Multi-Robot project

Stubborn consensus and stubborn rendez-vous

Paradiso Emiliano, 1940454

Piccione Brian, 1889051

Pisapia Vittorio, 1918590

Abstract

This work investigates consensus strategies that are robust to outliers, focusing on two approaches: stubborn consensus and weighted consensus. The first one converges to the harmonic mean of the agents' initial conditions, thereby effectively rejecting outliers, but its applicability is limited to positive states. Conversely, weighted consensus leverages weights inversely proportional to the agents' initial conditions, enabling robustness to outliers even in the presence of mixed states. Although it achieves slower convergence due to the smaller connectivity eigenvalue, it ensures reliable performance under more general conditions.

GitHub link: https://github.com/HatoKng/Stubborn-consensus-protocol

Contents

1 Related works

Consensus algorithms have been extensively studied in the context of multi-agent systems and distributed control. Under the standard consensus protocol [1], a **connected** interaction graph guarantees that all agents asymptotically reach agreement on a common state. In undirected graphs, and in directed balanced graphs that contain a rooted out-branching, the limit equals the arithmetic mean of the initial states. Typical applications include rendezvous tasks, where agents converge to the same position.

While this classical setting is elegant and well understood, it is also sensitive to the presence of heterogeneities and **outliers**. In practical scenarios, not all agents are equally cooperative or reliable: some nodes may have fixed stances (stubborn agents), while others may produce anomalous measurements due to faults, adversarial behavior, or noise. When such outliers are present, the arithmetic mean can be significantly biased, leading to consensus states that do not accurately reflect the majority of agents' intentions.

In this perspective, a natural question arises: can consensus protocols be adapted to neglect large outliers? Two main approaches have been investigated in this work. The first is to introduce a **weighted interaction graph**, where the edge weights depend on the initial conditions of the agents. This method attenuates the influence of agents with extreme values, but the resulting agreement is only semi-global, as it depends on the specific distribution of states. The second approach is the so-called **stubborn consensus**, where convergence is shown to the harmonic mean of initial conditions, yielding a global solution when all agents have positive states.

2 Proposed Approaches

In this work, we examined the previously discussed methods in order to make the consensus protocol work reliably even in the presence of outliers. The strategies considered are **weighted consensus** and **stubborn consensus**.

2.1 Weighted consensus

The objective is to develop a consensus protocol for connected graphs[1] that enables all agents to converge to a **weighted average** of their initial states.

To this end, we define the **non-symmetric weighted adjacency matrix W** as:

$$W(i,j) = A(i,j) \cdot w(j) \tag{1}$$

where A is the adjacency matrix of the graph and w is the weight vector, chosen as the absolute value of the inverse of the agents' initial conditions. This choice downweights the influence of agents with large initial values, thereby improving robustness to outliers.

Each row of W is then normalized to ensure the matrix is **row-stochastic**.

Using this weighted matrix, we define the corresponding graph Laplacian as:

$$L = D - W \tag{2}$$

where D is the diagonal matrix of row sums of W. Since W is row-stochastic, all row sums equal 1, and therefore D coincides with the identity matrix and $L\mathbf{1} = 0$.

The continuous-time consensus protocol is then modeled by the linear differential equation:

$$\dot{x} = -Lx \tag{3}$$

with solution:

$$x(t) = e^{-Lt}x(0) \tag{4}$$

A key assumption, is that $\text{rank}(L) = N - 1$, where N denotes the number of agents. Equivalently, the associated interaction graph admits a **rooted-out branching**. Under this assumption[2], the spectrum of L satisfies:

$$\lambda_1 = 0 \quad \text{and} \quad 0 < \Re\{\lambda_2\} \leq \cdots \leq \Re\{\lambda_N\} \tag{5}$$

[1]For directed graphs the presence of a rooted-out branching is required.

[2]It is worth noting that, in the case of undirected graphs, the existence of a rooted out-branching is equivalent to the graph being connected. In the case of directed graphs, Gershgorin's theorem applies.

And $\mathbf{1}$ is the unique vector spanning the right null-space of L.

In general, L is non-symmetric, hence diagonalizability is not guaranteed. However, one can adopt the Jordan decomposition of L:

$$L = PJ(\Lambda)P^{-1} \tag{6}$$

where $J(\Lambda)$ is the Jordan canonical form of L, the columns of P are the right eigenvectors p_i, and the rows of $Q = P^{-1}$ are the corresponding left eigenvectors q_i^T.

Substituting this decomposition, the solution becomes:

$$x(t) = Pe^{\Lambda t}P^{-1}x(0) = p_1 \cdot q_1^T e^{-J_{\text{block}}(\lambda_1)t} x_0 + \sum_{i=2}^{N} \Big(P_i \, e^{-J_{\text{block}}(\lambda_i)t} \, Q_i \Big) x_0 \tag{7}$$

Since p_1 and q_1 are the right and left eigenvectors associated with $\lambda_1 = 0$, and $p_1 = \mathbf{1}$ (the only vector spanning the right null-space of L), we deduce that the asymptotic value is:

$$\lim_{t\to\infty} x(t) = (p_1 \cdot q_1^T)x_0 \tag{8}$$

If we further impose the normalization condition $q_1^T p_1 = 1$, and introduce π as the normalized version of q_1 such that:

$$\pi^T \cdot \mathbf{1} = 1 \tag{9}$$

Moreover,

$$\pi^T L = \pi^T(I - W) = 0 \quad \implies \quad \pi^T W = \pi^T \tag{10}$$

Thus, π corresponds to the stationary distribution of W, and the consensus value is

$$\lim_{t\to\infty} x(t) = (\pi^T x_0) \cdot \mathbf{1} \tag{11}$$

This approach guarantees convergence to a weighted sum of the initial states, with the weights π reflecting the influence of each agent. By designing the weights $w(j)$ as the inverse of the initial values' magnitudes, the protocol naturally reduces the impact of extreme values, thus enhancing robustness against large outliers.

