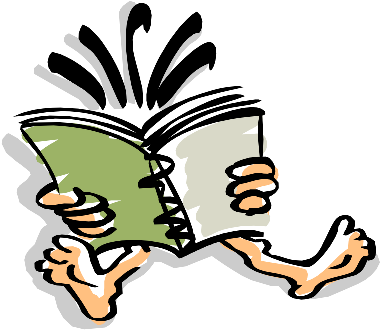
Analysis of Algorithms

Shortest Paths



Chapter 24

Shortest Path Problems

- How can we find the shortest route between two points on a road map?
- Model the problem as a graph problem:
  - Road map is a weighted graph:

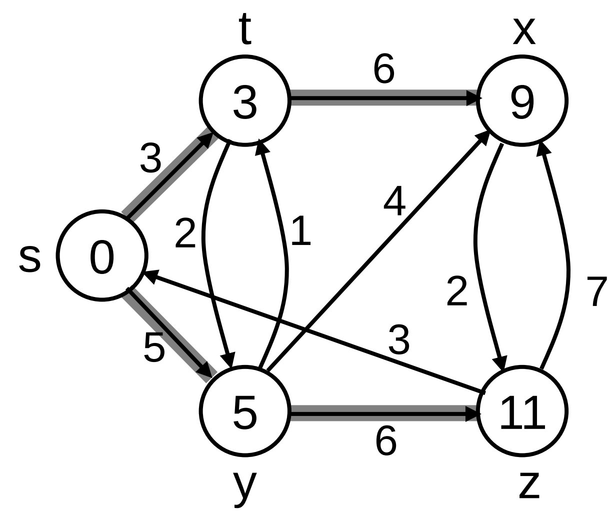
    vertices = cities

    edges = road segments between cities

    edge weights = road distances
  - Goal: find a shortest path between two vertices (cities)

Shortest Path Problem

- **Input:**
  - Directed graph G = (V, E)
  - Weight function w : E → **R**



- **Weight of path** p = $\langle v_0, v_1, \ldots, v_k \rangle$

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- **Shortest-path weight** from u to v:

$$\delta(u, v) = \begin{cases} \min \{ w(p) : u \overset{p}{\rightsquigarrow} v \} & \text{if there exists a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- **Note:** there might be <u>multiple shortest</u> paths from u to v

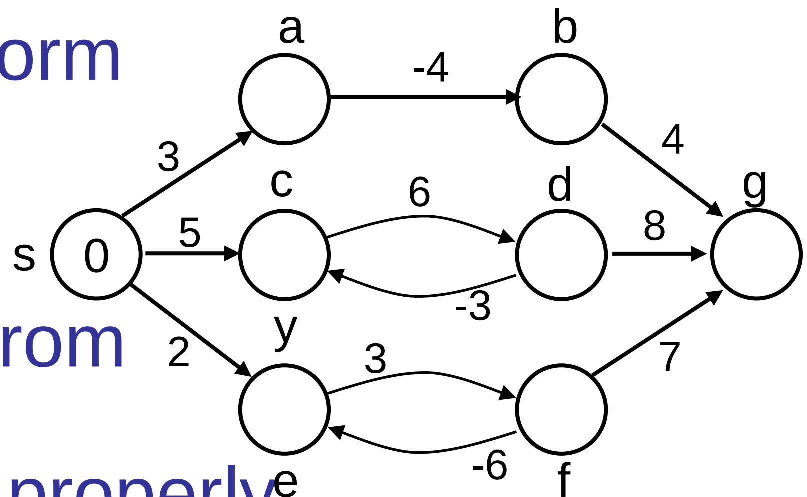
Variants of Shortest Path

- **Single-source shortest paths**
  - G = (V, E) $\Rightarrow$ find a shortest path from a given source vertex s to each vertex $v \in V$
- **Single-destination shortest paths**
  - Find a shortest path to a given destination vertex **t** from each vertex v
  - Reversing the direction of each edge $\Rightarrow$ single-source

Variants of Shortest Paths (cont'd)

- **Single-pair shortest path**
  - Find a shortest path from u to v for given vertices u and v
- **All-pairs shortest-paths**
  - Find a shortest path from u to v for every pair of vertices u and v

Negative-Weight Edges



- Negative-weight edges may form negative-weight cycles
- If such cycles are reachable from the source, then δ(s, v) is not properly defined!
  - Keep going around the cycle, and get w(s, v) = - ∞ for all v on the cycle

Negative-Weight Edges

- s $\rightarrow$ a: only one path

  $\delta(s, a) = w(s, a) = 3$

- s $\rightarrow$ b: only one path

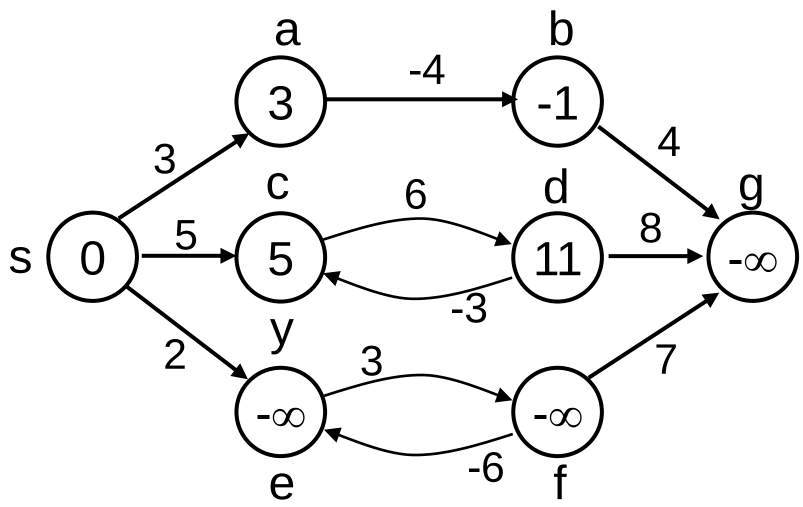
  $\delta(s, b) = w(s, a) + w(a, b) = -1$

- s $\rightarrow$ c: infinitely many paths

  $\langle s, c\rangle, \langle s, c, d, c\rangle, \langle s, c, d, c, d, c\rangle$

  cycle has positive weight (6 - 3 = 3)

  $\langle s, c\rangle$ is shortest path with weight $\delta(s, b) = w(s, c) = 5$

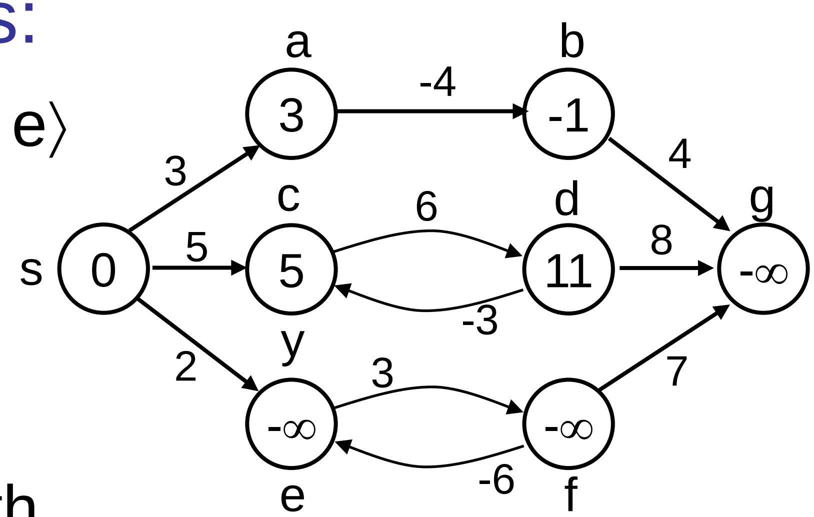


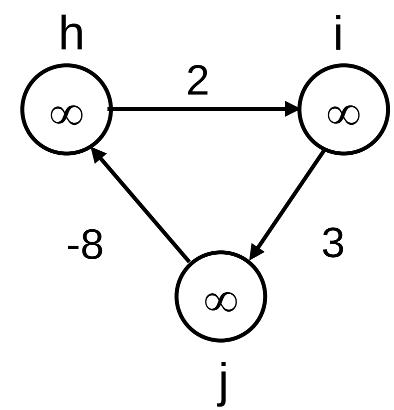
Negative-Weight Edges

- $s \rightarrow e$: infinitely many paths:
  - $\langle s, e\rangle, \langle s, e, f, e\rangle, \langle s, e, f, e, f, e\rangle$
  - cycle $\langle e, f, e\rangle$ has negative weight:

