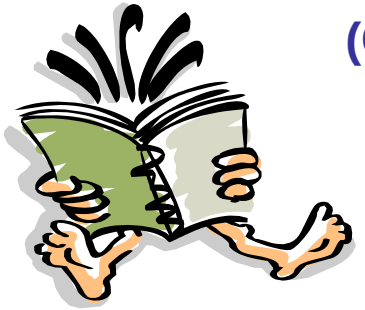
Analysis of Algorithms

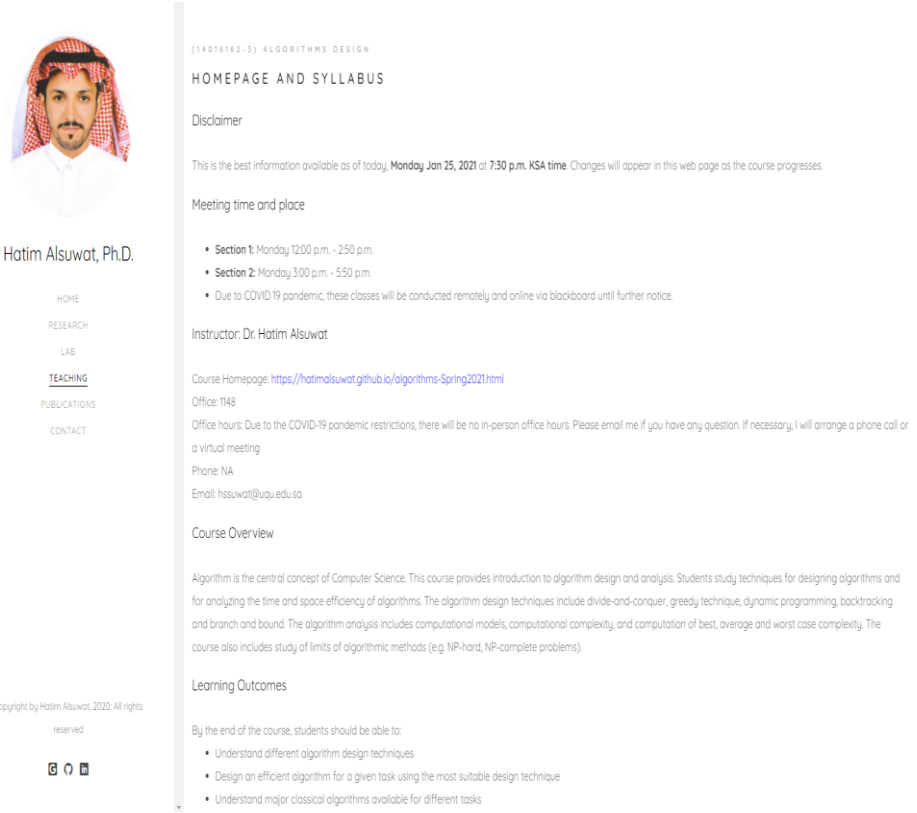
Asymptotic Analysis



(Chapter 3, Appendix A)

Course Information

https://hatimalsuwat.github.io/algorithms-3rdtrimester.html



Hatim Alsuwat, Ph.D.

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(1401616 2-3) ALGORITHMS DESIGN

HOMEPAGE AND SYLLABUS

Disclaimer

This is the best information available as of today, **Monday Jan 25, 2021** at **7:30 p.m. KSA time**. Changes will appear in this web page as the course progresses.

Meeting time and place

- **Section 1:** Monday 12:00 p.m. - 2:50 p.m.
- **Section 2:** Monday 3:00 p.m. - 5:50 p.m.
- Due to COVID 19 pandemic, these classes will be conducted remotely and online via blackboard until further notice.

Instructor: Dr. Hatim Alsuwat

Course Homepage: https://hatimalsuwat.github.io/algorithms-Spring2021.html
Office: 1148
Office hours: Due to the COVID-19 pandemic restrictions, there will be no in-person office hours. Please email me if you have any question. If necessary, I will arrange a phone call or a virtual meeting
Phone: NA
Email: hssuwat@uqu.edu.sa

Course Overview

Algorithm is the central concept of Computer Science. This course provides introduction to algorithm design and analysis. Students study techniques for designing algorithms and for analyzing the time and space efficiency of algorithms. The algorithm design techniques include divide-and-conquer, greedy technique, dynamic programming, backtracking and branch and bound. The algorithm analysis includes computational models, computational complexity, and computation of best, average and worst case complexity. The course also includes study of limits of algorithmic methods (e.g. NP-hard, NP-complete problems).

Learning Outcomes

By the end of the course, students should be able to:

- Understand different algorithm design techniques
- Design an efficient algorithm for a given task using the most suitable design technique
- Understand major classical algorithms available for different tasks

Communication:

- Announcements on webpage/ emails/ blackboard
- Questions? Email me.
- Staff email: hssuwat@uqu.edu.sa

Course technology:

- Website
- UQU Blackboard
- Regular homework
- Help us make it awesome!

