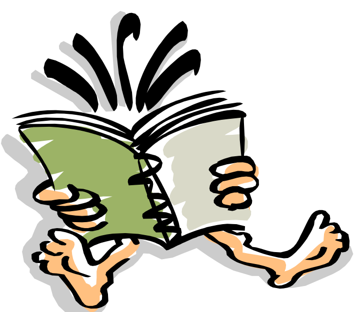
CS1101
Discrete Structures 1

Chapter 01
The Foundations: Proofs



Proofs Techniques (1/17)

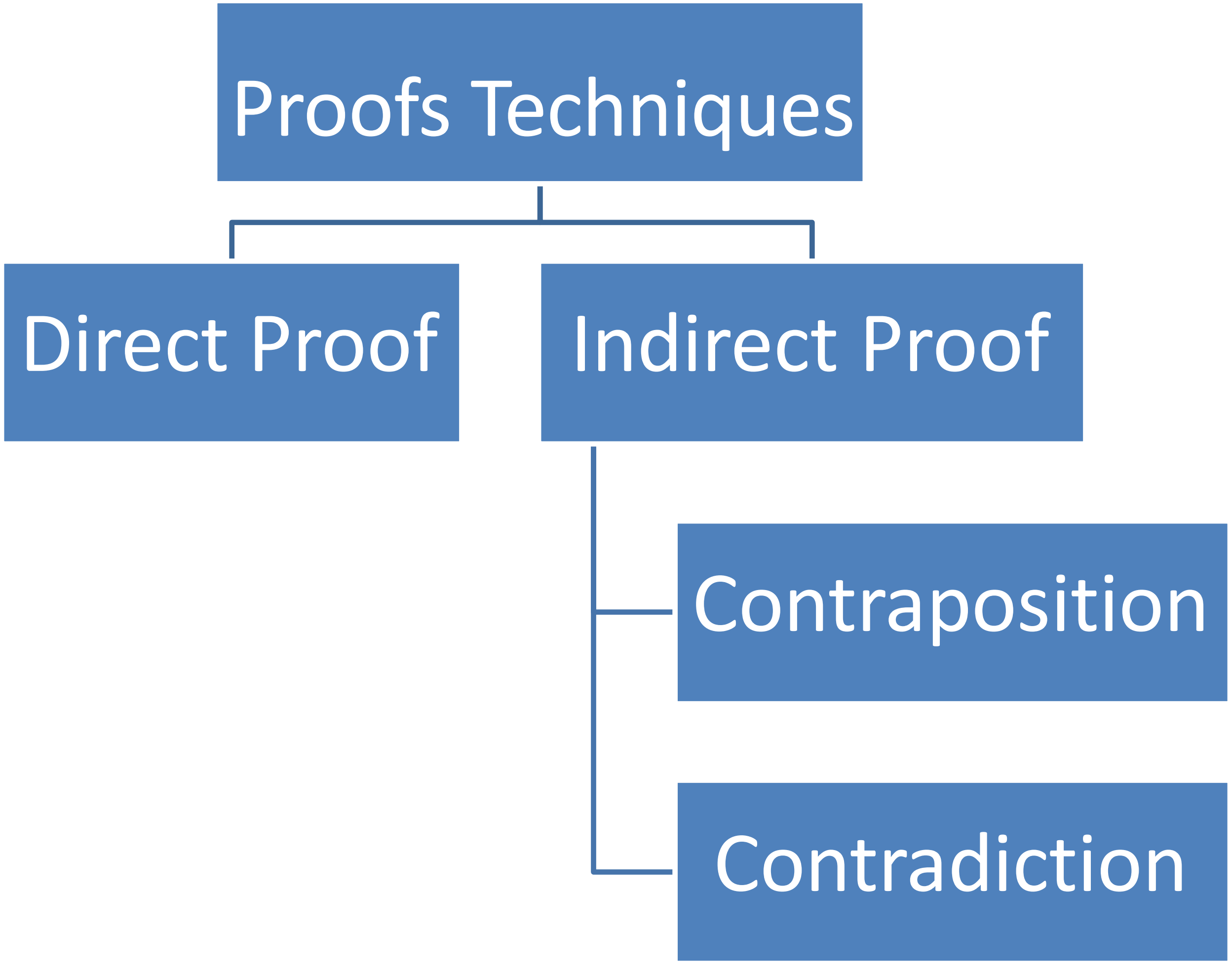
Some Terminology

Definition 1. A **theorem** (fact/result) is a statement that can be shown to be true. We demonstrate that a theorem is true with a proof.

Definition 2. A **proof** is a valid argument that establishes the truth of a theorem.