    $3 + (-6) = -3$
  - can find paths from *s* to *e* with arbitrarily large negative weights
  - $\delta(s, e) = -\infty \Rightarrow$ no shortest path exists between *s* and *e*
  - Similarly: $\delta(s, f) = -\infty$,

    $\delta(s, g) = -\infty$





h, i, j not reachable from s

$\delta(s, h) = \delta(s, i) = \delta(s, j) = \infty$

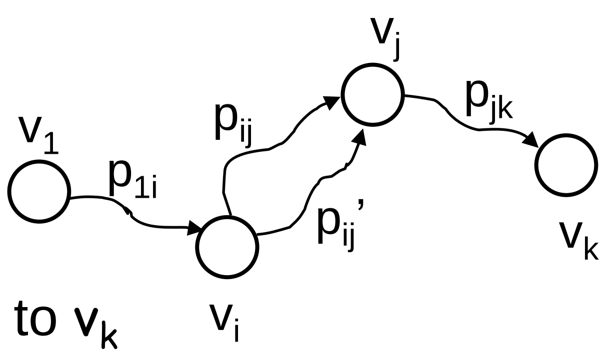
Cycles

- Can shortest paths contain cycles?
- Negative-weight cycles No!
  - Shortest path is not well defined
- Positive-weight cycles: No!
  - By removing the cycle, we can get a shorter path
- Zero-weight cycles
  - No reason to use them
  - Can remove them to obtain a path with same weight

Optimal Substructure Theorem

Given:

- A weighted, directed graph $G = (V, E)$
- A weight function $w: E \to \mathbf{R}$,
- A shortest path $p = \langle v_1, v_2, \ldots, v_k \rangle$ from v_1 to v_k
- A subpath of p: $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$, with $1 \le i \le j \le k$



Then: p_{ij} is a shortest path from v_i to v_j

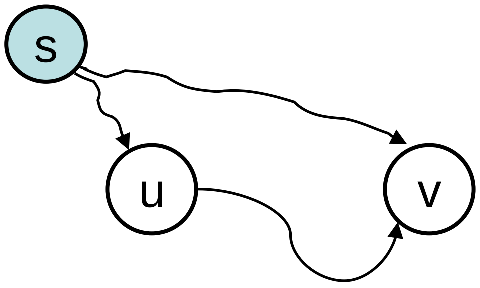
Proof: $p = v_1 \overset{p_{1i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$

$$w(p) = w(p_{1i}) + w(p_{ij}) + w(p_{jk})$$

Assume $\exists\, p_{ij}'$ from v_i to v_j with $w(p_{ij}') < w(p_{ij})$

$\Rightarrow w(p') = w(p_{1i}) + w(p_{ij}') + w(p_{jk}) < w(p)$ contradiction!

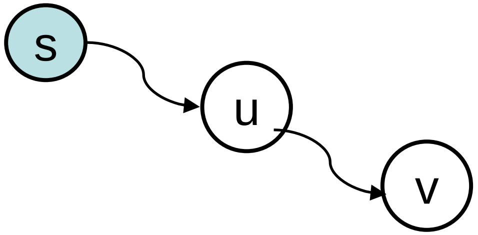
Triangle Inequality



For all $(u, v) \in E$, we have:

$$\delta(s, v) \leq \delta(s, u) + \delta(u, v)$$

- If u is on the shortest path to v we have the equality sign



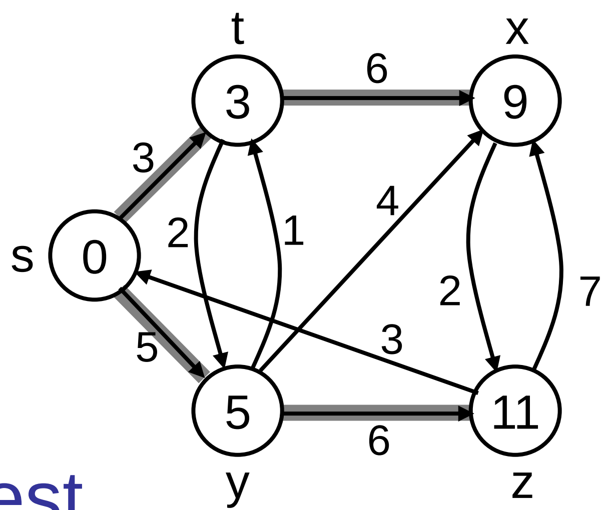
Algorithms

- Bellman-Ford algorithm
  - Negative weights are allowed
  - Negative cycles reachable from the source are not allowed.
- Dijkstra's algorithm
  - Negative weights are not allowed
- Operations common in both algorithms:
  - Initialization
  - Relaxation

Shortest-Paths Notation

For each vertex $v \in V$:

- $\delta(s, v)$: **shortest-path weight**
- d[v]: shortest-path weight **estimate**
  - Initially, d[v]=∞
  - d[v]→δ(s,v) as algorithm progresses
- π[v] = **predecessor** of v on a shortest path from *s*
  - If no predecessor, π[v] = NIL
  - π induces a tree—**shortest-path tree**



Initialization

Alg.: INITIALIZE-SINGLE-SOURCE(V, s)

1. **for** each $v \in V$

2. **do** $d[v] \leftarrow \infty$

3. $\pi[v] \leftarrow$ NIL

4. $d[s] \leftarrow 0$

- All the shortest-paths algorithms start with INITIALIZE-SINGLE-SOURCE

Relaxation Step

- **Relaxing** an edge (u, v) = testing whether we can improve the shortest path to v found so far by going through u

    If $d[v] > d[u] + w(u, v)$

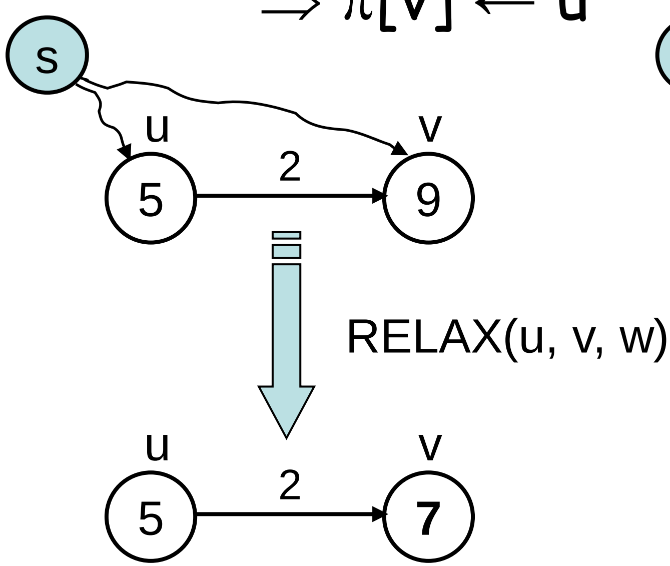
    we can improve the shortest path to v

