№ 2

$$T^{\mu}_{\ \nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial\left(\sqrt{|g|}\,\mathcal{L}\right)}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial\left(\sqrt{|g|}\,\mathcal{L}\right)}{\partial g_{\mu\nu,\lambda}} + \dots\right)$$

Нам понадобится соотношение из лин. алгебры о диф-нии симметр матрицы:

$$\frac{\partial}{\partial x}|G| = |G|\,\mathrm{tr}\left(G^{-1}\frac{\partial G}{\partial x}\right)$$

1). $\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} = -\frac{1}{4}g^{\alpha\eta}g^{\beta\sigma}F_{\eta\sigma}F_{\alpha\beta}$.

$$T^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial\sqrt{|g|}}{\partial g_{\mu\nu}}\mathcal{L} + \sqrt{|g|}\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\right) = -\frac{2}{\sqrt{|g|}}\left(\frac{|g|}{2\sqrt{|g|}}g^{\rho\varkappa}\frac{\partial g_{\rho\varkappa}}{\partial g_{\mu\nu}} \ast\right.$$

$$\ast\ \mathcal{L} - \frac{\sqrt{|g|}}{4}\left[\frac{\partial g^{\alpha\eta}}{\partial g_{\mu\nu}}g^{\beta\sigma}F_{\eta\sigma}F_{\alpha\beta} + g^{\alpha\eta}\frac{\partial g^{\beta\sigma}}{\partial g_{\mu\nu}}F_{\eta\sigma}F_{\alpha\beta}\right] =$$

$$= \Big/\!\Big/\ \frac{\partial g_{\rho\varkappa}}{\partial g_{\mu\nu}} = \delta^{\rho}_{\mu}\cdot\delta^{\varkappa}_{\nu}\,;\quad \frac{\partial\left(g^{\alpha\eta}g_{\alpha\varkappa}\right)}{\partial g_{\mu\nu}} = \frac{\partial g^{\alpha\eta}}{\partial g_{\mu\nu}}g_{\alpha\varkappa} + g^{\alpha\eta}\frac{\partial g_{\alpha\varkappa}}{\partial g_{\mu\nu}} = 0 \Rightarrow$$

$$\Rightarrow g_{\alpha\varkappa}\frac{\partial g^{\alpha\eta}}{\partial g_{\mu\nu}} = -g^{\alpha\eta}\delta^{\alpha}_{\mu}\delta^{\varkappa}_{\nu} = -g^{\mu\eta}\delta^{\varkappa}_{\nu} \Rightarrow \frac{\partial g^{\alpha\eta}}{\partial g_{\mu\nu}} = -g^{\mu\eta}g^{\alpha\nu}\ \Big/\!\Big/ =$$

$$= \frac{1}{4}g^{\mu\nu}g^{\alpha\eta}g^{\beta\sigma}F_{\eta\sigma}F_{\alpha\beta} - \frac{1}{2}g^{\alpha\mu}g^{\eta\nu}g^{\beta\sigma}F_{\eta\sigma}F_{\alpha\beta} - \frac{1}{2}g^{\alpha\eta}g^{\beta\mu}g^{\sigma\nu}F_{\eta\sigma}F_{\alpha\beta} =$$

$$= \frac{1}{4}\left(g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - 2g^{\alpha\mu}F^{\nu\beta}F_{\alpha\beta} - 2g^{\beta\mu}F^{\alpha\nu}F_{\alpha\beta}\right) =$$

$$= \frac{1}{4}\left(g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - 2F^{\nu\beta}F^{\mu}_{\ \beta} - 2\underbrace{F^{\alpha\nu}F_{\alpha}^{\ \mu}}_{=F^{\nu\alpha}F^{\mu}_{\ \alpha}}\right) =$$

$$= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\nu\alpha}F^{\mu}_{\ \alpha}.$$

2)

$$2).\quad \mathcal{L} = \frac{1}{2} g^{\mu\nu} G_{ab}(\phi)\,\partial_\mu\phi^a\,\partial_\nu\phi^b - U(\phi).$$

$$T^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial\sqrt{|g|}}{\partial g_{\mu\nu}}\mathcal{L} + \sqrt{|g|}\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\right)$$

нашли выше. найдено выше

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = \frac{1}{2}\partial_\eta\phi^\alpha\partial_\varkappa\phi^\beta G_{\alpha\beta}\frac{\partial g^{\eta\varkappa}}{\partial g_{\mu\nu}} = -\frac{G_{\alpha\beta}}{2}\partial_\eta\phi^\alpha\partial_\varkappa\phi^\beta g^{\mu\varkappa}g^{\nu\eta} \Rightarrow$$

$$\Rightarrow T^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left[\frac{|g|}{2\sqrt{|g|}}g^{\delta\varkappa}\frac{\partial g_{\delta\varkappa}}{\partial g_{\mu\nu}}\left(\frac{1}{2}g^{\eta\delta}G_{\alpha\beta}\partial_\eta\phi^\alpha\partial_\delta\phi^\beta - U(\phi)\right) + \right.$$

$$\left. + \sqrt{|g|}\left(-\frac{1}{2}\right)G_{\alpha\beta}\partial_\eta\phi^\alpha\partial_\varkappa\phi^\beta g^{\mu\varkappa}g^{\nu\eta}\right] = G_{\alpha\beta}\partial^\nu\phi^\alpha\partial^\mu\phi^\beta -$$

$$- g^{\mu\nu}\left(g^{\eta\varkappa}\frac{G_{\alpha\beta}}{2}\partial_\eta\phi^\alpha\partial_\varkappa\phi^\beta - U(\phi)\right)$$