2.2 Stubborn consensus

The goal is to design a consensus protocol for connected graphs such that all agents converge to the **harmonic mean** of their initial conditions instead of the arithmetic mean as in the standard consensus protocol. The proposed approach consists of introducing a **reciprocal transformation**:

$$y_i(t) = \frac{1}{x_i(t)} \tag{12}$$

Then a standard consensus protocol is applied to the transformed variables y_i. The dynamics of the i-th component of y can be written as

$$\dot{y}_i(t) = \sum_j (y_j - y_i) \tag{13}$$

or, in matrix form,

$$\dot{y}(t) = -Ly(t), \tag{14}$$

where $L = D - A$ is the Laplacian matrix of the graph. Since $y_i = 1/x_i$, differentiating with respect to time yields

$$\dot{y}_i = -\frac{1}{x_i^2}\,\dot{x}_i \tag{15}$$

Substituting (15) into (13) gives the dynamics for x_i:

$$\dot{x}_i = x_i^2 \sum_j \left(\frac{1}{x_i} - \frac{1}{x_j}\right) \tag{16}$$

From (14), the solution for $y(t)$ is exponential:

$$y(t) = e^{-Lt} y(0) \tag{17}$$

Rewriting[3] it in terms of x:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\,\frac{1}{x(0)}\right]_i} \tag{18}$$

[3]Where $[\cdot]_i$ represents the i-th component of $[\cdot]$.

Since our goal is to make y converge to the arithmetic mean, we proceed by considering two separate cases:

- **Undirected graphs:** In this case, the Laplacian L is symmetric and positive semidefinite. Hence, it admits the diagonalization by an orthogonal matrix U:

$$L = U\Lambda U^T, \qquad \Lambda = \mathrm{diag}(\lambda_i), \tag{19}$$

and the matrix exponential can be written as

$$e^{-Lt} = Ue^{-\Lambda t}U^T \tag{20}$$

Thus,

$$e^{-Lt}\frac{1}{x(0)} = \sum_{i=1}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} = u_1 u_1^T \frac{1}{x(0)} + \sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} \tag{21}$$

Since the graph is connected, $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_N > 0$. Therefore, as $t \to \infty$:

$$\sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} \quad \longrightarrow \quad 0 \tag{22}$$

Being $u_1 = \frac{1}{\sqrt{N}}\mathbf{1}$, the first term in (21) becomes

$$u_1 u_1^T \frac{1}{x(0)} = \frac{1}{N}\mathbf{1}\sum_{k=1}^{N}\frac{1}{x_k(0)} \tag{23}$$

Hence, for $t \to \infty$:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\frac{1}{x(0)}\right]_i} \quad \longrightarrow \quad \frac{1}{\frac{1}{N}\sum_{k=1}^{N}\frac{1}{x_k(0)}}, \qquad \forall i, \tag{24}$$

which corresponds to the **harmonic mean** of the initial conditions.

- **Directed graphs:** In this case, L is no longer symmetric, yet the property $L\mathbf{1} = 0$ still holds. If, in addition, the interaction graph is **balanced**, then also $\mathbf{1}^T L = 0$.

  Assuming that the graph admits a **rooted-out branching** we can, by analogy with the weighted consensus case, adopt the Jordan decomposition of L and write:

$$e^{-Lt}\frac{1}{x(0)} = p_1 q_1^T e^{-J_{\mathrm{block}}(\lambda_1)t}\frac{1}{x(0)} + \sum_{i=2}^{N}\left(P_i\, e^{-J_{\mathrm{block}}(\lambda_i)t}\, Q_i\right)\frac{1}{x(0)} \tag{25}$$

Here, both p_1 and q_1 lie in span($\mathbf{1}$). By imposing the normalization condition $p_1^T q_1 = 1$, we obtain for $t \to \infty$:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\frac{1}{x(0)}\right]_i} \longrightarrow \frac{1}{\frac{1}{N}\sum_{k=1}^{N}\frac{1}{x_k(0)}}, \qquad \forall i, \tag{26}$$

which again corresponds to convergence towards the **harmonic mean** of the initial conditions.

2.3 Comparison

This section presents a comparison of the proposed approaches with the standard case across different scenarios.

2.3.1 EXPERIMENT I.

- **Agent dimension:** 1D.
- **Setup:** 8 agents (all positive values) with 3 outlier.

$$x_0 = \begin{bmatrix} 1 & 1.2 & 0.9 & 1.1 & 1.3 & 27 & 23 & 19 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** fully connected, undirected graph, (Figure 1).

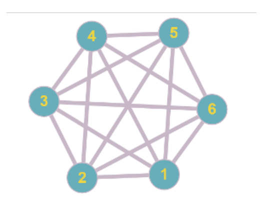


Figure 1: Graph used in Experiment I.

From the standard consensus plot in Figure 2, it can be observed that in the presence of a considerable number of outliers[4] (3 out of 8 agents), the performance of the standard consensus degrades significantly. The outliers strongly influence the arithmetic mean of the group(≈ 9.3125), leading to a noticeable shift in

