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Hermite and Laguerre 2D polynomials

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Abstract

We define Hermite 2D polynomials $H_{m,n}(U;x,y)$ and Laguerre 2D polynomials $L_{m,n}(U;z,\bar{z})$ as functions of two variables with an arbitrary 2D matrix U as parameter and discuss their properties and their explicit representation. Recursion relations and generating functions for these polynomials are derived. The advantage of the introduced Hermite and Laguerre 2D polynomials in comparison to the related usual two-variable Hermite polynomials is that they satisfy orthogonality relations in a direct way, whereas for the purpose of orthonormalization of the last, one has to introduce two different kinds of such polynomials which are biorthogonal to each other. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

The kinds of Hermite and Laguerre 2D polynomials which we introduce and discuss in this contribution (see also [14]) play a great role in two-dimensional problems which are related to the degenerate 2D harmonic oscillator. In [15], we introduced and discussed a special set of orthonormalized Laguerre 2D functions and in [16] the corresponding Laguerre 2D polynomials. They appear in many formulas of quantum optics connected with the two-dimensional phase space of one mode (quasiprobabilities, Fock-state representation and ordered moments [15,16]) but they find applications also in classical optics (e.g., wave propagation in paraxial approximation) and certainly in other problems. There are problems which need more general kinds of 2D polynomials related to Hermite and Laguerre polynomials (e.g., beam splitter, general light polarization, ordered moments in quantum optics, transformations of Gauss–Hermite and Gauss–Laguerre beams in paraxial approximation in classical optics [1,11,13]). The general Laguerre 2D polynomials involve an arbitrary 2D matrix as a parameter and make the transition to the special Laguerre polynomials [16] for the identity matrix. In case of general Hermite 2D polynomials, the transition to the identity matrix provides products of two usual Hermite polynomials. The considered Hermite and Laguerre 2D functions contain the weight factor for the orthonormalization of the corresponding polynomials and are eigenfunctions

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of the degenerate 2D harmonic oscillator (equal frequencies). The usual two-variable Hermite polynomials [2–8] possess the disadvantage that for their orthonormalization one needs a second kind of polynomials to get biorthogonality relations. They are therefore not as well appropriate as basis systems for expansions of functions of two variables as the here-considered 2D polynomials.

We introduce and discuss the Hermite 2D polynomials in Section 2 and the Laguerre 2D polynomials in Section 3 and establish the relations between them. In Section 4, we derive the recursion relations and in Section 5, we outline the orthonormality relations (more details and proof in [14]). In Section 6, we give the generating functions for the Hermite and Laguerre 2D polynomials and compare them with the generating functions for the usual two-variable Hermite polynomials that exposes the relations between them.

2. Definition of Hermite 2D polynomials

Before defining Hermite 2D polynomials, we give the following alternative definition of (usual) Hermite polynomials $H_m(x)$ of one variable x which plays a great role in our further considerations but is little known [9,10,12,17] (obviously some authors, including myself, found it independently)

$$H_m(x) \equiv \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^m = \sum_{k=0}^{[m/2]} \frac{(-1)^k m!}{k!(m-2k)!}(2x)^{m-2k}, \quad (m = 0, 1, \ldots). \tag{2.1}$$

The operator which acts in this definition onto the functions $(2x)^m$ (asymptotics of Hermite polynomials for fixed m) is a Gaussian convolution operator (or better deconvolution operator for the negative sign in the exponent) and the explicit form can easily be obtained by Taylor series expansion of the exponential function where this series in powers of $\partial^{2k}/\partial x^{2k}$ in application to $(2x)^m$ can be truncated at $k = [m/2]$ ($[\mu]$ is integer part of μ). The usual definition of Hermite polynomials is

$$H_m(x) \equiv (-1)^m \exp(x^2)\frac{\partial^m}{\partial x^m}\exp(-x^2) = \left(2x - \frac{\partial}{\partial x}\right)^m 1, \quad (m = 0, 1, \ldots),$$

$$\left(2x - \frac{\partial}{\partial x}\right)^m = \sum_{k=0}^{m} \frac{(-1)^k m!}{k!(m-k)!} H_{m-k}(x)\frac{\partial^k}{\partial x^k}. \tag{2.2}$$

We emphasize here that $H_m(x)$ in (2.1) and (2.2) considered as operators which act not only onto the function $f(x)=1$ in the definition of the Hermite polynomials such as written in the first line but onto arbitrary functions $f(x)$ are nonequivalent. The Laguerre polynomials, in analogy, also possess two similar equivalent definitions but it is reasonable to discuss this only after development of some formalism necessary for the present paper.

We now define a set of Hermite 2D polynomials $H_{m,n}(U; x, y)$ as polynomials of two independent real or, in general, complex variables (x, y) (2D vector) which depend on an arbitrary fixed 2D matrix U as a parameter in the following way:

$$H_{m,n}(U; x, y) \equiv \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(2x')^m(2y')^n, \quad (m, n = 0, 1, \ldots), \tag{2.3}$$

with linearly transformed 2D vector (x', y') written as column vectors by means of the 2D matrix U according to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_{xx}x + U_{xy}y \\ U_{yx}x + U_{yy}y \end{pmatrix}, \quad U \equiv \begin{pmatrix} U_{xx}, & U_{xy} \\ U_{yx}, & U_{yy} \end{pmatrix}, \quad |U| \equiv U_{xx}U_{yy} - U_{xy}U_{yx}. \tag{2.4}$$

The corresponding transformation of the partial derivatives written as row vectors is