№ 3

$$\Delta^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\lambda\left(\sqrt{|g|}\,\psi^{\mu\nu\lambda}\right),\quad \psi^{\mu\nu\lambda} \text{ — антисим.}$$

$$\nabla_\nu\left(\frac{1}{\sqrt{|g|}}\partial_\lambda\left(\sqrt{|g|}\,\psi^{\mu\nu\lambda}\right)\right) = \nabla_\nu\left(\partial_\lambda\log\sqrt{|g|}\,\psi^{\mu\nu\lambda} + \partial_\lambda\psi^{\mu\nu\lambda}\right) = \left\{\begin{array}{l}\text{из лекции 3}\\ \text{известно:}\\ \partial_\lambda\log\sqrt{|g|} = \Gamma^\mu_{\lambda\mu}\end{array}\right\} =$$

$$= \nabla_\nu\Gamma^\varkappa_{\lambda\varkappa}\psi^{\mu\nu\lambda} + \nabla_\nu\partial_\lambda\psi^{\mu\nu\lambda} = \partial_\nu\left(\Gamma^\varkappa_{\lambda\varkappa}\psi^{\mu\nu\lambda}\right) + \cancelto{0}{\Gamma^\mu_{\varkappa\nu}\Gamma^\varkappa_{\lambda\varkappa}\psi^{\varkappa\nu\lambda}} +$$

$$+ \cancelto{0}{\partial_\nu\partial_\lambda\psi^{\mu\nu\lambda}} + \cancelto{0}{\Gamma^\mu_{\varkappa\nu}\partial_\lambda\psi^{\varkappa\nu\lambda}} + \Gamma^\nu_{\varkappa\nu}\partial_\lambda\psi^{\mu\varkappa\lambda} = \partial_\nu\left(\Gamma^\varkappa_{\lambda\varkappa}\right)\psi^{\mu\nu\lambda} +$$

$$+ \left(\Gamma^\varkappa_{\lambda\varkappa}\partial_\nu + \Gamma^\varkappa_{\nu\varkappa}\partial_\lambda\right)\psi^{\mu\nu\lambda} = \left\{\begin{array}{l}\partial_\nu\partial_\lambda\log\sqrt{|g|} =\\ = \partial_\lambda\partial_\nu\log\sqrt{|g|} =\\ = \partial_\lambda(\Gamma^\varkappa_{\nu\varkappa})\end{array}\right\} \overset{\Rightarrow 0}{\text{сим.}} = \left(\Gamma^\varkappa_{\lambda\varkappa}\partial_\nu + \Gamma^\varkappa_{\nu\varkappa}\partial_\lambda\right)\psi^{\mu\nu\lambda} =$$

$= 0$ // все зачеркнутые слагаемые — свертки антисиммет. тензора и симметр. //

ч.т.д.

Док-во:

$$\frac{\partial}{\partial x}|G| = |G|\, tr\left(G^{-1}\frac{\partial G}{\partial x}\right)$$

G – приводится к диаг виду ортогональными преобразованиями. Посмотрим, что при переходе от G к D равенство остается.

$D = AGA^{-1}$, A – матрица перехода.

$$tr\left(G^{-1}\frac{\partial G}{\partial x}\right) \mapsto tr\left(A^{-1}D^{-1}A\frac{\partial(A^{-1}DA)}{\partial x}\right) = tr\left(A^{-1}D^{-1}A\left(\frac{\partial A^{-1}}{\partial x}DA + \right.\right.$$

$$\left.\left. + A^{-1}\frac{\partial}{\partial x}DA + A^{-1}D\frac{\partial A}{\partial x}\right)\right) = \Big/\!\!\Big/ \quad AA^{-1} = \mathbb{1},\ \frac{\partial A}{\partial x}A^{-1} + A\frac{\partial A^{-1}}{\partial x} = 0 \Rightarrow \Rightarrow \frac{\partial A^{-1}}{\partial x} = -A^{-1}\frac{\partial A}{\partial x}A^{-1} \Big/\!\!\Big/ =$$

$$= tr\left[A^{-1}D^{-1}A\left(-A^{-1}\frac{\partial A}{\partial x}A^{-1}DA + A^{-1}\frac{\partial D}{\partial x}A + A^{-1}D\frac{\partial A}{\partial x}\right)\right] = -tr\left(D^{-1}\frac{\partial A}{\partial x}A^{-1}D\right) +$$

$$+ tr\left(D^{-1}\frac{\partial D}{\partial x}\right) + tr\left(A^{-1}\frac{\partial A}{\partial x}\right) = tr\left(D^{-1}\frac{\partial D}{\partial x}\right)$$

Докажем для диагональной G:

$$G = diag(d_1, \dots, d_n) \Rightarrow \frac{\partial}{\partial x}|G| = \frac{\partial}{\partial x}\prod_{i=1}^{n} d_i$$

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$$|G|\, tr\left(G^{-1}\frac{\partial}{\partial x}G\right) = \prod_{i=1}^{n} d_i \sum_{j=1}^{n}\frac{1}{d_j}\frac{\partial d_j}{\partial x}$$