Course Information

- Course Website https://hatimalsuwat.github.io/algorithms-3rdtrimester.html
- Discussion:
  - Please ask any question during the lecture (don't be shy)
  - There is no such thing as a stupid question.
  - Answer others' questions - if you know the answer ;-)
  - Learn from others' questions and answers

Course Information

- **Assignments:**
  - ***Quizzes:*** there will be several quizzes randomly given
  - ***Homework assignments*:** there will be several homework assignments during the semester.
  - ***Exams*:** One Midterm Exam and One Final Exam. Closed book tests will cover the course material.
  - Assignments are always due on the announced day and time. Exams must be taken as scheduled except in cases of extenuating circumstances such as a documented emergency.
- Participation can help on margins

Course Information

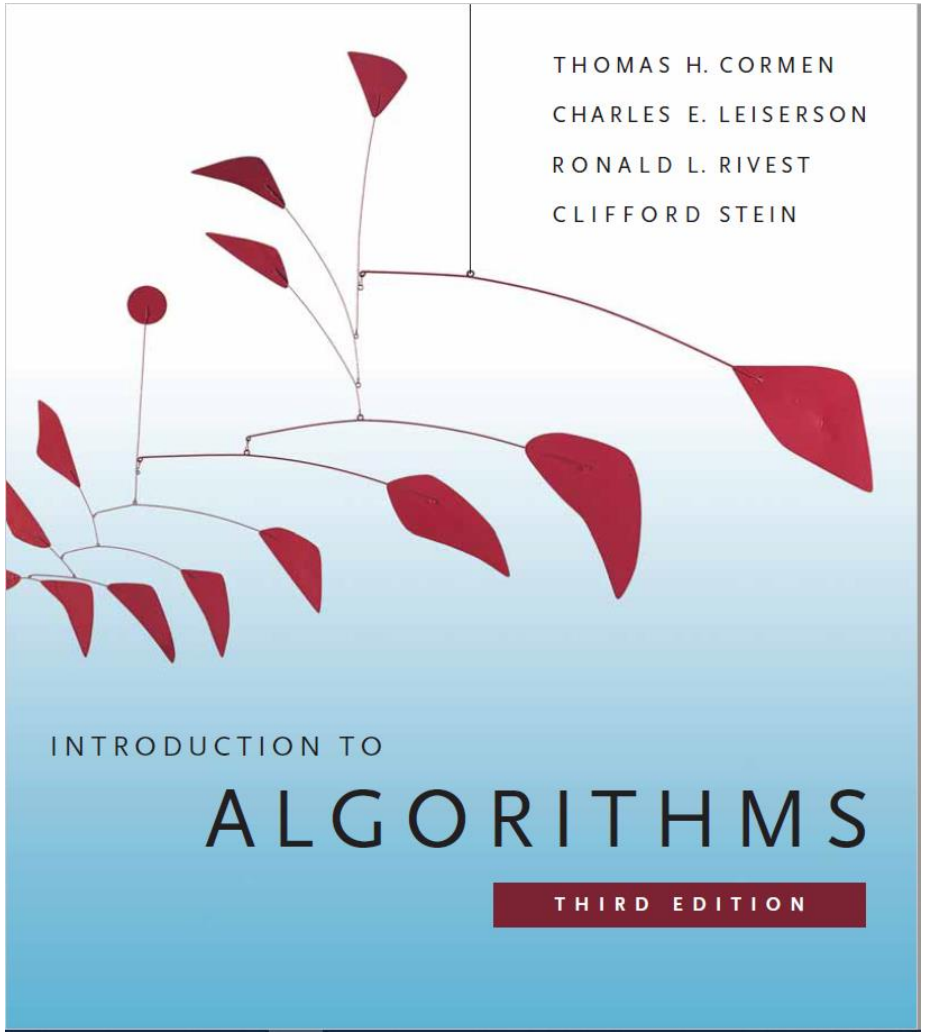
- **Grading:**
  - **Midterm Exam: 20%**
  - **Homework Assignments: 10%**
  - **Participation and Quizzes: 5%**
  - **Final Exam: 50%**

- **Total score that can be achieved: 85**

Course Information

- **Meeting time and place:**
  - **Office:** Department of Computer Science (office #1148)
  - **Office hours:** Please email me if you have any question. If necessary, I will arrange a phone call or in-person meeting
  - **Email:** Hssuwat@uqu.edu.sa

Textbook


THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN
INTRODUCTION TO
ALGORITHMS
THIRD EDITION

Course Information: Feedback

- Please give feedback positive or negative as early as you can via email.

Analysis of Algorithms

- An *algorithm* is a finite set of precise instructions for performing a computation or for solving a problem.
- What is the goal of analysis of algorithms?
  - To compare algorithms mainly in terms of running time but also in terms of other factors (e.g., memory requirements, programmer's effort etc.)
- What do we mean by running time analysis?
  - **Determine how running time increases as the size of the problem increases**.

Input Size

- Input size (number of elements in the input)
  - size of an array
  - polynomial degree
  - # of elements in a matrix
  - # of bits in the binary representation of the input
  - vertices and edges in a graph

Types of Analysis

- Worst case
  - Provides an upper bound on running time
  - An absolute guarantee that the algorithm would not run longer, no matter what the inputs are
- Best case
  - Provides a lower bound on running time
  - Input is the one for which the algorithm runs the fastest

$$Lower\ Bound \leq Running\ Time \leq Upper\ Bound$$

- Average case
  - Provides a prediction about the running time
  - Assumes that the input is random

How do we compare algorithms?

