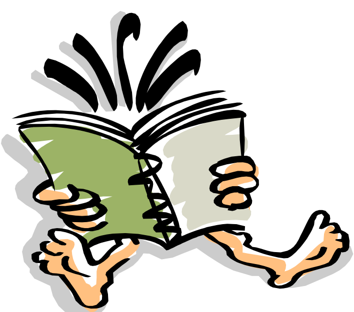
CS1101
Discrete Structures 1

Chapter 02
Basic Structures: Sets,



Sets (1/24)

A **set** is an unordered collection of objects.

The objects in a set are called the *elements*, or *members*, of the set. A set is said to contain its elements.

Sets (2/24)

$S = \{a, b, c, d\}$

We write $a \in S$ to denote that a is an element of the set S. The notation $e \notin S$ denotes that e is not an element of the set S.

Sets (3/24)

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

ellipses (…)

Sets (4/24)

Another way to describe a set is to use **set builder** notation.

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

$$\boxed{O = \{x \mid x \text{ is an odd positive integer less than } 10\},}$$

$$\boxed{O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}.}$$

Sets (5/24)

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all **natural numbers**

$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all **integers**

$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all **positive integers**

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$,
the set of all **rational numbers**

$\mathbf{R}$, the set of all **real numbers**

$\mathbf{R}^+$, the set of all **positive real numbers**

$\mathbf{C}$, the set of all **complex numbers**.

Interval Notation

Closed interval $[a, b]$
Open interval (a, b)

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}$$

Sets (7/24)

If A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$, if A and B are equal sets.

- The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements.

- $\{1, 3, 3, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

Empty Set

There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$.

The empty set can also be denoted by $\{\ \}$

Cardinality

The cardinality is the number of distinct elements in S. The cardinality of S is denoted by $|S|$.

Example1

$$S = \{a, b, c, d\}$$
$$|S| = 4$$

$$A = \{1, 2, 3, 7, 9\}$$

$$\emptyset = \{\ \}$$

Example1

$S = \{a, b, c, d\}$
$|S| = 4$

$A = \{1, 2, 3, 7, 9\}$
$|A| = 5$

$\emptyset = \{\ \}$
$|\emptyset| = 0$

Example2

$S = \{a, b, c, d, \{2\}\}$
$|S| =$

$A = \{1, 2, 3, \{2,3\}, 9\}$
$|A| =$

$\{\emptyset\} = \{\{\ \ \}\}$
$|\{\emptyset\}| =$

Example2

$$S = \{a, b, c, d, \{2\}\}$$
$$|S| = 5$$

$$A = \{1, 2, 3, \{2,3\}, 9\}$$
$$|A| = 5$$

$$\{\emptyset\} = \{\{\ \}\}$$
$$|\{\emptyset\}| = 1$$

Infinite

A set is said to be **infinite** if it is not finite.
The set of positive integers is infinite.

$$Z^{+} = \{1,2,3, \ldots\}$$

Subset

The set A is said to be a subset of B if and only if every element of A is also an element of B .

We use the notation $A \subseteq B$ to indicate that A is a subset of the set B .

$$\boxed{A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)}$$

Subset

The set A is said to be a subset of B if and only if every element of A is also an element of B .

We use the notation $A \subseteq B$ to indicate that A is a subset of the set B .

$$\boldsymbol{(A \subseteq B) \equiv (B \supseteq A)}$$

$$\boxed{A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)}$$

Subset

For every set S,

$(i)\ \emptyset \subseteq S$ **and** $(ii)\ S \subseteq S$.

To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Proper Subset

The set A is a subset of the set B but that $A \neq B$,
we write $A \subset B$
and say that A is a **proper subset** of B.

$$\boxed{A \subset B \leftrightarrow \big(\forall x(x \in A \to x \in B) \land \exists x(x \in B \land x \notin A)\big)}$$

Sets (15/24)

Example

For each of the following sets,
determine whether 3 is an element of that set.

