EEEN 415: Assignment 3

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Section A - Formative Questions

a

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \qquad poles = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ -2x_2 + x_3 + 6u \\ -2x_3 \end{bmatrix} \qquad y = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \end{bmatrix}$$

$\mathbf{x_2}$ is the only state that determines the last states of the system and the B matrix's last row is a 0, which would result in an uncontrollable state. Additionally, the input **u** will not be able to affect the $\mathbf{x_2}$ and its corresponding eigenvalue is $\mathbf{-2}$. The controllability of this system is further verified with the controllability matrix, indicating that there are only 2 controllable states. However, as the poles of this are all unstable and uncontrollable, the system is unstabilisable.

As illustrated in the output matrix; only $\mathbf{x_1}$ can initially be observed. However, as all the system states are coupled to $\mathbf{x_1}$, given by the state equation matrix the entire system is observable. The observability of this system is further verified with the observability matrix, further indicating that the system is observable. As all the unstable poles in the system are observable, (being outside the unit circle) the system is detectable.

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b

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad poles = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ -2x_2 + x_3 - 6u \\ -2x_3 \end{bmatrix} \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \end{bmatrix}$$

Similarly to the question **a**, $\mathbf{x_3}$ is the only state that determines the last states of the system and the B matrix's last row is a 0, which would result in an uncontrollable state. Additionally, the input **u** will not be able to affect the final state of the system and the corresponding eigenvalue for this state is **2**. The controllability of this system is further verified with the controllability matrix (Kalman's test), indicating that there are only 2 controllable states. As the state's corresponding pole is unstable, (being outside the unit circle). As a result, the system is not stabilisable.

As illustrated in the output matrix; only $\mathbf{x_1}$ can be observed. However, as all the system states are coupled to $\mathbf{x_1}$, given by the state equation matrix the entire system is observable. The observability of this system is further verified with the observability matrix (Kalman's test), indicating that the system is observable. As all the unstable poles in the system are observable, (being outside the unit circle) the system is detectable.

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c

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad poles = \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 3x_1 \\ -3x_2 + u_1 \\ x_3 - u_2 \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \end{bmatrix}$$

Looking at the state space equation matrix allowed us to analyse the relationship between each state. $\mathbf{x_3}$ is affected by $\mathbf{u_2}$ and $\mathbf{x_1}$,$\mathbf{x_2}$ is affected by $\mathbf{u_1}$ resulting in a fully controllable system. The controllability and observability are further verified with Kalman's test. All the unstable poles are controllable and as a result, this system is stabilisable.

However, the output matrix indicates that it can only observe $\mathbf{x_1}$ state, which is only coupled to $\mathbf{x_1}$ and $\mathbf{x_2}$ and not $\mathbf{x_3}$. As a result, we can only observe the first and second states and not the final state. This is also verified using the observability matrix. As not all unstable poles are observable in this system, thus, the system is not detectable. The

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d

$$A = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \quad poles = \begin{bmatrix} 0.7 \\ 0.8 \\ 0.9 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9x_1 + u \\ 0.8x_2 + u \\ 0.7x_3 + u \end{bmatrix} \quad y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 - 2x_3 \end{bmatrix}$$

As the B matrix allows the input to affect each state, the system is fully controllable. However, the states are not coupled. Additionally, this system is fully observable as $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$ can be observed in the output vector. The controllability and observability are further verified with Kalman's test.

However, there are no unstable poles in this system, and while being fully controllable and observable. As a result, the system will be both detectable and stabilisable.

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e

$$A = \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.1 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0.1 \end{bmatrix} \quad poles = \begin{bmatrix} 0.8 + 0.1i \\ 0.8 - 0.1i \\ 0.7 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.1 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.8x_1 + 0.1x_2 \\ 0.8x_2 - 0.1x_1 + u \\ 0.7x_3 + u \end{bmatrix} \quad y = \begin{bmatrix} 1 & -1 & 0.1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 0.1x_3 \end{bmatrix}$$

Inputs do not directly affect $\mathbf{x_1}$, but $\mathbf{x_1}$ is coupled to $\mathbf{x_2}$. As a result, that state is controllable. Furthermore, $\mathbf{x_3}$ can be affected by input and thus, the entire system is controllable. The entire system is also observable as all the states can be presented in the output equation matrix.

