Liquidity:

The third dimension in asset pricing*

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* We thank Prof. Heinz Zimmerman, Ph.D. Andreas Gottschling, Ph.D. Peter Weber and Ph.D. Philippe Mangold for their thorough review and valuable feedback to this paper.

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1. Introduction

The term liquidity is a widely used expression when it comes to the description of equity markets. In general, market liquidity describes the possibility to buy and sell financial products easily. Therefore, previous definitions of liquidity mostly pay special attention to the fact that significant quantities of a security can be traded *quickly* and at *acceptable prices*. To give consideration to the high importance of liquidity, it is the overriding mission of an exchange organization to build, maintain and grow liquidity pools. This holds true not only for the trading segment but also for the clearing as well as the settlement and custody services that an exchange organization offers. Liquidity is key throughout the entire value chain of an exchange organization.

But how does it come that some markets are more liquid than others? What exactly is the definition of liquidity? How can liquidity as a key figure in financial decisions be derived, measured and used? And what are the factors that determine liquidity in equity markets?[1]

The unique feature of this paper is to derive a definition of liquidity based on first principles by way of developing the concept from an abstract and general articulation to one that is concrete and individual; concrete in the sense of an applicable and empirically testable formulaic form, and individual in the sense of a specific asset, basket (e.g. indices) or market. We claim that four necessary elements in sum are sufficient to define the term liquidity and that its structure is determined by the relations between these elements. We subsequently use this to discuss the pricing of financial assets and argue that the liquidity factor λ should be a characteristic, similar to the market risk premium. The advantage of the choice of the general and abstract starting point is that all subsequent decisions in terms of adding concreteness are transparent and supported by a clear underlying rationale and concept. Consequently, one can easily adapt the specification to other security classes or trading arrangements – we demonstrate one application considering a stock market and we look at the orders and the matched trades in the Consolidated Limit Order Book (CLOB) of an exchange.

To underline our assumptions, we will start with some basic remarks about definitions. This perspective of a meta definition will be concretized and applied to our concept of the term liquidity. As mentioned, we will focus our deliberations on the liquidity of equity

[1] A recent paper by Diaz/Escribano provides a comprehensive overview on empirical literature about the measurement of liquidity in financial markets. However, it also reveals the shortcomings of previous approaches. Antonio Diaz, Ana Escribano. Measuring the multi-faceted dimension of liquidity in financial markets: a literature review. Research in International Business and Finance, 51 (2020), https://doi.org/10.1016/j.ribaf.2019.101079

markets. This is the basic starting point for the concretization and individualization of the term liquidity. It enables us to measure liquidity in concrete situations in the real world. Having reached the highest level of concretization and individualization we are now able to apply the liquidity factor λ as the measured ease to trade a certain stock. We end up with some working definitions which are core for the development and application of the formula. After these rather theoretical considerations we turn to the concrete calculation of the liquidity factor λ. We spend some thoughts on general aspects for the calculation of liquidity before we derive a concrete formula for its measurement. We analyze this liquidity factor from different angles and also take a look at related formulae before we conclude our paper with an outlook on further objects of investigation.

As to the methodology of this paper we will pursue a deductive approach. We will start with some general rules and characteristics on liquidity before we successively narrow the range under consideration until we draw our conclusions on liquidity for a certain stock under predefined premises. Nevertheless, the results we will obtain are equally applicable to other securities with other predefined premises: They are applicable to asset markets in general.

2. About Definitions

2.1 Definitions of liquidity

Before we start to define the term liquidity, we should make some general remarks about definitions. Each definition is a statement about a subject matter and its meaning, it is a statement of what a subject matter *is* and what it *means*. Statement in this context means the description of the essence and the explanation of the meaning. We therefore want to understand a definition in the following way:

DE-FINITION [sic!] means a clear delineation and a clear distinction of a subject matter by means of clarification, description and explanation.

A definition of a subject matter (SM) consists of / contains / is the sum of all the necessary elements (E) under one or several specified premise(s) *and* all the relations (R) between these elements (=internal relations [IR]). The elements have predicates and propositions and built relations to and with each other. This specifies, describes and creates the structure, the content *and* the scope of the SM.

	THE SUM OF <u>ALL</u>* NECESSARY ELEMENTS		THE SUM OF <u>ALL</u>* INTERNAL RELATIONS		
D =	E(Elements)	+	E(Internal Relations)	+	THE SCOPE
	ESSENTIALS/ NECESSITIES		STRUCTURE	+	(SPACE + TIME)

$$\boxed{D = \underbrace{\text{ESSENTIALS/NECESSITIES} \quad \text{STRUCTURE}}_{\text{CONTENT (= CHARACTERISTICS)}} + \text{SCOPE (= EXTENSION)}}$$

*<u>ALL</u> => Sufficient as sum

Fig. 2.1

Having said that, we can now come to a definition of the term **liquidity**:

In general, **liquidity is the ease to convert an asset into another asset**. Applied to our focal point, the liquidity of equity markets is the ease (reference: market/segment/index/basket) to convert (to buy or to sell) stocks into cash (=price) and vice versa. Note that liquidity is always a function of time.

Looking at it in a more formal way, we can derive the following equation:

$$\text{LIQUIDITY} = \text{Ease} \circ \text{Conversion} \circ \text{Asset 1} \circ \text{Asset 2}$$

$$\text{SM} = E1 \circ E2 \circ E3 \circ E4$$

Liquidity is the concatenation of Ease, Conversion, Asset 1 and Asset 2 and comprises the sum of all internal relations between these elements. With these four necessary and in sum sufficient elements we describe the simplest possible definition of liquidity.

2.2 Schematic overview and types of relations

To make our previous explanations more descriptive we can also transfer them into a schematic overview. The following chart shows the connection between a subject matter and different internal and external elements.

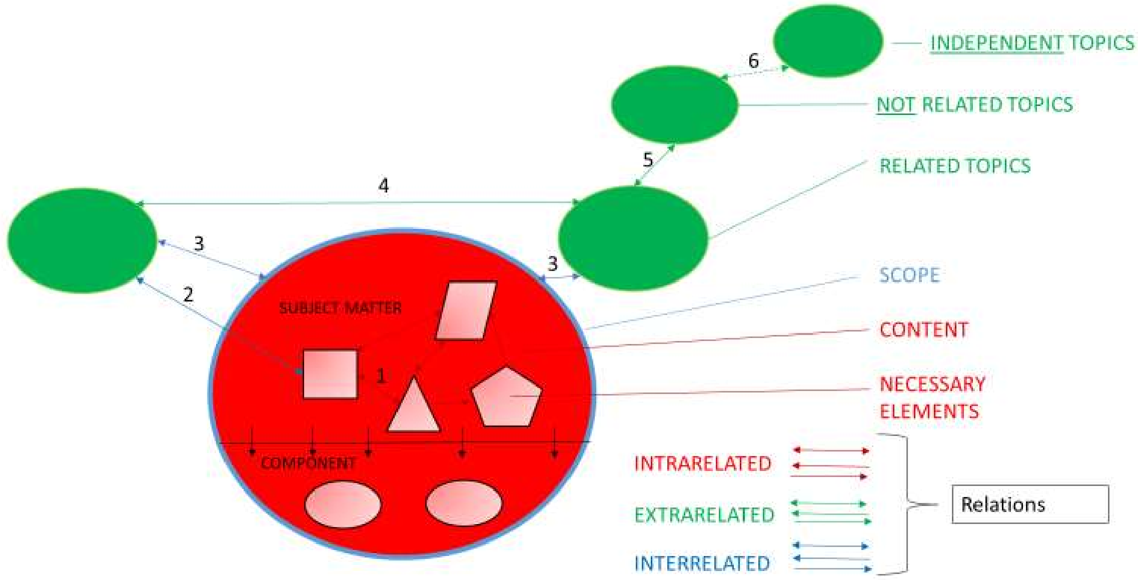


Fig. 2.2a

The blue elliptic line defines the scope of a subject matter. The light red elements in the upper part of the ellipse are the necessary constitutive elements of the subject matter. The light red elements in the lower part of the ellipse are components which are derived from the subject matter, however do not form part of the definition of the subject matter. The green elliptic forms outside of the scope are external elements some of which affect the subject matter and some of which don't.