$\{1,2,3,4\}$

$\{\{1\},\{2\},\{3\},\{4\}\}$

$\{1,2,\{1,3\}\}$

Venn Diagram

$A = \{1,2,3,4,7\}$
$B = \{0,3,5,7,9\}$
$C = \{1,2\}$

Venn Diagram

$A = \{1,2,3,4,7\}$
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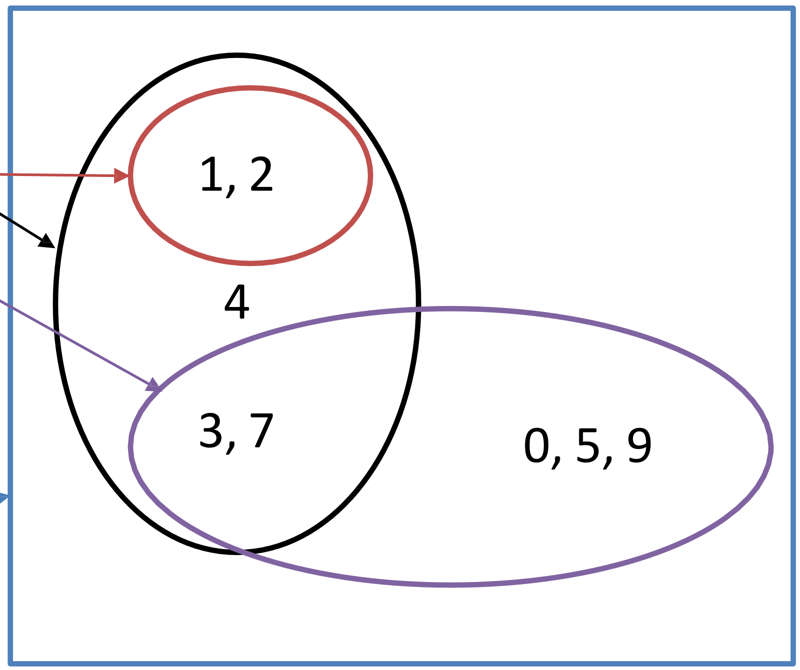
$C = \{1,2\}$

Universal Set



Power Set

The set of all subsets.

If the set is S. The power set of S is denoted by $P(S)$.

The number of elements in the power set is $2^{|S|}$

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The set of all subsets.

If the set is S. The power set of S is denoted by $P(S)$.

The number of elements in the power set is $2^{|S|}$

$S = \{1,2,3\}$

$\boldsymbol{|P(S)| = 2^3 = 8}$ **elements**

$P(S) = 2^S$

$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Example1

What is the power set of the empty set?

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What is the power set of the empty set?

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

Example2

What is the power set of the set $\{\emptyset\}$?

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What is the power set of the set $\{\emptyset\}$?

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

The ordered n-tuple

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, … , and a_n as its nth element.

In particular, ordered 2-tuples are called ordered pairs (e.g., the ordered pairs (a, b))

Cartesian Products

Let A and B be sets.

The Cartesian product of A and B, denoted by $A \times B$,

is the set of all ordered pairs (a, b), where $a \in A$ and

$b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$.

Cartesian Products - Example

Let $A = \{1,2\}$, and $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

$|A \times B| = |A| * |B| = 2 * 3 = 6$

Cartesian Products - Example

Let $A = \{1,2\}$, and $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$

$|A \times B| = |A| * |B| = 2 * 3 = 6$

Find $B \times A$?

The Cartesian product of more than two sets.

The Cartesian product of the sets $A_1, A_2, \dots,$ A_n, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \dots, n$. In other words,

$$\boxed{A_1 \times A_2 \times \cdots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n \}.}$$

Sets (24/24)

Example:

$A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$

$$\boxed{\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \end{aligned}}$$

Set Operations (1/7)

Union

Let A and B be sets. The **union** of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.

$$\boxed{A \cup B = \{x \mid x \in A \vee x \in B\}}$$

Set Operations (1/7)

Union

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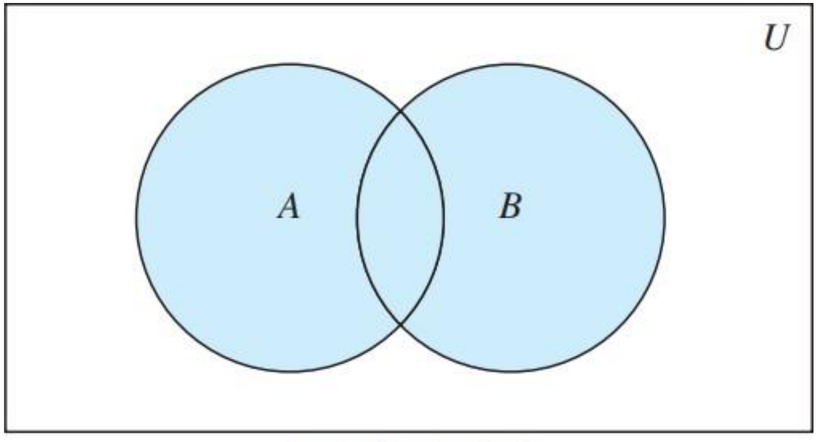

$A \cup B$ is shaded.