[4]The inliers are clustered near zero, while the outliers take on larger values

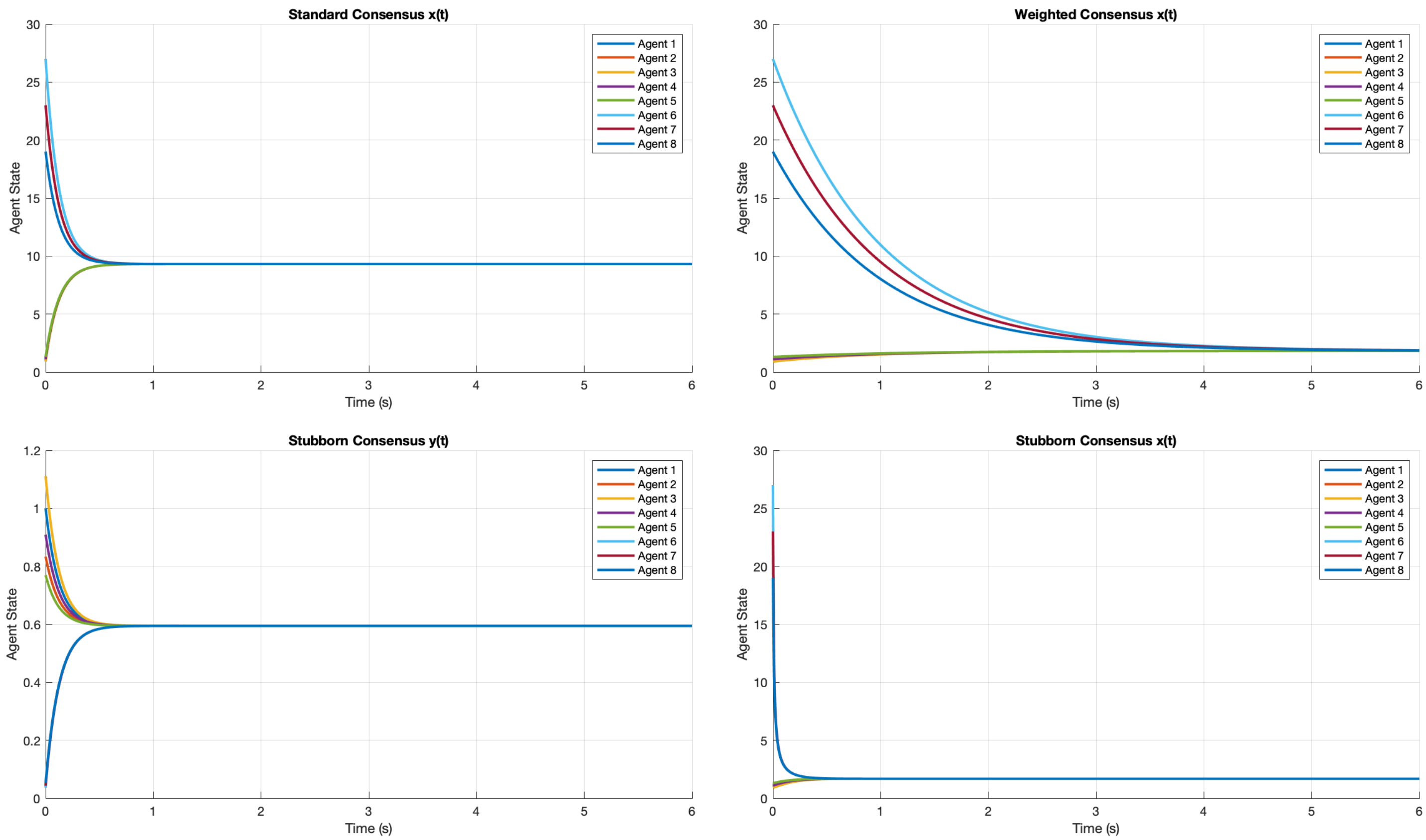


Figure 2: Consensus plots for Experiment I.

the convergence value. In contrast, the stubborn consensus effectively ignores the outliers (agents 6,7 and 8), converging instead to the harmonic mean (≈ 1.6821). Similarly, the weighted consensus also mitigates the effect of the outliers by assigning weights inversely proportional to the agents' initial values, thereby diminishing the influence of the large values. The convergence value is ≈ 1.8226, significantly closer to the inliers compared to the standard consensus case.

It is interesting to analyze the convergence rate of the different algorithms, which is determined by the connectivity eigenvalue, namely the second smallest eigenvalue of the Laplacian matrix. In the standard and weighted consensus cases, these values are respectively $\lambda_2^{std} = 8.00$ and $\lambda_2^{wgt} = 1.0083$. As predicted by theory, the standard consensus exhibits faster convergence due to its larger eigenvalue (leading to a quicker exponential decay), while the weighted consensus converges more slowly because of its smaller eigenvalue.

In the stubborn consensus case, the Laplacian matrix coincides with that of the standard consensus, and thus the convergence rate of the transformed states (y_i) matches the standard case. Once these transformed states converge to the arithmetic mean of their initial conditions, the original states necessarily settle at the consensus value corresponding to the harmonic mean. However, since the transient dynamics of the agents in the stubborn consensus are governed by equation (18), the resulting convergence rate does not exactly follow that of the standard consensus; in fact, it is observed to be faster.

2.3.2 EXPERIMENT II.

- **Agent dimension:** 1D
- **Setup:** 6 agents (positive and negative values) with 2 outlier.

$$x_0 = \begin{bmatrix} -1 & -1.2 & -0.9 & -1.1 & 0.3 & 8 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** fully connected, undirected graph (Figure 1).

As in the previous case, the standard consensus (Figure 3) converges to the arithmetic mean (≈ 0.6833), which is strongly affected by the outliers. However, the harmonic mean is not well-defined when both positive and negative values are present, making it unsuitable in these scenarios. In such cases, the stubborn consensus converges to ≈ -15.1821, and during convergence the outliers with opposite sign introduce discontinuities. In contrast, the weighted consensus remains robust even in the presence of negative agents, converging to -0.3688 by effectively discounting the outlier and thus providing a stable outcome. The same considerations regarding the convergence rate discussed in *Experiment I* still apply here.