- We need to define a number of objective measures.

(1) Compare execution times?

Not good: times are specific to a particular computer !!

(2) Count the number of statements executed?

Not good: number of statements vary with the programming language as well as the style of the individual programmer.

Ideal Solution

- Express running time as a function of the input size *n* (i.e., *f(n)*).
- Compare different functions corresponding to running times.
- Such an analysis is independent of machine time, programming style, etc.

Example

- Associate a "cost" with each statement.
- Find the "total cost“ by finding the total number of times each statement is executed.

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	...
arr[N-1] = 0;	c_1

$c_1+c_1+...+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 =$
$(c_2 + c_1) \times N + c_2$

Another Example

- ***Algorithm 3*** *Cost*

```
sum = 0;                          c1
for(i=0; i<N; i++)                c2
   for(j=0; j<N; j++)             c2
        sum += arr[i][j];         c3
                                  ----------
```

$c_1 + c_2 \times (N+1) + c_2 \times N \times (N+1) + c_3 \times N^2$

Asymptotic Analysis

- To compare two algorithms with running times *f(n)* and *g(n)*, we need a **rough measure** that characterizes **how fast each function grows.**
- *Hint:* use *rate of growth*
- Compare functions in the limit, that is, **asymptotically!**

  (i.e., for large values of *n*)

Rate of Growth

- Consider the example of buying *elephants* and *goldfish:*

  Cost: cost_of_elephants + cost_of_goldfish

  Cost ~ cost_of_elephants (approximation)

- The low order terms in a function are relatively insignificant for **large** *n*

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

i.e., we say that $n^4 + 100n^2 + 10n + 50$ and n^4 have the same **rate of growth**

Asymptotic Notation

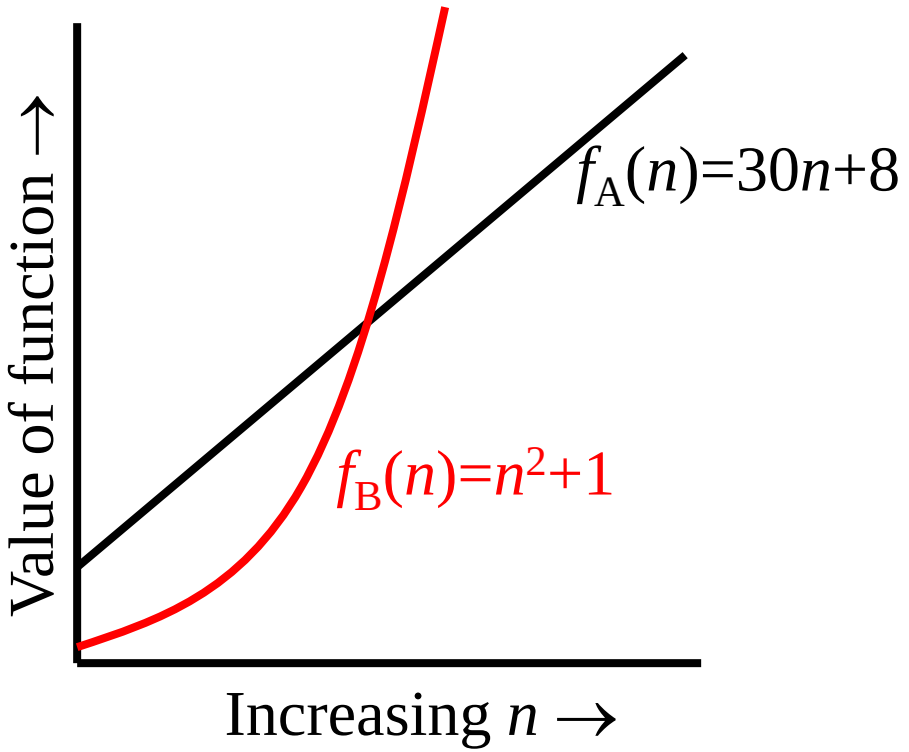
- O notation: asymptotic “less than”:
  - f(n)=O(g(n)) implies: f(n) “≤” g(n)
- Ω notation: asymptotic “greater than”:
  - f(n)= Ω (g(n)) implies: f(n) “≥” g(n)
- Θ notation: asymptotic “equality”:
  - f(n)= Θ (g(n)) implies: f(n) “=” g(n)

Big-O Notation

- We say $f_A(n)=30n+8$ is *order n*, or O (n)
  It is, at most, roughly *proportional* to *n*.
- $f_B(n)=n^2+1$ is *order* n^2, or O(n^2). It is, at most, roughly proportional to n^2.
- In general, any O(n^2) function is faster-growing than any O(*n*) function.

Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



More Examples …

- $n^4 + 100n^2 + 10n + 50$ is $O(n^4)$
- $10n^3 + 2n^2$ is $O(n^3)$
- $n^3 - n^2$ is $O(n^3)$
- constants
  - 10 is $O(1)$
  - 1273 is $O(1)$

Back to Our Example

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	
arr[N-1] = 0;	c_1

$c_1+c_1+...+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 = (c_2 + c_1) \times N + c_2$

- Both algorithms are of the same order: *O(N)*

Example (cont'd)

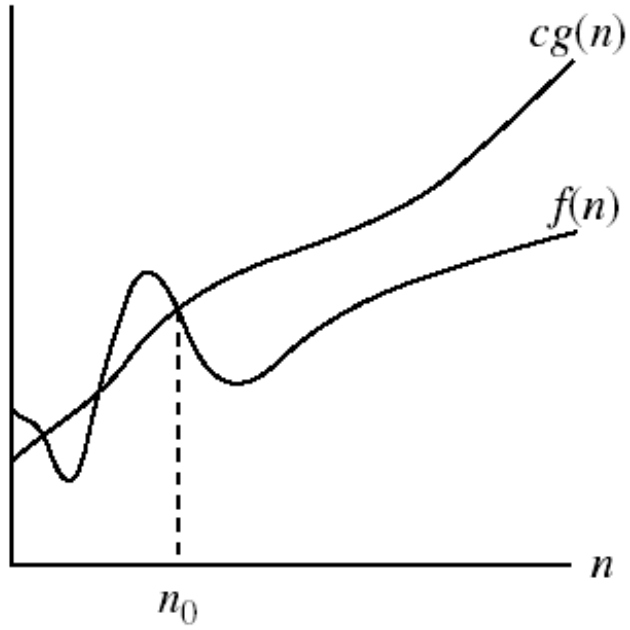
Algorithm 3	***Cost***
sum = 0;	c_1
for(i=0; i<N; i++)	c_2
for(j=0; j<N; j++)	c_2
sum += arr[i][j];	c_3