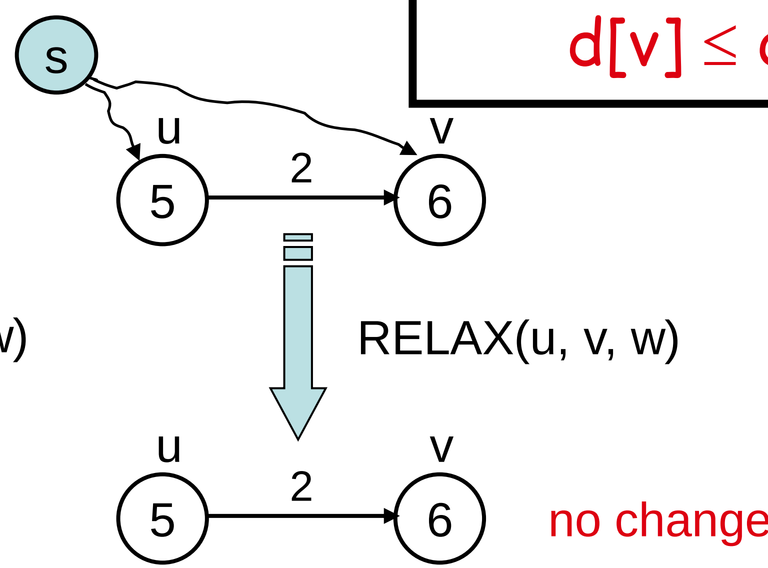
    $\Rightarrow d[v]=d[u]+w(u,v)$

    $\Rightarrow \pi[v] \leftarrow u$

After relaxation:

$d[v] \leq d[u] + w(u, v)$





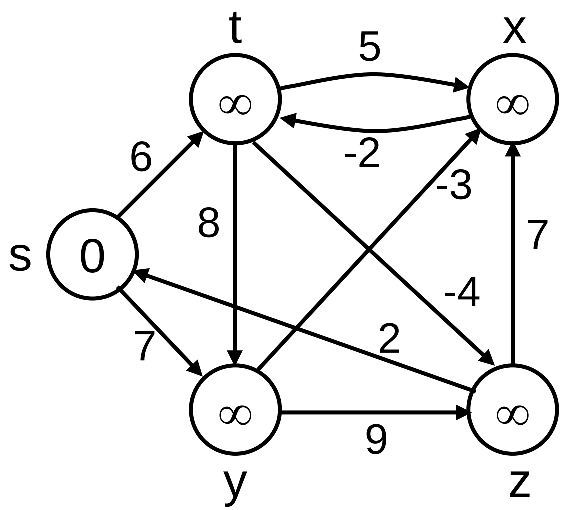
Bellman-Ford Algorithm

- Single-source shortest path problem
  - Computes $\delta(s, v)$ and $\pi[v]$ for all $v \in V$
- Allows negative edge weights - can detect negative cycles.
  - Returns TRUE if no negative-weight cycles are reachable from the source s
  - Returns FALSE otherwise $\Rightarrow$ no solution exists

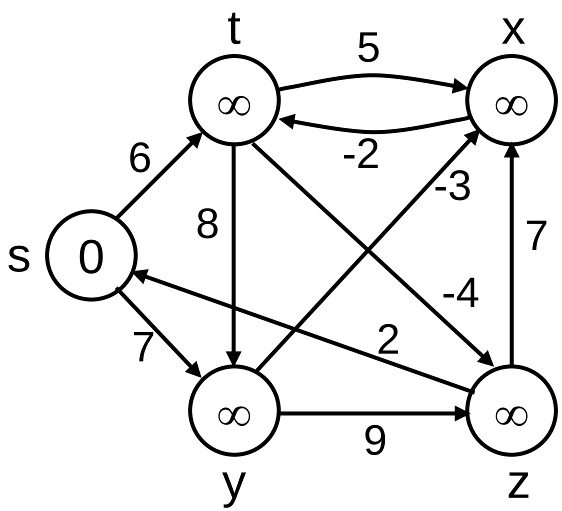
Bellman-Ford Algorithm (cont'd)

- Idea:
  - Each edge is relaxed |V–1| times by making |V-1| passes over the whole edge set.
  - To make sure that each edge is relaxed exactly |V – 1| times, it puts the edges in an unordered list and goes over the list |V – 1| times.

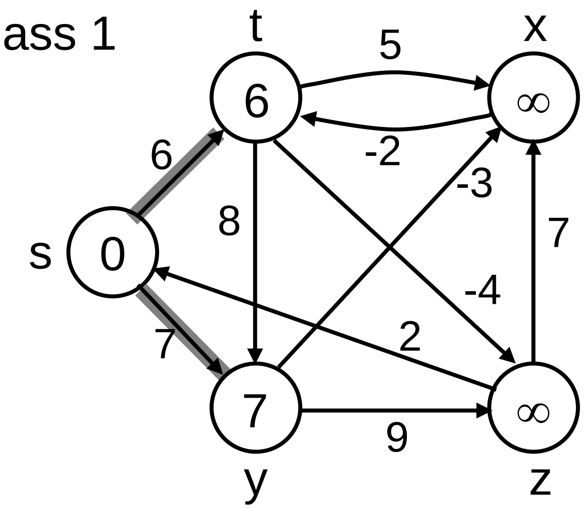
(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)



BELLMAN-FORD(V, E, w, s)



Pass 1

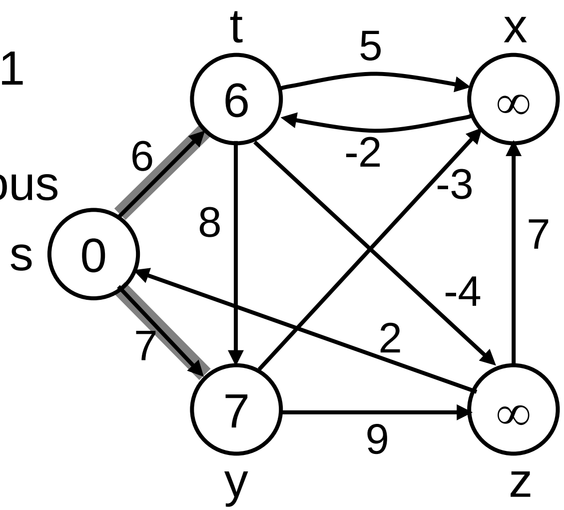


E: (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

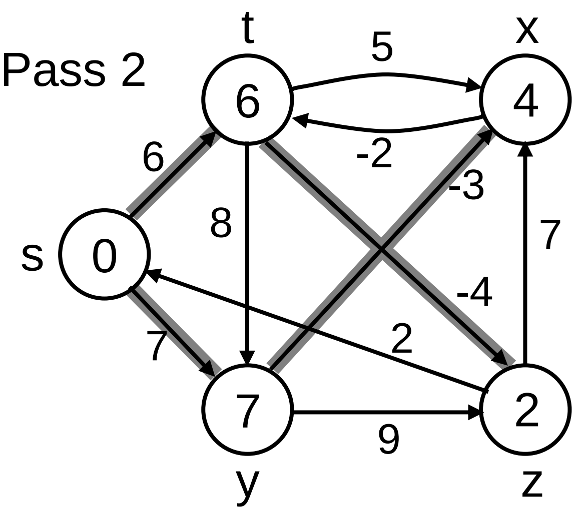
Example

(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

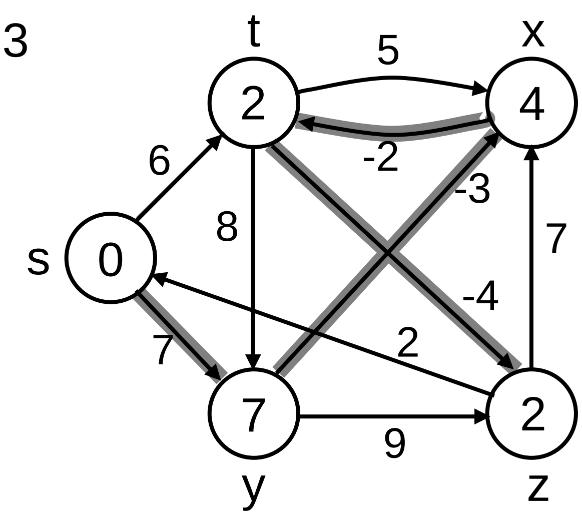
Pass 1
(from
previous
slide)