We can distinguish three kinds of relations between the different elements in the above schematic overview: intrarelated, extrarelated and interrelated types of relation.

	A	B	A ∧ B
Relation Type / Grade	INTRARELATED	EXTRARELATED	INTERRELATED
1 = First Grade	X		
2 = Second Grade			X
3 = Third Grade			X
4 = Fourth Grade		X	
5 = Fith Grade		X	
6 = Sixth Grade		(Ø)	

Fig. 2.2b

Intra-relations exist between the internal elements of the subject matter. In the schematic overview they are indicated by red arrows. They represent first-grade relations. Inter-relations can occur between internal elements of the subject matter and external elements. The blue arrow shows this kind of relation which are either second-grade relations when they exist between a constitutive element of the subject matter and an external element or third-grade relations when they exist between the subject matter as a whole and an external element. The third type of relation, so called extra-relations can also occur in various ways. As a fourth-grade relation they represent a relation between two external elements both of which have a third-grade relation to the subject matter. A fifth-grade relation is a connection between an external element with third-grade relation to the subject matter and an independent external element without any connection to the subject matter. Sixth-grade relations are those which have no direct connection to the subject matter.

2.3 Liquidity: a four-constituent relationship

When we apply the schematic overview on the definition of liquidity in equity markets, we receive the following relations:

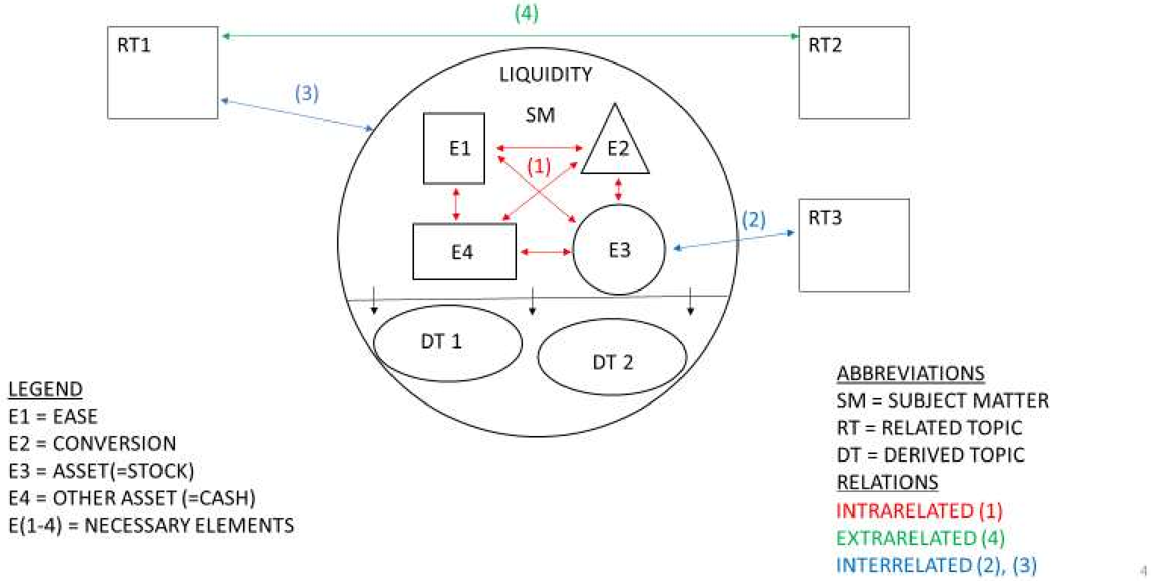


Fig. 2.3

The subject matter, liquidity in equity markets, comprises four internal elements, i.e. the ease (E1) to convert (to buy or to sell) (E2) the asset class stocks (E3) into the asset class cash (E4) or vice versa. As we said, these elements are necessary and in sum sufficient to define liquidity in equity markets. If asset (E3) is stock and asset (E4) is cash it comes to a sell-transaction, if asset (E3) is cash and asset (E4) is stock, a buy-transaction takes place. However, liquidity may also contain other elements which are the derived topics DT1 and DT2 in the above overview. The external elements are so called related topics as they stand in relation to the subject matter or its internal elements. The arrows and lines mark the intra-, extra- and interrelated connections between the elements.

2.4 Analysis: tools and method

So far, we have analyzed the subject matter of liquidity to obtain a structured basis for further investigations. Analysis, as we understand it here is the target-oriented break up of a subject matter expressed by a term into all essential and relevant elements to clarify its meaning and significance in order to create a fruitful and consistent relation of and amongst the specified elements. The relation we identified is

- the basis how to use this term
- the basis for the definition

- the basis for the applications and
- the basis to deal with other related subject matters

In the next chapter we will further elaborate on the most simple and basic definition of liquidity, the so-called conceptual definition of liquidity.

3. The concretization of liquidity

3.1 Schematic overview and relations

In chapter 2.1 we have already outlined the most basic definition of liquidity. The broad approach for the definition was required in order to identify the constitutive elements which are necessary and in sum sufficient to define liquidity. We will come to a further concretization of liquidity now.

(1) Liquidity = the ease to convert an asset into another asset

(2) Liquidity = the ease to convert (buy or sell) an asset into cash and vice versa

(3) Liquidity for equity markets = the **ease** to buy or sell (= **convert**) **stocks** (asset 1) against **cash** (asset 2)

Liquidity definition (3) is the basis for further analysis.