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}\right) U = \left(U_{xx}\frac{\partial}{\partial x'} + U_{yx}\frac{\partial}{\partial y'}, U_{xy}\frac{\partial}{\partial x'} + U_{yy}\frac{\partial}{\partial y'}\right). \tag{2.5}$$

Definition (2.3) is made in such a way that the identity matrix I provides products of two usual Hermite polynomials $H_m(x)H_n(y)$ as a special set of Hermite 2D polynomials

$$\begin{aligned} H_{m,n}(I; x, y) &\equiv \exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(2x)^m(2y)^n = H_m(x)H_n(y) \\ &= \left(2x - \frac{\partial}{\partial x}\right)^m \left(2y - \frac{\partial}{\partial y}\right)^n 1, \quad I \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}. \end{aligned} \tag{2.6}$$

The added equivalent definition on the right-hand side in analogy to (2.2) suggests that there is also in the general case of an arbitrary matrix U a possible equivalent definition of Hermite 2D polynomials starting from (2.2) but this has to be considered carefully because their definition is already given by (2.3), and one cannot automatically use the transformations (2.4) and (2.5) for this purpose (see [14]). From definition (2.3), we find that the multiplication of the matrix U by an arbitrary factor λ leads only to the multiplication of the obtained Hermite 2D polynomial $H_{m,n}(U; x, y)$ by a factor λ^{m+n} that means to 2D polynomials which are not essentially different from the considered one. Therefore, one can restrict oneself in most cases to the considerations of matrices U with determinant $|U|$ equal $|U| = 1$ (unimodular matrices). Exceptions form the cases with vanishing determinant $|U| = 0$ which we treat separately and which lead to usual Hermite polynomials with arguments being linear combinations of (x, y). The set of 2D unimodular matrices forms the three-parameter group SL(2, C) in case of complex arguments with many possible subgroups of matrices. Therefore, the definition of Hermite 2D polynomials by (2.3) can be considered as the SL(2, C) generalization of products of two Hermite polynomials.

We now derive two very different explicit representations of the Hermite 2D polynomials. The first of these representations is obtained by writing the powers $x'^m y'^n$ in (2.3) as superpositions of powers $x^k y^l$ of the primary variables in the first step. This leads to the following relation:

$$x'^m y'^n = \left(\sqrt{|U|}\right)^{m+n} \sum_{j=0}^{m+n} \left(\frac{U_{xy}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{yy}}{\sqrt{|U|}}\right)^{n-j} P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{xy}U_{yx}}{|U|}\right) x^j y^{m+n-j}, \tag{2.7}$$

where $P_j^{(\alpha,\beta)}(u)$ denotes the Jacobi polynomials defined in a standard way (e.g., [8], see also [14–16]). The comparison of the explicit form of the coefficients in front of $x^j y^{m+n-j}$ with the explicit representation of the Jacobi polynomials proves this. In the second step, we insert (2.7) into (2.3) and apply the alternative definition of Hermite polynomials given in (2.1). This leads to the following explicit representation of the Hermite 2D polynomials by superposition of products of usual Hermite polynomials $H_j(x)H_{m+n-j}(y)$ with constant sum $m+n$ of the indices

$$H_{m,n}(U;x,y) = \left(\sqrt{|U|}\right)^{m+n} \sum_{j=0}^{m+n} \left(\frac{U_{xy}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{yy}}{\sqrt{|U|}}\right)^{n-j} \times P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{xy}U_{yx}}{|U|}\right) H_j(x)H_{m+n-j}(y). \tag{2.8}$$

We see here that all terms with the matrix U in the interior part of the sum can be reduced to corresponding unimodular matrices ($|U| = 1$) without changing this sum and only the factor $(\sqrt{|U|})^{m+n}$ in front of the sum indicates the determinant of the matrix U.

A second essentially different representation of the Hermite 2D polynomials can be derived by using transformation (2.6) and by writing definition (2.3) in the first step

$$H_{m,n}(U;x,y) = 2^{m+n}\exp\left\{-\frac{1}{4}\left((U_{xx}^2 + U_{xy}^2)\frac{\partial^2}{\partial x'^2} + (U_{yx}^2 + U_{yy}^2)\frac{\partial^2}{\partial y'^2} + 2(U_{xx}U_{yx} + U_{xy}U_{yy})\frac{\partial^2}{\partial x'\partial y'}\right)\right\} x'^m y'^n. \tag{2.9}$$

One of the operations in this expression has a structure which can be carried out by Taylor series expansion of the exponential function as follows ($\{m,n\} \equiv \mathrm{Min}(m,n)$):

$$\exp\left(-\lambda\frac{\partial^2}{\partial u \partial v}\right) u^m v^n = \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!}(-\lambda)^j u^{m-j} v^{n-j}. \tag{2.10}$$

The right-hand side possesses a structure which is related to Laguerre polynomials and can be represented by them. If we use this relation in correspondingly substituted form in (2.9) and if we apply the remaining convolution operators onto the arising products $x'^{m-j} y'^{n-j}$ that leads to products of Hermite polynomials and if we finally make the transition back from the variables (x', y') to the primary variables (x, y) according to (2.4), we obtain

$$H_{m,n}(U;x,y) = \left(\sqrt{U_{xx}^2 + U_{xy}^2}\right)^m \left(\sqrt{U_{yx}^2 + U_{yy}^2}\right)^n \times \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(-2\frac{U_{xx}U_{yx} + U_{xy}U_{yy}}{\sqrt{(U_{xx}^2 + U_{xy}^2)(U_{yx}^2 + U_{yy}^2)}}\right)^j \times H_{m-j}\left(\frac{U_{xx}x + U_{xy}y}{\sqrt{U_{xx}^2 + U_{xy}^2}}\right) H_{n-j}\left(\frac{U_{yx}x + U_{yy}y}{\sqrt{U_{yx}^2 + U_{yy}^2}}\right). \tag{2.11}$$

The above considerations prove the unknown identity of the expressions on the right-hand sides of (2.8) and (2.11) which in special cases (e.g., vanishing determinant of U) makes the transition to known identities.