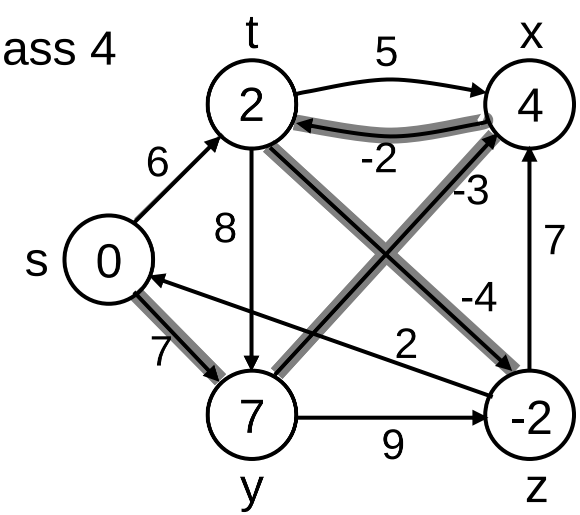
Pass 2



Pass 3

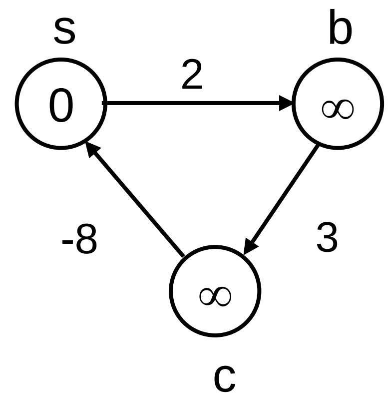


Pass 4

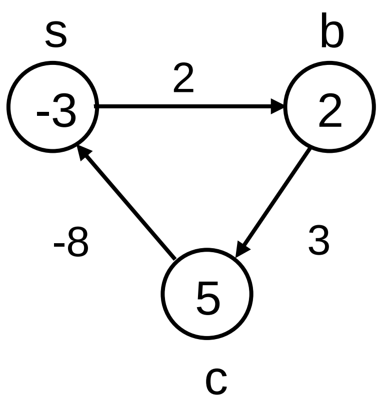


Detecting Negative Cycles (perform extra test after V-1 iterations)

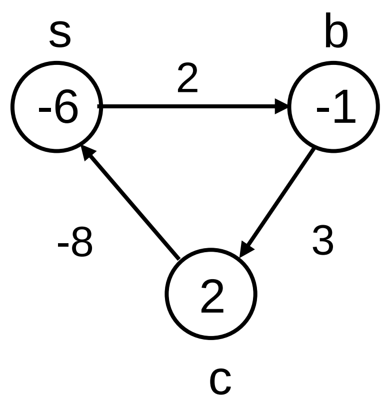
- **for** each edge $(u, v) \in E$
- **do if** $d[v] > d[u] + w(u, v)$
- **then return** FALSE
- **return** TRUE



1st pass



2nd pass



(s,b) (b,c) (c,s)

Look at edge (s, b):

$d[b] = -1$
$d[s] + w(s, b) = -4$

$\Rightarrow d[b] > d[s] + w(s, b)$

BELLMAN-FORD(*V*, *E*, *w*, *s*)

INITIALIZE-SINGLE-SOURCE(V, s) ← $\Theta(V)$

for i ← 1 to |V| - 1 ← $O(V)$

  do for each edge (u, v) $\in$ E ← $O(E)$

    do RELAX(u, v, w)

O(VE)

for each edge (u, v) $\in$ E ← $O(E)$

  do if d[v] > d[u] + w(u, v)

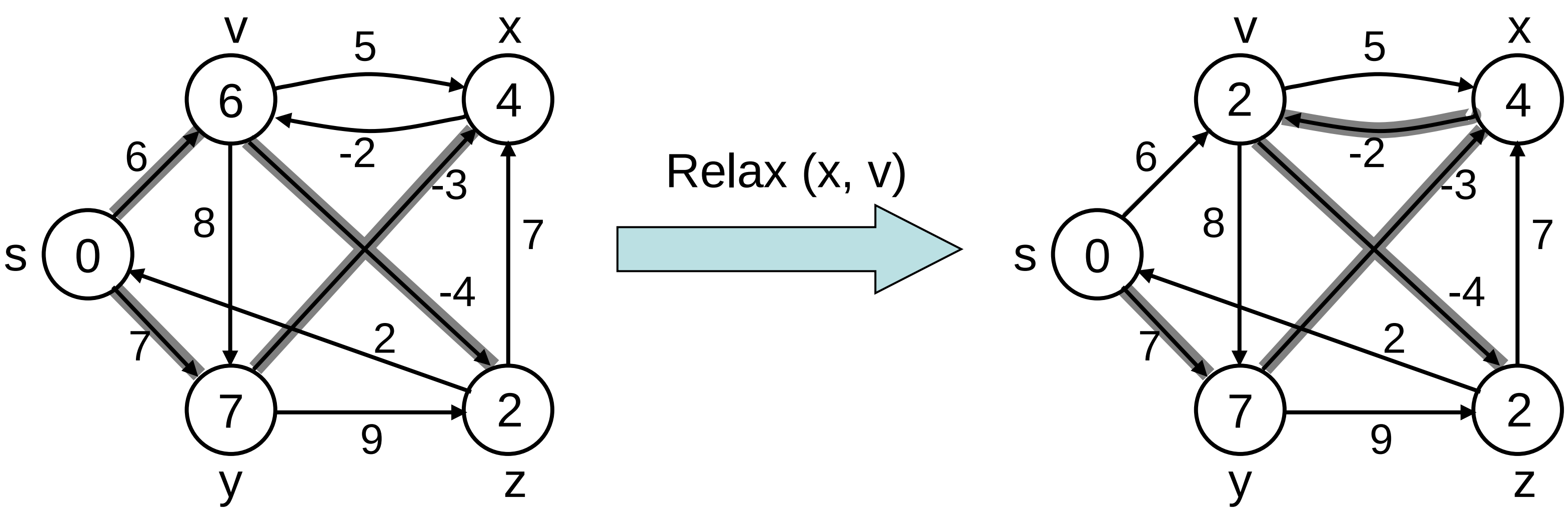
    then return FALSE

return TRUE

Running time: O(V+VE+E)=O(VE)

Shortest Path Properties

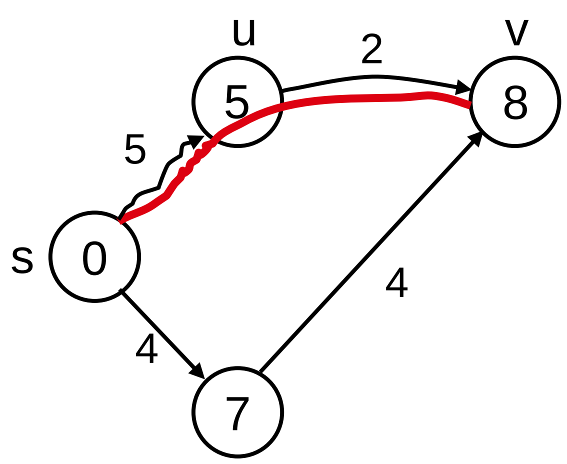
- **Upper-bound property**
  - We always have $d[v] \geq \delta(s, v)$ for all v.
  - The estimate never goes up – relaxation only lowers the estimate



Shortest Path Properties

- **Convergence property**

If $s \leadsto u \rightarrow v$ is a shortest path, and if $d[u] = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $d[v] = \delta(s, v)$ at all times after relaxing (u, v).