2.3.3 EXPERIMENT III.

- **Agent dimension:** 1D

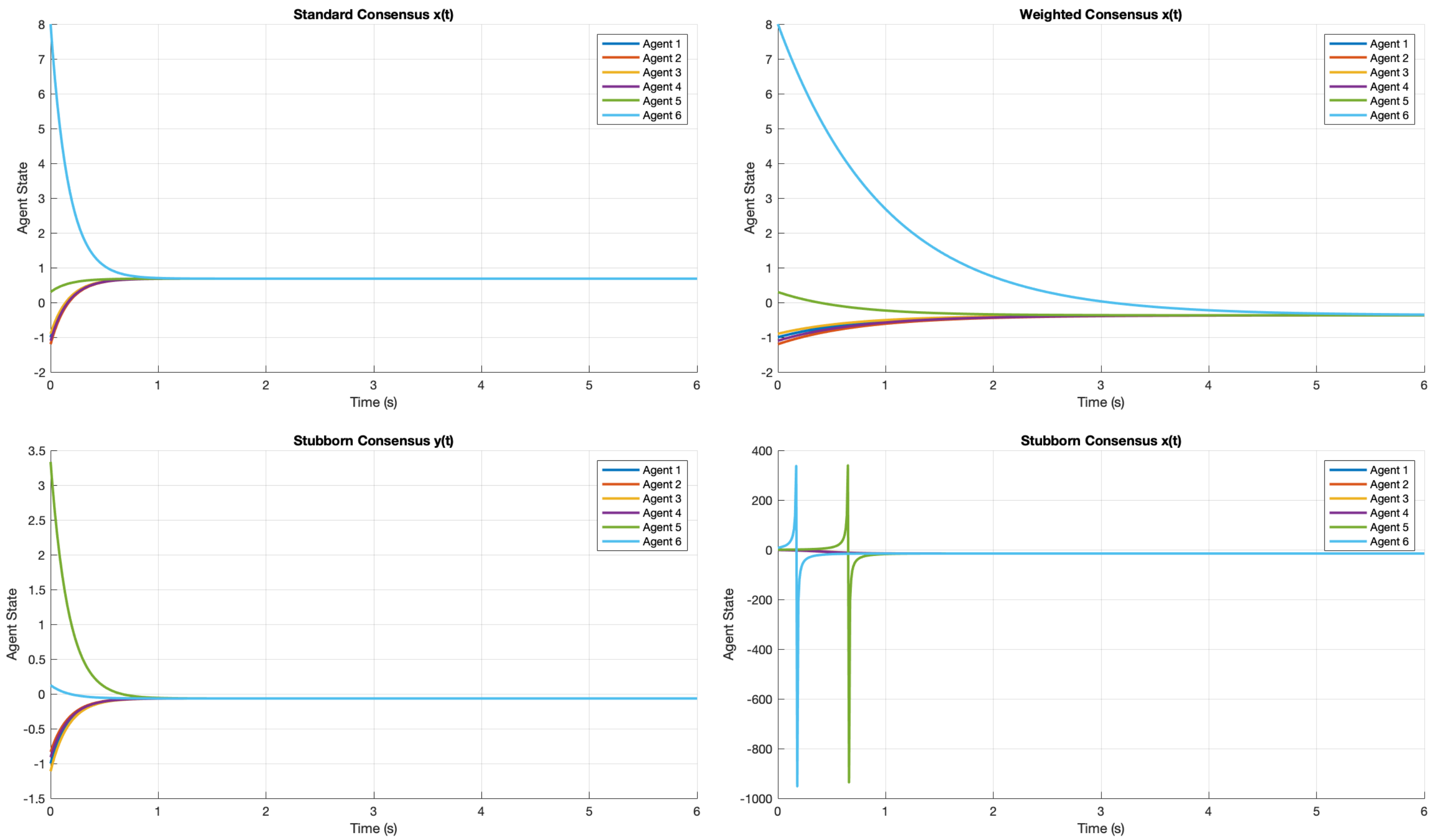


Figure 3: Consensus plots for Experiment II.

- **Setup:** 6 agents with 1 outlier.

$$x_0 = \begin{bmatrix} 0.5 & 0.2 & 0.4 & 0.1 & 0.35 & 20 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** balanced, directed graph (Figure 4).

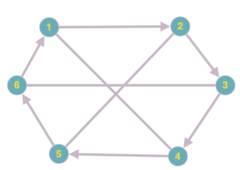


Figure 4: Graph used in Experiment III.

