Multi-Robot project

Stubborn consensus and stubborn rendez-vous

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Abstract

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GitHub link: https://github.com/HatoKng/Stubborn-consensus-protocol

Contents

1 Related works

Consensus algorithms have been extensively studied in the context of multi-agent systems and distributed control. Under the standard consensus protocol [1], a **connected** interaction graph guarantees that all agents asymptotically reach agreement on a common state. In undirected graphs, and in directed balanced graphs that contain a rooted out-branching, the limit equals the arithmetic mean of the initial states. Typical applications include rendezvous tasks, where agents converge to the same position.

While this classical setting is elegant and well understood, it is also sensitive to the presence of heterogeneities and **outliers**. In practical scenarios, not all agents are equally cooperative or reliable: some nodes may have fixed stances (stubborn agents), while others may produce anomalous measurements due to faults, adversarial behavior, or noise. When such outliers are present, the arithmetic mean can be significantly biased, leading to consensus states that do not accurately reflect the majority of agents' intentions.

In this perspective, a natural question arises: can consensus protocols be adapted to neglect large outliers? Two main approaches have been investigated in this work. The first is to introduce a **weighted interaction graph**, where the edge weights depend on the initial conditions of the agents. This method attenuates the influence of agents with extreme values, but the resulting agreement is only semi-global, as it depends on the specific distribution of states. The second approach is the so-called **stubborn consensus**, where convergence is shown to the harmonic mean of initial conditions, yielding a global solution when all agents have positive states.

2 Proposed Approaches

In this work, we examined the previously discussed methods in order to make the consensus protocol work reliably even in the presence of outliers. The strategies considered are **weighted consensus** and **stubborn consensus**.

2.1 Weighted consensus

The objective is to develop a consensus protocol for connected graphs that enables all agents to converge to a **weighted average** of their initial states.

To this end, we define the **non-symmetric weighted adjacency matrix W** as:

$$W(i,j) = A(i,j) \cdot w(j) \tag{1}$$

where A is the adjacency matrix of the graph and w is the weight vector, chosen as the absolute value of the inverse of the agents' initial conditions. This choice downweights the influence of agents with large initial values, thereby improving robustness to outliers.

Each row of W is then normalized to ensure the matrix is **row-stochastic**.

Using this weighted matrix, we define the corresponding graph Laplacian as:

$$L = D - W \tag{2}$$

where D is the diagonal matrix of row sums of W. Since W is row-stochastic, all row sums equal 1, and therefore D coincides with the identity matrix.

The continuous-time consensus protocol is then modeled by the linear differential equation:

$$\dot{x} = -Lx \tag{3}$$

with solution:

$$x(t) = e^{-Lt}x(0) \tag{4}$$

A key assumption in the following is that $\text{rank}(L) = N - 1$, where N denotes the number of agents. Equivalently, the associated interaction graph admits a **rooted-out branching**. Under this assumption, together with the connectivity of the graph, the spectrum of L satisfies:

$$\lambda_1 = 0 \quad \text{and} \quad 0 < \Re\{\lambda_2\} \leq \cdots \leq \Re\{\lambda_N\} \tag{5}$$

And $\mathbf{1}$ is the unique vector spanning the right null-space of L.

In general, L is non-symmetric, hence diagonalizability is not guaranteed. However, one can adopt the Jordan decomposition of L:

$$L = PJ(\Lambda)P^{-1} \tag{6}$$

where $J(\Lambda)$ is the Jordan canonical form of L, the columns of P are the right eigenvectors p_i, and the rows of $Q = P^{-1}$ are the corresponding left eigenvectors q_i^T.

Substituting this decomposition, the solution becomes:

$$x(t) = Pe^{\Lambda t}P^{-1}x(0) = p_1 \cdot q_1^T e^{-J_{\text{block}}(\lambda_1)t} x_0 + \sum_{i=2}^{N} \left(P_i \, e^{-J_{\text{block}}(\lambda_i)t} \, Q_i \right) x_0 \tag{7}$$

