combinations (11/16)

Example 4:

How many bit strings of length n contain exactly r 1s?

Location	1	2	3	4		…		$n-2$	$n-1$	n
Bit										

This is just asking us to choose r out of n slots to place 1's in.

Permutations and Combinations (11/16)

Example 4:

How many bit strings of length n contain exactly r 1s?

Solution:

The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $\boxed{C(n, r)}$ bit strings of length n that contain exactly r 1s.

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10
Bit										

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10
Bit	1	1	1	1	0	0	0	0	0	0

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10
Bit	1	1	1	1	0	0	0	0	0	0

Location	1	2	3	4	5	6	7	8	9	10
Bit	0	0	0	1	1	1	1	0	0	0

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10
Bit										

This is just asking us to choose 4 out of 10 slots to place 1’s in.

Permutations and Combinations (12/16)

Example 5:

How many bit strings of length 10 contain exactly four 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10
Bit										

This is just asking us to choose 4 out of 10 slots to place 1's in.

$$C(10,4) = \frac{10!}{4! \times 6!} = 210$$

Permutations and Combinations (13/16)

Example 6:

How many bit strings of length 10 contain at most four 1s?

Permutations and Combinations (13/16)

Example 6:

How many bit strings of length 10 contain at most four 1s?

Solution:

We add up the number of bit strings of length 10 that contain zero 1s, one 1, two 1s, three 1s, and four 1s.

$$C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$$

$$= 1 \quad + 10 \quad + 45 \quad + 120 \quad + 210 \quad = 386$$

Permutations and Combinations (14/16)

Example 7:

How many bit strings of length 10 contain at least four 1s?

Permutations and Combinations (14/16)

Example 7:

How many bit strings of length 10 contain at least four 1s?

4, 5, 6, 7, 8, 9 or 10 1s.

Permutations and Combinations (14/16)

Example 7:

How many bit strings of length 10 contain at least four 1s?

Solution:

4, 5, 6, 7, 8, 9 or 10 1s.

We subtract from the total number of bit strings of length 10 those that have only 0, 1, 2 or 3 1s.

Permutations and Combinations (14/16)

Example 7:

How many bit strings of length 10 contain at least four 1s?

Solution:

4, 5, 6, 7, 8, 9 or 10 1s.

We subtract from the total number of bit strings of length 10 those that have only 0, 1, 2 or 3 1s.

$$= 2^{10} - \left(C(10,0) + C(10,1) + C(10,2) + C(10,3)\right)$$

$$= 1024 - (1 \quad + 10 \quad + 45 \quad + 120 \quad) = 848$$

Permutations and Combinations (15/16)

Example 8:

How many bit strings of length 10 contain an equal number of 0s and 1s?

Permutations and Combinations (15/16)

Example 8:

How many bit strings of length 10 contain an equal number of 0s and 1s?

Solution:

Choose 5 out of 10 slots to place 1s (the remaining 5 slots are filled with 0s):

$$C(10,5) = \frac{10!}{5! \times 5!} = 252$$

Permutations and Combinations (16/16)

Example 9:

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Permutations and Combinations (16/16)

Example 9:

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Bit																			

Permutations and Combinations (16/16)

Example 9:

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Bit																			

‘011’ for five 0s, then we use 15 bits, and the remainder 4 bits contains 1s.

Choose 5 out of 9 locations
(<u>Note:</u> 9 locations are: 5 locations for ‘011’ and 4 locations for ‘1’)

Permutations and Combinations (16/16)

Example 9:

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution:

Location	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Bit																			

Choose 5 out of 9 locations

$$C(9,5) = \frac{9!}{5! \times 4!} = 126$$