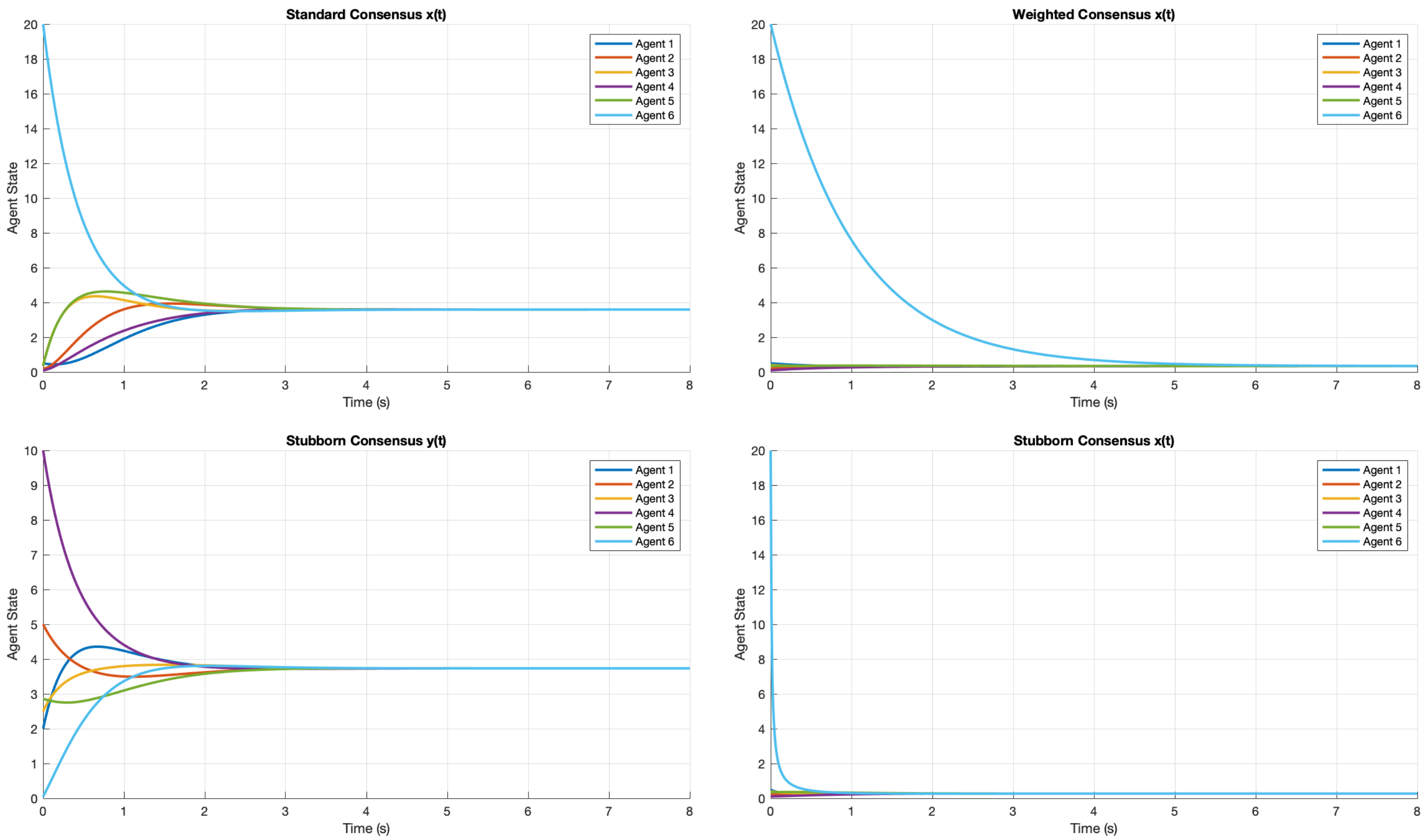


Figure 5: Consensus plots for Experiment III.

From the plots in Figure 5, we can conclude that the proposed approaches remain effective even for directed graphs, where the connectivity is weaker than in the previous experiments. When the key assumptions (existence of a rooted out-branching and a balanced structure) are satisfied, the algorithms behave as expected, just as in the undirected case. Specifically, the standard consensus converges to ≈ 3.5917, the weighted consensus successfully mitigates the outlier effect converging to 0.3429, and the stubborn consensus converges to the harmonic mean of the initial conditions, 0.2678.

In this setting, the non-symmetric nature of the graph implies that the Laplacian may admit complex eigenvalues. Indeed, the second smallest eigenvalue is $\lambda_2^{std} = 1.5000 - 0.8660i$ for the standard consensus and $\lambda_2^{wgt} = 1.0000 - 0.0928i$ for the weighted case. Theoretical predictions are confirmed: since $Re\{\lambda_2^{std}\} > Re\{\lambda_2^{wgt}\}$, the standard consensus converges faster than the weighted consensus. The presence of a nonzero imaginary part explains the oscillatory transient observed in the standard consensus, while in the weighted consensus the oscillations are barely noticeable, as the imaginary component is small in magnitude. As in the previous cases, the stubborn consensus exhibits the fastest convergence rate.

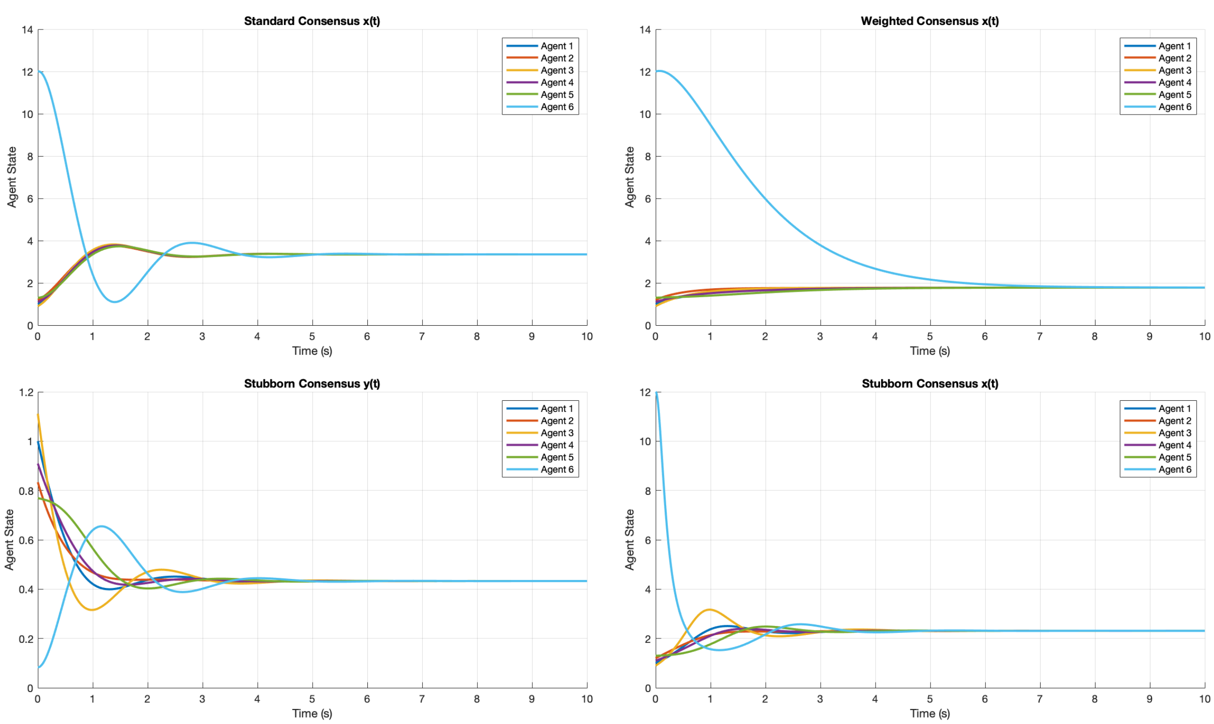
2.3.4 EXPERIMENT IV.