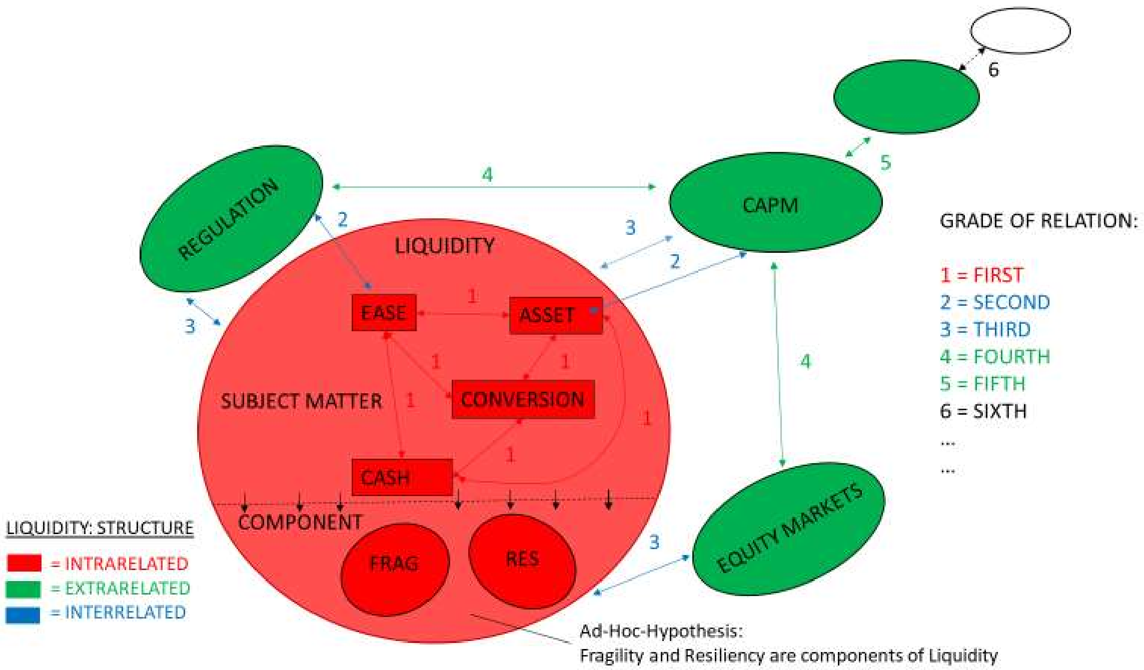


Fig. 3.1

The above schematic overview shows the four necessary and in sum sufficient elements of liquidity, i.e. ease, conversion (= buy or sell), the first asset (stock) and the second asset (cash). Also, within the scope of the subject matter are the two components fragility (FRAG) and resiliency (RES). Our assumption is, that they both are components of liquidity however not as constitutive but as derived elements. We will come back to this later. As to the external elements some examples are shown here like other equity markets, regulation policies or the Capital Asset Pricing Model (CAPM) which determines a theoretically appropriate required rate of return of an asset. Between all the different elements we can note different relationships: The sum of all first-grade relations defines the intrarelated structure of liquidity. The sum of all second-grade relations specifies the interrelated structure of liquidity as they show the connection between a constitutive element of liquidity (like the asset stock) and an external element (like the CAPM). The third-grade relations describe the interdependencies of liquidity because they show the connection between liquidity as a whole and an external element (like the CAPM). The fourth-grade relationships show the relationships between two external factors which both have an impact on liquidity. Fifth-grade relations describe indirect effects on liquidity. Sixth-grade relations are those without any impact on liquidity.

3.2 The constitutive elements of liquidity

In this chapter we will take a closer look at the different constitutive elements of liquidity. Starting with element E1 – the Ease – we must already recognize that this might be the most difficult element to explain at the current stage. We will come back to it in more detail when we turn from the pure concept view to the concrete and individual perspective. However, what we can state already here is that the ease is of major importance for the overall structure of liquidity. We can break down the ease into three sub categories to cover its whole impact, i.e. quantity, quality and time. Quantity refers to the volume traded, quality is reflected in the execution price and time is expressed by the duration of execution of all the transactions of the package. With these three sub categories, which will finally become the parameters of the liquidity formula, we can comprise all aspects of the ease at this conceptual stage of the L1-definition of liquidity. The same holds true for Conversion as the second element E2 of liquidity. Also here we can consider quantity, quality and time. Of major interest in this case is the quality which is strongly affected by price determination. The quality of conversion depends on different factors like the implemented market model, the governing rules and regulations and the design of the trading platform. Concerning the assets E3 and E4, we will concentrate on two fungibles assets: on the one hand a specific stock of the cash market and on the other hand cash as currency. We will elaborate in more detail how the different elements can be specified when we leave the conceptual stage of liquidity in Chapter 4.

3.3 Basic definition of liquidity

We can summarize the characteristics of this very basic and **conceptual definition of liquidity** as follows:

- Liquidity is the ease to convert an asset into another asset
- Quantity, quality and time matter for ease and for conversion
- The two assets to be converted have to be fungible assets
- The four elements **ease**, **conversion**, and for equity markets the asset-classes **stock** and **cash** are necessary and in sum sufficient to define liquidity
- The conversion in equity markets is carried out by the buy- and the sell-side (two-sided market)

Therefore, the conceptual definition of liquidity in equity markets is the ease to convert stocks into cash and vice versa.

4. The Liquidity-Matrix

4.1 The Matrix

“The ease to convert stocks into cash and vice versa” is the conceptual definition of liquidity in equity markets. This definition of liquidity

1. contains all the characteristic and necessary elements. These elements are
    - in sum sufficient to describe the essence and explain the meaning
    - independent from each other.
2. is the simplest possible and complete definition to guarantee consistency: As much as necessary, as little as possible! [2]
3. has the biggest scope with the most focused content.

[2] Wilhelm von Ockham: “Plurality is never to be posited without necessity.”

4. is a clear delineation and description of the subject matter in ideality.
5. describes the four-number-relation of and in between its elements.

This liquidity definition is in its nature general and abstract. However, other definitions of liquidity exist and must be specified, e.g. a concrete and individual definition of liquidity in equity markets. The following chart shows the position of the different definitions of liquidity in a matrix.

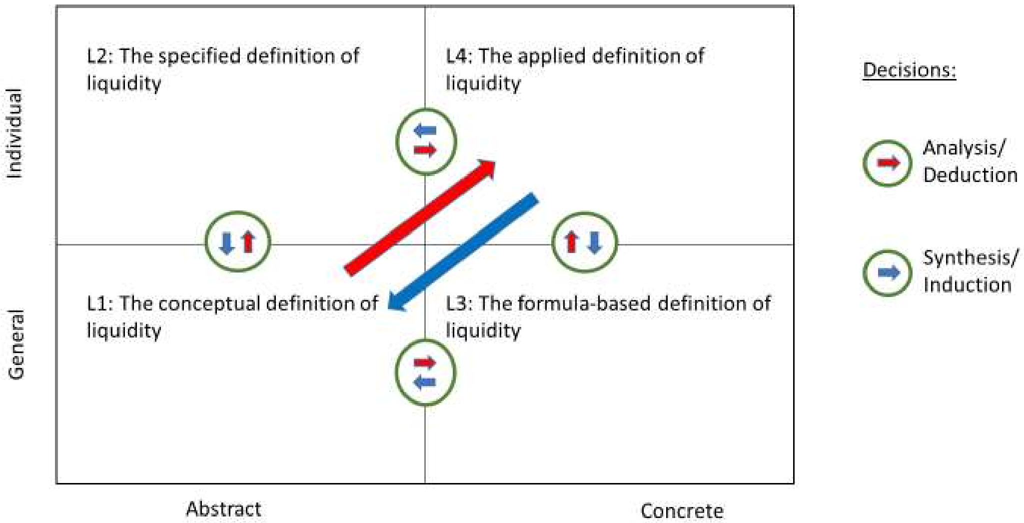


Fig. 4.1

The matrix is characterized and defined as follows:

1. There are two dimensions
   i. Abstract – Concrete
   ii. General – Individual

   which are resulting in a cluster of four possible definitions of a subject matter:

   <u>L1</u> General/Abstract

   <u>L2</u> Individual/Abstract

   <u>L3</u> General/Concrete

   <u>L4</u> Individual/Concrete

2. One **definition** for each L
3. From one L to another L only through **decision**
4. **Decisions**

   1) TOP-DOWN: Analysis/Deduction => Objective for equity markets: get highest possible turnover!