$c_1 + c_2 \times (N+1) + c_2 \times N \times (N+1) + c_3 \times N^2 = O(N^2)$

Asymptotic notations

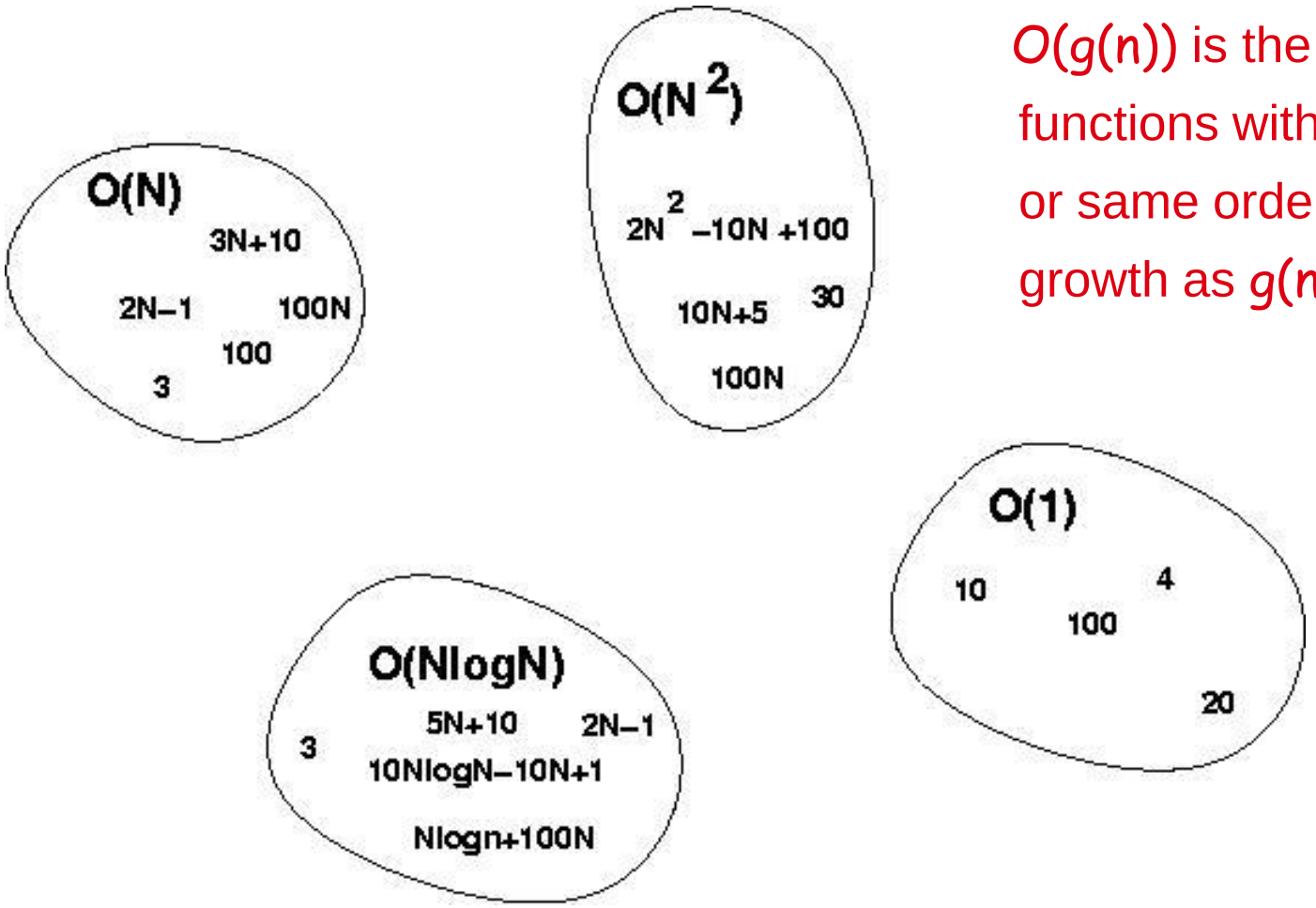
- *O-notation*

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}.$$



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Big-O Visualization



$O(g(n))$ is the set of functions with smaller or same order of growth as $g(n)$

Examples

- $2n^2 = O(n^3)$: $2n^2 \leq cn^3 \Rightarrow 2 \leq cn \Rightarrow c = 1$ and $n_0 = 2$
- $n^2 = O(n^2)$: $n^2 \leq cn^2 \Rightarrow c \geq 1 \Rightarrow c = 1$ and $n_0 = 1$
- $1000n^2+1000n = O(n^2)$:

$1000n^2+1000n \leq 1000n^2+ n^2 = 1001n^2 \Rightarrow c=1001$ and $n_0 = 1000$

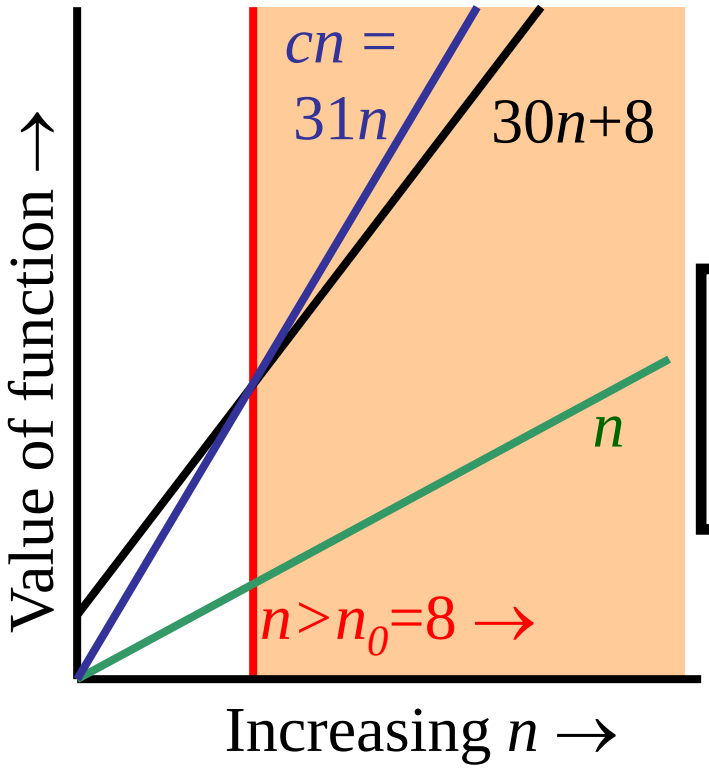
- $n = O(n^2)$: $n \leq cn^2 \Rightarrow cn \geq 1 \Rightarrow c = 1$ and $n_0 = 1$

More Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c, n_0$: $30n+8 \leq cn$, $\forall n > n_0$.
    - Let $c=31$, $n_0=8$. Assume $n>n_0=8$. Then $cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.

Big-O example, graphically

- Note 30*n*+8 isn't less than *n* *anywhere* (*n*>0).
- It isn't even less than 31*n* *everywhere*.
- But it *is* less than 31*n* everywhere to the right of *n*=8.



$30n+8 \in O(n)$

No Uniqueness

- There is no unique set of values for n_0 and c in proving the asymptotic bounds
- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \leq 100n + n = 101n \leq 101n^2$

    for all $n \geq 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \leq 100n + 5n = 105n \leq 105n^2$

    for all $n \geq 1$

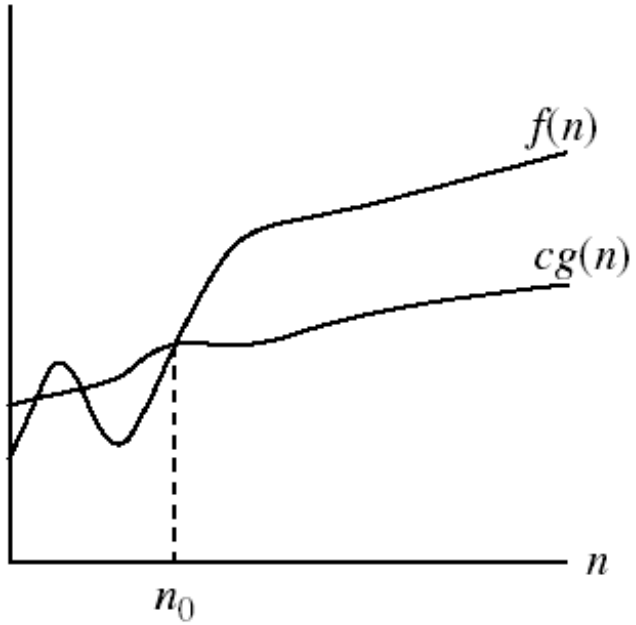
    $n_0 = 1$ and $c = 105$ is also a solution

Must find **SOME** constants c and n_0 that satisfy the asymptotic notation relation

Asymptotic notations (cont.)