- **Agent dimension:** 1D
- **Setup:** 6 agents with 1 outlier.

$$x_0 = \begin{bmatrix} 1 & 1.2 & 0.9 & 1.1 & 1.3 & 12 \end{bmatrix}^T$$

- **Dynamics:** double integrator.
- **Topology:** fully connected, undirected graph (Figure 1).

■ **Consensus behaviour n.1:** the agents converge to the same constant position.



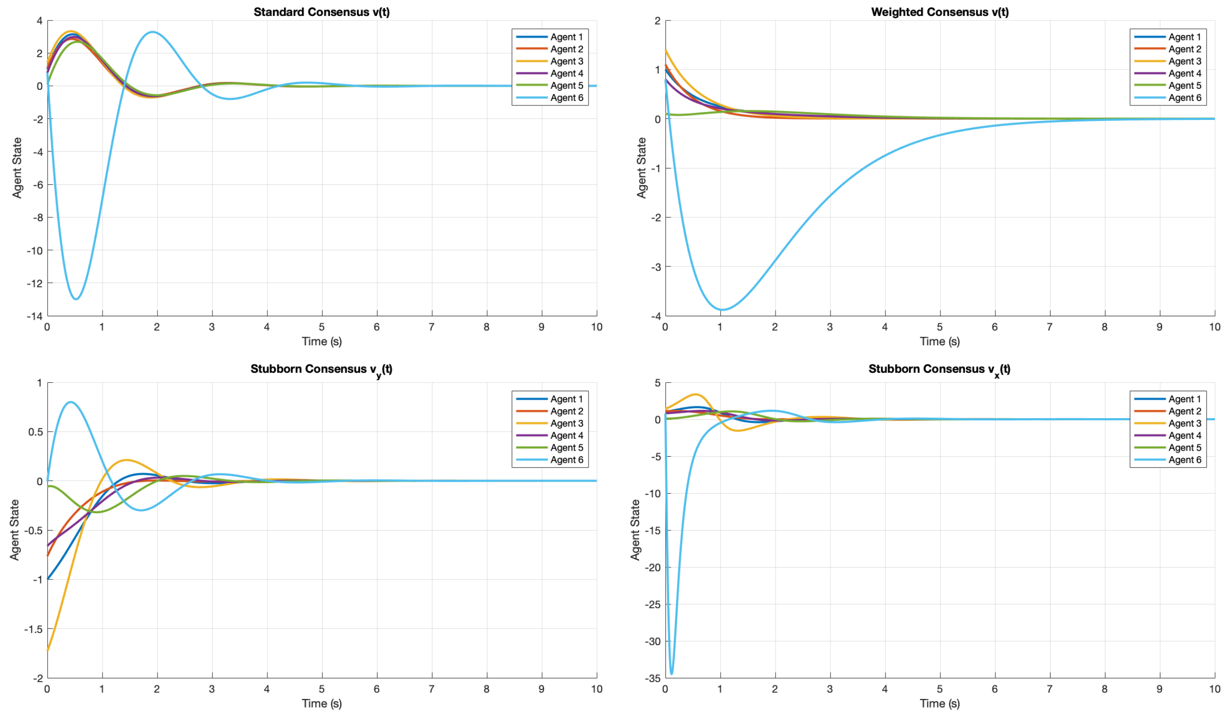


Figure 6: Consensus plots for Experiment IV (behavior n.1), with $v_0 = \begin{bmatrix} 1 & 1.1 & 1.4 & 0.8 & 0.1 & 0.8 \end{bmatrix}^T$.

In the general case, with the initial velocity vector set to $v_0 = \begin{bmatrix} 1 & 1.1 & 1.4 & 0.8 & 0.1 & 0.8 \end{bmatrix}^T$, the results shown in Figure 6 indicate that the standard consensus converges to 3.3500, the weighted consensus to 1.7783, and the stubborn consensus to 2.3103. In contrast, under the particular condition where $v_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, the outcomes in Figure 7 show that the standard consensus converges to the arithmetic mean of the agents' initial conditions (≈ 2.3103), while the stubborn consensus converges to the harmonic mean (≈ 1.2749). These results confirm that, even in double-integrator dynamics, the proposed approaches remain effective, correctly filtering out the outlier.