   2) BOTTOM-UP: Synthesis/Induction => Objective for equity markets: get the liquidity measured!

   Through **decisions** this matrix leads from the general, abstract level L1 to the individualized, concrete definition L4 through **intermediate steps** L2/L3 and vice versa: **Analysis and Synthesis**.
5. "The ease to convert stocks into cash and vice versa" is the L1 definition of liquidity in stock equity markets: L1 provides a complete and consistent basis and therefore a clear presentation of the subject matter liquidity for further analysis.
6. The subject matter, liquidity, is always a function of time.

4.2 Working Definitions L1 – L4

Our deliberations so far show that liquidity occurs in various ways. For our purpose we distinguish four different definitions of liquidity L1 – L4. We will now introduce some working definitions which help us to explain the decision process leading from L1 to L4 and vice versa.

	Abstract	Concrete
Individual	L2: THE SPECIFICATION LIQUIDITY = EASE TO CONVERT STOCKS AT A (STOCK) EXCHANGE INTO CASH (AND VICE VERSA)	L4: THE APPLICATION OF THE FORMULA LIQUIDITY = λ FOR A STOCK, A MARKET OR A MARKET SEGMENT AS A CALCULATED NUMBER λ_x -> THE CALCULATED λ IS THE MEASURED LIQUIDITY
General	L1: THE CONCEPT LIQUIDITY = EASE TO CONVERT STOCKS INTO CASH (AND VICE VERSA)	L3: THE FORMULA LIQUIDITY = THE **EASE** (= FORMULA FOR λ) TO **CONVERT** (BUY/SELL) **STOCKS** THROUGH A SPECIFIED PRICE DETERMINATION (CLOB, MATCHING AND EXECUTION RULES [PRICE/CURRENCY])

NOTE: LIQUIDITY IS ALWAYS A FUNCTION OF TIME

Fig. 4.2

Along the lines of our matrix we start in the general/abstract quadrant with the basic **L1-definition** of liquidity. Liquidity is neither more nor less than the ease to convert stocks into cash and vice versa. Making the first decision to concretize the concept brings us to the **L3-definition** of liquidity (general/concrete). This is the concrete formula for λ which measures the ease to convert any stock through a specified price determination process. Another decision, that we also take on the basis of L1 is the specification decision. It brings us to the individual/abstract quadrant and provides us with the **L2-definition** of liquidity. According to this, liquidity is the ease to convert stocks at a stock exchange into cash and vice versa. From this position we can again take a decision for higher concretization in order to move to liquidity as a concrete number for a specified stock under predefined premises. This brings us to the **L4-definition** of liquidity which is a λ for a stock, a market or a market segment as a calculated number λ_x. In the next chapter we will demonstrate how we move through the matrix by taking certain decisions.

4.3 From concretization and individualization to application through decisions ① - ④

The following matrix shows the different stages of liquidity as well as the respective decisions from another perspective.

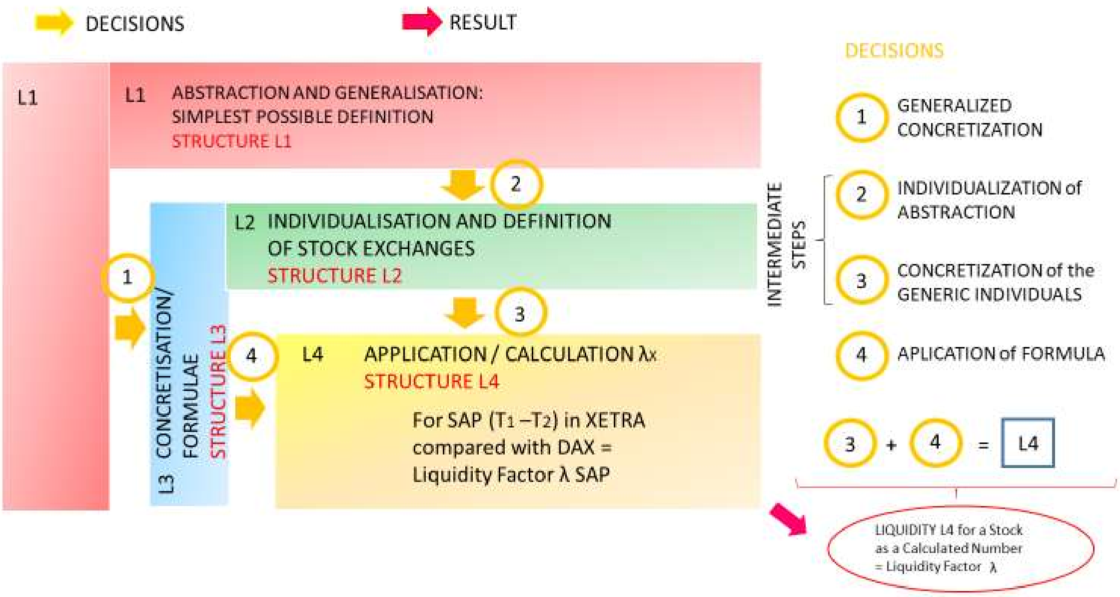


Fig. 4.3a

Starting from the conceptual stage at L1 certain decisions are required to reach the next levels L2 and L3. The first decision – indicated by the yellow arrow 1 in the chart above – leads from L1 to L3. Therefore, it is required to develop a concrete formula to measure the liquidity of stocks based on the L3-definition. This formula provides us with the liquidity factor λ which measures the liquidity of equity markets in general: the ease (E1) to convert (= buy or sell) stocks into cash and vice versa. The second decision (yellow arrow 2) individualizes our basic liquidity concept insofar as it determines some necessary parameters (e.g. the question, which asset-class [stocks, cash] to convert, which market model and market place to choose for the transaction). Having made these decisions, we are now able to apply the concrete formula to our individual transaction and we will receive the Liquidity L4 (measured by the liquidity factor λ) for a certain stock under predefined conditions. Note that it is very important to make a clear distinction between definition and decision.