- If $d[v] > \delta(s, v) \Rightarrow$ after relaxation:

  $d[v] = d[u] + w(u, v)$

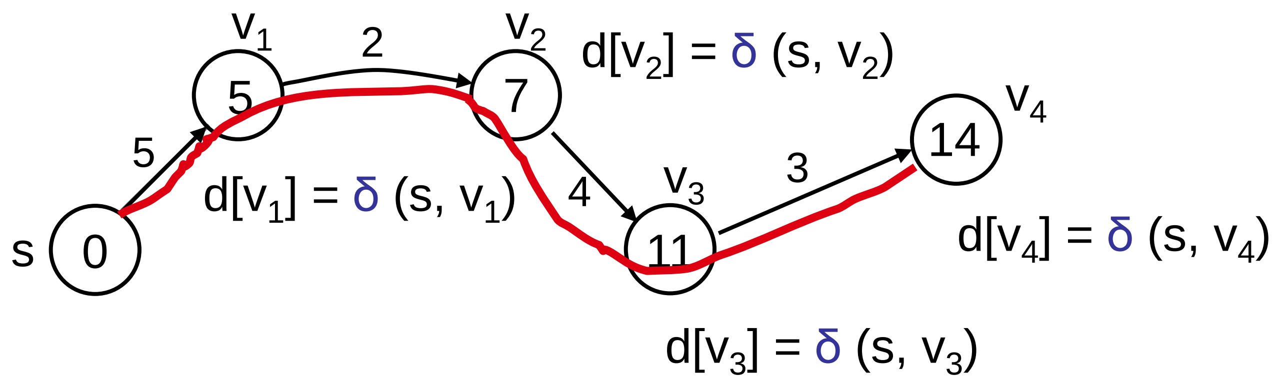
  $d[v] = 5 + 2 = 7$

- Otherwise, the value remains unchanged, because it must have been the shortest path value

Shortest Path Properties

- **Path relaxation property**

  Let $p = \langle v_0, v_1, \ldots, v_k\rangle$ be a shortest path from $s = v_0$ to v_k. If we relax, in order, $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, even intermixed with other relaxations, then $d[v_k] = \delta(s, v_k)$.



Correctness of Belman-Ford Algorithm

- **Theorem:** Show that $d[v] = \delta(s, v)$, for every v, after |V-1| passes.

  Case 1: G does not contain negative cycles which are reachable from s

  - Assume that the shortest path from s to v is
    $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $s=v_0$ and $v=v_k$, $k \leq |V-1|$
  - Use mathematical induction on the number of passes i to show that:

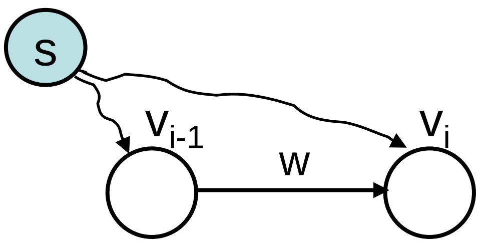
$$d[v_i] = \delta(s, v_i),\ i=0,1,\ldots,k$$

Correctness of Belman-Ford Algorithm (cont.)

Base Case: $i=0$ $d[v_0] = \delta(s, v_0) = \delta(s, s) = 0$

Inductive Hypothesis: $d[v_{i-1}] = \delta(s, v_{i-1})$

Inductive Step: $d[v_i] = \delta(s, v_i)$



$d[v_{i-1}] = \delta(s, v_{i-1})$

After relaxing (v_{i-1}, v_i):

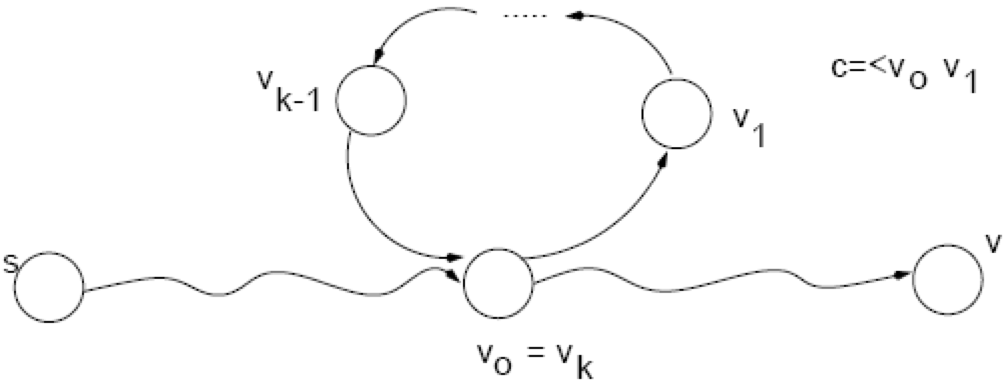
$d[v_i] \le d[v_{i-1}] + w = \delta(s, v_{i-1}) + w = \delta(s, v_i)$

From the upper bound property: $d[v_i] \ge \delta(s, v_i)$

Therefore, $d[v_i] = \delta(s, v_i)$

Correctness of Belman-Ford Algorithm (cont.)

- <u>Case 2:</u> G contains a negative cycle which is reachable from s



c=<v_o v_1 v_k> is a negative cycle

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Proof by Contradiction: suppose the algorithm returns a solution

After relaxing (v_{i-1}, v_i): $\mathbf{d}[v_i] \le \mathbf{d}[v_{i-1}] + w(v_{i-1}, v_i)$

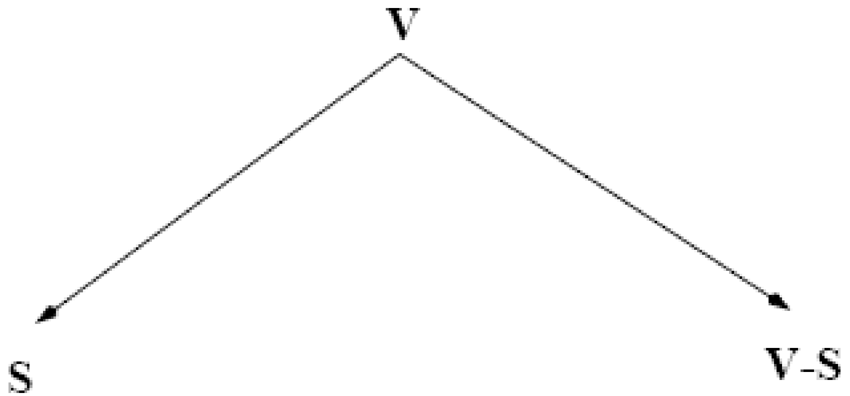
or $\sum_{i=1}^{k} \mathbf{d}[v_i] \le \sum_{i=1}^{k} \mathbf{d}[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$

or $\sum_{i=1}^{k} w(v_{i-1}, v_i) \ge 0$ $(\sum_{i=1}^{k} \mathbf{d}[v_i] = \sum_{i=1}^{k} \mathbf{d}[v_{i-1}])$

Contradiction!

Dijkstra's Algorithm

- Single-source shortest path problem:
  - No negative-weight edges: $w(u, v) > 0, \ \forall (u, v) \in E$
- Each edge is relaxed **only once!**
- Maintains two sets of vertices:



$d[v]=\delta(s, v)$ $\qquad$ $d[v]>\delta(s, v)$

Dijkstra's Algorithm (cont.)