Since p_1 and q_1 are the right and left eigenvectors associated with $\lambda_1 = 0$, and $p_1 = \mathbf{1}$ (the only vector spanning the right null-space of L), we deduce that the asymptotic value is:

$$\lim_{t\to\infty} x(t) = (p_1 \cdot q_1^T)x_0 \tag{8}$$

If we further impose the normalization condition $q_1^T p_1 = 1$, and introduce π as the normalized version of q_1 such that:

$$\pi^T \cdot \mathbf{1} = 1 \tag{9}$$

Moreover,

$$\pi^T L = \pi^T(I - W) = 0 \quad \implies \quad \pi^T W = \pi^T \tag{10}$$

Thus, π corresponds to the stationary distribution of W, and the consensus value is

$$\lim_{t\to\infty} x(t) = (\pi^T x_0) \cdot \mathbf{1} \tag{11}$$

This approach guarantees convergence to a weighted sum of the initial states, with the weights π reflecting the influence of each agent. By designing the weights $w(j)$ as the inverse of the initial values' magnitudes, the protocol naturally reduces the impact of extreme values, thus enhancing robustness against large outliers.

2.2 Stubborn consensus

The goal is to design a consensus protocol for connected graphs such that all agents converge to the **harmonic mean** of their initial conditions instead of the arithmetic mean as in the standard consensus protocol.

The proposed approach consists of introducing a **reciprocal transformation**:

$$y_i(t) = \frac{1}{x_i(t)} \tag{12}$$

Then a standard consensus protocol is applied to the transformed variables y_i. The dynamics of the i-th component of y can be written as

$$\dot{y}_i(t) = \sum_j \left(y_j - y_i\right) \tag{13}$$

or, in matrix form,

$$\dot{y}(t) = -Ly(t), \tag{14}$$

where $L = D - A$ is the Laplacian matrix of the graph. Since $y_i = 1/x_i$, differentiating with respect to time yields

$$\dot{y}_i = -\frac{1}{x_i^2}\,\dot{x}_i \tag{15}$$

Substituting (15) into (13) gives the dynamics for x_i:

$$\dot{x}_i = x_i^2 \sum_j \left(\frac{1}{x_i} - \frac{1}{x_j}\right) \tag{16}$$

From (14), the solution for $y(t)$ is exponential:

$$y(t) = e^{-Lt} y(0) \tag{17}$$

Rewriting it in terms of x:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\,\frac{1}{x(0)}\right]_i} \tag{18}$$

Since our goal is to make y converge to the arithmetic mean, we proceed by considering two separate cases:

- **Undirected graphs:** In this case, the Laplacian L is symmetric and positive semidefinite. Hence, it admits the diagonalization by an orthogonal matrix U:

  $$L = U\Lambda U^T, \qquad \Lambda = \mathrm{diag}(\lambda_i), \tag{19}$$

  and the matrix exponential can be written as

  $$e^{-Lt} = Ue^{-\Lambda t}U^T \tag{20}$$

Thus,

$$e^{-Lt}\frac{1}{x(0)} = \sum_{i=1}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} = u_1 u_1^T \frac{1}{x(0)} + \sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} \tag{21}$$

Since the graph is connected, $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_N > 0$. Therefore, as $t \to \infty$:

$$\sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t}\frac{1}{x(0)} \quad \longrightarrow \quad 0 \tag{22}$$

Being $u_1 = \frac{1}{\sqrt{N}}\mathbf{1}$, the first term in (21) becomes

$$u_1 u_1^T \frac{1}{x(0)} = \frac{1}{N}\mathbf{1}\sum_{k=1}^{N}\frac{1}{x_k(0)} \tag{23}$$

Hence, for $t \to \infty$:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\frac{1}{x(0)}\right]_i} \quad \longrightarrow \quad \frac{1}{\frac{1}{N}\sum_{k=1}^{N}\frac{1}{x_k(0)}}, \qquad \forall i, \tag{24}$$

which corresponds to the **harmonic mean** of the initial conditions.

- **Directed graphs:** In this case, L is no longer symmetric, yet the property $L\mathbf{1} = 0$ still holds. If, in addition, the interaction graph is **balanced**, then also $\mathbf{1}^T L = 0$.

  Assuming that the graph admits a **rooted-out branching** and exploiting the resulting connectivity property, we can, by analogy with the weighted consensus case, adopt the Jordan decomposition of L and write:

$$e^{-Lt}\frac{1}{x(0)} = p_1 q_1^T e^{-J_{\text{block}}(\lambda_1)t}\frac{1}{x(0)} + \sum_{i=2}^{N}\left(P_i\, e^{-J_{\text{block}}(\lambda_i)t}\, Q_i\right)\frac{1}{x(0)} \tag{25}$$

  Here, both p_1 and q_1 lie in span($\mathbf{1}$). By imposing the normalization condition $p_1^T q_1 = 1$, we obtain for $t \to \infty$:

$$x_i(t) = \frac{1}{\left[e^{-Lt}\frac{1}{x(0)}\right]_i} \quad \longrightarrow \quad \frac{1}{\frac{1}{N}\sum_{k=1}^{N}\frac{1}{x_k(0)}}, \qquad \forall i, \tag{26}$$

  which again corresponds to convergence towards the **harmonic mean** of the initial conditions.