The following chart makes the four decision processes even more transparent:

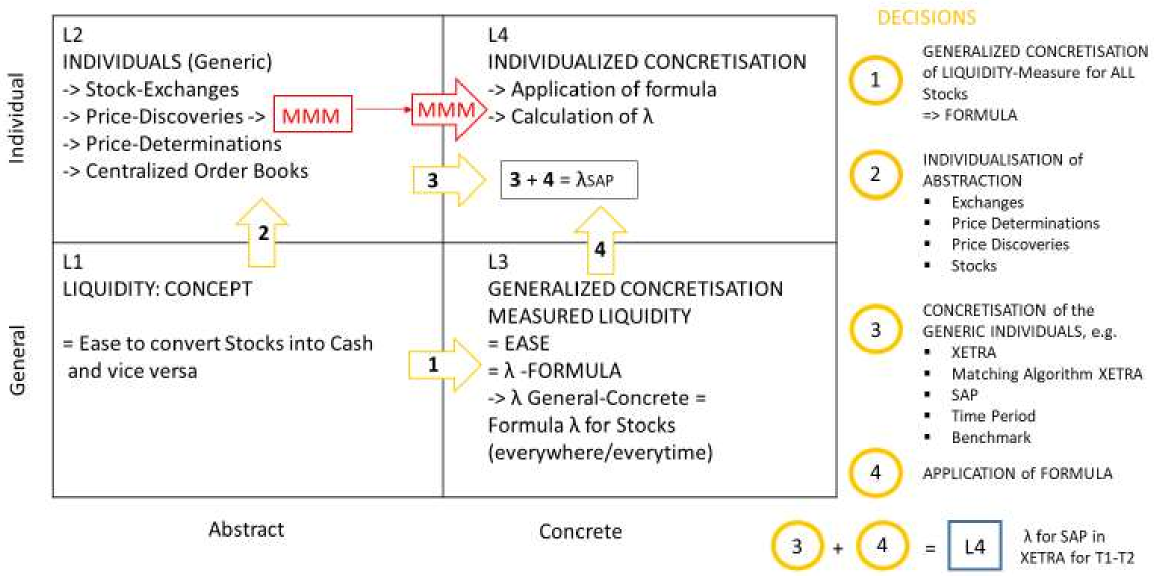


Fig. 4.3b

The **first decision** (yellow arrow 1) implies the generalized concretization of liquidity by transferring the basic definition into a concrete formula. This brings us from L1 to L3. The **second decision** (yellow arrow 2) individualizes the basic definition of liquidity insofar as it sharpens the scope of the conversion from one asset into another. In this step we decide about the asset (e.g. stocks), the market segment (e.g. blue-chips), the trading form (e.g. call auction) and certain trading parameters (e.g. order types). The bandwidth of these decisions can be estimated along the different parts of the modular market model (MMM)[3]. They are reflected in the sub-categories of quality, quantity and time which we applied to the constitutive elements E1 (ease) and E2 (conversion) in our basic definition of liquidity. The **third decision** (yellow arrow 3) focuses the scope of the conversion even more as it determines which stock (e.g. SAP, segment, benchmark) to trade on which market place (e.g. XETRA) at what time. As soon as these parameters are fixed, the concrete formula for the liquidity of a certain stock under the predefined conditions can be applied, which is the **fourth decision** (yellow arrow 4): The combined application of decision 3 and 4 leads to λ which is the measure of the

[3] A modular market model specifies the market microstructure into different modules which interact with each other but also allow for a sequential set-up from one module to the next. The modules comprise "Securities/markets", "Market segments/stocks", "Trading forms" and "Trading parameters". For further information see: Reto Francioni, Robert A. Schwartz, *Equity markets in action – the value chain, price discovery, regulation and beyond*, Springer International, Cham 2017, p. 37.

liquidity of a stock, a liquidity pool, a market segment or a customized benchmark within a predefined time period.
With this decision process, which takes us from L1 to L4, we proceed from the pure concept to the reality of liquidity by means of deduction. Of course, the decision process could also be pursued the other way round, i.e. from L4 to L1 which would be an inductive approach. We will now take a closer look at the heart of liquidity measurement: the liquidity factor λ.

5. Calculation of the Liquidity Factor λ

5.1 General aspects for the calculation of Liquidity

Liquidity per se cannot be represented by a unique, single number. The ease of a conversion always depends on three key parameters or dimensions: the time that is available for getting the cash or the stock, the price you get or you have to pay and finally the volume to be traded.

If you have plenty of time or the price would not matter at all, then the liquidity would be very high. And on the other hand, if you have a big number of stocks (compared to the free float of that stock) you have to sell in a short period of time, then it will generally be difficult to sell at a reasonable price – the liquidity is almost zero.

Mathematically, liquidity is a three-dimensional function (sic!) with the parameters PRICE, TIME and VOLUME:

(Form. 1): ***Liquidity = f (P, T, Vol).***

In practice, this is difficult to handle, and comparing and positioning with other stocks would be painful and arduous and only possible with the help of powerful computer software. That's why we are working with a comprehensible, easy applicable approach by introducing a factor which relates to the corresponding market (usually the most liquid market of a stock: the Home Market!): we call it the λ-factor.

Based on the framework and the logical systematics set up in the previous chapters, we now define the mechanism to calculate a comprehensive, realistic and comparable factor for the liquidity – comparable to the β–Factor of the CAPM-Model:

LIQUIDITY AT EQUITY <u>STOCK</u> EXCHANGES with: - Listed stocks - fully fledged electronic trading-platforms - CLOBs - Price Discoveries and Determinations - Continuous Trading / Call Auctions / Hybrids **Application of the Modular Market Model** L2	Specific Liquidity factor λ for Stock x or a benchmark with Parameters T and P L4
LIQUIDITY in Equity Markets as «The Ease to convert a Stock into Cash and vice versa. » L1	General Formula: λ (the ease to convert) as the key measure for stocks or a stock in relation to the Liquidity of the benchmark defined in L2 L3

Fig. 5.1

5.1.1 Liquidity at L1 level

The key word in the L1 definition is "Ease". How do we measure this? As we have three parameters that depend on each other, we can work with two free parameters and one that will depend on them. We could choose any of the three as the bound parameter, and we choose the Volume parameter as it is the most descriptive one. The "Ease to convert…" is translated into the question "How many shares can I convert into cash within price range P and time frame T" – or vice versa. This number will be the indicator for the ease.

5.1.2 Liquidity at L2 level

It is important to be aware that we are looking at specific situations at Stock Exchanges with the criteria defined in the table above. For the calculation of λ based on decision 3, we refer to the information contained in the corresponding Centralized Limit Order Book (CLOB). In order to get a meaningful factor, a “lively” market with sufficient trading activity driven for example by a trading-platform with continuous trading and auctions is required.[4]

For the derivation of the Formula, the following facts are important:

For a proper application of the Formula, we need to know the whole order book, actual and historical data. But in real situations, this is not necessarily the case. Therefore:

- We have to consider the consequences of reduced information.
- We can define the requirements for the data providers and market operators in order to achieve a meaningful λ.
- and we can have a look at additional opportunities to extend the information offering of market makers and information providers

5.1.3 Liquidity at L3 Level: Deriving a formula for measuring liquidity

For the calculation of λ, we have a look at the CLOB (defined by decision 3) for a certain period of time (Time Frame T). During this time, we consider all trades taking place, and in between the trades (t_{i+1}-t_i), we analyse in addition the dynamics of the order book: how fast does it recover after a trade (→ Resiliency!), and what volume is added to the order book during this period. Here we look only at increases of the order book and not reductions![5] For this analysis, we consider all orders within a price range of +-p percent of the last execution price (bandwidth of price variation). This is a dynamic and not only a static view.