We now consider the degenerate case of vanishing determinant of U. In this case, x' and y' become linearly dependent according to

$$|U| = U_{xx}U_{yy} - U_{xy}U_{yx} = 0 \quad \Rightarrow \quad y' = \frac{U_{yx}}{U_{xx}}x' = \frac{U_{yy}}{U_{xy}}x'. \tag{2.12}$$

This leads according to (2.3) and (2.1) to usual Hermite polynomials with transformed argument as follows:

$$(H_{m,n}(U;x,y))_{|U|=0} = \left(\frac{U_{yx}}{U_{xx}}\right)^n H_{m+n,0}(U;x,y) = \left(\frac{U_{xy}}{U_{yy}}\right)^m H_{0,m+n}(U;x,y). \tag{2.13}$$

According to (2.11) this can be explicitly represented by

$$(H_{m,n}(U;x,y))_{|U|=0} = \left(\frac{U_{yx}}{U_{xx}}\right)^n (\sqrt{U_{xx}^2 + U_{xy}^2})^{m+n} H_{m+n}\left(\frac{U_{xx}x + U_{xy}y}{\sqrt{U_{xx}^2 + U_{xy}^2}}\right), \tag{2.14}$$

or by a second equivalent form which we do not write down. The Jacobi polynomials as coefficients in (2.8) reduce in the corresponding case ($m \to m+n, n \to 0$) and limiting case of argument to binomial coefficients because x'^{m+n} can be represented by powers of (x,y) by using the binomial formula. Therefore, representation (2.8) simplifies in this case to

$$(H_{m,n}(U;x,y))_{|U|=0} = \left(\frac{U_{yx}}{U_{xx}}\right)^n \sum_{j=0}^{m+n} \frac{(m+n)!}{j!(m+n-j)!} U_{xx}^j U_{xy}^{m+n-j} H_j(x) H_{m+n-j}(y). \tag{2.15}$$

The identity of the right-hand sides of (2.14) and (2.15) is a known identity (addition theorem for Hermite polynomials) obtained here as a subsidiary result.

3. Definition of Laguerre 2D polynomials

Definition (2.3) for Hermite 2D polynomials can be generalized in an obvious way to a definition of Hermite νD polynomials in the ν-dimensional case. However, the two-dimensional case shows a peculiarity connected with the transition to a pair of complex conjugated variables $(z,\bar{z})$ which we represent according to (in physical context, the notation z^* instead of $\bar{z}$ is mostly used but in pure mathematical context this notation is never(?) used)

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} x + \mathrm{i}y \\ x - \mathrm{i}y \end{pmatrix} = (1-\mathrm{i})Z\begin{pmatrix} x \\ y \end{pmatrix}, \quad Z \equiv \frac{1}{2}\begin{pmatrix} 1+\mathrm{i}, & -1+\mathrm{i} \\ 1+\mathrm{i}, & 1-\mathrm{i} \end{pmatrix} = \bar{Z}^{-1}, \qquad |Z| = 1. \tag{3.1}$$

The matrix Z is introduced as a unimodular matrix and, in addition, it became automatically a unitary matrix ($Z^{-1} = \bar{Z}$). The operators of partial differentiation are related by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{3.2}$$

where additionally are given the two representations of the 2D Laplace operator.

The considerations suggest to define in analogy to (2.3) the following set of Laguerre 2D polynomials

$$L_{mn}(U;z,\bar{z}) \equiv \exp\left(-\frac{\partial^2}{\partial z \partial \bar{z}}\right) z'^m \bar{z}'^n, \quad (m,n = 0,1,\ldots), \tag{3.3}$$

where the 2D matrix U is given by

$$\begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = U \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} U_{zz}z + U_{z\bar{z}}\bar{z} \\ U_{\bar{z}z}z + U_{\bar{z}\bar{z}}\bar{z} \end{pmatrix}, \quad U \equiv \begin{pmatrix} U_{zz}, & U_{z\bar{z}} \\ U_{\bar{z}z}, & U_{\bar{z}\bar{z}} \end{pmatrix}. \tag{3.4}$$

In the special case of the identity matrix $U = I$, definition (3.3) provides explicitly

$$\begin{aligned} L_{m,n}(I; z, \bar{z}) &= \exp\left(-\frac{\partial^2}{\partial z \partial \bar{z}}\right) z^m \bar{z}^n = \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!}(-1)^j z^{m-j}\bar{z}^{n-j} \\ &= (-1)^n n! z^{m-n} L_n^{m-n}(z\bar{z}) = (-1)^m m! \bar{z}^{n-m} L_m^{n-m}(z\bar{z}), \end{aligned} \tag{3.5}$$

where $L_n^{\nu}(u)$ denotes the generalized Laguerre polynomials in their modern definition (e.g., [8]). These representations justify the chosen name "Laguerre 2D polynomials". It is even this special case $U = I$ in (3.3) which plays an important role for the different representations of quasiprobabilities in quantum optics [15,16]. In analogy to usual Hermite and to Hermite 2D polynomials, the special Laguerre 2D polynomials (3.5) possess a possible alternative definition according to

$$L_{m,n}(I; z, \bar{z}) \equiv (-1)^{m+n} \exp(z\bar{z}) \frac{\partial^{m+n}}{\partial \bar{z}^m \partial z^n} \exp(-z\bar{z}) = \left(z - \frac{\partial}{\partial \bar{z}}\right)^m \left(\bar{z} - \frac{\partial}{\partial z}\right)^n 1. \tag{3.6}$$

With some care, this definition can also be generalized to the definition of $L_{m,n}(U; z, \bar{z})$ [14]. In a similar way, the usual Laguerre polynomials possess two fully equivalent definitions.