2.3 Comparison

This section presents a comparison of the proposed approaches with the standard case across different scenarios.

2.3.1 EXPERIMENT I.

- **Agent dimension:** 1D
- **Setup:** 6 agents (all positive values) with 1 outlier.

$$x_0 = \begin{bmatrix} 1 & 1.2 & 0.9 & 1.1 & 1 & 8 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** fully connected, undirected graph, (Figure 1).

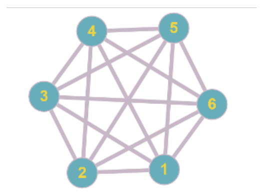

Figure 1: Graph used in Experiment I.

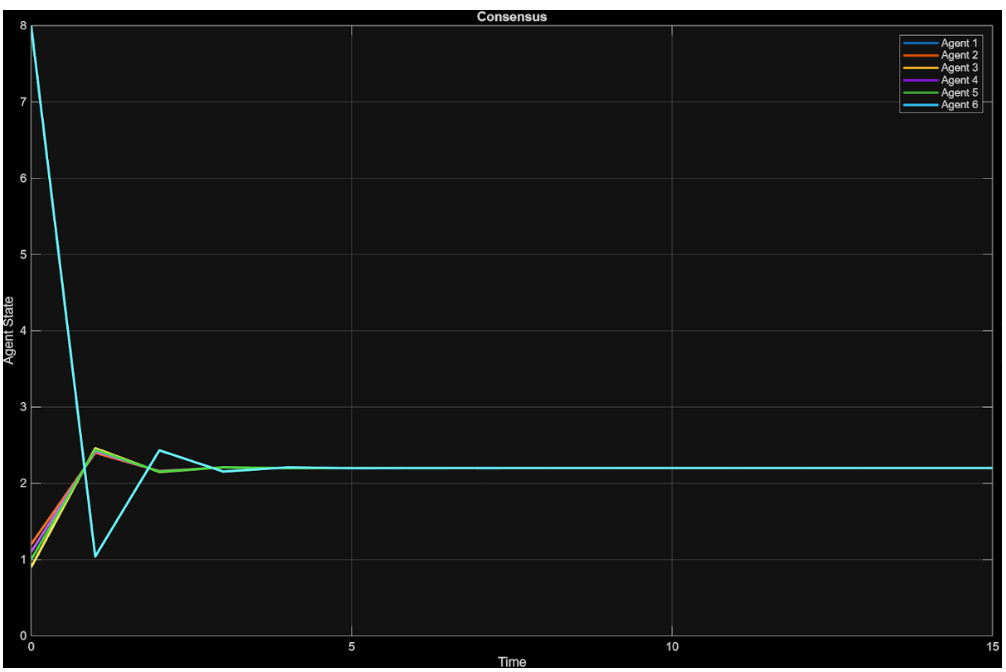

Figure 2: Standard consensus for Experiment I.

