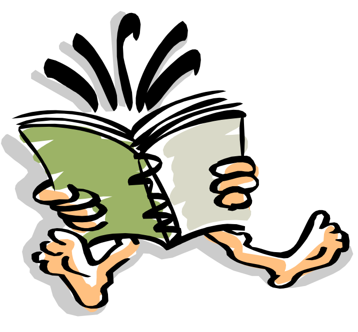
CS1101
Discrete Structures 1

Chapter 02
Sequences and Summations



Sequences (1/13)

Definition

- A sequence is a set of things (usually numbers) that are in order.
  - For example, 1 , 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81, . . . , 30, . . . is an infinite sequence.
- We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
- We use the notation $\{a_n\}$ to describe the sequence.

$$\{a_n\} = \{a_1, a_2, a_3, \dots\}$$

Example

- Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

Starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Sequences (3/13)

Geometric

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the $\boxed{\textit{initial term } a}$ and the $\boxed{\textit{common ratio } r}$ are real numbers.

$$2, 10, 50, 250, \ldots$$

Geometric – Example1

$$1, -1, 1, -1, 1, \ldots;$$

$$\{ar^n\}, \qquad n = 0,1,2,\ldots$$

$$a = 1$$
$$r = -1$$

Geometric – Example2

$$2, 10, 50, 250, 1250, \ldots ;$$

$$\{ar^n\}, \qquad n = 0,1,2,\ldots$$

$$a = 2$$

$$r = 5$$

Geometric – Example3

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$$

$\{ar^n\}, \qquad n = 0,1,2, \ldots$

$a = 6$

$r = 1/3$

Geometric – Example4

Find $\boldsymbol{a}, \boldsymbol{r}$? $\{\mathbf{3} * \mathbf{4}^{\boldsymbol{n}}\}$, $\boldsymbol{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$

$$\{ar^n\}, \qquad n = 0,1,2, \ldots$$

$$a = 3$$

$$r = 4$$

Geometric – Example5

Find $\boldsymbol{a}, \boldsymbol{r}?\ \{\mathbf{3} * \mathbf{4}^{\boldsymbol{n}}\},\ \ \boldsymbol{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$

$$a = 12$$
$$r = 4$$

Arithmetic

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the $\boxed{\textit{initial term } a}$ and the $\boxed{\textit{common difference } d}$ are real numbers.

Sequences (10/13)

Arithmetic – Example1

$$-1, 3, 7, 11, \ldots,$$

$$\{a + nd\}, \qquad n = 0,1,2, \ldots$$

$$a = -1$$
$$d = 4$$

Arithmetic – Example2

$$7, 4, 1, -2, \ldots$$

$$\{a + nd\}, \qquad n = 0,1,2, \ldots$$

$$a = 7$$

$$d = -3$$

Sequences (12/13)

Notes:

- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?
- Are there cycles among the terms?

Fibonacci Sequence

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$
is defined by the initial conditions $f_0 = 0, f_1 = 1$, and
the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

Summations (1/8)

Next, we introduce **summation notation.**
We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j)

to represent

Here, the variable j is called the **index of summation**

$$a_m + a_{m+1} + \cdots + a_n.$$

Summations (1/8)

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

Here, the index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n. A large uppercase Greek letter sigma, $\sum$, is used to denote summation.

Example 1

Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for $n = 1, 2, 3, \ldots$

Example 1

Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for $n = 1, 2, 3, \ldots$

Answer

$$\sum_{n=1}^{100} 1/n$$

Summations (4/8)

Example 2

What is the value of $\sum_{j=1}^{5} j^2$?

Summations (4/8)

Example 2

What is the value of $\sum_{j=1}^{5} j^2$?

Answer

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

Summations (5/8)

Example 3

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Example 3

What is the value of $\sum_{s \in \{0,2,4\}} s$?

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

Summations (6/8)

Example 4

Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

but want the index of summation to run between 0 and 4

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$$

It is easily checked that both sums are $1 + 4 + 9 + 16 + 25 = 55$.

Double Summation

Find $$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

Double Summation

Find

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$