The Laguerre 2D polynomials are closely related to Hermite 2D polynomials and vice versa according to

$$\begin{aligned} L_{m,n}(U; x + \mathrm{i}y, x - \mathrm{i}y) &= \left(\frac{1 - \mathrm{i}}{2}\right)^{m+n} H_{m,n}(UZ, x, y), \\ H_{m,n}(U; x, y) &= (1 + \mathrm{i})^{m+n} L_{m,n}(U\bar{Z}, x + \mathrm{i}y, x - \mathrm{i}y). \end{aligned} \tag{3.7}$$

It is, however, not appropriate to abandon the definition of Laguerre 2D polynomials because it is mostly better suited for the representation of results by pairs of complex conjugated variables $(z, \bar{z})$.

In analogy to Hermite 2D polynomials, one finds two equivalent explicit representations of the Laguerre 2D polynomials, first by special Laguerre 2D polynomials with Jacobi polynomials as coefficients

$$\begin{aligned} L_{m,n}(U; z, \bar{z}) = \left(\sqrt{|U|}\right)^{m+n} \sum_{j=0}^{m+n} &\left(\frac{U_{z\bar{z}}}{\sqrt{|U|}}\right)^{m-j} \left(\frac{U_{\bar{z}\bar{z}}}{\sqrt{|U|}}\right)^{n-j} \\ &\times P_j^{(m-j,n-j)}\left(1 + 2\frac{U_{z\bar{z}}U_{\bar{z}z}}{|U|}\right) L_{j,m+n-j}(I; z, \bar{z}), \end{aligned} \tag{3.8}$$

and, second by superposition of products of Hermite polynomials with transformed arguments

$$\begin{aligned} L_{m,n}(U; z, \bar{z}) = (\sqrt{U_{zz}U_{z\bar{z}}})^m (\sqrt{U_{\bar{z}z}U_{\bar{z}\bar{z}}})^n &\sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \\ &\times \left(-\frac{U_{zz}U_{\bar{z}\bar{z}} + U_{z\bar{z}}U_{\bar{z}z}}{\sqrt{U_{zz}U_{z\bar{z}}U_{\bar{z}z}U_{\bar{z}\bar{z}}}}\right)^j H_{m-j}\left(\frac{U_{zz}z + U_{z\bar{z}}\bar{z}}{2\sqrt{U_{zz}U_{z\bar{z}}}}\right) H_{n-j}\left(\frac{U_{\bar{z}z}z + U_{\bar{z}\bar{z}}\bar{z}}{2\sqrt{U_{\bar{z}z}U_{\bar{z}\bar{z}}}}\right). \end{aligned} \tag{3.9}$$

The degenerate case of vanishing determinant of U in Laguerre 2D polynomials can be dealt with in analogy to the degenerate case for Hermite 2D polynomials. This leads to

$$(L_{m,n}(U;z,\bar{z}))_{|U|=0} = \left(\frac{U_{\bar{z}z}}{U_{zz}}\right)^n L_{m+n,0}(U;z,\bar{z}), \left(\frac{U_{z\bar{z}}}{U_{\bar{z}\bar{z}}}\right)^m L_{0,m+n}(U;z,\bar{z}). \tag{3.10}$$

With the explicit form of $L_{m+n,0}(U;z,\bar{z})$ taken from (3.9), one finds

$$(L_{m,n}(U;z,\bar{z}))_{|U|=0} = \left(\frac{U_{\bar{z}z}}{U_{zz}}\right)^n (\sqrt{U_{zz}U_{z\bar{z}}})^{m+n} H_{m+n}\left(\frac{U_{zz}z + U_{z\bar{z}}\bar{z}}{2\sqrt{U_{zz}U_{z\bar{z}}}}\right). \tag{3.11}$$

With the representation of the Jacobi polynomials in the considered case by binomial coefficients, one obtains the following representation by special Laguerre 2D polynomials:

$$(L_{m,n}(U;z,\bar{z}))_{|U|=0} = \left(\frac{U_{\bar{z}z}}{U_{zz}}\right)^n \sum_{j=0}^{m+n} \frac{(m+n)!}{j!(m+n-j)!} U_{zz}^j U_{z\bar{z}}^{m+n-j} L_{j,m+n-j}(I;z,\bar{z}), \tag{3.12}$$

where explicit forms of $L_{j,m+n-j}(I;z,\bar{z})$ can be taken from (3.5).

