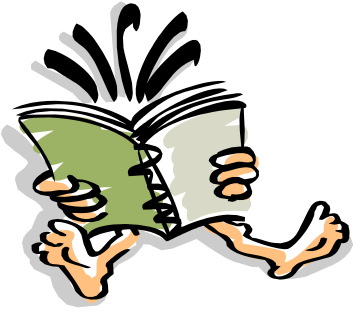
CS1101
Discrete Mathematics 1

Chapter 01
The Foundations: Logic



Today's Topics

- 1.4 Predicates and Quantifiers

Predicates and Quantifiers (1/22)

Predicate:

x is greater than 3

Predicates and Quantifiers (1/22)

Predicate:

$\boxed{x}$ $\boxed{\text{is greater than 3}}$

The subject

The predicate

We can denote the statement "x *is greater than* 3" by $\boldsymbol{P(x)}$

where $\boldsymbol{P}$ denotes the predicate "*is greater than* 3" and $\boldsymbol{x}$ is the variable.

The statement $\boldsymbol{P(x)}$ is also said to be the value of the **propositional function** $\boldsymbol{P}$ at $\boldsymbol{x}$. Once a value has been assigned to the variable $\boldsymbol{x}$, the statement $\boldsymbol{P(x)}$ becomes a proposition and has a truth value.

Predicates and Quantifiers (2/22)

Example1:

Let $P(x)$ denote the statement "$x > 3$."

What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement *P(4)* by setting x = 4 in the statement "x > 3." Hence, *P(4)*, which is the statement "4 > 3," is true. However, *P(2)*, which is the statement "2 > 3," is false.

Predicates and Quantifiers (2/22)

Example1:

Let $P(x)$ denote the statement "$x > 3$."

What are the truth values of $P(4)$ and $P(2)$?

T **F**

Example2:

Let $Q(x, y)$ denote the statement "$x = y + 3$."

What are the truth values of the propositions

$Q(1, 2)$ and $Q(3, 0)$?

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F **T**

Example3:

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?

   a) $P(0)$ b) $P(4)$ c) $P(6)$

2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?

   a) P(orange) b) P(lemon)
   c) P(true) d) P(false)

Predicates and Quantifiers (4/22)

Example3:

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?

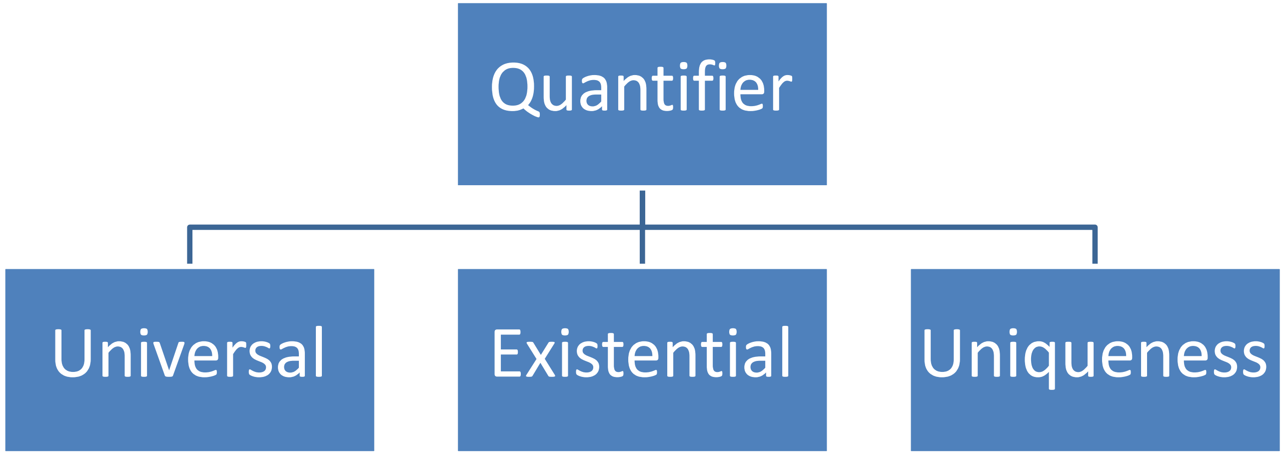
   a) $P(0)$ **T** b) $P(4)$ **T** c) $P(6)$ **F**

2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?

   a) $P(\text{orange})$ **T** b) $P(\text{lemon})$ **F**
   c) $P(\text{true})$ **F** d) $P(\text{false})$ **T**