However, there are no unstable poles in this system, and while being fully controllable and observable. As a result, the system will be both detectable and stabilisable. Furthermore, the controllability and observability of this system are further verified with Kalman's test.

f

$$A = \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.1 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0.1 \end{bmatrix} \quad poles = \begin{bmatrix} 0.8 + 0.1i \\ 0.8 - 0.1i \\ 0.8 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.1 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.8x_1 + 0.1x_2 \\ 0.8x_2 - 0.1x_1 + u \\ 0.8x_3 + u \end{bmatrix} \quad y = \begin{bmatrix} 1 & -1 & 0.1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 0.1x_3 \end{bmatrix}$$

For the same reasons as **e**, the entire system is controllable and observable. Similar to **e** and **d**, there are no unstable poles in this system. This system will be detectable and stabilisable.

2

$$A = \begin{bmatrix} -2 & -10 & 0 & 0 & 0 \\ 10 & -2 & 0 & 0 & 0 \\ 0 & 40 & -0.4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 30 \\ 0 & 0 & 0 & -30 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0.2 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -2ax_1 - 10x_2 \\ 10ax_1 - 2x_2 + u \\ 40x_2 - 0.4x_3 + u \\ 30x_5 - x_4 \\ -30x_4 - x_5 0.2u \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

a

The first observer should go to $\mathbf{x_3}$, as $\mathbf{x_2}$ is coupled to it. Additionally, $\mathbf{x_2}$ can be influenced by $\mathbf{x_1}$, whereas $\mathbf{x_3}$ can only be is influenced by $\mathbf{x_2}$ and itself. As a result, to observe as many states with limited resources one of the 2 sensors should be allocated to $\mathbf{x_3}$. The second observer should be allocated $\mathbf{x_4}$ or $\mathbf{x_5}$ as both states influence each other. As a result, $\mathbf{x_4}$ was picked. The allocation of these 2 observers will be able to provide sufficient coverage of the system. The observability of the system was further verified by analysing its controllability matrix. Its rank matches the number of states in the system.

b

$$poles = \begin{bmatrix} -0.4 \\ -2+10i \\ -2-10i \\ -1+30i \\ -1-30i \end{bmatrix} \qquad ITAE_{poles} = 3 \begin{bmatrix} -0.8955 \\ -0.3764+1.2920j \\ -0.3764-1.2920j \\ -0.5758+0.5339j \\ -0.5758-0.5339j \end{bmatrix} \qquad L = \begin{bmatrix} -0.0563 \\ 0.2562 \\ 0.0613 \\ -1.9384 \\ -33.0494 \end{bmatrix}$$

The poles of the observer can be determined by multiplying the system's pole by a magnitude of 3-5, as the observer must be faster than the compensator of the system. So that the estimate becomes meaningful before the compensator acts, the observer poles must be faster than the compensator poles. As the compensator is 1, the magnitude of the observer can be 4. As the observer is arranged in an ITAE pole configuration, a 5th-order ITAE pole will be multiplied by 4 to generate an observer for the open loop system and the results can be shown in fig 1.

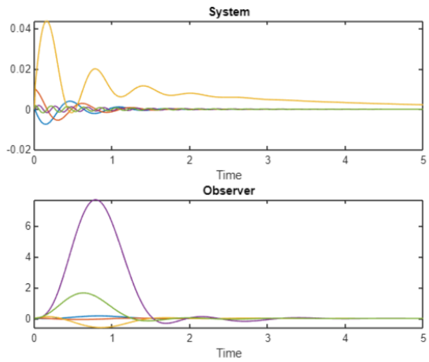

Figure 1: state output vs observer output

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c

As the observer is created, it can generate an estimate of the state vector so that it can modify the compensator design to estimate the state vector rather than the real vector. This results in a dynamic system. The 50% increase in the fastest mode in the system resulted in the same system state changes over time. However, the observer detects the changes, this is because observers can susceptible to high-frequency noise in the measurement as shown in fig 2 below compared to fig 1.