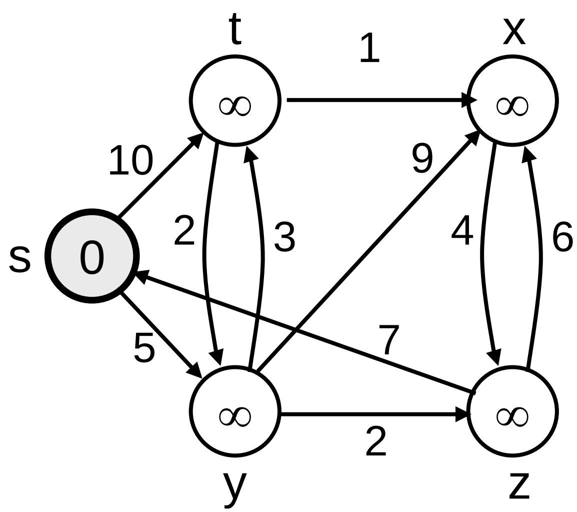
- Vertices in V – S reside in a min-priority queue
  - Keys in Q are estimates of shortest-path weights d[u]
- Repeatedly select a vertex u ∈ V – S, with the minimum shortest-path estimate d[u]
- Relax all edges leaving u

- **Steps**

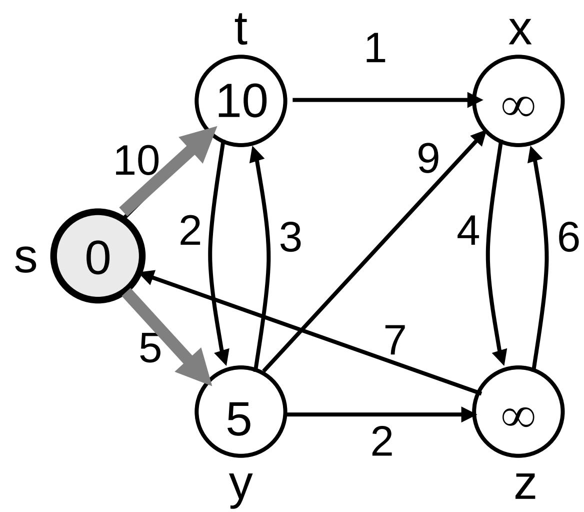
  1) Extract a vertex u from Q (i.e., u has the highest priority)
  2) Insert u to S
  3) Relax all edges leaving u
  4) Update Q

Dijkstra (G, w, s)

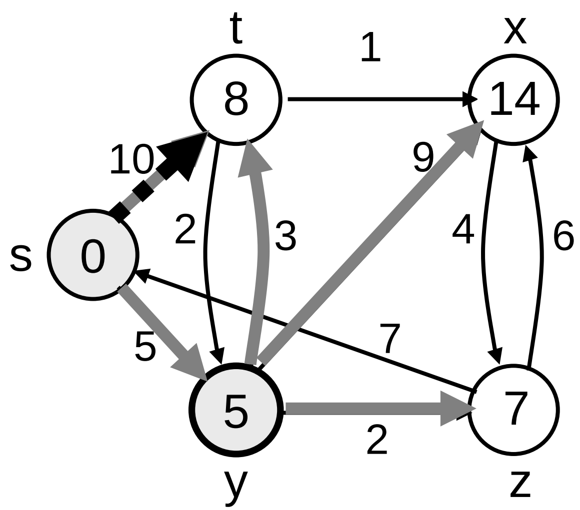
S=<> Q=<s,t,x,z,y>



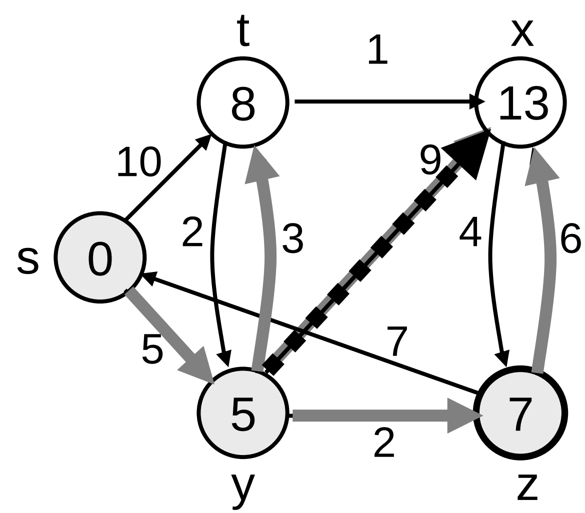
S=<s> Q=<y,t,x,z>



Example (cont.)



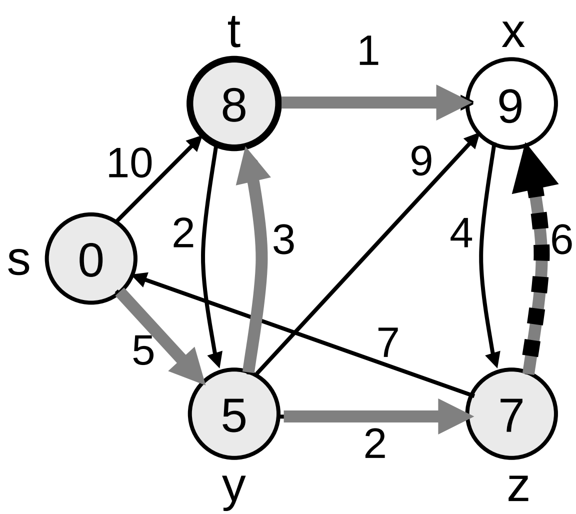
S=<s,y> Q=<z,t,x>



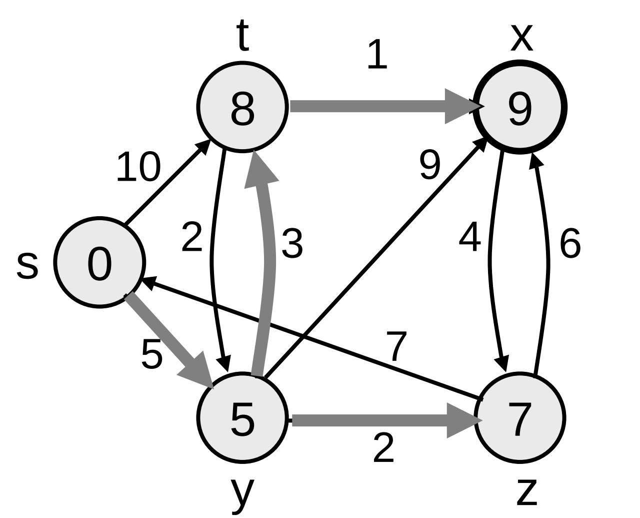
S=<s,y,z> Q=<t,x>

Example (cont.)

S=<s,y,z,t> Q=<x>



S=<s,y,z,t,x> Q=<>



Dijkstra (G, w, s)

```
INITIALIZE-SINGLE-SOURCE(V, s)              ← Θ(V)
S ← ∅
Q ← V[G]                                    ← O(V) build min-heap
while Q ≠ ∅                                 ← Executed O(V) times  } O(VlgV)
    do u ← EXTRACT-MIN(Q)                   ← O(lgV)               }
        S ← S ∪ {u}
        for each vertex v ∈ Adj[u]          ← O(E) times (total)   } O(ElgV)
            do RELAX(u, v, w)
            Update Q (DECREASE_KEY)         ← O(lgV)               }
```

Running time: $O(V\lg V + E\lg V) = O(E\lg V)$

Binary Heap vs Fibonacci Heap

Running time depends on the implementation of the heap

	EXTRACT-MIN	DECREASE-KEY	Total
binary heap	O(lgV)	O(lgV)	O(ElgV)
Fibonacci heap	O(lgV)	O(1)	O(VlgV + E)