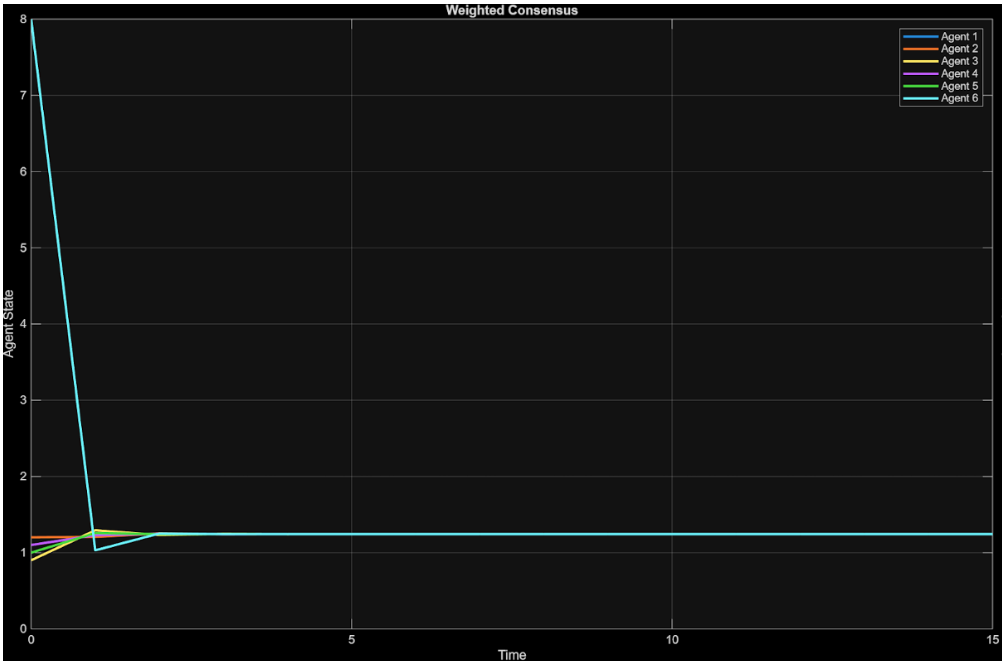

Figure 3: Weighted consensus for Experiment I.

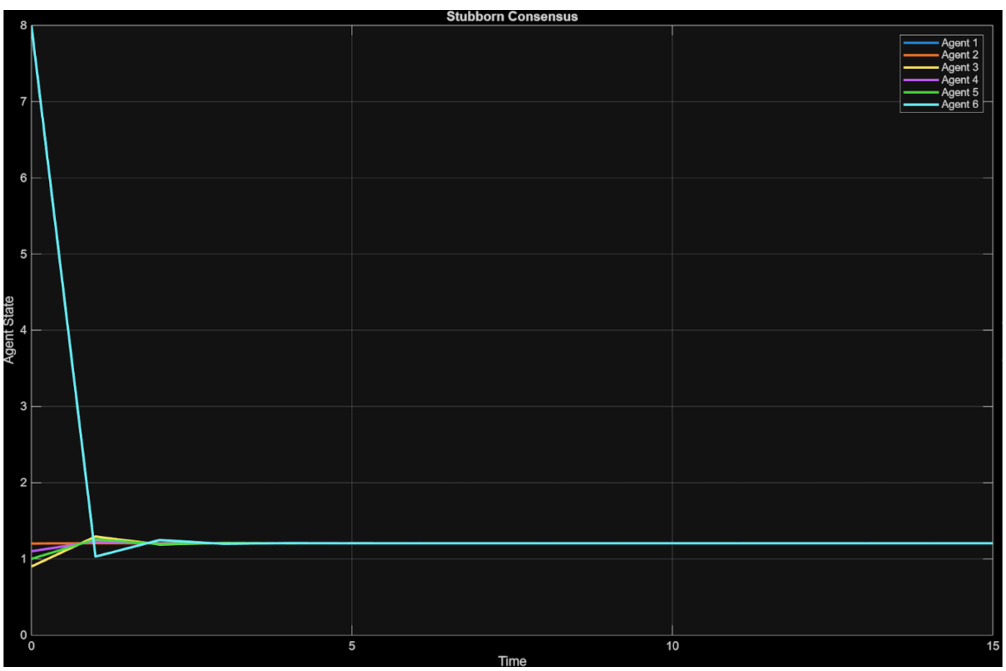

Figure 4: Stubborn consensus for Experiment I.

From the standard consensus plot, it can be observed that the outlier strongly influences the arithmetic mean of the group(≈ 2.200), leading to a noticeable shift in the convergence value. In contrast, the stubborn consensus effectively ignores the outlier (agent 8), converging instead to the harmonic mean (≈ 1.2052). Similarly, the weighted consensus also mitigates the effect of the outlier by assigning weights inversely proportional to the agents' initial values, thereby diminishing the influence of the large values. The convergence value is ≈ 1.2437.

2.3.2 EXPERIMENT II.

- **Agent dimension:** 1D
- **Setup:** 6 agents (positive and negative values) with 1 outlier.

$$x_0 = \begin{bmatrix} -1 & -1.2 & 0.9 & 1.1 & 1 & 8 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** fully connected, undirected graph (Figure 1).

