Discrete Mathematics 6 Supervisior: Marton Havasi
Tryhard solution 23/02/2018

1. Prove that if A and B are sets such that $A \cong \mathbb{R}$ and $B \cong \mathbb{R}$ then $A \cup B \cong \mathbb{R}$.

   Note: I tried to come up with a proof that does not prove the Schröder–Bernstein theorem at the same time but I could not. In set theory, the Schröder–Bernstein theorem states that, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$. Check out the theorem here.

   Assume that a and b are bijections between A and $\mathbb{R}$ and B and $\mathbb{R}$ respectively.

   Consider $f(x) = e^x$. $f : \mathbb{R} \to \mathbb{R}^+$ is a bijection between $\mathbb{R}$ and $\mathbb{R}^+$.

   Consider $i : A \cup B \to \mathbb{R}$:

   $$i(x) = \begin{cases} a(x) & \text{if } x \in A \\ b(x) & \text{otherwise} \end{cases} .$$

   i is injective.

   Consider $j : \mathbb{R} \to A \cup B$:

   $$j(x) = \begin{cases} (f^{-1} \circ a^{-1})(x) & \text{if } x > 0 \\ (f^{-1} \circ b^{-1})(-x) & \text{if } x < 0 \\ a^{-1}(0) & \text{otherwise} \end{cases} .$$

   j is injective.

   Define the equivalence relations

   $$e_{\mathbb{R}}(x, y) = \exists\, n \geq 0.\big((i \circ j)^n(x) = y \land (i \circ j)^n(y) = x\big)$$

   $$e_{A\cup B}(u, v) = \exists\, n \geq 0.\big((j \circ i)^n(u) = v \land (j \circ i)^n(v) = u\big)$$

   Now we are ready to construct the bijection between $\mathbb{R}$ and $A \cup B$. By definition, the equivalence classes partition their respective sets. Every element is in exactly one equivalence class.

   We will construct h by exhibiting a bijection between $[x]_{e_{\mathbb{R}}}$ and $[j(x)]_{e_{A\cup B}}$ for arbitrary x in $\mathbb{R}$. The resulting relation is a bijection, since it also bijects the elements of $[u]_{e_{A\cup B}} = [j(i(u))]_{e_{A\cup B}}$ and $[i(u)]_{e_{\mathbb{R}}}$ for arbitrary u in $A \cup B$.

   - Case 1 $[x]_{e_{\mathbb{R}}}$ has an element y such that $\forall\, u \in A \cup B. i(u) \neq y$. In this case, j is a bijection between $[x]_{e_{\mathbb{R}}}$ and $[j(x)]_{e_{A\cup B}}$.
   - Case 2 $[x]_{e_{\mathbb{R}}}$ does not have an element y such that $\forall\, u \in A \cup B. i(u) \neq y$. In this case, i is a bijection between $[j(x)]_{e_{A\cup B}}$ and $[x]_{e_{\mathbb{R}}}$.