Master thesis TODO

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30/08/2023

Random notes

Definition 0.1. Let G be a group. Let R be a G algebra over the field K, i.e. $R = KG$ with the free module structure, and normal multiplication of group and field elements.

Then the "stable module category over R", denoted $\mathcal{T} := \mathrm{StMod}(R)$ is defined in the following way:

1. $\mathrm{Obj}(\mathcal{T}) := \mathrm{Obj}(\mathrm{Mod}(R))$.
2. $\mathrm{Hom}_{\mathcal{T}}(A, B) := \mathrm{Hom}_R(A, B)/\,\{\text{maps that factor through a projective}\}$

Teorem 0.2. *The definition in Definition 0.1 is well defined, and it is an additive category.*

Proof. First, need to check that the set of maps that factor through a projective is an R submodule of $\mathrm{Hom}_R(A, B)$.

Let, f and g be two maps that factor through the projectives P and Q respectively. Then we have the following diagrams:

$$A \xrightarrow{f_1} P \xrightarrow{f_2} B$$

Where $f_2 \circ f_1 = f$, and

$$A \xrightarrow{g_1} Q \xrightarrow{g_2} B$$

Where $g_2 \circ g_1 = g$.

Can then construct the map

$$A \xrightarrow{(f_1, g_1)^T} P \oplus Q \xrightarrow{(f_2, g_2)} B$$

Composing these two maps, one gets the map $f_2 \circ f_1 + g_2 \circ g_1 = f + g$. This maps factors thorugh $P \oplus Q$, which is projective since it's a direct sum of projective modules.

Therefore, the set of homomorphisms that factor through a projective is closed under addition. And multiplying with a ring element still factors through the same projective, since every map is an R homomorphism. Therefore the set of maps that factor through a projective is an R submodule.

Therefore $\mathrm{Hom}_{\mathcal{T}}(A,B)$ is an abelian group, and the outstanding properties of an additive category is inherited from $\mathrm{Mod}(R)$ as well. $\square$

Definition 0.3. Let $A \in \mathrm{Obj}(\mathcal{T})$.

Let Σ be an endofunctor on $\mathcal{T}$. Where $\Sigma(A)$ is given by choosing a monomorphism from A into an injective module, I, denoted by ι_A and taking the cokernel of that map, for every A. I.e $\Sigma(A) = \mathrm{coker}(\iota_A)$.

Let Ω an endofunctor on $\mathcal{T}$. Where $\Omega(A)$ is given by choosing a projecive module P for every A with an endomorphism π_A from P to A, and taking the kernel of that map. I.e $\Omega(A) = \ker(\pi_A)$.

Remark 0.4. From the definition of Ω, $\Omega(f)$ is constructed as follows:

Looking at the following commutative diagram:

$$\begin{tikzcd}
\Omega(A) \arrow[r, hook, "\iota_A"] \arrow[d, "\Omega(f)"'] & P_A \arrow[r, two heads, "\pi_A"] \arrow[d, "h_f"] & A \arrow[d, "f"] \\
\Omega(B) \arrow[r, hook, "\iota_B"] & P_B \arrow[r, two heads, "\pi_B"] & B
\end{tikzcd}$$

One has that for a map $f : A \to B$, one gets the map h_f from the lifitng property of projective modules. Please note that this map is *not neccesarily* unique.

Furthermore, since $\pi_B \circ h_f \circ \iota_A = f \circ \pi_A \circ \iota_A = f \circ 0 = 0$, one has from the universal kernel property that there is a *unique* map $\Omega(f)$ from $\Omega(A)$ to $\Omega(B)$, which is the map defined by the functor.

Lemma 0.5. *One has that Ω is a functor.*

Proof. First want to show that Ω is functorial. Let $A, B, C \in \mathrm{Obj}(\mathcal{T})$. Then one can create the following diagram using the notation from before:

$$\begin{tikzcd}
\Omega(A) \arrow[r, hook] \arrow[d, dashed, "\Omega(g)"'] \arrow[dd, bend left, "\Omega(f\circ g)"] & P_A \arrow[r, two heads] \arrow[d, "h_g"] \arrow[dd, bend left, "h_{f\circ g}"] & A \arrow[d, "g"] \arrow[dd, bend left, "f\circ g"] \\
\Omega(B) \arrow[r, hook] \arrow[d, "\Omega(f)"'] & P_B \arrow[r, two heads] \arrow[d, "h_f"'] & B \arrow[d, "f"] \\
\Omega(C) \arrow[r, hook] & P_C \arrow[r, two heads] & C
\end{tikzcd}$$

Then, one gets the following commuting diagram:

$$\begin{tikzcd}
\Omega(A) \arrow[r, hook] \arrow[d, "\Omega(f\circ g)-\Omega(f)\circ\Omega(g)"'] & P_A \arrow[rr, two heads] \arrow[d, "h_{f\circ g}-h_f\circ h_g"] \arrow[dl, dashed, "\phi"'] & & A \arrow[d, "f\circ g-f\circ g=0"] \\
\Omega(C) \arrow[r, hook] & P_C \arrow[rr, two heads] & & C
\end{tikzcd}$$

