Discrete Mathematics 4 Supervisior: Marton Havasi
Lectures 10-12 07/01/2018

Practice questions

1. What is the number of divisors of $x = 2 \cdot 2 \cdot 2 \cdot 7 \cdot 11 \cdot 11 \cdot 17 \cdot 19$?
2. Prove, using mathematical induction, that $n^3 \leq 3^n$ for $n \in \mathbb{N}$.
3. State the Reflexivity, Transitivity and Antisymmetry properties of the subset ($\subseteq$) operator.

Core questions

1. Exercise sheet 4.1.2

   Prove that, for any positive integer n, a 2^n by 2^n square grid with any one square removed can be tiled with L-shaped pieces consisting of 3 squares.

2. Prove that for every positive integer n there exists an n digit number divisible by 5^n all of whose digits are odd.

   (Hint: using induction on n. If $a_1a_2 \ldots a_n$ is divisible by 5^n show that one of $1a_1a_2 \ldots a_n$, $3a_1a_2 \ldots a_n$, $5a_1a_2 \ldots a_n$, $7a_1a_2 \ldots a_n$, $9a_1a_2 \ldots a_n$ is divisible by 5^{n+1}.)

3. 2015 Paper 2 Question 9 part b, Link

4. Excercise sheet 4.3.2 The set of (univariate) polynomials (over the rationals) on a variable x is defined as that of arithmetic expressions equal to those of the form $\sum_{i=0}^{n} a_i x^i$, for some $n \in \mathcal{N}$ and some $a_1, \ldots a_n \in \mathbb{Q}$.

   (a) Show that if $p(x)$ and $q(x)$ are polynomials then so are $p(x) + q(x)$ and $p(x)q(x)$.

   (b) Deduce as a corollary that, for all $a, b \in \mathbb{Q}$, the linear combination $ap(x) + bq(x)$ of two polynomials $p(x)$ and $q(x)$ is a polynomial.

   (c) Show that there exists a polynomial $p_2(x)$ such that $p_2(n) = \sum_{i=0}^{n} i^2 = 0^2 + 1^2 + \cdots + n^2$ for every $n \in \mathbb{N}$.
   Hint: Note that for every $n \in \mathbb{N}$,
   $$(n+1)^3 = \sum_{i=0}^{n} (i+1)^3 - \sum_{i=0}^{n} i^3$$

   (d) Show that, for every $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ such that, for all $n \in \mathbb{N}$, $p_k(n) = \sum_{i=0}^{n} i^k = 0^k + 1^k + \cdots + n^k$.
   Hint: Generalise
   $$(n+1)^2 = \sum_{i=0}^{n} (i+1)^2 - \sum_{i=0}^{n} i^2$$
   and the hint in part (c) above.

5. 2017 Paper 2 Question 8, Link

Tryhard questions

1. Let a, b, c be positive integers. If $gcd(a, b, c)lcm[a, b, c] = abc$, prove that $gcd(a, b) = gcd(b, c) = gcd(c, a) = 1$. ($lcm$ means *least common multiple*)
2. Find all positive integers n such that $3^{n-1} + 5^{n-1} | 3^n + 5^n$.