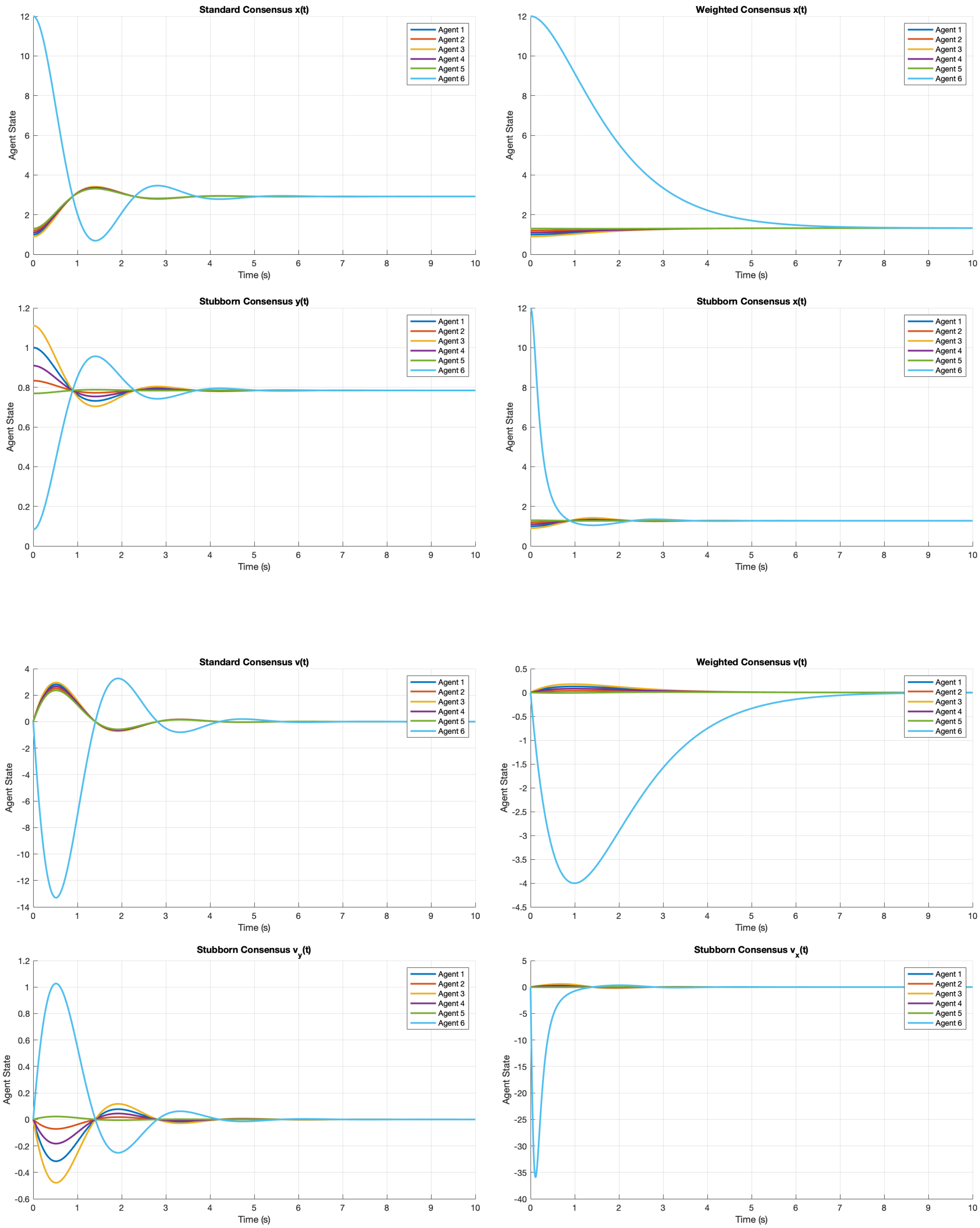


Figure 7: Consensus plots for Experiment IV (behavior n.1), with $v_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

■ **Consensus behaviour n.2:** the agents achieve the same velocity and travel all together.

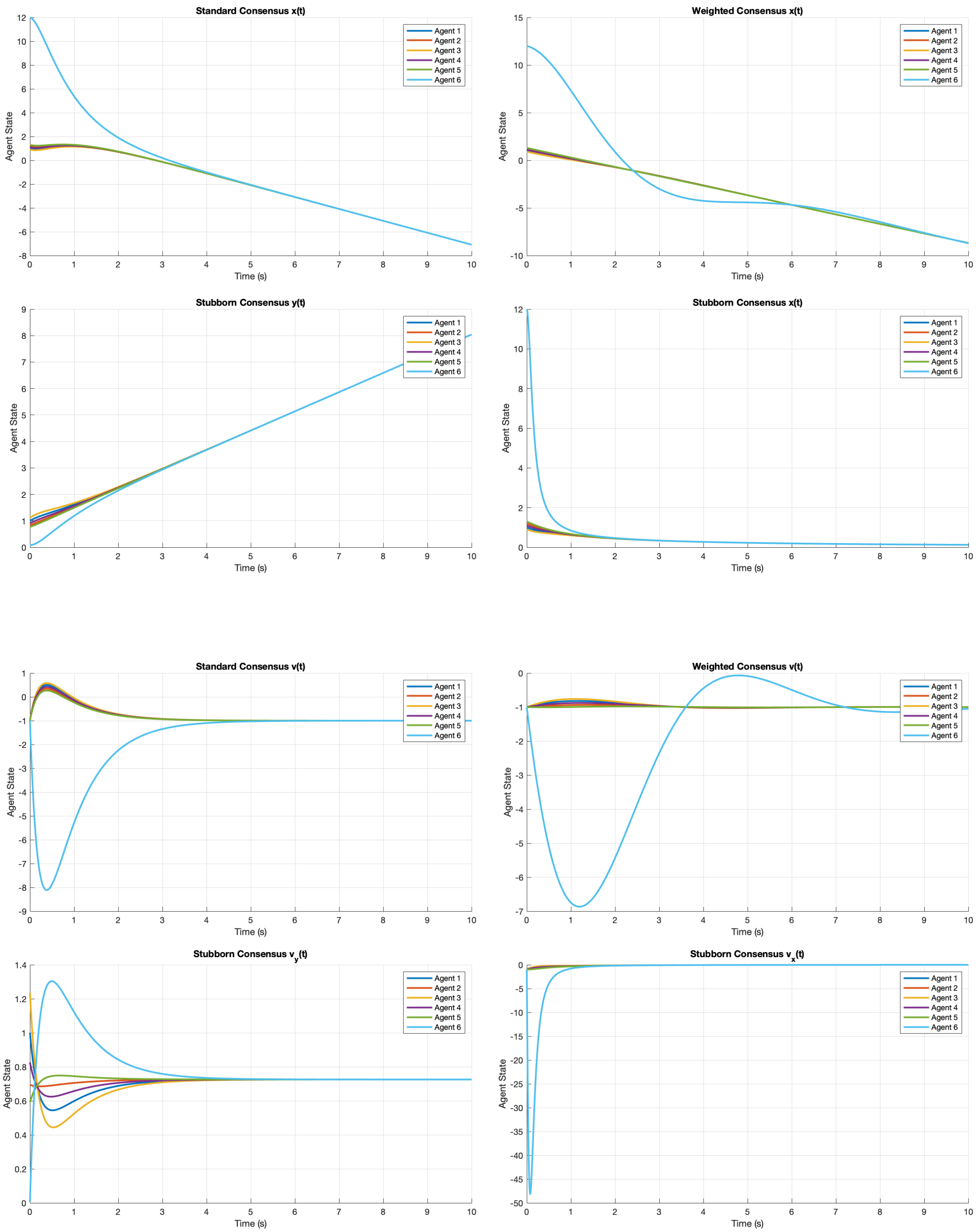


Figure 8: Consensus plots for Experiment IV (behavior n.2), with v_{02}.