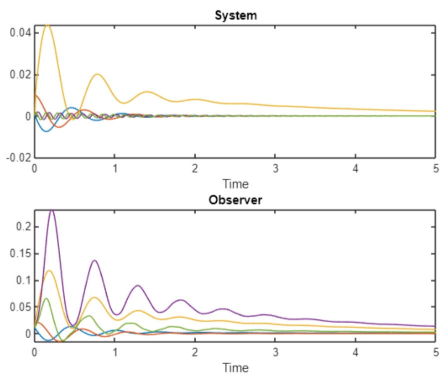

Figure 2: state output vs observer output (50% increase in system pole)

Section B - Summative Questions

$$\hat{x} = \begin{bmatrix} 0.945 & 0.05 & 0 & 1 \\ 0.05 & 0.9 & 0.05 & 0 \\ 0 & 0.05 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0.945x_1 + 0.05x_2 + x_4 + 0.05u_1 \\ 0.05x_1 + 0.9x_2 + 0.05x_3 \\ 0.05x_2 + 0.9x_3 + 0.05u_2 \\ x_4 + 0.01u_3 \end{bmatrix}$$

a

To modify the code so that a one-unit heating signal can be applied to the heater between t = 30s and t = 60s, the 3rd column of B must be influenced as it allows the input to directly control x_4 which is the temperature state. This can be seen in the state matrix equation shown above. As a result, the u matrix's 3rd column must be modified to have that respective input signal. The results can be seen in fig 3.

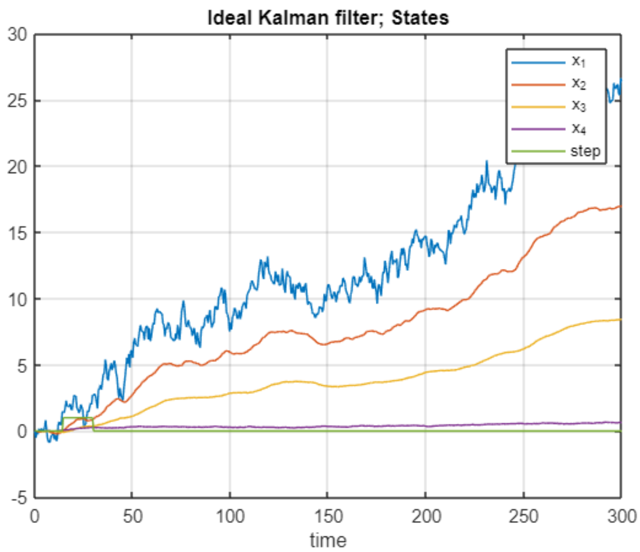

Figure 3: states plotted against time

As expected, the estimated optimal Kalman filter causes some variance when provided a step input between the 30s and 60s. This is due to the estimating nature of the Kalman filter being an observer. Since the optimal Kalman filter's initial estimate is at 1, there will be expected volatility in the estimation as it starts to correct with each iteration by changing its observer poles to achieve greater certainty as it tends towards the real value of the temperature, illustrated in fig 4. x_4 is subtracted from itself and the Kalman filter to better evaluate its efficacy (this applies for all questions onwards).

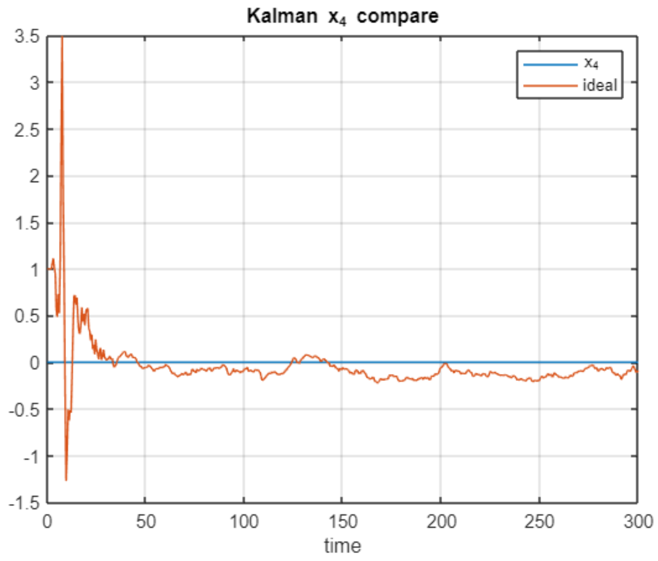

Figure 4: Optimal Kalman filter vs x_4

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b

A steady-state Kalman filter can be created through the `Kalman` function of MatLab. This provides a set of poles that the Kalman filter can operate on. A steady-state Kalman filter will not have the benefits of dynamic pole adjustment like the ideal/optimal Kalman filter. As a result, it will not perform as well as the optimal Kalman filter. As observed in fig 5, the ideal Kalman filter reaches the temperature state value at approximately t = 30s compared to the steady-state, which achieves the real state at approximately t = 135 s.