Set Operations (1/7)

Union

Let A and B be sets. The union of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$

Set Operations (2/7)

- Intersection

- Let A and B be sets. The intersection of the sets A and

- B , denoted by $A \cap B$, is the set that contains

those elements that are in both A and B.

$$\boxed{A \cap B = \{x \mid x \in A \land x \in B\}}$$

Set Operations (2/7)

- Intersection
- Let A and B be sets. The intersection of the sets A and
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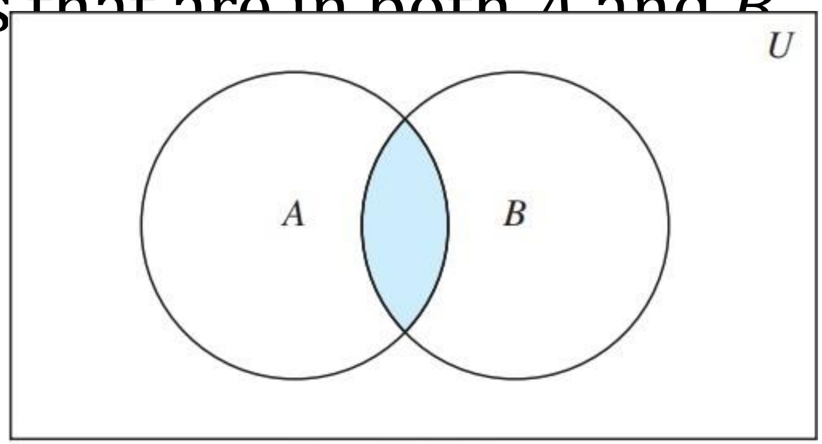


$A \cap B$ is shaded.

Set Operations (2/7)

- Intersection

- Let A and B be sets. The intersection of the sets A and
- B , denoted by $A \cap B$, is the set that contains

those elements that are in both A and B.

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$

is the set $\{1, 3\}$

Set Operations (3/7)

Disjoint

Two sets are called disjoint if their intersection is the empty set.

$$A \cap B = \emptyset$$

Difference

Let A and B be sets. The difference of A and B , denoted by $A - B$, is the set containing those elements that are in A but not in B.

$$\boxed{A - B = \{x \mid x \in A \wedge x \notin B\}}$$

Set Operations (4/7)

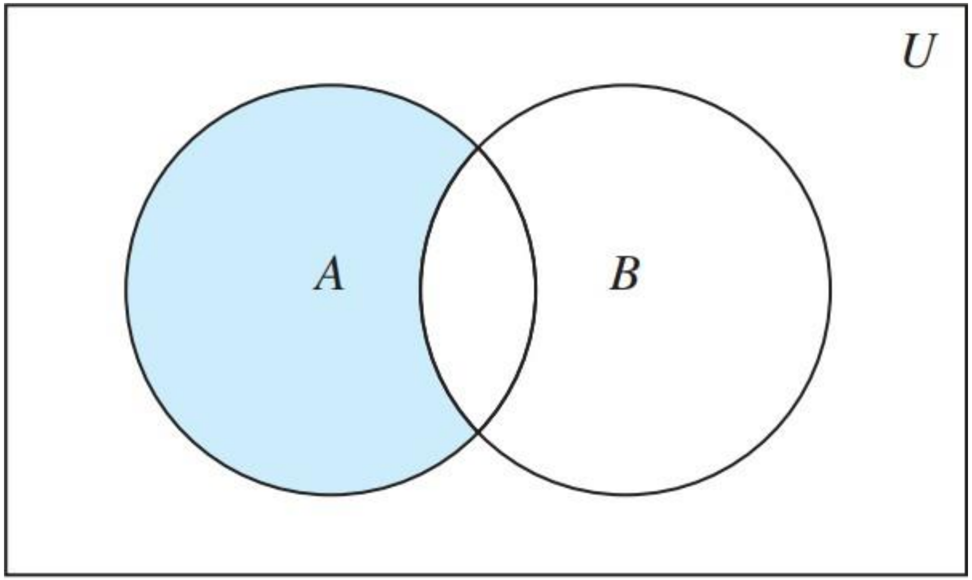
Difference

Let A and B be sets. The difference of A and B , denoted by $A - B$, is the set containing those elements that are in A but not in B.

$$A = \{1,3,5\}, \qquad B = \{1,2,3\}$$

$$A - B = \{5\}$$

Difference



$A - B$ is shaded.