Definition 3. A **lemma** is a ‘helping theorem’ or a result which is needed to prove a theorem. Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually.

Proofs Techniques (2/17)



Proofs Techniques (3/17)

Even Integer

$$2 * (Any\ Integer\quad) = even$$

if a is an even number, so you can write it as follows:

$$a = 2n, \qquad \text{where } n \text{ is integer}$$

Proofs Techniques (3/17)

Even + 1 = Odd
Odd + 1 = Even

¬Even = Odd
¬Odd = Even

Proofs Techniques (4/17)

Odd Integer

if a is an odd number, so you can write it as follows:

$$\boxed{a = 2m + 1, \qquad \text{where } m \text{ is integer}}$$

Prefect Square

if a is a prefect square, so you can write it as follows:

$$\boxed{a = (n)^2, \qquad \text{where } n \text{ is integer}}$$

Proofs Techniques (6/17)

Rational Number

if a is a $rational\ number$, so you can write it as follows:

$$a = \frac{n}{m},$$ where n, and m are integers

with NO common factor, and $m \neq 0$

¬Rational = Irrational

¬Irrational = Rational

Proofs Techniques (7/17)

Direct Proof

$p \to q$

1. We assume that p is true
2. We try to prove that q is also true
3. Then $p \to q$ is true.

Example1

Give a direct proof of the theorem
"If n is an odd integer, then n^2 is odd."

Proofs Techniques (8/17)

Example1

Give a direct proof of the theorem

"If $\underbrace{n \text{ is an odd integer,}}_{p}$ then $\underbrace{n^2 \text{ is odd.}}_{q}$"

$p \rightarrow q$

Proofs Techniques (8/17)

Example1

Give a direct proof of the theorem

$$\text{"If } \underbrace{n \text{ is an odd integer,}}_{p} \text{ then } \underbrace{n^2 \text{ is odd."}}_{q}$$

$p \rightarrow q$

1. We assume that p is true
2. We try to prove that q is also true
3. Then $p \rightarrow q$ is true.

Example1

Give a direct proof of the theorem

"If $\underbrace{n \text{ is an odd integer}}_{p}$, then $\underbrace{n^2 \text{ is odd}}_{q}$."

$2 * (Any\ Integer\ ) = even$

1. We assume that p is true

   $n = 2m + 1,$ where m is integer.

Example1

Give a direct proof of the theorem
"If $\underbrace{n \text{ is an odd integer}}_{p}$, then $\underbrace{n^2 \text{ is odd}}_{q}$."

$2 * (Any\ Integer) = even$

1. We assume that p is true
   $n = 2m + 1,$ where m is integer.
2. We try to prove that q is also true

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= even + 1 = odd \end{aligned}$$

Example1

Give a direct proof of the theorem
"If n is an odd integer, then n^2 is odd."

1. We assume that p is true

   $n = 2m + 1,$ where m is integer.

2. We try to prove that q is also true

$$
\begin{aligned}
n^2 &= (2m+1)^2 \\
&= 4m^2 + 4m + 1 \\
&= 2\ (2m^2 + 2m) + 1 \\
&= \mathit{even} \qquad\qquad + 1 = \mathit{odd}
\end{aligned}
$$

3. $\therefore p \rightarrow q$ is true.

Proofs Techniques (9/17)

Indirect Proof (Contraposition)

$$p \rightarrow q$$
$$\neg q \rightarrow \neg p$$

1. We assume that $\neg q$ is true
2. We try to prove that $\neg p$ is also true
3. Then $\neg q \rightarrow \neg p$ is true.
4. The $p \rightarrow q$ is alsotrue.

Example1

Prove by contraposition that if n is an integer and $3n + 2$ is odd, then n is odd.

Example1

Prove by contraposition that if n is an integer and $\underbrace{3n + 2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

$\neg q \rightarrow \neg p$ contraposition

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Prove by contraposition that if n is an integer and $\underbrace{3n + 2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

$$\boxed{\neg q \to \neg p}$$

contraposition

$\neg q$ is (n is even)

$\neg p$ is ($3n + 2$ is even)

Proofs Techniques (10/17)

Example1

Prove by contraposition that if n is an integer and $\underbrace{3n + 2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

$\boxed{\neg q \rightarrow \neg p}$

1. we assume that (n is even)
   $n = 2m$, where m is integer.