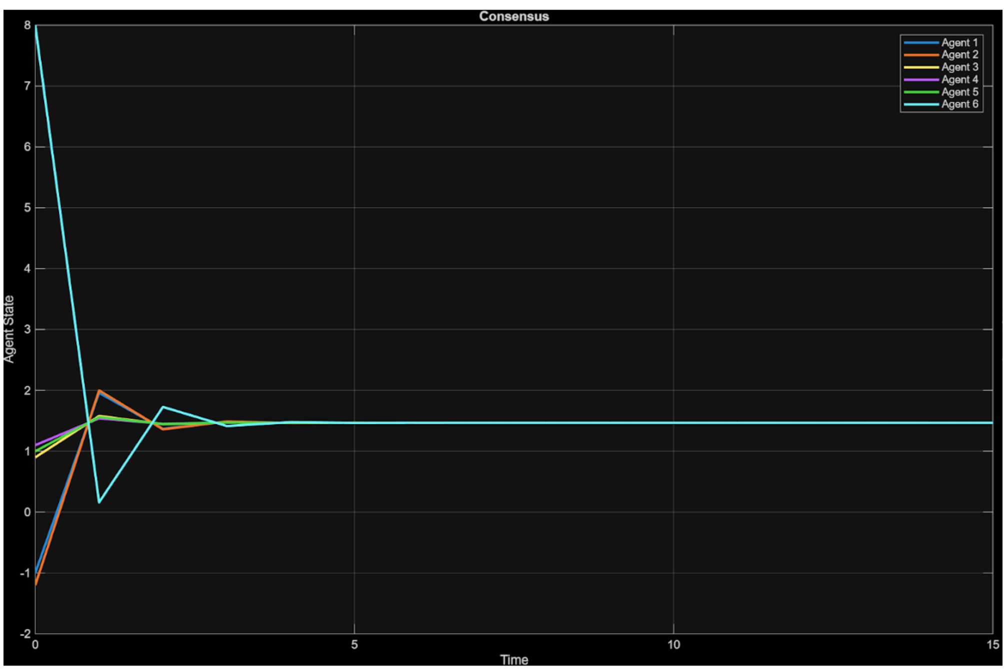

Figure 5: Standard consensus for Experiment II.

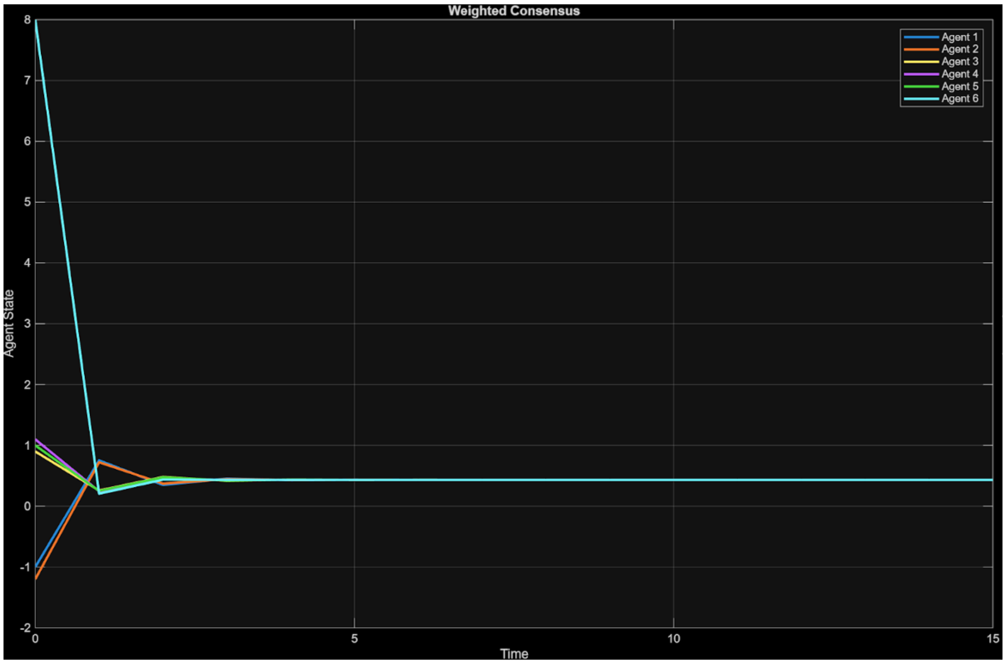

Figure 6: Weighted consensus for Experiment II.