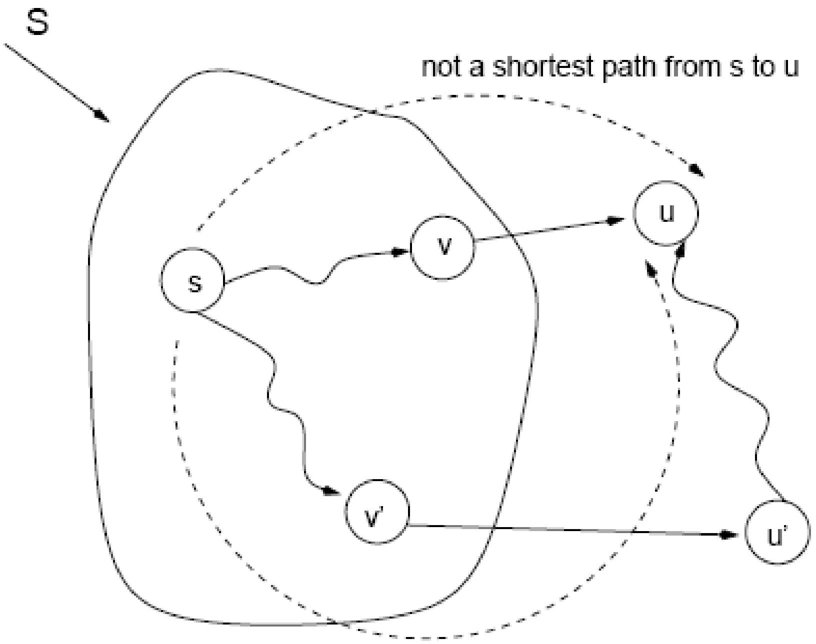
Correctness of Dijskstra's Algorithm

- For each vertex $u \in V$, we have $d[u] = \delta(s, u)$ at the time when u is added to S.

Proof:

- Let u be the first vertex for which $d[u] \neq \delta(s, u)$ when added to S
- Let's look at a true shortest path p from s to u:

Correctness of Dijskstra’s Algorithm



What is the value of d[u]?

$d[u] \le d[v] + w(v,u) = \delta(s,v) + w(v,u)$

What is the value of d[u’]?

$d[u'] \le d[v'] + w(v',u') = \delta(s,v') + w(v',u')$

Since u’ is in the shortest path of u: $d[u'] < \delta(s,u)$

Using the upper bound property: $d[u] > \delta(s,u)$

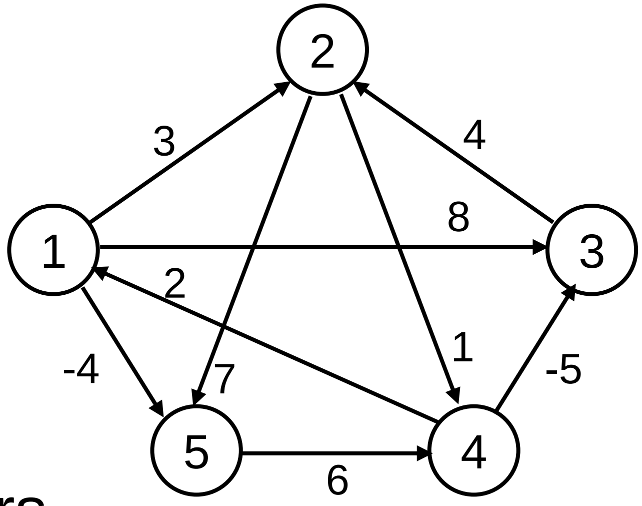
$d[u'] < d[u]$

Contradiction!

Priority Queue Q: <u, …, u’, ….> (i.e., $d[u] < \ldots < d[u'] < \ldots$)

All-Pairs Shortest Paths

- **Given:**
  - Directed graph G = (V, E)
  - Weight function $w : E \rightarrow \mathbf{R}$
- **Compute:**
  - The shortest paths between all pairs of vertices in a graph
  - Result: an $n \times n$ matrix of shortest-path distances $\delta(u, v)$



All-Pairs Shortest Paths - Solutions

- Run **BELLMAN-FORD** once from each vertex:
  - $O(V^2E)$, which is $O(V^4)$ if the graph is dense ($E = \Theta(V^2)$)
- If no negative-weight edges, could run **Dijkstra's** algorithm once from each vertex:
  - $O(VE\lg V)$ with binary heap, $O(V^3\lg V)$ if the graph is dense
- We can solve the problem in $O(V^3)$, with no elaborate data structures

Application: Feasibility Problem

- **Linear Programming**

$$\max c_1x_1 + c_2x_2 + \cdots + c_nx_n \text{ (objective function)}$$

$$\text{subject to } Ax \leq b \text{ (constraints)}$$

  \- *Simplex* is a common approach used to solve the above problem

- **Feasibility problem**

  \- Find x such that $Ax \leq b$

Application: Feasibility Problem (cont.)

- **Special case of fesibility problem**

  - All constraints have the form $x_j - x_i \le b_k$

$$x_1 - x_2 \le 3$$

$$x_2 - x_3 \le -2 \quad \text{or} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

$$x_1 - x_3 \le 2$$

Application: Feasibility Problem (cont.)

- **Constraint graph**

  - Assign one vertex per variable

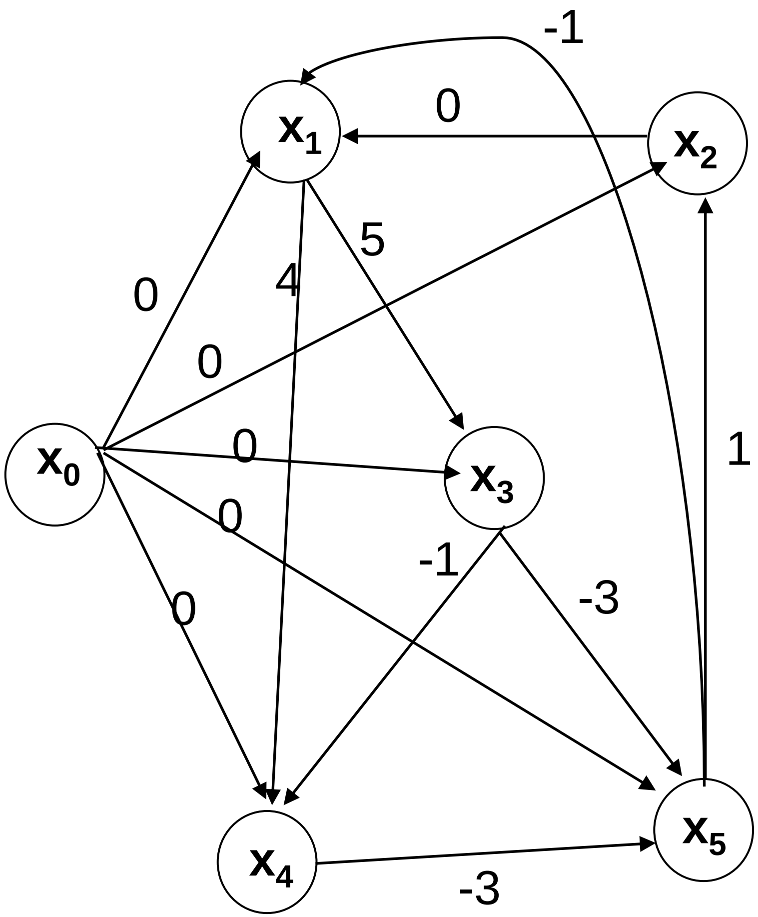
  - Assign one edge per constraint with weight b_k

If $X_j - X_i <= b_k$ then $V_i \xrightarrow{W_{ij} = b_k} V_j$

- Include an extra vertex and edges from this vertex to every other vertex

- Set the weights of the extra edges to zero

Application: Feasibility Problem (cont.)