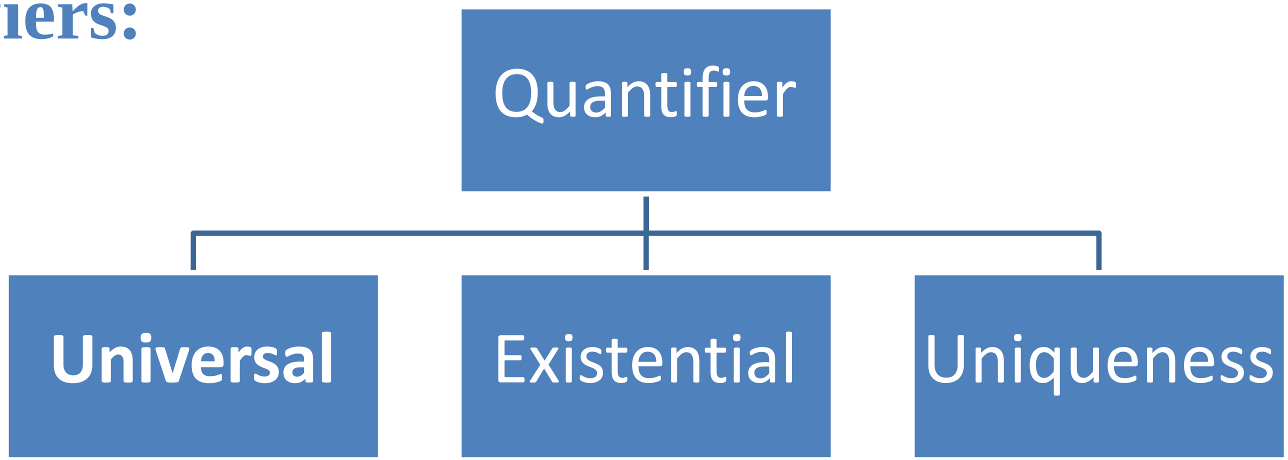
Quantifiers:

Expresses the extent to which a predicate is true over a **range** of elements.



Predicates and Quantifiers (5/22)

Quantifiers:

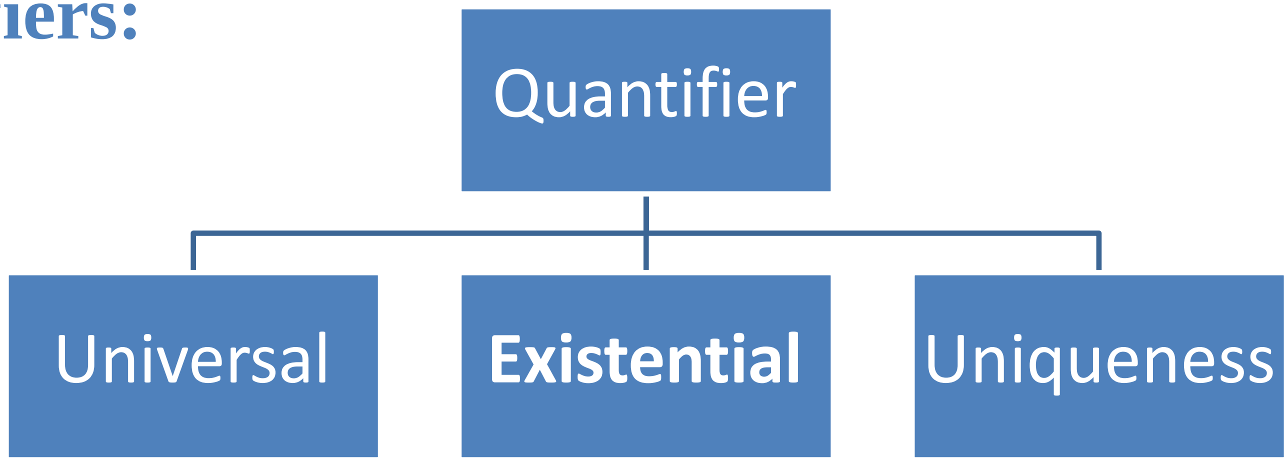


$\forall x\, P(x)$

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

Quantifiers:



$$\exists x\, P(x)$$

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Predicates and Quantifiers (7/22)

Quantifiers:

- Quantifier
  - Universal
  - Existential
  - **Uniqueness**

$$\exists! x P(x)$$

$$\exists_1 x P(x)$$

"There exists a unique x such that $P(x)$ is true."

Predicates and Quantifiers (8/22)

Quantifiers:

- Quantifier
  - Universal: $\forall x P(x)$
  - Existential: $\exists x P(x)$
  - Uniqueness: $\exists! x P(x)$, $\exists_1 x P(x)$

TABLE 1 Quantifiers.

Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Predicates and Quantifiers (9/22)

Translate into English – Example1:

Express the statement “Every student in this class has studied calculus.

Solution $P(x)$: x has studied calculus.

$S(x)$: x is in this class.

The statement can be expressed as $\boxed{\forall x(S(x) \rightarrow P(x))}$

Example2:

Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

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Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true.

Predicates and Quantifiers (10/22)

Example3:

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Example3:

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

Example3:

Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

Predicates and Quantifiers (11/22)

Example3:

Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance,

when $x = 4$—the existential quantification of $P(x)$,

which is $\exists x P(x)$, is true.

Predicates and Quantifiers (12/22)

Example4:

What is the truth value of $\exists x P(x)$,

where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Example5:

What is the truth value of $\exists x P(x)$,

where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$,

the proposition $\exists x P(x)$ is the same as the disjunction

$P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because $P(4)$, which is the statement "$4^2 > 10$," is true,

it follows that $\exists x P(x)$ is true.