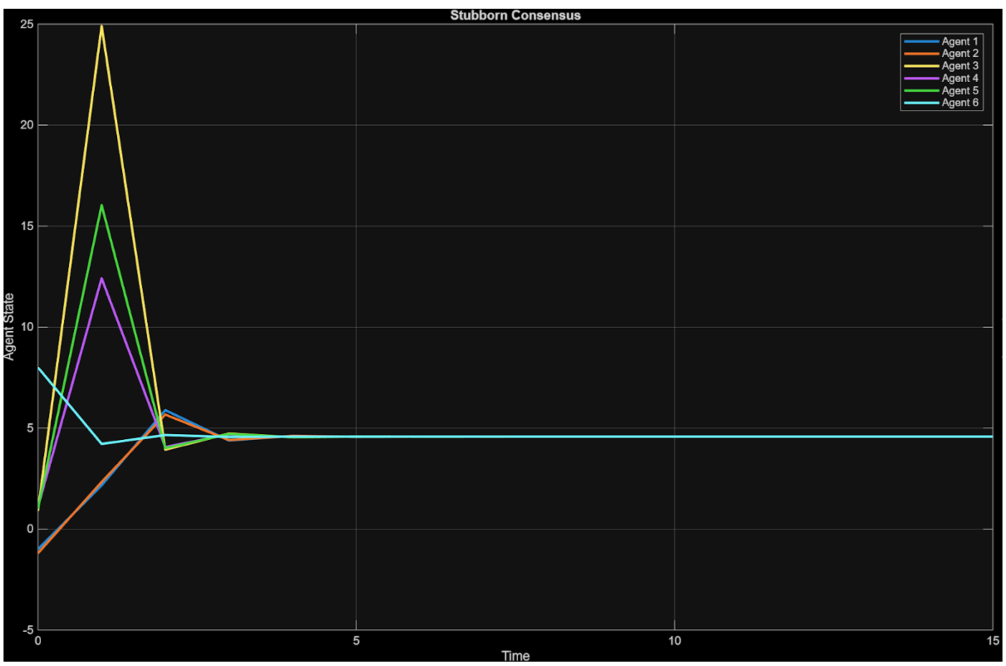

Figure 7: Stubborn consensus for Experiment II.

Similarly to the previous case, the standard consensus converges to the arithmetic mean (≈ 1.4667), which is heavily influenced by the outlier (Agent 8). Since the harmonic mean is only well-defined for positive values, its use becomes problematic in the presence of negative agents (i.e., agents with negative states). In this scenario, the stubborn consensus converges to 4.5736 (a value again strongly affected by the outlier) and most agents exhibit pronounced overshooting during convergence. In contrast, the weighted consensus remains robust even in the presence of negative agents: it converges to 0.4319, effectively disregarding the outlier and ensuring a stable result.

2.3.3 EXPERIMENT III.

- **Agent dimension:** 1D
- **Setup:** 6 agents (positive) with 3 outliers.

$$x_0 = \begin{bmatrix} 1 & 1.2 & 0.9 & 10 & 9 & 8 \end{bmatrix}^T$$

- **Dynamics:** simple integrator.
- **Topology:** fully connected, undirected graph (Figure 1).