And based on the working definition of L3 we have three parameters or dimensions. Therefore, there is not a single λ, but λ depends on time frame and price range: λ (T, P). We can calculate λ for a set of parameters and get as a result a discrete set of values. This is on the one hand necessary due to the multi-dimensional character of

[4] Important: the kind of conversion does not matter! It could be auction only, auction and continuous trading, market making, single or multiple markets and hybrids.

[5] This is important to prevent manipulation and fraud and shows the real potential of the market.

the liquidity, on the other hand, it is also realistic in practice: a long-term investor, a fund manager or a day trader have different perspectives on liquidity, so they can choose the λ with the appropriate parameters.

Usually, the sell side is different from the buy side. So we have to calculate λ for both sides separately.

The liquidity factor λ is a liquidity indicator for a specific share compared to the relevant market (Home Market) and the defined benchmark in general.

For practical reasons, it makes sense not to consider the total market, but a benchmark with a representative set of shares like DAX, SMI of S&P 500 or relevant baskets.

5.1.4 Liquidity at L4 level

There are big differences between different shares in one market place: some of them are traded several times a minute or even seconds, while others are traded only few times a day or even less. This is reflected in the breadth, depth and dynamics of the order book, and this has of course implications for the calculation of λ. To get feasible results, T should be chosen that n (T) > m, where n= Number of trades within T and m=minimum number of trades required[6] to get a reasonable λ.

5.2 Derivation of the formula for λ

5.2.1 Symbols

Symbol	**Meaning**	**Comments**
x	Specific Share	
T	Time Interval	
P	Price Range	± % of last executed price (LEP)

[6] The minimum number of trades required has to be determined empirically.

G	Benchmark where x is part of	Can be the total market, an index or a basket
n(T)	Number of Trades within Time T	A basic parameter for the calculation. Can be varied.
$\Delta CLOB^P$	Increase of CLOB volume within P range	We consider only INCREASES of the CLOB during (t_i, T_{i+1})
(t_i, t_{i+1})	Time segment between two trades	
FF	Free Float of share x or total market G	

Tab. 5.2.1

5.2.2 Elements of the formula

There is no single indicator that could represent a liquidity factor: aspects like spread, attributes of the order book or resiliency are **only partial aspects** of the topic. And as the characteristic for each share can differ very much (companies with big and with small Market Cap, many trades per minute vs. few per day etc.), we have to consider these different aspects and include them in one comprehensive, general formula in order to facilitate a comparison of different shares.

There are two main influence factors for the formula:

One is the volume of trades executed within time period T. The more a share is traded, the higher the liquidity. This term looks like

(Form. 2):
$$\sum_{i=0}^{n(T)} Trades_x$$

In addition, the dynamics of the order book is the second main factor for the calculation. The dynamics of the increase of the order book between two trades is also affecting the liquidity factor (→ Resiliency!):

(Form. 3):
$$\sum_{i=0}^{n(T)} \Delta CLOB^P_{x(t_{i,}\, t_{i+1})}$$

As we want to express liquidity as a relative number and not as an isolated value, we also have to set these parameters in relation to the free float FF_x and the corresponding values for the relevant (home) market.

Description/Specification/Key Parameters	**Notions**
Volume of Trades of a specific share (x) or total market (G) within a Time Period	$\sum_{i=0}^{n(T)} Trades_x$
Change of Order Book (CLOB) within Price Range based on executed Prices (EP): 1 Buy-Side 2 Sell-Side	$\sum_{i=0}^{n(T)} \Delta CLOB_{x(t_{i,}\, t_{i+1})}^{P}$
Total available share-volume: Free float vgl. -> 3.1 Origin of Liquidity =< FREE FLOAT GROSS	FF_x
Total Market: Consolidated Entirety of Market segment (e.g. represented by DAX for German Blue Chips, SIX for Swiss Blue Chips etc.) or basket	

Tab. 5.2.2

5.2.3 Basic Formula

The basic idea is to determine the number of shares that can be bought/sold in a defined time and price bandwidth. The higher this number is, the easier it is to buy or to sell this asset → the higher the EASE! This number is set in relation to the free float of the stock and the corresponding factor of the overall market segment.

- **Step 1**: We analyze the trades and the changes of the CLOB during a defined time frame T. Each trade triggers a new event with a specific trade volume (number of shares multiplied by the execution prize). And between two trades, we consider increases of the order book within a bandwidth of n percent of the last execution price.

Note: the order book changes indicate the degree of resiliency! Resiliency is **not** representing Liquidity per se, but it is only one component. This is reflected in the formula.

- **Step 2**: The sum of the trades and the CLOB increases during T represent the potential volume that could be traded within a price range P. This number is divided by the free float of the stock, and the result expresses the percentage of available shares that could be acquired or sold within T at a price range P.

(Form. 4):

$$\frac{(\sum_{i=0}^{n(T)} Trades_x + \sum_{i=0}^{n(T)} \Delta CLOB_{x(t_i, t_{i+1})}^{PS})}{\boldsymbol{FF_x}}$$

are you missing a term that captures the initial STOCK of the order book here?

This factor represents the percentage of shares in relation to the free float that are potentially available during T.

- **Step 3**: Finally, by dividing this factor by the corresponding factor for the total market/segment (or an index representing this market)

(Form. 5):

$$\frac{(\sum_{i=0}^{n(T)} Trades_G + \sum_{i=0}^{n(T)} \Delta CLOB_{G(t_i, t_{i+1})}^{PS})}{\boldsymbol{FF_G}}$$

we get the liquidity factor λ. This is an important characteristic of λ: thereby, we eliminate general influence factors outside of the stock by normalizing it to the overall market.

The final result looks as follows:

(Form. 6): Sell Side: $\boldsymbol{\lambda_x^S(T,p)} =$

$$\frac{(\sum_{i=0}^{n(T)} Trades_x + \sum_{i=0}^{n(T)} \Delta CLOB_{x(t_i, t_{i+1})}^{pS}) * \boldsymbol{FF_G}}{(\sum_{i=0}^{n(T)} Trades_G + \sum_{i=0}^{n(T)} \Delta CLOB_{G(t_i, t_{i+1})}^{pS}) * \boldsymbol{FF_x}}$$

(Form. 7): Buy Side: $\lambda_x^B(\boldsymbol{T}, \boldsymbol{p}) =$

$$\frac{(\sum_{i=0}^{n(T)} Trades_x + \sum_{i=0}^{n(T)} \Delta CLOB_{x(t_i, t_{i+1})}^{pB}) * \boldsymbol{FF_G}}{\sum_{i=0}^{n(T)} Trades_G + \sum_{i=0}^{n(T)} \Delta CLOB_{G(t_i, t_{i+1})}^{pB}) * \boldsymbol{FF_x}}$$

Important: Sell Side ≠ Buy Side! So we have two different values: λ_S and λ_B.

The average of the two values can be used as a general liquidity factor:

(Form. 8): $$\lambda_T = \frac{\lambda_S + \lambda_B}{2}$$

Alternative approach:

Instead of choosing a fixed bandwidth P (as percentage of last execution price) to calculate the CLOB changes during an interval between two trades, we could consider all orders in the order book, but weigh them depending on the delta of the execution price to the offer price of the order.