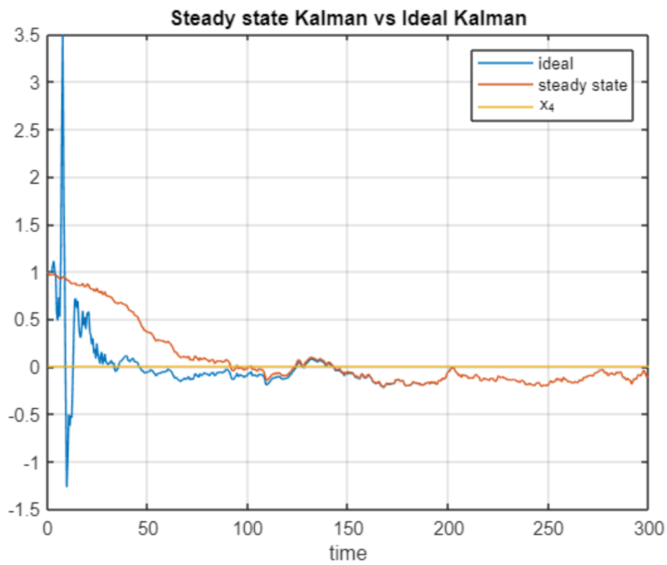

Figure 5: steady-state Kalman filter vs Optimal Kalman filter

However, the advantage of the steady-state Kalman filter is the reduction in volatility when there's greater uncertainty, such as in the beginning. Additionally, the steady-state Kalman filter will eventually meet with the optimal Kalman filter as the optimal Kalman filter's observer poles tend towards its steady-state Kalman's poles.

c

```
%c

a = C*x(:,k) + sqrt(R)*randn(1,1);  % Measure the output.
b = C*x(:,k) + sqrt(R)*randn(1,1);  % Measure the output. use different noise
y_c(:,k) = (a+b)/2;
```

Figure 6: Matlab code for having **2 identical sensors**

The Kalman filter, in a sense, works better the more sensors there are. To utilise **2 identical sensors**. 2 parts to the output can be created and an average of the 2 sensors' outputs will be taken to best determine future states. Although the sensors will have identical C matrices, the measurement noise R must be varied to provide a more realistic simulation as no 2 sensors will have the same noise.

As the 2 sensors are identical we can trust the result of both sensors equally, thus, the summation of the 2 sensor's output will be divided by 2. As more information is provided, the certainty of the **2 identical sensors** would improve. The MatLab implementation of the 2 sensors can be seen in fig 6, where the Y value for the Kalman filter will be the average of the 2 sensors.

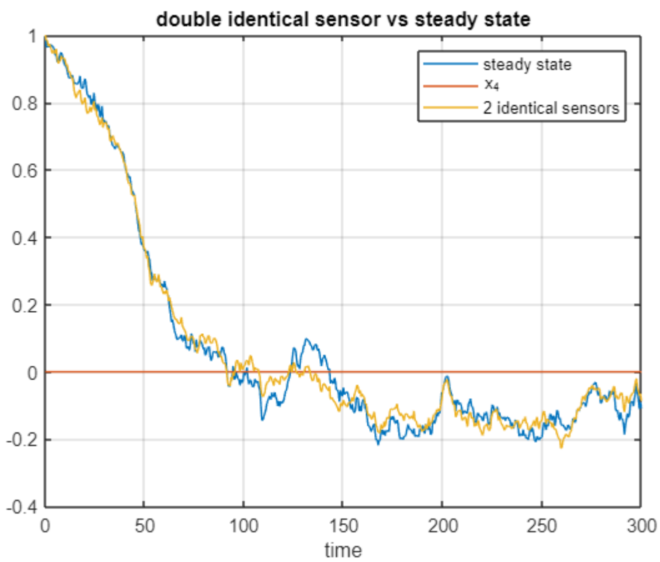

Figure 7: 2 identical sensors vs steady-state Kalman

As observed in fig 7. The Kalman filter with 2 identical sensors improved the certainty, resulting in less noise compared to the steady-state Kalman filter while achieving the same temperature state value time. Thus, it can be confirmed that 2 identical sensors work better than 1 sensor when predicting the value of the temperature state.