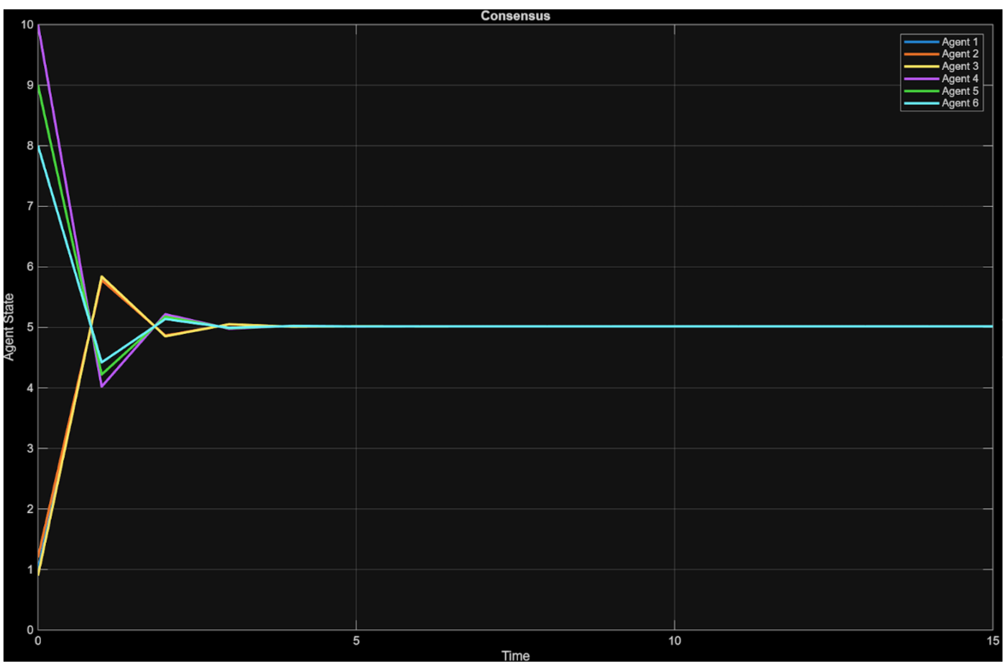

Figure 8: Standard consensus for Experiment III.

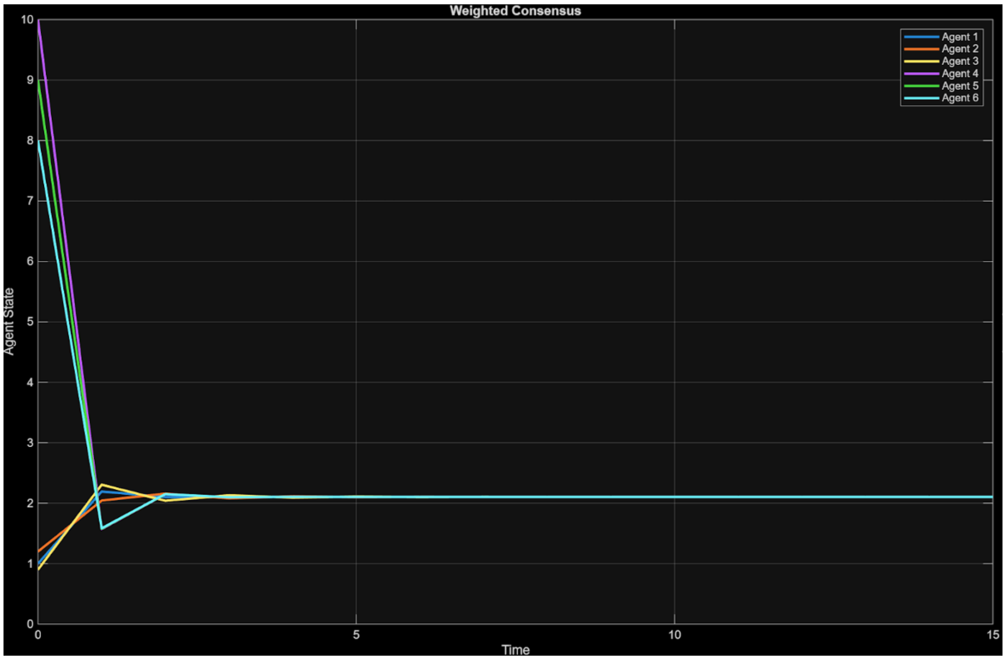

Figure 9: Weighted consensus for Experiment III.