4. Recursion relations

By differentiation of the defining relations (2.3) for Hermite 2D polynomials with regard to x and y, one obtains relations which can be linearly combined to the following relations:

$$\frac{\partial}{\partial x'} H_{m,n}(U;x,y) = \frac{1}{|U|}\left(U_{yy}\frac{\partial}{\partial x} - U_{yx}\frac{\partial}{\partial y}\right) H_{m,n}(U;x,y) = 2mH_{m-1,n}(U;x,y),$$

$$\frac{\partial}{\partial y'} H_{m,n}(U;x,y) = \frac{1}{|U|}\left(-U_{xy}\frac{\partial}{\partial x} + U_{xx}\frac{\partial}{\partial y}\right) H_{m,n}(U;x,y) = 2nH_{m,n-1}(U;x,y). \tag{4.1}$$

Thus, we have obtained the lowering operators of the indices of the Hermite 2D polynomials. The corresponding raising operators can be obtained by using the commutation relations

$$\exp\left\{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}(x,y)\exp\left\{\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\} = \left(x - \frac{1}{2}\frac{\partial}{\partial x}, y - \frac{1}{2}\frac{\partial}{\partial y}\right). \tag{4.2}$$

This leads to

$$2\left\{U_{xx}\left(x - \frac{1}{2}\frac{\partial}{\partial x}\right) + U_{xy}\left(y - \frac{1}{2}\frac{\partial}{\partial y}\right)\right\} H_{m,n}(U;x,y) = H_{m+1,n}(U;x,y),$$

$$2\left\{U_{yx}\left(x - \frac{1}{2}\frac{\partial}{\partial x}\right) + U_{yy}\left(y - \frac{1}{2}\frac{\partial}{\partial y}\right)\right\} H_{m,n}(U;x,y) = H_{m,n+1}(U;x,y), \tag{4.3}$$

that determines the explicit form of the raising operators of the indices of the Hermite 2D polynomials. By elimination of the differentiations of the Hermite 2D polynomials in (4.3) by means of (4.2), one obtains the following recursion relations:

$$H_{m+1,n}(U;x,y) = 2(U_{xx}x + U_{xy}y)H_{m,n}(U;x,y) - 2m(U_{xx}^2 + U_{xy}^2)H_{m-1,n}(U;x,y)$$
$$-2n(U_{xx}U_{yx} + U_{xy}U_{yy})H_{m,n-1}(U;x,y),$$

$$H_{m,n+1}(U;x,y) = 2(U_{yx}x + U_{yy}y)H_{m,n}(U;x,y) - 2n(U_{yx}^2 + U_{yy}^2)H_{m,n-1}(U;x,y)$$
$$-2m(U_{xx}U_{yx} + U_{xy}U_{yy})H_{m-1,n}(U;x,y). \quad (4.4)$$

The nondiagonal elements U_{xy} and U_{yx} of the 2D matrix U couple these two sets of recursion relations.

The lowering operators for the Laguerre 2D polynomials are determined by

$$\frac{\partial}{\partial z'}L_{m,n}(U;z,\bar{z}) = \frac{1}{|U|}\left(U_{\bar{z}\bar{z}}\frac{\partial}{\partial z} - U_{\bar{z}z}\frac{\partial}{\partial \bar{z}}\right)L_{m,n}(U;z,\bar{z}) = mL_{m-1,n}(U;z,\bar{z}),$$
$$\frac{\partial}{\partial \bar{z}'}L_{m,n}(U;z,\bar{z}) = \frac{1}{|U|}\left(-U_{z\bar{z}}\frac{\partial}{\partial z} + U_{zz}\frac{\partial}{\partial \bar{z}}\right)L_{m,n}(U;z,\bar{z}) = nL_{m,n-1}(U;z,\bar{z}). \quad (4.5)$$

By using the commutation relations

$$\exp\left(-\frac{\partial^2}{\partial z \partial \bar{z}}\right)(z,\bar{z})\exp\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right) = \left(z - \frac{\partial}{\partial \bar{z}}, \bar{z} - \frac{\partial}{\partial z}\right), \quad (4.6)$$

one obtains in an analogous way to (4.3) the raising operators

$$\left\{U_{zz}\left(z - \frac{\partial}{\partial \bar{z}}\right) + U_{z\bar{z}}\left(\bar{z} - \frac{\partial}{\partial z}\right)\right\}L_{m,n}(U;z,\bar{z}) = L_{m+1,n}(U;z,\bar{z}),$$
$$\left\{U_{\bar{z}z}\left(z - \frac{\partial}{\partial \bar{z}}\right) + U_{\bar{z}\bar{z}}\left(\bar{z} - \frac{\partial}{\partial z}\right)\right\}L_{m,n}(U;z,\bar{z}) = L_{m,n+1}(U;z,\bar{z}) \quad (4.7)$$

and in analogy to (4.4) the recursion relations

$$L_{m+1,n}(U;z,\bar{z}) = (U_{zz}z + U_{z\bar{z}}\bar{z})L_{m,n}(U;z,\bar{z}) - 2mU_{zz}U_{z\bar{z}}L_{m-1,n}(U;z,\bar{z})$$
$$-n(U_{zz}U_{\bar{z}\bar{z}} + U_{z\bar{z}}U_{\bar{z}z})L_{m,n-1}(U;z,\bar{z}),$$

$$L_{m,n+1}(U;z,\bar{z}) = (U_{\bar{z}z}z + U_{\bar{z}\bar{z}}\bar{z})L_{m,n}(U;z,\bar{z}) - 2nU_{\bar{z}z}U_{\bar{z}\bar{z}}L_{m,n-1}(U;z,\bar{z})$$
$$-m(U_{zz}U_{\bar{z}\bar{z}} + U_{z\bar{z}}U_{\bar{z}z})L_{m-1,n}(U,z,\bar{z}). \quad (4.8)$$

In comparison to (4.4), these two sets of recursions relations show a different type of coupling.