During the experiments it was observed that, in the stubborn consensus for double integrator dynamics, discontinuities may arise in the original agents (x_i) whenever the transformed agents (y_i) cross zero, due to the transformation defined in Eq. (12). For this reason, the initial velocities are chosen so as to prevent the transformed states from crossing zero, thereby avoiding divergence in x. The plots in Figure 8 correspond to the choice of initial velocities $v_{02} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}^T$.

While the standard and weighted consensus exhibit similar behaviors (namely, the agents converge to a common point and then move together at constant velocity) the stubborn consensus shows a comparable trend only for the transformed state, with an opposite slope as a result of the sign inversion of the initial velocities in Eq. (15). For the original state, however, the evolution differs: in particular, as $y_i \to \infty$, one obtains $x_i \to 0$, which explains the convergence of the stubborn consensus toward zero, while still preserving the characteristic "traveling together" behavior defined as behavior n.2.

From these results, if we interpret the influence of the outliers as a shift in the starting point of the convergence line, pulling the agents' trajectories toward the outliers, we can conclude that although the standard consensus is affected by outliers, the weighted and stubborn consensus maintain their robustness, with convergence trajectories that remain much closer to the contribution of the inliers alone.

3 Conclusion

In conclusion, the experimental results confirm that the two proposed approaches enhance the robustness of consensus in the presence of outliers. Specifically, the stubborn consensus converges to the harmonic mean of the agents' initial conditions, effectively ignoring the outliers. Although it relies on the same Laplacian matrix as the classical consensus, its behavior differs and exhibits faster convergence. Nonetheless, for directed graphs, convergence to the harmonic mean requires the graph to be balanced, and since the harmonic mean is not properly defined for mixed values, this method cannot be applied when the agents' states include both positive and negative values (mixed agents). In contrast, the weighted consensus employs weights inversely proportional to the agents' initial conditions, which allows it to properly ignore outliers even when agents have mixed (positive and negative) states. However, due to the smaller connectivity eigenvalue, convergence is slower compared to the standard consensus.

4 Future works

In this work, the proposed consensus approaches have been evaluated in simple yet representative scenarios, allowing us to highlight their fundamental properties and convergence behavior. However, further investigation is required to validate their applicability in more complex and realistic settings. Future extensions of this study may involve incorporating **non-holonomic constraints**. Additionally, the framework can be extended to handle **higher-dimensional state spaces**, overcoming the current limitation where each agent's dynamics are restricted to a single spatial dimension.

Moreover, the current formulation is limited by a strict assumption regarding the classification of inliers and outliers: only agents with small initial distances from the origin of the reference frame are treated as inliers, while agents located farther away are considered outliers. This assumption inherently excludes the presence of negative-valued agents, as it relies on the use of the harmonic mean, which becomes undefined or misleading in such cases.

This limitation also prevents the correct identification of inliers in scenarios where they are farther from the origin than the outliers (e.g., an initial condition vector like: $x_0 = \begin{bmatrix} 7000 & 7001 & 7002 & 8 \end{bmatrix}^T$).

In such situations, the use of a **RANSAC strategy** is proposed to robustly identify the true inlier set, irrespective of their distance from the origin. Once the inliers have been detected, a homogeneous transformation of the reference frame can be applied (when feasible) to reposition the inliers near the origin, thereby restoring the validity of the initial assumption. An illustrative example of this approach is shown in Figure 9.

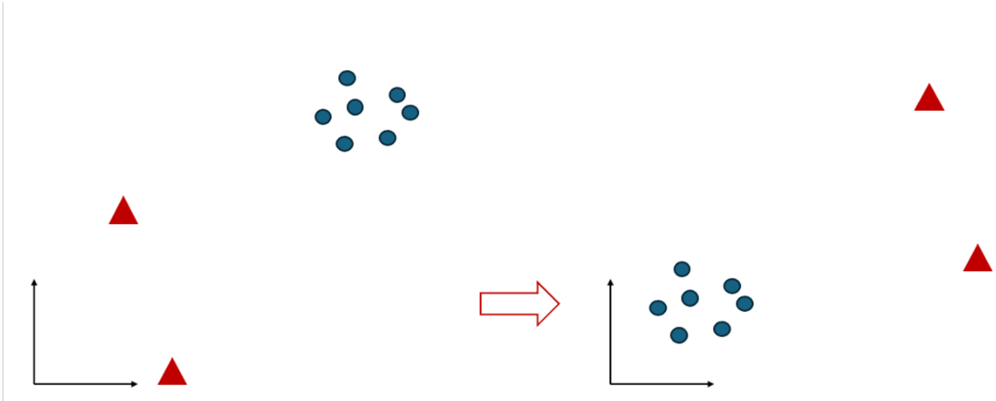


Figure 9: An illustrative example of how an homogeneous transformation of the original reference frame can solve the problem of ”large” outliers definition.

References

[1] R. Olfati-Saber , R. M. Murray, “Consensus Problems in Networks of Agents With Switching Topology and Time-Delays”, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 9, SEPTEMBER 2004.