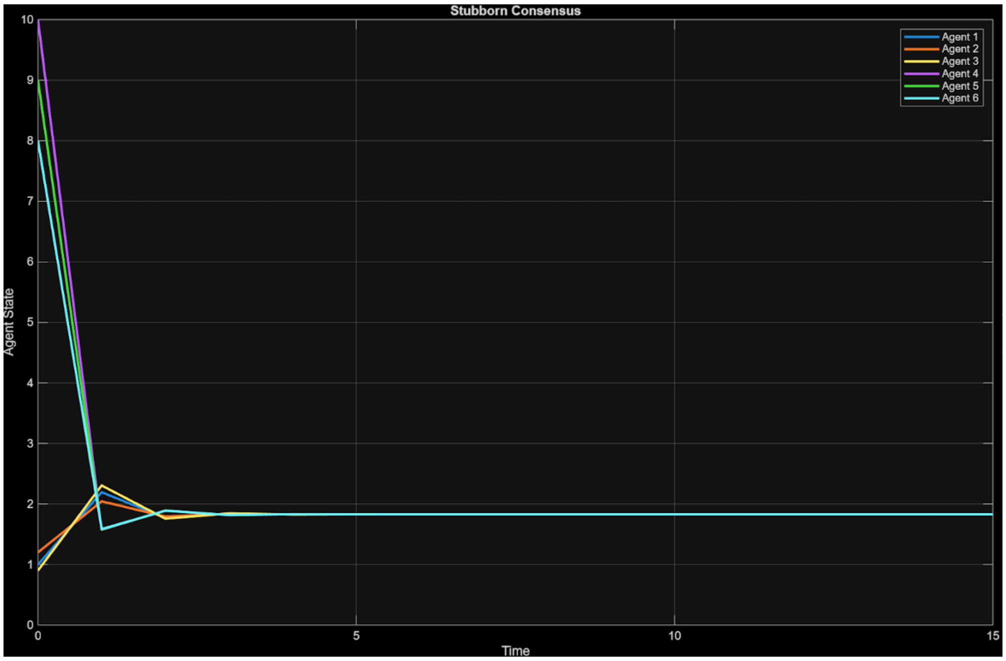

Figure 10: Stubborn consensus for Experiment III.

In the presence of a larger number of outliers[1] (3 out of 6 agents), the performance of the standard consensus degrades significantly. When agents 4, 5, and 6 are treated as outliers, the system converges to the arithmetic mean of 5.0167, which is heavily biased by their influence. In contrast, both the stubborn consensus and the weighted consensus successfully mitigate the influence of the outliers, converging instead to 1.8290 (harmonic mean) and 2.1043 (weighted mean), respectively.

2.3.4 EXPERIMENT IV.

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3 Conclusion

4 Future works

In this work, the proposed consensus approaches have been evaluated in simple yet representative scenarios, allowing us to highlight their fundamental properties and convergence behavior. However, further investigation is required to validate their applicability in more complex and realistic settings. Future extensions of this study may involve incorporating **non-holonomic constraints**. Additionally, the framework can be extended to handle **higher-dimensional state spaces**, overcoming the current limitation where each agent's dynamics are restricted to a single spatial dimension.

Moreover, the current formulation is limited by a strict assumption regarding the classification of inliers and outliers: only agents with small initial distances from the origin of the reference frame are treated as inliers, while agents located farther away are considered outliers. This assumption inherently excludes the presence of negative-valued agents, as it relies on the use of the harmonic mean, which becomes undefined or misleading in such cases.

This limitation also prevents the correct identification of inliers in scenarios where they are farther from the origin than the outliers (e.g., an initial condition vector like: $x_0 = \begin{bmatrix} 7000 & 7001 & 7002 & 8 \end{bmatrix}^T$).

[1]The inliers are clustered near zero, while the outliers take on larger values

In such situations, the use of a **RANSAC strategy** is proposed to robustly identify the true inlier set, irrespective of their distance from the origin. Once the inliers have been detected, a homogeneous transformation of the reference frame can be applied (when feasible) to reposition the inliers near the origin, thereby restoring the validity of the initial assumption. An illustrative example of this approach is shown in Figure 11.

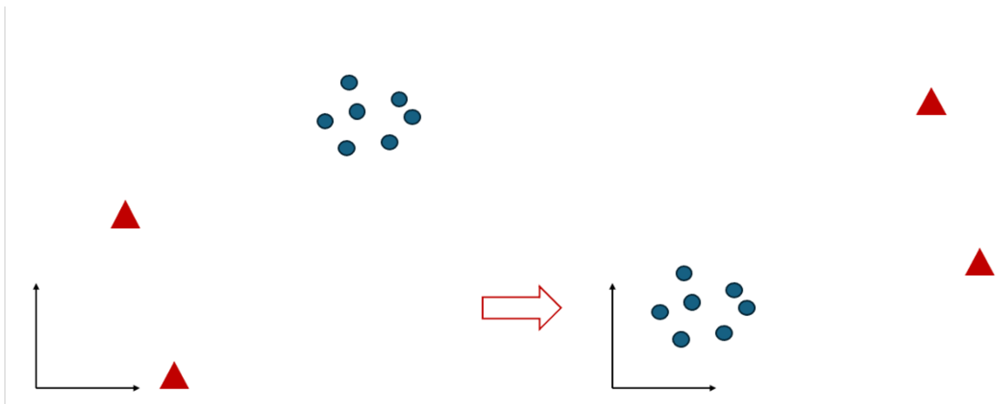

Figure 11: An illustrative example of how an homogeneous transformation of the original reference frame can solve the problem of "large" outliers definition.

References

[1] R. Olfati-Saber , R. M. Murray, "Consensus Problems in Networks of Agents With Switching Topology and Time-Delays", IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 9, SEPTEMBER 2004.