Complement

Let U be the universal set.

The complement of the set A , denoted by AЂ

An element x belongs to U if and only if $x \notin A$.

$$\boxed{\overline{A} = \{x \in U \mid x \notin A\}}$$

Complement

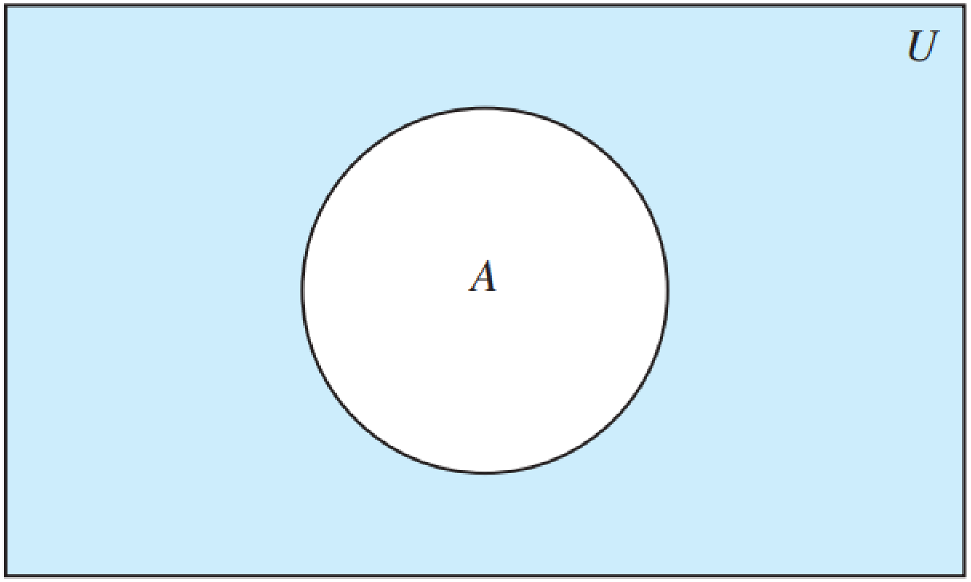
Let U be the universal set.

The complement of the set A , denoted by A'

An element x belongs to U if and only if $x \notin A$.

$$\boxed{\begin{array}{c} U = \{1,2,3,4,5\}, \qquad A = \{1,3\} \\ \bar{A} = \{2,4,5\} \end{array}}$$

Complement



$\overline{A}$ is shaded.

Set Operations (6/7)