As weighting function, we use an inverse square function – the inverse value of the square of the range between execution price and the price of the corresponding value in the order book plus 1:

$$WF = 1 + \left(\frac{POB - EP}{H\ x\ EP}\right)^2$$

H: the leverage parameter (at the price of H percent plus or minus to the EP, the weight value will be 0,5).

POB: the corresponding price in the order book.

In the term

$$\boldsymbol{\Delta} CLOB^{PS}_{x(t_{i,}\, t_{i+1})}$$

in the above formulas, we use all the orders in the CLOB instead of using only those within price bandwidth p. However, the order price will be adjusted by the weight factor WF:

$$WV_{CLOB,tj} = \sum_{i=1}^{n} \frac{POBi}{1 + \left(\frac{POBi - EP}{H\ x\ EP}\right)^2}$$

Where

N: number of orders in the CLOB at time t_j

$WV_{CLOB,t}$: weighted value of orderbook at time t_j.

The weighted value WV will be calculated each time when there is a change in the CLOB entries, and $\boldsymbol{\Delta} CLOB_{x(t_{i,}\, t_{i+1})}$ is the biggest change of WV within timeframe ($t_{i,}$ t_{i+1})

Instead of the square approach to the weight function, we could also choose a linear approach. To decide which of these alternatives to choose, we will rely on empirical results based on historical CLOB data to determine the model that fits best the needs of practical portfolio management.

5.2.4 Remarks on the formula

The magnitude of λ is an indicator: if this factor is 1, it behaves exactly like the market. Higher values mean higher liquidity.

The reference price used for the calculation (the last executed price EP) is not static, but it floats over time. Each time a trade is executed, the new reference price is set to the new EP. By doing this, we eliminate the influence of volatility which is contained in the ß of the CAPM. So we have two independent dimensions: β and λ. They represent an orthogonal coordinate system.

is orthogonality obvious?

Also important: we consider only **increases** of the CLOB during T. Otherwise, this factor could be manipulated easily by adding and removing orders and thereby pretending a much higher resiliency and liquidity.

And like ß, λ is an important factor to determine the price one is willing to pay for an asset: it justifies a surcharge or a discount to the technical value of a share.

The ratio for the number of shares available within the defined parameters is set in relation to the free float and not to the Market Cap!

The CLOB does not show virtual orders[7] (see Slide 4.3 and 4.3.1). This implies these orders cannot be considered directly in the formula. But as we observe a longer time period, the effect of these virtual orders will be reflected in the results (resiliency).

[7] The virtual liquidity of a stock (e.g. SAP) consists of all conditional orders outside the CLOB of an exchange (e.g. XETRA, Frankfurt) with the following characteristics:

i. These orders are either in the OB of the member of the exchange or in the OB of an investor or trader with direct access to a member of the exchange.
ii. These orders are affected by the price of the respective stock determined and disseminated by this exchange.

The price information of the exchange triggers the conversion of these orders into market or limit orders and the subsequent routing into the (SAP-) CLOB of the exchange (XETRA, Frankfurt).

5.3 Practical application of λ

If one has access to the data, he can calculate λ with every parameter he is interested in. But as detailed data for all CLOB entries is usually not available for everybody, and as it is sufficient in most cases to have an approximate value for λ in order to compare it to the benchmark set, the basic idea for the practical application of the λ-Concept is to calculate a set of characteristic λs for each stock with a few predefined values for T and p%: we could choose for instance T with short, mid- and long-term-values, and a small, medium (?) and wide range for p%. If we take two values for P and three values for T, we get 6 results for λ, and if we use three values for P, then we get 9.[8]

In order to facilitate the comparison between different stocks, these values must be the same for all stocks. They are defined once and serve as a basis for an additional valuation of a stock. By doing this, we can reduce the complex three-dimensional function which can hardly be handled in practice by a simple set of few representative values. This calculation can be done by market place operators or service companies, and as this set of λs are based on the same parameters, they allow a fast and easy comparison of a single stock with the benchmark set of stocks.

6. Analysis of λ

The technical and technological possibilities that are available today provide many new opportunities, not only for a complex, comprehensive calculation of λ, but also for a detailed analysis of λ over longer time periods. Today, we have "unlimited" storage capacity, Big Data technology, number crunching and fast data communication all over the world. We can easily collect data of thousands of stocks with trades and order book transaction in seconds or even faster, we can store it almost unlimited in quantity and time, and we can make all kinds of calculation and analysis within a very short time – all this with inexpensive equipment. This allows us to do additional analysis and comparison with all kinds of parameters, formulae and data sets.

This detailed analysis allows to make additional conclusions regarding a specific stock, a basket or the market as a whole. So, based on a complete data set of all CLOB

[8] The decision about the proper determination for these values should be made based on an empirical approach by calculating a huge set of λs based on historical data.

entries[9] of several month or even years, we can calculate an almost infinite number of values for the λ-function in order to make this further analysis. Again - not for the calculation of λ itself (here we restrict ourselves to very few values!), but for deepened analysis purpose.

This is different from the analysis of the basic chart of a stock, as with the λ formula, we are now on an aggregated level with the static and dynamic nature of the Order Book. We can get more sophisticated results by working on this meta-level. The order book always represents the actual offers at a certain point in time and reflects **only** the actual orders of a stock. The virtual orders are not visible in the CLOB as this represents a static view on the offering. But if we look at time series, the changes of the CLOB within a certain time period, then we can see the dynamic picture showing the manifestation of the virtual orders deployed from sources **outside** the Order Book. This approach allows us to recognize important aspects of the order structure that are normally hidden if we look at the available data in the usual way!

This further analysis is not part of this paper but will be explicated in an additional paper covering these aspects. Here we give some short examples to give an idea what can be done:

6.1 Sell Side vs Buy Side

Important: Sell Side ≠ Buy Side! So we have two different values: λ_S and λ_B.

The average of the two values can be used as a general liquidity factor:

(Form. 9):

$$\lambda_T = \frac{\lambda_S + \lambda_B}{2}$$

For further analysis, we can look at the delta of the two values:

(Form. 10):

$$\Delta\lambda_{S,B} = \lambda_S - \lambda_B$$

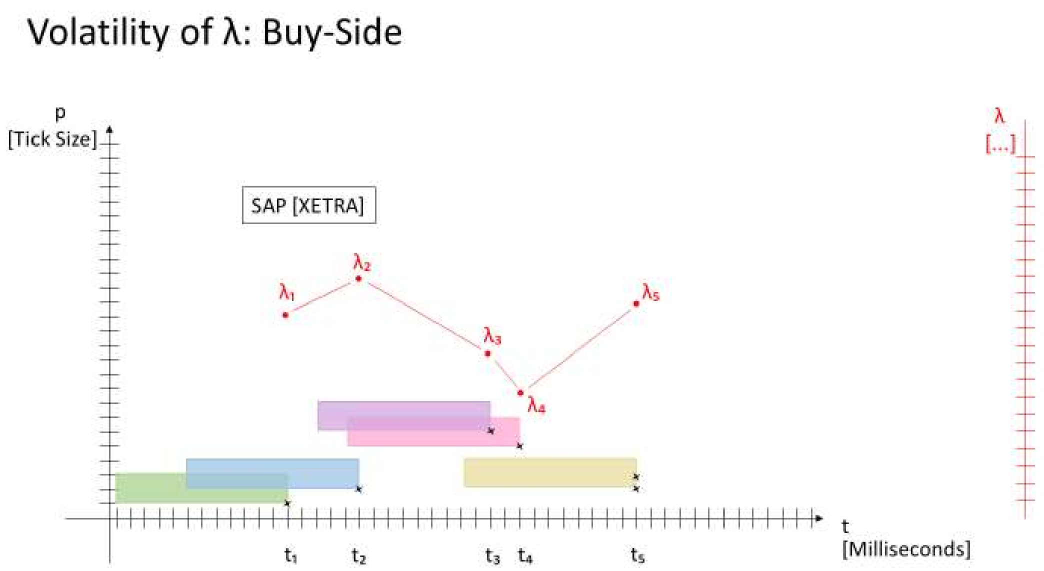
Key Parameters for the interpretation are the magnitude of Δ and the sign of the difference (+ or -). Important is the trend over time. There will always be a small difference