- Ω - *notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



$\Omega(g(n))$ is the set of functions with larger or same order of growth as $g(n)$

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples

- $5n^2 = \Omega(n)$

  $\exists\ c, n_0$ such that: $0 \le cn \le 5n^2 \Rightarrow cn \le 5n^2 \Rightarrow c = 1$ and $n_0 = 1$

- $100n + 5 \neq \Omega(n^2)$

  $\exists\ c, n_0$ such that: $0 \le cn^2 \le 100n + 5$

  $100n + 5 \le 100n + 5n\ (\forall\ n \ge 1) = 105n$

  $cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

  Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$

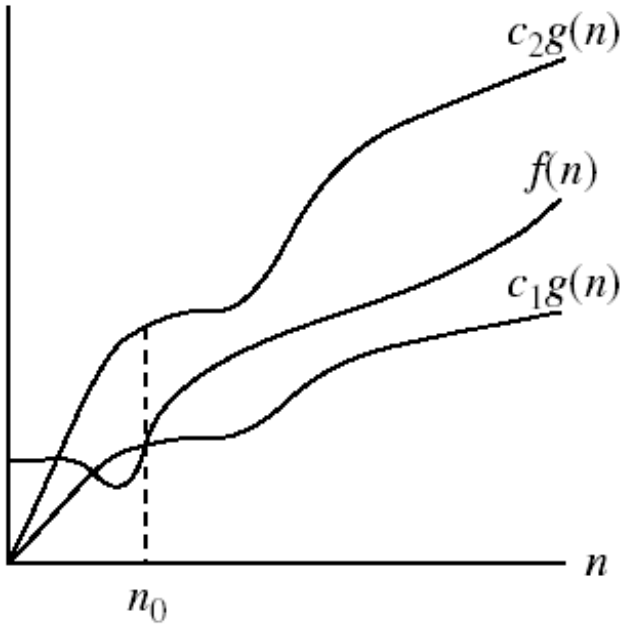
  $\Rightarrow$ contradiction: n cannot be smaller than a constant

- $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

Asymptotic notations (cont.)

- *Θ-notation*

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .$$



Θ(g(n)) is the set of functions with the same order of growth as g(n)

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Examples

- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2}\, n^2 - \frac{1}{2}\, n \leq \frac{1}{2}\, n^2 \;\forall n \geq 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2}\, n^2 - \frac{1}{2}\, n \geq \frac{1}{2}\, n^2 - \frac{1}{2}\, n * \frac{1}{2}\, n \;(\; \forall n \geq 2\;) = \frac{1}{4}\, n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \neq \Theta(n^2)$: $c_1\, n^2 \leq n \leq c_2\, n^2$

  $\Rightarrow$ only holds for: $n \leq 1/c_1$

Examples

- $6n^3 \neq \Theta(n^2)$: $c_1 n^2 \leq 6n^3 \leq c_2 n^2$

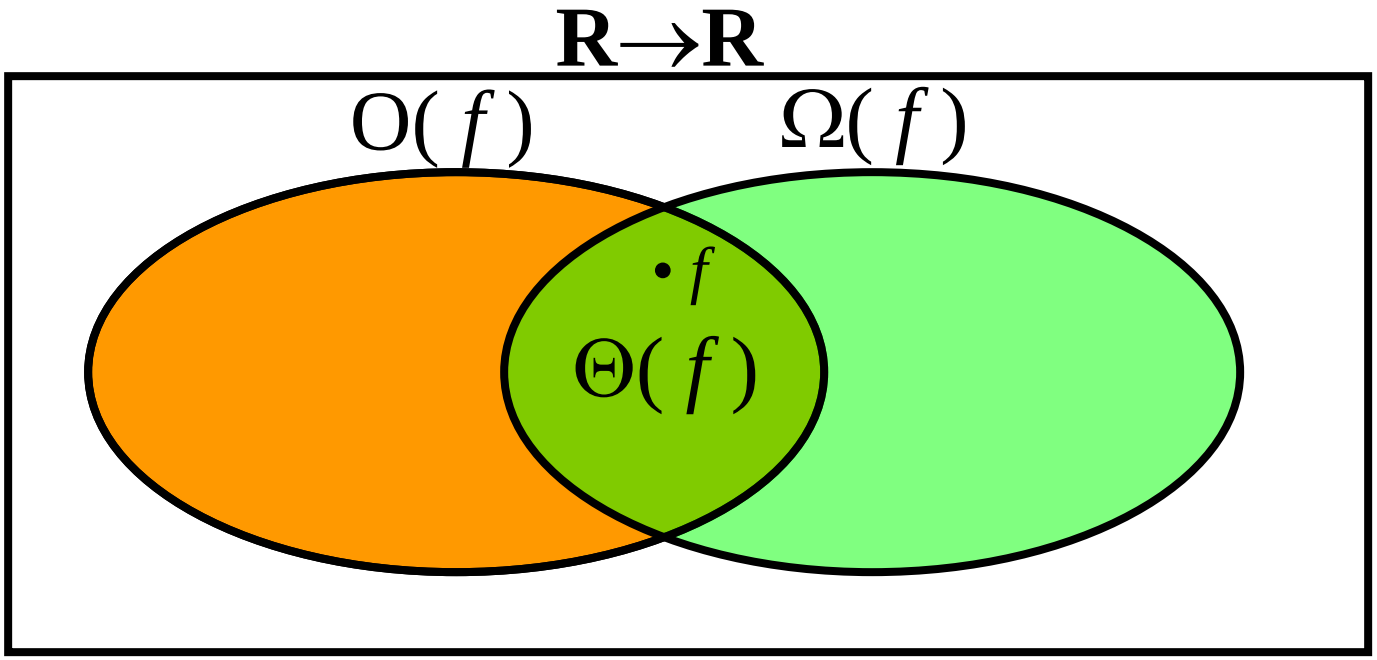
  $\Rightarrow$ only holds for: $n \leq c_2 / 6$