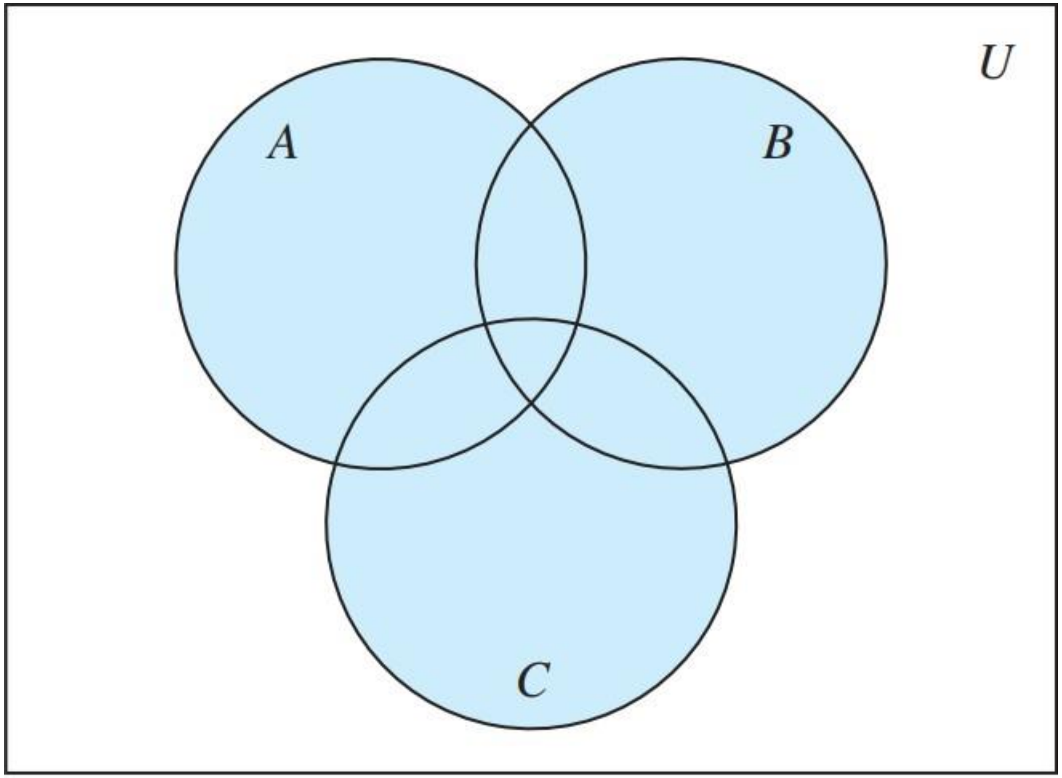
Generalized Unions

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Generalized Unions



$A \cup B \cup C$ is shaded.

Set Operations (7/7)

Generalized Intersections

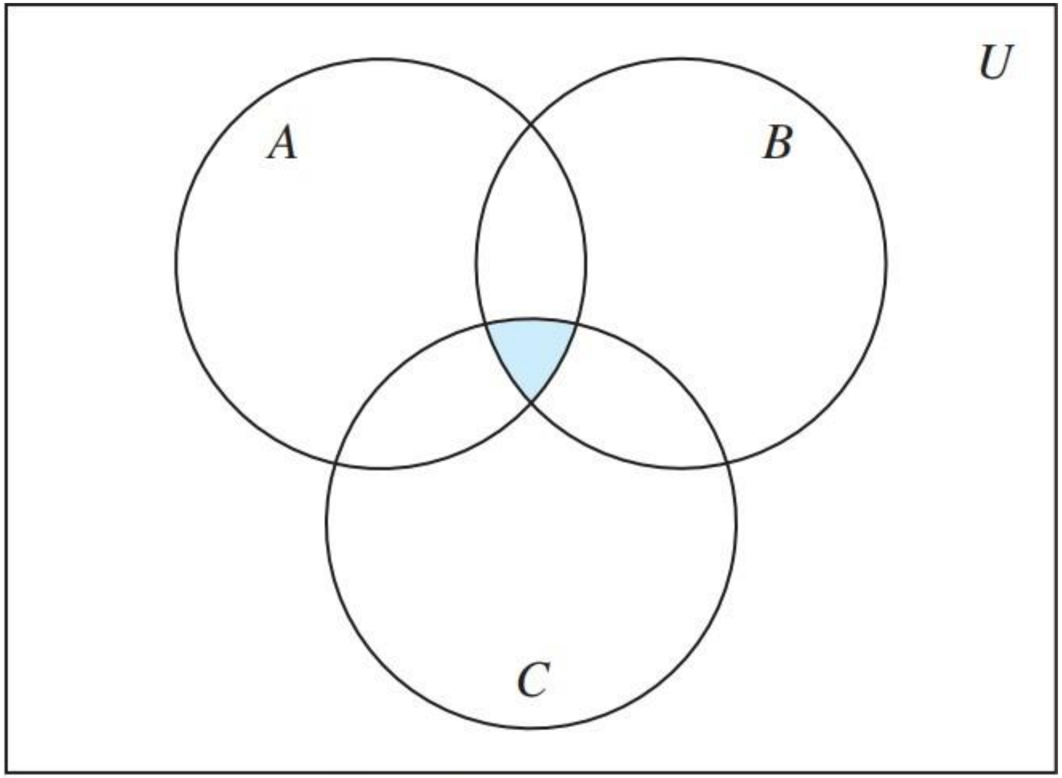
We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$.

Set Operations (7/7)

Generalized Intersections



$A \cap B \cap C$ is shaded.

Set Identities (1/8)

TABLE Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Set Identities (2/8)

TABLE Set Identities.	
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities (3/8)

Example1

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Set Identities (4/8)

Example1 – Answer

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Set Identities (5/8)

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Set Identities (6/8)

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set Identities (7/8)

Example2

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Set Identities (8/8)

Example2 – Answer

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$