5. Orthonormalization and completeness relations

Besides the Hermite 2D polynomials, $H_{m,n}(U;x,y)$, and Laguerre 2D polynomials, we introduce Hermite 2D functions $h_{m,n}(U;x,y)$ and Laguerre 2D functions in the following way:

$$h_{m,n}(U;x,y) \equiv \frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2+y^2}{2}\right)\frac{H_{m,n}(U;x,y)}{\sqrt{2^{m+n}m!n!}},$$
$$l_{m,n}(U;z,\bar{z}) \equiv \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z\bar{z}}{2}\right)\frac{L_{m,n}(U;z,\bar{z})}{\sqrt{m!n!}}. \quad (5.1)$$

It can be checked in connection with the action of the lowering and raising operators given in the preceding section that these functions satisfy the following eigenvalue equations for the degenerate 2D harmonic oscillator

$$\left\{\frac{x^2+y^2}{2}-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}h_{m,n}(U;x,y)=(m+n+1)h_{m,n}(U;x,y),$$

$$\left\{\frac{z\bar{z}}{2}-2\frac{\partial^2}{\partial z\partial\bar{z}}\right\}l_{m,n}(U;z,\bar{z})=(m+n+1)l_{m,n}(U;z,\bar{z}). \tag{5.2}$$

The main purpose of the introduction of the Hermite and Laguerre 2D functions in the above form is their orthonormalization and completeness. We do not present here their derivation [14] and give only the final results together with some remarks. The orthonormalization relations for general 2D matrices U are ($\mathrm{d}x\wedge\mathrm{d}y=(\mathrm{i}/2)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$ area element of plane, $\tilde{U}$ transposed matrix U)

$$\int \mathrm{d}x\wedge\mathrm{d}y\, h_{k,l}(\tilde{U}^{-1};x,y)h_{m,n}(U;x,y)=\delta_{k,m}\delta_{l,n},$$

$$\int \frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\, l_{k,l}(\tilde{U}^{-1};\bar{z},z)l_{m,n}(U;z,\bar{z})=\delta_{k,m}\delta_{l,n}. \tag{5.3}$$

According to (5.2), the Hermite and Laguerre 2D functions are eigenfunctions of a Hermitean operator to eigenvalues $m+n+1$ for arbitrary matrix U. This means that Hermite and Laguerre 2D functions to, in general, different matrices U and V are orthogonal to each other for different sums $m+n$. The remaining part of the proof can be obtained from an addition theorem for the Jacobi polynomials which appear as coefficients in representations (2.8) and (3.8). By considering the product $W=UV$ of two 2D matrices U and V, one obtains from the composition of transformations of powers of the two variables (x,y) or $(z,\bar{z})$ written by involving the Jacobi polynomials in the coefficients a composition identity for Jacobi polynomials which we call addition theorem and which is derived in [14] (Appendix A). By specialization to $W=UV=I$ and applied to the left-hand sides of (5.3) in their explicit representation, this leads finally to (5.3). The corresponding completeness relations obtained by transformation of the known completeness relations for the special case of the identity matrix $U=I$ in connection with the orthonormality relations are

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{m,n}(U;x,y)h_{m,n}(\tilde{U}^{-1};x',y')=\delta(x-x')\delta(y-y'),$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} l_{m,n}(U;z,\bar{z})l_{m,n}(\tilde{U}^{-1};\bar{z}',z')=\delta(z-z',\bar{z}-\bar{z}'), \tag{5.4}$$

where $\delta(z,\bar{z})=\delta(x)\ \delta(y)$ denote the two-dimensional delta functions. Relations (5.3) and (5.4) allow to make expansions of two-dimensional functions into sets of Hermite or Laguerre 2D functions and to determine the coefficients in these expansions.

In most applications, U is a unitary matrix $U^{-1}=\bar{\tilde{U}}$ and the Hermite and Laguerre 2D functions in (5.3) and (5.4) with the matrix U^{-1} as the parameter can be related in this case to complex conjugation of analogous functions with U as the parameter that is then similar to the usual form of such relations in the one-dimensional case. We do not explicitly write down this. Whereas in the definitions of the Hermite and Laguerre 2D polynomials the variables (x,y) can be extended to their own complex planes and $(z,\bar{z})$ can be extended to an independent pair (z,w) of complex variables, in relations (5.3) and (5.4) they are necessarily pairs of real or complex conjugated variables, respectively.

We mention here that one can introduce annihilation and creation operators for lowering and raising the indices of the Hermite and Laguerre 2D functions [15]. One finds from (4.1) and (4.3) for the Hermite 2D functions

$$\begin{aligned}
&\frac{1}{\sqrt{2}\,|U|}\left\{+U_{yy}\left(x+\frac{\partial}{\partial x}\right)-U_{yx}\left(y+\frac{\partial}{\partial y}\right)\right\}h_{m,n}(U;x,y)=\sqrt{m}\,h_{m-1,n}(U;x,y),\\
&\frac{1}{\sqrt{2}\,|U|}\left\{-U_{xy}\left(x+\frac{\partial}{\partial x}\right)+U_{xx}\left(y+\frac{\partial}{\partial y}\right)\right\}h_{m,n}(U;x,y)=\sqrt{n}\,h_{m,n-1}(U;x,y),\\
&\frac{1}{\sqrt{2}}\left\{U_{xx}\left(x-\frac{\partial}{\partial x}\right)+U_{xy}\left(y-\frac{\partial}{\partial y}\right)\right\}h_{m,n}(U;x,y)=\sqrt{m+1}\,h_{m+1,n}(U;x,y),\\
&\frac{1}{\sqrt{2}}\left\{U_{yx}\left(x-\frac{\partial}{\partial x}\right)+U_{yy}\left(y-\frac{\partial}{\partial y}\right)\right\}h_{m,n}(U;x,y)=\sqrt{n+1}\,h_{m,n+1}(U;x,y)
\end{aligned}\tag{5.5}$$