- $n \neq \Theta(\log n)$: $c_1 \log n \leq n \leq c_2 \log n$

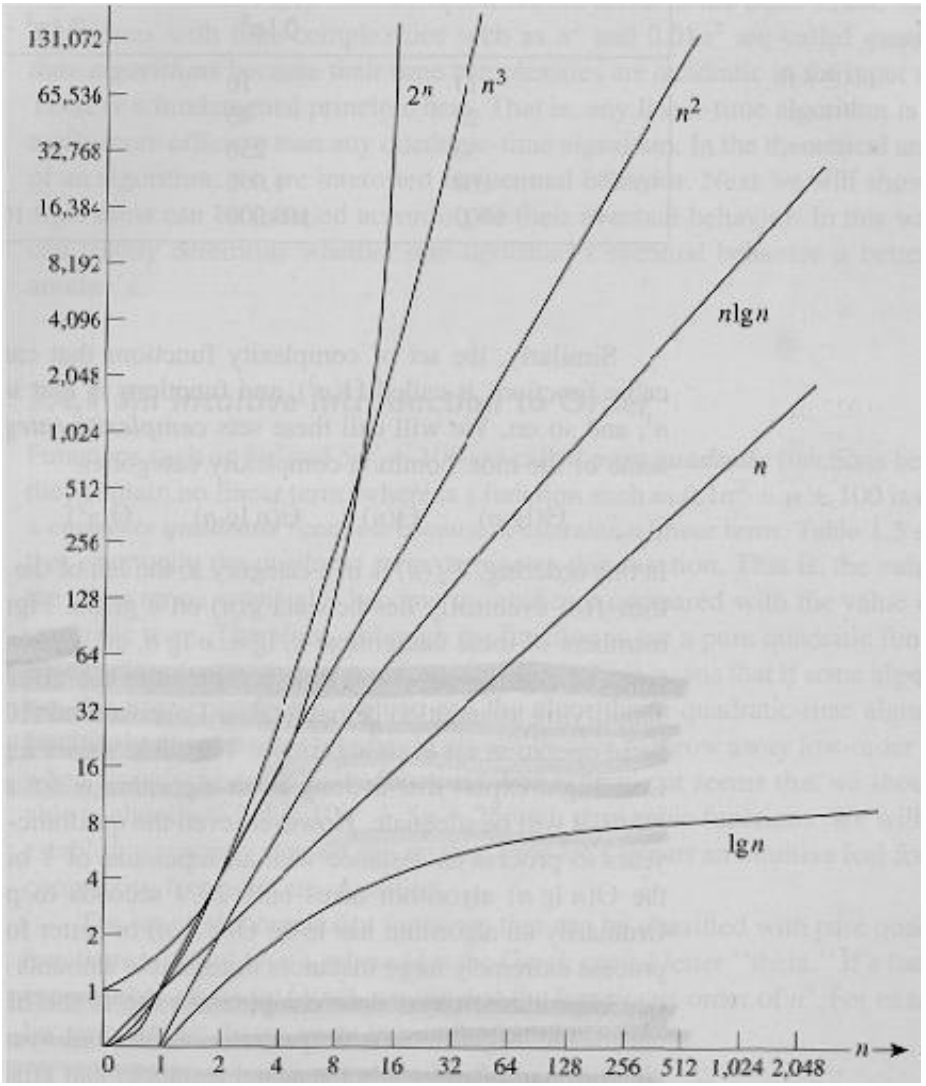
  $\Rightarrow c_2 \geq n/\log n, \forall n \geq n_0$ – impossible

Relations Between Different Sets

- Subset relations between order-of-growth sets.



Common orders of magnitude



Common orders of magnitude

Table 1.4 Execution times for algorithms with the given time complexities

n	$f(n) = \lg n$	$f(n) = n$	$f(n) = n \lg n$	$f(n) = n^2$	$f(n) = n^3$	$f(n) = 2^n$
10	0.003 μs*	0.01 μs	0.033 μs	0.1 μs	1 μs	[illegible] μs
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	8 μs	[illegible] ms†
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	27 μs	[illegible] s
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	64 μs	18.3 min
50	0.005 μs	0.05 μs	0.282 μs	2.5 μs	[illegible]25 μs	13 days
10^2	0.007 μs	0.10 μs	0.664 μs	10 μs	1 ms	4×10^{15} years
10^3	0.010 μs	1.00 μs	9.966 μs	1 ms	1 s	
10^4	0.013 μs	[illegible]0 μs	130 μs	100 ms	16.7 min	
10^5	0.017 μs	0.10 ms	1.67 ms	10 s	11.6 days	
10^6	0.020 μs	1 ms	19.93 ms	16.7 min	31.7 years	
10^7	0.023 μs	0.01 s	0.23 s	1.16 days	31,709 years	
10^8	0.027 μs	0.10 s	2.66 s	115.7 days	3.17 $\times$ [illegible] years	
10^9	0.030 μs	1 s	29.90 s	31.7 years		

*1 μs = 10^{-6} second.
†1 ms = 10^{-3} second.