[9] We can even use multiple CLOBs if a stock is listed or traded at different market places.

in one or the other direction, but if one side is dominating over a longer period, this is an indicator of a basic order book imbalance.

6.2 Volatility of λ

λ is usually not constant, but it will vary over time[10] → we have a volatility of λ!



If λ is quite constant over time (a straight line), then the liquidity of the share has the same fluctuation as the market. A detailed analysis of the above function allows us in-depth analysis of the characteristics of the share.

7. Related Formulae

Aspects like Spread, Resiliency, Fragility, Immediacy and Breadth and Depth of the Order Book CAN NOT be direct measures for the Liquidity. They only give an isolated

[10] Here we don't talk about T, the time window for the calculation of λ, but about t as a specified point in time (year, month, day, and daytime) when the calculation of λ starts.

indication, but can never be the denominator for λ on its own![11] Nevertheless they are included as necessary parts in our formula for λ. If we adapt our formula derivation in order to isolate other indicators for a share we get the following derived formulae:

7.1 Resiliency

In order to measure the resiliency of a stock in the CLOB, we use some of the term of our basic formula: resiliency is about the ability of a stock price and the available volume to recover after a trade. So we sum up all the order book increases and set this in relation to the total aggregated trade volume within this period:

(Form. 11):

$$\boldsymbol{Res_x^{B,S} = \frac{\sum_{i=0}^{n(T)} \Delta CLOB^{p}_{x(t_{i},\, t_{i+1})}}{\sum_{i=0}^{n(T)} Trades_x}}$$

Meaning of values: 0 = no resiliency at all

1 = perfect resiliency

7.2 Fragility

By definition, Fragility is the vulnerability of a financial system (a market or part of a market) to a financial crisis like information shocks or other events that affect whole industries or markets.

Here we look at the singular fragility of a stock, defined as the sensitivity of a stock offering over time. We can derive this from the resiliency: if the resiliency does not fluctuate too much over time t, then fragility is low:

(Form. 12):

$$\boldsymbol{Frag_x = \sigma\ (_{i=0}^{n(T)} Res_{x,i})}$$

σ = Variance

i = Individual Trades (Price Discovery)

[11] This will be shown in the next chapter.

Alternatively, we could look at the behavior of a single stock in the event of a general crisis. Does it react like the average of the market or does it have a significantly different behavior than the market?

7.3 Immediacy

The immediacy reflects the offering (trading volume) at a specific moment → immediate. How many shares can I get **right now**? Of course, this depends on the price I am willing to accept, so immediacy has the price range p% as a parameter:

(Form. 13): $$\sum_{i=EP}^{i=EP+p\%*EP} Entry(CLOB^{i}_{x(t)})$$

8. Impact of λ on the CAPM

The CAPM is used as a model and the standard approach to determine the expected return of a security or a portfolio. Its prominent characteristic is the parameter β which reflects the systemic risk of the volatility of a share as a factor. The higher the volatility, the higher the risk premium and vice versa:

(Form. 14): $$E(R_i) = R_f + \beta_i\,(E(R_m) - R_f)$$

where:

$E(R_i)$ is the expected return on the capital asset

R_f is the risk-free rate of interest

$E(R_m)$ is the expected return of the market

As a characteristic, liquidity has a comparable but inverse influence on the valuation of an asset relative to volatility: the lower the liquidity, the higher the risk premium and vice versa. This is one of the key benefits of the work presented in this paper: the systematic approach combined with a concrete formula allows to calculate specific values for specific assets. And these values can be used to extend the CAPM formula:

In analogy to β, we use λ to calculate a risk premium based on liquidity, thus adding an additional characteristic and dimension to risk and return:

(Form. 15): $\frac{1}{\lambda i} \times \Phi_L \times (E(R_m) - R_f)$

Φ_L represents the market price for Illiquidity. This factor will be determined within the scope of further analysis.

The following properties of β and λ are important:

- based on our definition of λ and as mentioned before, the liquidity definition is decoupled from the influence of the volatility as the reference price in the formula is not fixed to the execution price and the beginning of period T, but it is updated to the actual execution price at each trade. Otherwise, volatility would be considered twice in the extended CAPM formula.
- The period during which β and λ are calculated plays an important role. There are many empirical analyses on β that show significantly different values depending whether one takes for example 255 trading days, 2 years or 5 years. For λ, empirical values are not yet available (this is one of the next steps in pursuing the work described in this paper), but we expect a similar effect on λ.
- Due to the fluctuation of the two parameters, depending on the underlying time period, it is important to choose the same time interval for both of them in order to get a meaningful, consistent result.

The CAPM model works well with values close to 1. The same can be expected to hold true for λ: as it constitutes the denominator of the equation, the formula does not give meaningful results with values close to zero. Or in other words: a very low value means that the asset is not liquid at all, and in this case, the formula is not applicable, and an asset manager for example has to use other mechanisms to determine the expected return for his investment.

For "exotic" values of β, a lot of theoretical and empirical work has been done to use β in a reasonable manner by investors or portfolio managers. An important source for this is the empirical work based on real numbers of the past. Similar consideration must be done for λ and particularly for the combination of the two factors. In the future, empirical results calculating λ on the basis of actual order book data will lead us to more concrete insights on how λ behaves in the valuation of an asset.

9. Discussion

9.1 Reliability and manipulation

As soon as one defines new key indicators, we have to be aware of the risk of manipulation – even more if the calculation and derivation is transparent and known. One could e.g. manipulate the order book. For instance, one could put new orders in the order book after each trade to balance the reduction of the CLOB entries very fast and by this pretending a high resiliency of the share. Or he just could inflate the CLOB with additional offers to increase the breadth and the depth. If you would use now these indicators as the defining factor for Liquidity, you would get a wrong, manipulated result.

With our approach, this will not happen as we look not only at one parameter, but our λ is based on the trades as well as on the changes of the order book, and we look at the dynamic picture and not only a static view. This again is a proof that these indicators cannot serve as a measure for the Liquidity!

9.2 Outlook

This paper only reveals the tip of the iceberg concerning liquidity in equity markets and its measurement. Further investigations need to be conducted and elaborated in another context. This will also include the reconciliation of existing theories regarding this topic as well as the conduction of empirical studies and calculations.