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d

```
%d

d = C*x(:,k) + sqrt(R)*randn(1,1);  % Measure the output.
e = C*x(:,k) + 2*sqrt(R)*randn(1,1);  % Measure the output. use different noise but double it
y_d(:,k) = (d+e)/3;
```

Figure 8: Matlab code for having 2 identical sensors, one with more noise

To double the measurement variance and observe the Kalman filter's efficacy. The second sensor's measurement noise covariance will need to be doubled and a weighted average of the sensor outputs will need to be taken. Due to having to trust the noisier sensor, less, the sum of the 2 sensors will need to be divided by 3 as one of the sensor's measurement noise covariances has doubled, producing a weighted average.

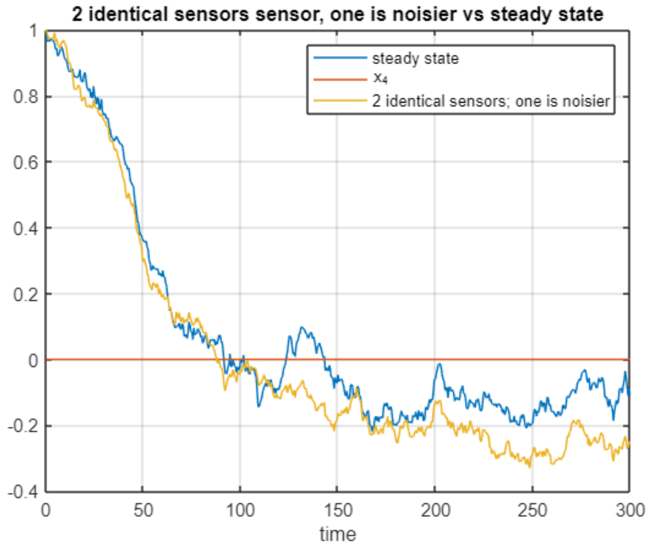

Figure 9: 2 identical sensors one with more noise vs steady-state Kalman filter

This new double-sensor output will be referred to as the Noisy sensor for convenience. The MatLab code observed in fig 8 describes this process. As illustrated in fig 9, although the new Noisy sensor output reaches the temperature state value before the steady-state Kalman, the Noisy sensor Kalman prediction tends to have a large steady-state error compared to the steady-state Kalman filter.

---

e

```
f = C*x(:,k) + sqrt(R)*randn(1,1);  % Measure the output.
g = [0 0 0 1]*x(:,k) + 2*sqrt(R)*randn(1,1);  % Measure the output.
y_e(:,k) = (f+g)/3;
```

Figure 10: Matlab code for having 2 different sensors, one with more noise

Similarly to question **d**, a weighted average must be obtained for the varying noisy second sensor. However, this time, the second sensor's C matrix will be augmented to observe the heater temperature x_4. From the state matrix equation, the temperature state value is determined only by the state and the input, hence a sensor there will only be able to observe the state and the input rather than the entire system.

Although having 2 sensors does decrease the overall noise of the observer, the output of this variation of the Kalman filter will have a greater steady-state error compared to the prior variations of the steady-state Kalman filter. As a result, create incorrect predictions on the system's temperature as shown in fig 11.