But this implies that $\pi_C \circ (h_{f\circ g} - h_f \circ h_g) = 0$, which inducec a map by the kernel property $\phi : P_A \to \Omega(C)$. Such that the lower triangle commutes. And since, ι_C is a monomorphism, one gets that the upper triangle also commutes. And therefore $\Omega(f \circ g) - \Omega(f) \circ \Omega(g)$ factors through a projective, and therefore $\Omega(f \circ g) \sim \Omega(f) \circ \Omega(g)$.

Second, need to show that $\Omega(\mathrm{Id}_A) = \mathrm{Id}_{\Omega(A)}$ in $\mathcal{T}$.

By the same argument like above, one can see that every square and triangle in the following diagram also commutes:

$$\begin{tikzcd}
\Omega(A) \arrow[r, hook] \arrow[d, "\Omega(Id_A)-Id_{\Omega(A)}"'] & P_A \arrow[r, two heads] \arrow[d, "h_{Id_A}-Id_{P_A}"] \arrow[dl, dashed, "\phi"'] & A \arrow[d, "Id_A-Id_A=0"] \\
\Omega(A) \arrow[r, hook] & P_A \arrow[r, two heads] & A
\end{tikzcd}$$

And therefore $\Omega(\mathrm{Id}_A) \sim Id_{\Omega(A)}$.

□

Lemma 0.6. *Let $A, B \in \mathcal{T}$, then for $f, g \in \mathrm{Hom}_{\mathcal{T}}(A, B)$, one has that $\Omega(f+g) = \Omega(f)+\Omega(g)$ in $\mathcal{T}$. I.e. Ω is additive.*

Proof. Want to show that $\Omega(f+g) \sim \Omega(f)+\Omega(g)$.

One has that for any morphisms $f, g \in \mathrm{Hom}_{\mathcal{T}}(A, B)$, from the definition of $\mathcal{T}$, that $f = g$ in $\mathcal{T}$ if $f - g$ factors through a projective.

With that in mind, look at the following diagram:

$$\begin{tikzcd}
\Omega(A) \arrow[r, hook, "\iota_A"] \arrow[d, "\Omega(f+g)-\Omega(f)-\Omega(g)"'] & P_A \arrow[r, two heads, "\pi_A"] \arrow[d, "h_{f+g}-h_f-h_g"] \arrow[dl, dashed, "\phi"'] & A \arrow[d, "f+g-f-g=0"] \\
\Omega(B) \arrow[r, hook, "\iota_B"] & P_B \arrow[r, two heads, "\pi_B"] & B
\end{tikzcd}$$

Starting from the leftmost side, want to show that $\Omega(f+g) - \Omega(f) - \Omega(g)$ factors through a projective.

Firstly, one can observe that $\iota_B \circ (\Omega(f+g) - \Omega(f) - \Omega(g)) = \iota_B \circ \Omega(f+g) - \iota_B \circ \Omega(f) - \iota_B \circ \Omega(g) = h_{f+g} \circ \iota_A - h_f \circ \iota_A - h_g \circ \iota_A = (h_{f+g} - h_f - h_g) \circ \iota_A$. So the map $h_{f+g} - h_f - h_g : P_A \to P_B$ makes the left square commute.

Secondly, one can see that $\pi_B \circ (h_{f+g} - h_f - h_g) = \pi_B \circ h_{f+g} - \pi_B \circ h_f - \pi_B \circ h_g = (f+g) \circ \pi_A - f \circ \pi_A - g \circ \pi_A = (f+g-f-g) \circ \pi_A = 0 \circ \pi_A = 0$.

But then from the kernel property there is an induced and unique map $\phi : P_A \to \Omega(B)$ such that $\iota_B \circ \phi = h_{f+g} - h_f - h_g$. But from the commutativity of the left square, one has that $\iota_B \circ (\Omega(f+g) - \Omega(f) - \Omega(g)) = \iota_B \circ \phi \circ \iota_A$. Furthermore, since ι_A is a monomorphism, one gets that $\Omega(f+g) - \Omega(f) - \Omega(g) = \phi \circ \iota_A$.

But that implies that $\Omega(f+g) - \Omega(f) - \Omega(g)$ factors thorugh a projective, and therefore $\Omega(f+g) \sim \Omega(f) + \Omega(g)$. □

Lemma 0.7. *The definition of Ω is well defined.*

Proof. □

Teorem 0.8. *Let Σ be as above. Let Δ TODO*

Then $\mathrm{StMod}(R)$ *is a triangulated category with Σ as the suspension and Δ as the distinguished triangles.*

Proof. TODO $\square$