Logarithms and properties

- In algorithm analysis we often use the notation "log n" without specifying the base

Binary logarithm $\lg n = \log_2 n$

Natural logarithm $\ln n = \log_e n$

$\lg^k n = (\lg n)^k$

$\lg \lg n = \lg(\lg n)$

$\log x^y = y \log x$

$\log xy = \log x + \log y$

$\log \frac{x}{y} = \log x - \log y$

$a^{\log_b x} = x^{\log_b a}$

$\log_b x = \frac{\log_a x}{\log_a b}$

More Examples

- For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) = Θ(g(n)). Determine which relationship is correct.
  - $f(n) = \log n^2$; $g(n) = \log n + 5$ — $f(n) = \Theta(g(n))$
  - $f(n) = n$; $g(n) = \log n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = \log \log n$; $g(n) = \log n$ — $f(n) = O(g(n))$
  - $f(n) = n$; $g(n) = \log^2 n$ — $f(n) = \Omega(g(n))$
  - $f(n) = n \log n + n$; $g(n) = \log n$ — $f(n) = \Omega(g(n))$
  - $f(n) = 10$; $g(n) = \log 10$ — $f(n) = \Theta(g(n))$
  - $f(n) = 2^n$; $g(n) = 10n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = 2^n$; $g(n) = 3^n$ — $f(n) = O(g(n))$

Properties

- *Theorem:*

$$f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n)) \text{ and } f = \Omega(g(n))$$

- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Asymptotic Notations in Equations

- On the right-hand side
  - $\Theta(n^2)$ stands for some anonymous function in $\Theta(n^2)$

  $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means:

  There exists a function $f(n) \in \Theta(n)$ such that

  $2n^2 + 3n + 1 = 2n^2 + f(n)$
- On the left-hand side

  $2n^2 + \Theta(n) = \Theta(n^2)$

  No matter how the anonymous function is chosen on the left-hand side, there is a way to choose the anonymous function on the right-hand side to make the equation valid.

Common Summations

- Arithmetic series:

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

- Geometric series:

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \left(x \neq 1\right)$$

  - Special case: $|x| < 1$:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

- Harmonic series:

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \ln n$$

- Other important formulas:

$$\sum_{k=1}^{n} \lg k \approx n \lg n$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \ldots + n^p \approx \frac{1}{p+1} n^{p+1}$$

Mathematical Induction

- A powerful, rigorous technique for proving that a statement S(*n*) is true for *every* natural number *n*, no matter how large.
- Proof:
  - **Basis step**: prove that the statement is true for $n = 1$
  - **Inductive step:** assume that $S(n)$ is true and prove that $S(n+1)$ is true for all $n \geq 1$
- Find case n "within" case $n+1$

Example

- Prove that: $2n + 1 \leq 2^n$ for all $n \geq 3$
- **Basis step:**
  - $n = 3$: $2 * 3 + 1 \leq 2^3 \Leftrightarrow 7 \leq 8$ TRUE
- **Inductive step:**
  - Assume inequality is true for n, and prove it for (n+1):

$2n + 1 \leq 2^n$ must prove: $2(n + 1) + 1 \leq 2^{n+1}$

$2(n + 1) + 1 = (2n + 1) + 2 \leq 2^n + 2 \leq$

$\leq 2^n + 2^n = 2^{n+1}$, since $2 \leq 2^n$ for $n \geq 1$

Summations – Review

Review on Summations

- Why do we need summation formulas?

For computing the running times of iterative constructs (loops). (CLRS – Appendix A)

Review on Summations

- **Constant Series:** For integers a and b, $a \le b$,

$$\sum_{i=a}^{b} 1 = b - a + 1$$

- **Linear Series (Arithmetic Series):** For $n \ge 0$,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Quadratic Series:** For $n \ge 0$,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Review on Summations

- **Cubic Series:** For $n \geq 0$,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Geometric Series:** For real $x \neq 1$,

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

For $|x| < 1$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Review on Summations

- **Linear-Geometric Series:** For $n \geq 0$, real $c \neq 1$,

$$\sum_{i=1}^{n} ic^i = c + 2c^2 + \cdots + nc^n = \frac{-(n+1)c^{n+1} + nc^{n+2} + c}{(c-1)^2}$$

- **Harmonic Series:** nth harmonic number, $n \in \mathbb{I}^+$,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

Review on Summations

- **Telescoping Series:**

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

- **Differentiating Series:** For $|x| < 1$,

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Review on Summations

- **Approximation by integrals:**
  - For monotonically increasing $f(n)$

$$\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$$

  - For monotonically decreasing $f(n)$

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

- **How?**

Review on Summations

- ***n*th harmonic number**

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1)$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} = \ln n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1$$