and from (4.5) and (4.7) for the Laguerre 2D functions

$$\begin{aligned}
&\frac{1}{|U|}\left\{+U_{\bar{z}\bar{z}}\left(\frac{\bar{z}}{2}+\frac{\partial}{\partial z}\right)-U_{\bar{z}z}\left(\frac{z}{2}+\frac{\partial}{\partial \bar{z}}\right)\right\}l_{m,n}(U;z,\bar{z})=\sqrt{m}\,l_{m-1,n}(U;z,\bar{z}),\\
&\frac{1}{|U|}\left\{-U_{z\bar{z}}\left(\frac{\bar{z}}{2}+\frac{\partial}{\partial z}\right)+U_{zz}\left(\frac{z}{2}+\frac{\partial}{\partial \bar{z}}\right)\right\}l_{m,n}(U;z,\bar{z})=\sqrt{n}\,l_{m,n-1}(U;z,\bar{z}),\\
&\left\{U_{zz}\left(\frac{z}{2}-\frac{\partial}{\partial \bar{z}}\right)+U_{z\bar{z}}\left(\frac{\bar{z}}{2}-\frac{\partial}{\partial z}\right)\right\}l_{m,n}(U;z,\bar{z})=\sqrt{m+1}\,l_{m+1,n}(U;z,\bar{z}),\\
&\left\{U_{\bar{z}z}\left(\frac{z}{2}-\frac{\partial}{\partial \bar{z}}\right)+U_{\bar{z}\bar{z}}\left(\frac{\bar{z}}{2}-\frac{\partial}{\partial z}\right)\right\}l_{m,n}(U;z,\bar{z})=\sqrt{n+1}\,l_{m,n+1}(U;z,\bar{z}).
\end{aligned}\tag{5.6}$$

One can check that the set of operators in front of the 2D functions in (5.5) and in (5.6) form in both cases a two-mode Heisenberg–Weyl algebra (compare with special case in [15]).

6. Generating functions

By using definition (2.3) of the Hermite 2D polynomials, one obtains the following simplest generating function:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(U;x,y)\\
&\quad=\exp\{2(sU_{xx}+tU_{yx})x+2(sU_{xy}+tU_{yy})y-(sU_{xx}+tU_{yx})^2-(sU_{xy}+tU_{yy})^2\}.
\end{aligned}\tag{6.1}$$

This can be written in a concise form by using the following notations of row or column vectors in dependence on the position in the relations (we do not distinguish this by a symbol for transposition) and corresponding indices of the elements of U

$$x \equiv (x_1, x_2), \quad s \equiv (s_1, s_2) = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad n \equiv (n_1, n_2), \quad \tilde{U} = \begin{pmatrix} U_{11}, & U_{21} \\ U_{12}, & U_{22} \end{pmatrix}, \tag{6.2}$$

where $\tilde{U}$ denotes the transposed matrix to U. Then, we can represent (6.1) in the form

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{s_1^{n_1} s_2^{n_2}}{n_1! n_2!} H_{n_1,n_2}(U; x_1, x_2) = \exp(2sUx - sU\tilde{U}s). \tag{6.3}$$

The generalization to the vD case is obvious by generalization of the vectors and matrices to this dimension.

The form (6.3) of the generating function of Hermite 2D polynomials enables us to compare it with the generating function for the usual two-variable Hermite polynomials $H^R_{n_1,n_2}(x_1, x_2)$ which commonly serves as the definition of these polynomials [8]

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t_1^{n_1} t_2^{n_2}}{n_1 n_2!} H^R_{n_1,n_2}(x'_1, x'_2) = \exp\left(tRx' - \frac{1}{2} tRt\right), \quad R = \tilde{R}, \tag{6.4}$$

where R is a symmetric 2D matrix. By comparison with the generating function (6.3), we see the following. Starting from (6.3), the parameters of the Hermite 2D polynomials can be expressed by the parameters of two-variable Hermite polynomials, for example, according to $t = \sqrt{2}\,s$, $R = U\tilde{U}$, $\sqrt{2}\,\tilde{U}x' = x$ that involves an argument transformation. The main difference, however, is that the same symmetric matrix R in the two-variable Hermite polynomials is involved in both terms in the exponent on the right-hand side of (6.4), whereas for the Hermite 2D polynomials (6.3) the first term of the exponent involves only the matrix U which is the square root of the matrix R in the sense $U\tilde{U} = R$. This square root problem does not have, in general, a unique solution for U but a one-parameter set of solutions [14]. The appearance of different matrices U and $U\tilde{U}$ in the two terms in the exponent of the generating functions for the Hermite 2D polynomials is the main difference to the usual two-variable Hermite polynomials and the trick which leads to their orthonormalizability, whereas two kinds of two-variable Hermite polynomials satisfy biorthogonality relations.

For the generating function for the Laguerre 2D polynomials, one finds analogously

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s_1^m s_2^n}{m!n!} L_{m,n}(U; z, \bar{z}) = \exp\left(s\,Ux - \frac{1}{2} s\,U\sigma_1\tilde{U}s\right), \tag{6.5}$$

with similar notations as given in (6.2) but here with $x \equiv (z, \bar{z})$. The matrix σ_1 is the first of the Pauli spin matrices and its appearance is due to

$$Z\tilde{Z} = \mathrm{i}\,\sigma_1, \quad \sigma_1 \equiv \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \tag{6.6}$$

where Z is the matrix for the transition from real to complex variables according to (3.1).