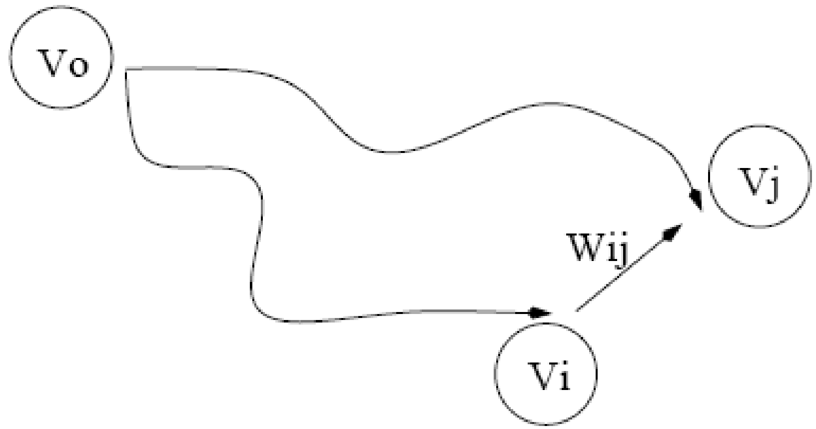


$$x_1 - x_2 \le 0$$
$$x_1 - x_5 \le -1$$
$$x_2 - x_5 \le 1$$
$$x_3 - x_1 \le 5$$
$$x_4 - x_1 \le 4$$
$$x_4 - x_3 \le -1$$
$$x_5 - x_3 \le -3$$
$$x_5 - x_4 \le -3$$

(feasible solution: -5, -3, 0, -1, -4)

Application: Feasibility Problem (cont.)

Theorem: If G contains no negative cycles, then $(\delta(v_0,v_1), \delta(v_0,v_2),\ldots, \delta(v_0,v_n))$ is a feasible solution.



For every (v_i, v_j): $\delta(v_0,v_j) \le \delta(v_0,v_i)+w(v_i,v_j)$

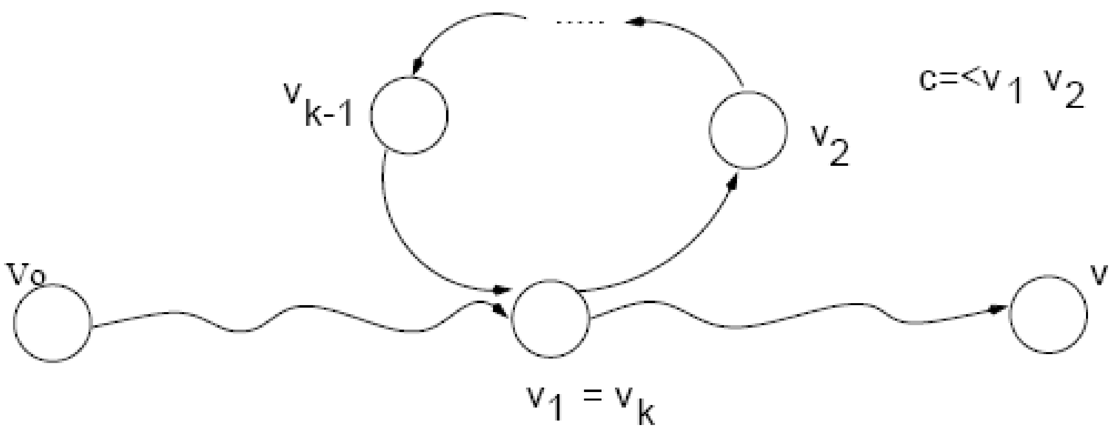
or $\delta(v_0,v_j) - \delta(v_0,v_i) \le w(v_i,v_j)$

Setting $x_i= \delta(v_0,v_i)$ and $x_j= \delta(v_0,v_j)$, we have

$x_j-x_i \le w(v_i,v_j)$

Application: Feasibility Problem (cont.)

- **Theorem:** If G contains a negative cycle, then there is no feasible solution.



$c = \langle v_1\ v_2\ \ldots\ v_k \rangle$ is a negative cycle

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Proof by contradiction: suppose there exist a solution, then:

$$\begin{aligned} x_2 - x_1 &\le w(v_1, v_2) \\ x_3 - x_2 &\le w(v_2, v_3) \\ &\ldots\ldots\ldots\ldots \\ x_k - x_{k-1} &\le w(v_{k-1}, v_k) \\ x_1 - x_k &\le w(v_k, v_1) \end{aligned}$$

- Add them up:

$$0 \le \sum_{i=1}^{k-1} w(v_i, v_{i+1}) \quad \textbf{Contradiction !!}$$

Application: Feasibility Problem (cont.)

- **Size of the constraint graph**

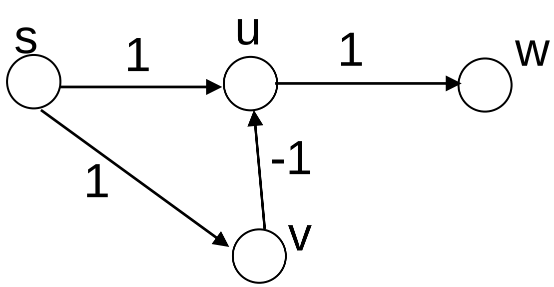
  - If we have m constraints with n unknowns ($Ax \le b$, A is m x n)

  $V = n + 1$ and $E = m + n$

  - Running time: $O(VE) = O((n+1)(m+n)) = O(n^2 + nm)$

Problem 1

Write down weights for the edges of the following graph, so that Dijkstra's algorithm would not find the correct shortest path from *s* to *t*.



1st iteration

$d[s]=0$
$d[u]=1$
$d[v]=1$

$S=\{s\}$ $Q=\{u,v,w\}$

2nd iteration

$d[w]=2$

$S=\{s,u\}$ $Q=\{v,w\}$

3rd iteration

$d[u]=0$

$S=\{s,u,v\}$ $Q=\{w\}$

4th iteration

$S=\{s,u,v,w\}$
$Q=\{\}$

- d[w] is not correct!
- d[u] should have converged when u was included in S!

Problem 2

- **(Exercise 24.3-4, page 600)** We are given a directed graph G=(V,E) on which each edge (u,v) has an associated value r(u,v), which is a real number in the range $0 \le r(u,v) \le 1$ that represents the reliability of a communication channel from vertex u to vertex v.
- We interpret r(u,v) as the probability that the channel from u to v will not fail, and we assume that these probabilities are independent.
- Give an efficient algorithm to find the most reliable path between two given vertices.

Problem 2 (cont.)

- Solution 1: modify Dijkstra's algorithm

  – Perform relaxation as follows:

  if d[v] < d[u] w(u,v) then

  d[v] = d[u] w(u,v)

  – Use "EXTRACT_MAX" instead of "EXTRACT_MIN"

Problem 2 (cont.)

- Solution 2: use Dijkstra's algorithm without any modifications!
  - r(u,v)=Pr(channel from u to v will not fail)
  - Assuming that the probabilities are independent, the reliability of a path $p=<v_1,v_2,\ldots,v_k>$ is:

$$r(v_1,v_2)r(v_2,v_3)\ \ldots\ r(v_{k-1},v_k)$$

  - We want to find the channel with the highest reliability, i.e.,

$$\max_p \prod_{(u,v)\in p} r(u,v)$$

Problem 2 (cont.)

- But Dijkstra's algorithm computes

$$\min_p \sum_{(u,v)\in p} w(u,v)$$

- Take the lg

$$\lg(\max_p \prod_{(u,v)\in p} r(u,v)) = \max_p \sum_{(u,v)\in p} \lg(r(u,v))$$

Problem 2 (cont.)

- Turn this into a minimization problem by taking the negative:

$$-\min_p \sum_{(u,v)\in p} \lg(r(u,v)) = \min_p \sum_{(u,v)\in p} -\lg(r(u,v))$$

- Run Dijkstra's algorithm using

$$w(u,v) = -\lg(r(u,v))$$