Predicates and Quantifiers (13/22)

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$

d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

Predicates and Quantifiers (13/22)

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ **T**　　**b)** $P(1)$ **T**　　**c)** $P(2)$ **F**

d) $P(-1)$ **F**　　**e)** $\exists x P(x)$ **T**　　**f)** $\forall x P(x)$ **F**

Translate into English – Example2:

Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$ into English, where $C(x)$ is "x has a computer", $F(x, y)$ is "x and y are friends," and both x and y is the set of all students in your school.

Solution

Every student in your school has a computer or has a friend who has a computer.

Translate into English – Example3:

Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$

Answer

a) Every comedian is funny.

Translate into English – Example3:

Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

b) $\forall x(C(x) \land F(x))$

Answer

b) Every person is a funny comedian.

Translate into English – Example3:

Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

c) $\exists x(C(x) \rightarrow F(x))$

Answer

c) There exists a person such that if she or he is a comedian, then she or he is funny.

Translate into English – Example3:

Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

d) $\exists x(C(x) \wedge F(x))$

Answer

d) Some comedians are funny.

Predicates and Quantifiers (17/22)

Translate into Logical Expression – Example1:

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

$\boxed{\text{Let } P(x) \text{ be “}x \text{ is perfect”}}$ $\boxed{\text{let } F(x) \text{ be “}x \text{ is your friend”}}$

the domain be all people

a) No one is perfect.

Answer

a) $\forall x \neg P(x)$

Predicates and Quantifiers (17/22)

Translate into Logical Expression – Example1:

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

Let $P(x)$ be "x is perfect" let $F(x)$ be "x is your friend"

the domain be all people

b) Not everyone is perfect.

Answer

b) $\neg\forall x\, P(x)$

Predicates and Quantifiers (17/22)

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Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

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the domain be all people

c) All your friends are perfect.

Answer

c) $\forall x(F(x) \rightarrow P(x))$

Predicates and Quantifiers (17/22)

Translate into Logical Expression – Example1:

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

$\boxed{\text{Let } P(x) \text{ be "}x\text{ is perfect"}}$ $\boxed{\text{let } F(x) \text{ be "}x\text{ is your friend"}}$

the domain be all people

d) At least one of your friends is perfect.

Answer

d) $\exists x(F(x) \wedge P(x))$

Predicates and Quantifiers (18/22)

Precedence of Quantifiers:

The quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators from propositional calculus.

For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.

In other words,

it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Logical Equivalences Involving Quantifiers:

Show that $\forall x(P(x) \land Q(x))$ and $\forall x P(x) \land \forall x Q(x)$ are logically equivalent

Predicates and Quantifiers (18/22)

Logical Equivalences Involving Quantifiers:

Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent

1) We assume that $\forall x \left(P(x) \wedge Q(x)\right)$ is true for all values x in the domain.
2) Then, $P(x)$ is true for all values x in the domain. And $Q(x)$ is true for all values x in the domain.
3) Then, $\forall x P(x)$ is true. And $\forall x Q(x)$ is true. $\left(\forall x P(x) \wedge \forall x Q(x)\right)$ is true.

---

1. We assume that $\forall x P(x) \wedge \forall x Q(x)$ is true for all values x in the domain.
2. Then, $\forall x P(x)$ is true. And $\forall x Q(x)$ is true. Then, $P(x)$ is true for all values x in the domain. And $xQ(x)$ is true for all values x in the domain.
3. Then, $P(x) \wedge Q(x)$ is true for all values x in the domain $\forall x \left(P(x) \wedge Q(x)\right)$ is true.

Predicates and Quantifiers (19/22)

Negating Quantified Expressions:

$P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class.

$\forall x P(x)$ **:**

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$\forall \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x})$ **:**

"Every student in your class has taken a course in calculus"

Predicates and Quantifiers (19/22)

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"There is at least one student in your class who has not taken a course in calculus"

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$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

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"At least one student in your class has taken a course in calculus"

Predicates and Quantifiers (20/22)

Negating Quantified Expressions:

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Predicates and Quantifiers (20/22)

Negating Quantified Expressions:

$P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class.

$\exists x P(x)$ **:**

“At least one student in your class has taken a course in calculus"

<u>The negation of this statement is</u>

"Every student in this class has not taken calculus"

Negating Quantified Expressions:

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“At least one student in your class has taken a course in calculus"

<u>The negation of this statement is</u>

"Every student in this class has not taken calculus"

$$\neg\exists \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x}) \equiv \forall \boldsymbol{x} \neg \boldsymbol{P}(\boldsymbol{x})$$

Example1:

What are the negations of the statements

$$\forall x(x^2 > x)$$

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What are the negations of the statements

$$\forall x(x^2 > x)$$

$$\neg\forall x(x^2 > x) \equiv \exists x \neg(x^2 > x)$$

$$\exists x\,(x^2 \leq x)$$

Example2:

What are the negations of the statements

$$\exists x(x^2 = 2)$$

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What are the negations of the statements

$$\exists x(x^2 = 2)$$

$$\neg\exists x(x^2 = 2) \equiv \forall x \neg(x^2 = 2)$$

$$\forall x\,(x^2 \neq 2)$$

Next Class

- 1.5 Nested Quantifiers