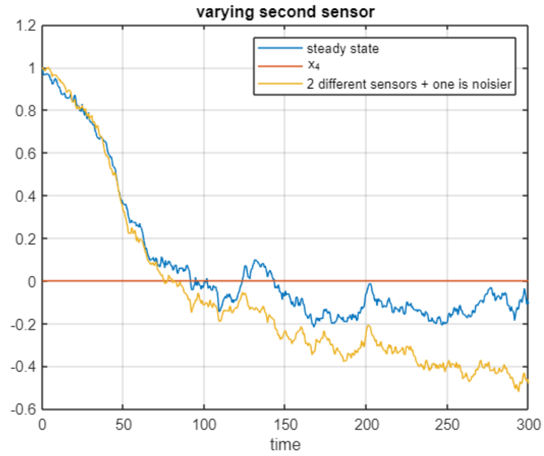

Figure 11: 2 different sensors, one with more noise vs steady-state Kalman filter

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f

$$L = \begin{bmatrix} 0.1252 \\ 0.0731 \\ 0.0319 \\ 0.0031 \end{bmatrix} \qquad L_s = \begin{bmatrix} 0.5634 \\ 0.3288 \\ 0.1434 \\ 0.014 \end{bmatrix} \qquad L_f = \begin{bmatrix} 0.0278 \\ 0.0162 \\ 0.0071 \\ 0.0007 \end{bmatrix}$$

To decrease the speed of the Kalman observer poles would mean increasing its pole values so that it is closer to the unit circle. The pole must be stable, as such, the pole values cannot be greater than one. Consequently, to increase the speed of the Kalman observer poles would mean decreasing its value so that it's closer to the origin, as close as it can get to 0.

To observe the effects of a fast/slow Kalman observer, these poles will be based on the steady-state Kalman poles and will be decreased/increased by 7.9 respectively. Since 7.9 is the largest value L can be multiplied by before becoming unstable, it was chosen as the magnitude. Furthermore, decreasing the pole values by that magnitude would reflect the inverse effects of the slow Kalman observer.

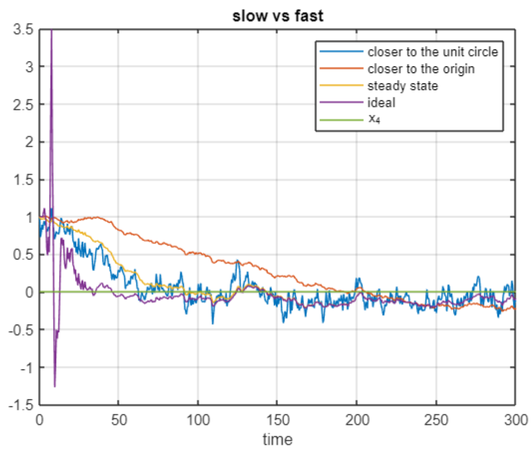

Figure 12: fast Kalman vs slow Kalman vs steady-state Kalman filter

As shown in fig 12, the slow poles (blue) creates a noisier signal, but achieves the state temperature value sooner than the steady-state Kalman filter. Contrary, the fast poles (orange) reach steady-state later than the steady-state Kalman filter but have less noise.

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g

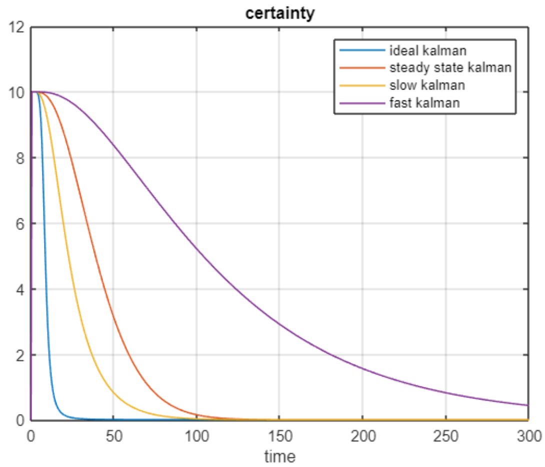

Figure 13: Certainty of different variations of the Kalman filter

As shown in fig 13, the certainty of different Kalman filters, (optimal, steady-state, slow and fast) reflects how fast it achieves steady-state. The certainty of the temperature state value x_4 can be obtained by observing the $P_{(4,4)}$ of each variation of the Kalman filter over time. Although the slow Kalman filter achieves certainty better than the steady-state Kalman filter, it comes at the cost of greater noise. Consequently, the fast Kalman filter achieves a steady-state much later than the steady-state Kalman filter but has a lot less noise, as demonstrated in fig 12. Finally, the certainty of the ideal Kalman filter performs the best as the poles of the observer dynamically change.