$\neg q$ is (n is even)
$\neg p$ is ($3n + 2$ is even)

Proofs Techniques (10/17)

Example1

Prove by contraposition that if n is an integer and $\underbrace{3n + 2 \text{ is odd,}}_{p}$ then $\underbrace{n \text{ is odd.}}_{q}$

$$\boxed{\neg q \rightarrow \neg p}$$

1. we assume that $(n$ is even $)$
   $n = 2m,$ where m is integer.
2. $(3n + 2) = (3(2m) + 2)$
   $= (6m + 2) = 2(3m + 1) =$ even
   $\therefore (3n + 2)$ is even

$\neg q$ is $(n$ is even $)$
$\neg p$ is $(3n + 2$ is even $)$

Example1

Prove by contraposition that if n is an integer and $\underbrace{3n + 2 \text{ is odd,}}_{p}$ then $\underbrace{n \text{ is odd.}}_{q}$

$\boxed{\neg q \rightarrow \neg p}$

1. we assume that (n is even)
   $n = 2m$, where m is integer.

$\neg q$ is (n is even)
$\neg p$ is ($3n + 2$ is even)

2. $(3n + 2) = (3(2m) + 2)$
   $= (6m + 2) = 2(3m + 1) = \text{even}$
   $\therefore (3n + 2)$ is even

3. $\therefore \neg q \rightarrow \neg p$ is true, then $p \rightarrow q$ is also true.

Indirect Proof (Contradiction)

A – We want to prove p.

We show that:

1. $\neg p \rightarrow F$ (i.e., a False statement)
2. We conclude that $\neg p$ is False since (1) is True and therefore p is True.

Indirect Proof (Contradiction)

B – We want to show $p \to q$

1. Assume the negation of the conclusion, i.e., $\neg q$
2. Show that $(p \wedge \neg q) \to F$
3. Since $((p \wedge \neg q) \to F) \Leftrightarrow (p \to q)$ we are done

(why?)

$$((p \wedge \neg q) \to F) \Leftrightarrow \neg(p \wedge \neg q)$$

$$\Leftrightarrow p \to q$$

Proofs Techniques (13/17)

Example1

Give a proof by contradiction of the theorem "If $3n+2$ is odd, then n is odd".

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Give a proof by contradiction of the theorem
"If $\underbrace{3n+2 \text{ is odd,}}_{p}$ then $\underbrace{n \text{ is odd}}_{q}$".

$p \wedge \neg q$

$\neg q$ is (n is even)
p is ($3n+2$ is odd)

Example1

Give a proof by contradiction of the theorem "If $\underbrace{3n+2 \text{ is odd,}}_{p}$ then $\underbrace{n \text{ is odd}}_{q}$".

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$\neg q$ is $(n$ is even $)$
p is $(3n+2$ is odd $)$

1. we assume that $(n$ is even $)$
   $n = 2m,$ where m is integer.
2. $(3n+2) = (3(2m)+2)$
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Give a proof by contradiction of the theorem
"$\underbrace{\text{If } 3n+2 \text{ is odd,}}_{p}$ then $\underbrace{n \text{ is odd}}_{q}$".

$p \wedge \neg q$

$\neg q$ is (n is even)
p is ($3n+2$ is odd)

1. we assume that (n is even)
   $n = 2m,$ where m is integer.

2. $(3n+2) = (3(2m)+2)$
   $= (6m+2) = 2(3m+1) = \text{even}$
   $\therefore$ $(3n+2)$ is even, then p is false.

3. $\because p \wedge \neg q$ is false, then by contradiction $p \to q$ is true.

Example2

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Let p the proposition "$\sqrt{2}$ is irrational". Suppose that $\neg p$ is true $\sqrt{❼}$ then2 is rational

Proofs Techniques (16/17)

Rational Number

if a is a $rational\ number$, so you can write it as follows:

$$a = \frac{n}{m},$$ where n, and m are integers

with NO common factor, and $m \neq 0$

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Proofs Techniques (17/17)

Example2 Answer:

Let p the proposition "$\sqrt{2}$ is irrational".

1) To start a proof by contradiction, we suppose that $\neg p$ is true, then $\sqrt{2}$ is rational.
2) $\sqrt{2} = \frac{a}{b}$, where a and b are intergers without common factor and $b \neq 0$
3) $2 = \frac{a^2}{b^2}$
4) $a^2 = 2b^2$, then a is even, and you can write $a = 2m$, where m is integer.
5) $(2m)^2 = 2b^2$, then $4m^2 = 2b^2$.
6) $b^2 = 2m^2$, then b is also even.
7) Because a and b are even, then a and b have a common factor 2.
8) Therefore, $\sqrt{2} = \frac{a}{b}$ is not rational, then $\neg p$ is false.
9) Therefore, p is true and $\sqrt{2}$ is irrational.