7. Conclusion

We defined Hermite and Laguerre 2D polynomials and considered shortly some of their properties (more details, proofs and references in [14]). These 2D polynomials play a great role in many applications, for example, in quantum optics. The main advantage of our approach in comparison to the existing two-variable Hermite polynomials is their orthonormalizability with Gaussian weight factors.

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Appendix A. Equivalent definitions of Laguerre polynomials

The case of Hermite polynomials discussed at the beginning suggests that also the usual Laguerre polynomials possess two equivalent possible definitions. Since this is almost unknown, we will shortly discuss this here (see [16,18]). The special Laguerre 2D polynomials $L_{m,n}(z,\bar{z})$ in which the usual Laguerre polynomials $L_n^{m-n}(u)$ appear with the argument $u \equiv z\bar{z}$ suggest to consider modified polar coordinates (u,φ) as follows:

$$u = z\bar{z}, \quad e^{i\varphi} = \sqrt{\frac{z}{\bar{z}}}, \quad z = \sqrt{u}e^{i\varphi}, \quad \bar{z} = \sqrt{u}e^{-i\varphi} \tag{A.1}$$

with the following relations of the partial derivatives:

$$\frac{1}{\bar{z}}\frac{\partial}{\partial z} = \frac{\partial}{\partial u} - \frac{i}{2u}\frac{\partial}{\partial \varphi}, \quad \frac{1}{z}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial u} + \frac{i}{2u}\frac{\partial}{\partial \varphi} \quad \Rightarrow \quad \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial u}u\frac{\partial}{\partial u} + \frac{1}{4u}\frac{\partial^2}{\partial \varphi^2}. \tag{A.2}$$

Since $L_n(u) \equiv L_n^0(u)$ as a function in the variables (u,φ) does not depend on φ, one can apply (3.5) in the special case $m = n$ by omitting the part with derivatives on φ. This leads to

$$L_n(u) = \frac{(-1)^n}{n!}\exp\left(-\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right)u^n. \tag{A.3}$$

By Taylor series expansion of the exponential function with the operator in the argument, one can easily check that this leads to the right explicit representation of the Laguerre polynomials. On the other side, the equivalent definition (3.6) leads to

$$L_n(u) = \frac{(-1)^n}{n!}\exp(u)\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right)^n \exp(-u). \tag{A.4}$$

For the transition from $L_n(u)$ to the generalized Laguerre polynomials $L_n^\nu(u)$, one has the following two equivalent ways [16]:

$$L_n^\nu(u) = \left(1 - \frac{\partial}{\partial u}\right)^\nu L_n(u) \tag{A.5}$$

and [8]

$$L_n^\nu(u) = \left(-\frac{\partial}{\partial u}\right)^n L_{n+\nu}(u), \tag{A.6}$$

from which the second is well known. We mention here that the operator in (A.4) can be represented in the following disentangled form (or normal ordering):

$$\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right)^n = \sum_{j=0}^{n} \frac{n!^2}{j!(n-j)!^2} u^{n-j} \frac{\partial^{2n-j}}{\partial u^{2n-j}}, \tag{A.7}$$

that can be proved by complete induction. It is easy to derive from these formulas lowering and raising operators for the indices in the generalized Laguerre polynomials. There is still another representation of generalized Laguerre polynomials. For the purpose to prove it, we consider the following action of a generalized operator (A.7) onto functions $f(u) = u^n$:

$$\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u} + v\frac{\partial}{\partial u}\right)^j u^n = \frac{(n+v)!n!}{(n+v-j)!(n-j)!} u^{n-j}, \tag{A.8}$$

that can be proved by complete induction. By means of this relation, one easily verifies that the Taylor series expansion of the exponential function in the following relation leads to:

$$\begin{aligned} L_n^v(u) &= \frac{(-1)^n}{n!} \exp\left\{-\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u} + v\frac{\partial}{\partial u}\right)\right\} u^n \\ &= \frac{(-1)^n}{n!} \sum_{j=0}^{\{n+v,n\}} \frac{(-1)^j (n+v)!n!}{j!(n+v-j)!(n-j)!} u^{n-j}, \end{aligned} \tag{A.9}$$

that is the right explicit representation for $L_n^v(u)$.

We mention additionally that the three operators

$$K_- \equiv \frac{\partial}{\partial u}u\frac{\partial}{\partial u} + v\frac{\partial}{\partial u}, \quad K_0 \equiv \frac{1}{2}\left(\frac{\partial}{\partial u}u + u\frac{\partial}{\partial u} + v\right), \quad K_+ \equiv u, \tag{A.10}$$

form a basis of a three-dimensional Lie algebra due to the commutation relations

$$\begin{aligned} &[K_0, K_-] = -K_-, \quad [K_0, K_+] = +K_+, \quad [K_-, K_+] = 2K_0, \\ &[C, K_\mp] = [C, K_0] = 0, \quad C \equiv (K_0)^2 - \tfrac{1}{2}(K_-K_+ + K_+K_-) = \frac{v^2-1}{4}. \end{aligned} \tag{A.11}$$

This is a realization of the Lie algebra $su(1,1) \sim su(2,\mathrm{R}) \sim sp(2,\mathrm{R})$ by differentiation and multiplication operators with the Casimir operator C and many relations of operator ordering and disentanglement of operators can be obtained from the group-theoretical treatment.

Behind our considerations to usual Laguerre polynomials $L_n^v(u)$ is the observation that these polynomials appear in applications very often in two-dimensional problems, where the argument $u=r^2=z\bar{z}$ is connected with modified polar coordinates $(u = r^2, \varphi)$ or pairs of complex conjugated variables $(z, \bar{z})$.

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